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Solutions practice problems 1

If you think you've spotted a typo or mistake, please let us know!

Problem 1

There are 11 students in a class: 4 boys and 7 girls. We need to form a group of 5 people.

(a) *How many different groups can you make?*

That is simply $\binom{11}{5}$.

(b) *How many different groups are possible with at least 4 girls?*

There are $N_4 = \binom{7}{4} \times 4 = 140$ groups with exactly 4 girls and $N_5 = \binom{7}{5} = 21$ groups with exactly 5 girls. That gives $n_D = N_4 + N_5 = 161$ groups with at least 4 girls.

(c) *If you pick the group (uniformly) at random, what is the probability that there are at least 3 boys in the group?*

There are $N_3 = \binom{3}{4} \times \binom{2}{7} = 84$ groups with exactly 3 boys and $N_4 = 1 \times 7 = 7$ groups with exactly 4 boys, giving a total of $N_D = N_3 + N_4 = 91$ groups. The total number of 5-student groups is $N_T = \binom{11}{5} = 462$, hence the probability asked for is $p_D = {}^{N_D}/_{N_T} \approx 0.197$.

Problem 2: Words

(a) *How many 'words' of length 5 can you make using each letter of the alphabet at most once?*

There are 26 letters to use, which gives ${}_{26}P_5 = {}^{26!}/_{(26-5)!} = 26 \times 25 \times 24 \times 23 \times 22 = 7893600$ 'words'.

(b) *And how many if the order of the letters is irrelevant? (I.e., if we treat 'words' and 'sword' as the same word.)*

Now we're looking at *combinations* rather than *permutations*: ${}_{26}C_5 = \binom{26}{5} = 65780$.

(c) *And how many words can you make if you can use every letter as many times as you like?*

For each of the 5 positions you have 26 choices, so 26^5.

(d) *In how many unique ways can the letters in the word 'error' be arranged?*

First, there are $\binom{5}{3}$ ways to position the r's. Then you can place the e in one on the 2 remaining positions, and the o goes in the remaining spot. So $\binom{5}{3} \times 2 \times 1 = 20$ words.

(e) *Consider a word of n letters in which two letters occur more than once: p and q times respectively. How many unique 'words' of the same length can you make of the n letters?*

Let r be the number of letters that occur only once (so $n = p + q + r$) and let's call the reoccuring letters P and Q. We follow the argument of the previous exercise. First, we can position the P's in $\binom{n}{p}$ places, after which the Q's can be placed in $\binom{n-p}{q}$ slots. The remaining r letters can be ordered in $r!$ ways, so in total there are

$$N_T = \binom{n}{p} \times \binom{n-p}{q} \times r!$$

unique words we can make of these letters.

Simplify this expression suggests a simpler argument. Expanding the binomial coefficients and writing $r = n - p - q$ gives

$$N_T = \frac{n!}{p!(n-p)!} \times \frac{(n-p)!}{q!(n-p-q)!} \times (n-p-q)! = \frac{n!}{p!q!}$$

So a simpler argument would be something like: there are $n!$ possible orderings of n letters, but $p!$ of those are indistinguishable as they only shuffle P's and a further $q!$ are identical because they only shuffled Q's, hence $N_T = {}^{n!}/_{p!q!}$.

Problem 3: Books

You have 3 books on complexity theory, 2 on probability theory, and 1 novel.

(a) *In how many ways can the books be arranged?*

The books are distinguishable, so there are $6! = 720$ ways to order these 6 books.

(b) *And what if the books on complexity theory must be together but the other books can be arranged in any order?*

Now there are $3! = 6$ ways to order the complexity-group, and $4! = 24$ ways to order the complexity group and the remaining 3 books. This gives a total of $24 \times 6 = 144$ orderings.

Problem 4: Poker hands

Calculate the probability of drawing each of these poker hands.

(a) **Two-pair** *Two cards have one rank, two cards have another rank, and the remaining card has a third rank.*

First we choose two ranks in $\binom{13}{2}$ ways. For each of those ranks we choose 2 suits in $\binom{4}{2}^2$ ways. For the last card, we pick one of 11 ranks and one of 4 suits. In total, this gives

$$N_D = \binom{13}{2} \times \binom{4}{2}^2 \times 11 \times 4 = 123552$$

ways to draw a two-pair. Since there are $N_T = \binom{52}{5} = 2598960$ possible hands, the probability of drawing a two-pair is

$$p_D = \frac{N_D}{N_T} = \frac{123552}{2598960} \approx 0.0475$$

(b) **Three-of-a-kind** *Three cards have one rank and the remaining two cards have two other ranks.*

First we pick one of 13 ranks, and then three suits in $\binom{3}{4}$ ways. Next we pick two ranks for the remaining cards in one of $\binom{12}{2}$ ways and two suits in 4^2 ways. This gives

$$N_D = 13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2 = 54912$$

and as a result

$$p_D = \frac{N_D}{N_T} = \frac{54912}{2598960} \approx 0.0211$$