Master of Logic, University of Amsterdam, 2018 —
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Practice problem set 3

This week's exercises deal with discrete random variables. You do not have to hand these exercises in; they are optional and for practicing only. If you have questions about them, please post them to the discussion forum and try to help each other. We will also keep an eye on that.

Problem 1

Let X be a random variable taking any of the values $0, 5, -5$ with probabilities $P(X = -5) = P(X = 0) = 0.3$ and $P(X = 5) = 0.4$. What is $E[X^2]$?

Problem 2

Compute the pmf of the number of tails for 7 subsequent coin tosses.

Problem 3

Calculate the variance of a loaded 6-sided die that has a probability of $\frac{1}{6}$ for all odd numbered sides and $P(\{2\}) = P(\{4\}) = 0.1$ and $P(\{6\}) = 0.3$.

Problem 4: Independence

Three events A, B, and C are pairwise independent if each pair is independent. They are mutually independent if they are pairwise independent and in addition

$$P(A \cap B \cap C) = P(A)P(B)P(C)\,. \tag{1}$$

(a) Suppose we roll two 6-sided dice. Consider the events:

$$D = \text{'odd on die 1'} \quad E = \text{'odd on die 2'} \quad F = \text{'odd sum'}$$

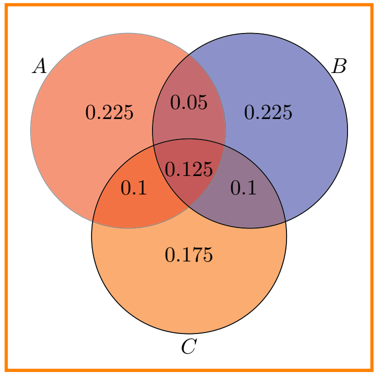

Figure 1: Venn diagram for problem 4 (b)

Are D, E, and F pairwise independent? Are they mutually independent?

(b) Consider the Venn diagram in figure **??**. A, B and C are the overlapping circles and the probabilities of each region are as marked. Does equation **??** hold? Are the events A, B, C mutually independent?

(c) For families with n children, the events "the family has children of both sexes" and "there is at most one girl" are independent. What is n?

Problem 5: Trees of cards

There are 8 cards in a hat:

$$\{1\heartsuit, 1\spadesuit, 1\diamondsuit, 1\clubsuit, 2\heartsuit, 2\spadesuit, 2\diamondsuit, 2\clubsuit\}.$$

You draw one card at random. If its rank is 1 you draw one more card; if its rank is two you draw two more cards. Let X be the sum of the ranks on all the cards you have drawn. Find $E(X)$.

Problem 6: Seating arrangements

A total of n people take their seats around a circular table with n chairs. No two people have the same height. What is the expected number of people who are shorter than both of their immediate neighbours?

Problem 7: Negative binomial distribution

Assume that a number of independent trials, each with a probability of success of p, $0 < p < 1$, are performed until q successes are registered. Let X be equal to the number of trials required, then

$$P(X = n) = \binom{n-1}{q-1} p^q (1-p)^{n-q} \qquad n = q, q+1, \ldots$$

Any RV X whose probability distribution is given by the above is said to be a *negative binomial RV* with parameters (q, p). Compute the expectation and variance of this RV with parameters (q, p).

CREDITS Some questions are from MIT course 18-05 by Jeremy Orloff and Jonathan Bloom, see ocw.mit.edu.