BASIC PROBABILITY: THEORY

Master of Logic, University of Amsterdam, 2018 —

TEACHERS Alexandre Cremers and Jakub Dotlačil TA Bas Cornelissen

Board questions set 7

Problem 1: The normal distribution

Recall that the normal distribution with parameters μ and σ has support over $\mathbb{R}$ and the pdf $f(x) = K \exp(\frac{-(x-\mu)^2}{2\sigma^2})$, where $K = (\sqrt{2\pi\sigma^2})^{-1}$ is the normalization constant. In this exercise we look at a normal random variable X for which $\sigma = 1$, so its density is

$$f_X(x) = K \cdot \exp\Big(-\frac{1}{2}(x-\mu)^2\Big), \qquad \text{where } K = \frac{1}{\sqrt{2\pi}}$$

(a) Show that when $\mu = 0$, then $\mathrm{E}[X] = 0$.

(b) Show that in general, $\mathrm{E}[X] = \mu$.
Hint: You can show that $\mathrm{E}[X-\mu] = 0$ *and use the linearity of* E.

In general, if we have n independent samples from a continuous random variable, we define the likelihood as:

$$f(x_1, \dots, x_n | \theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

(c) Now consider n independent Normal$(\mu, 1)$-distributed random variables. Show that the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a sufficient statistics for μ
Hints: The factorization theorem applies to the pdf. The following decomposition might help: $\sum(x_i - \mu)^2 = \sum\left[(x_i - \bar{x}) + (\bar{x} - \mu)\right]^2$

(d) Show that the MLE for μ is $\bar{x}$.

Cheat sheet:

$$\begin{aligned}
\mathrm{e}^{x+y} &= \mathrm{e}^x \mathrm{e}^y \\
\ln(\mathrm{e}^x) &= x \\
[(x-\mu)^2]' &= 2(x-\mu) \\
\int_a^b g'(x) \cdot \mathrm{e}^{g(x)}\, dx \;&=\; \left[\mathrm{e}^{g(x)}\right]_a^b \;=\; \mathrm{e}^{g(b)} - \mathrm{e}^{g(a)}
\end{aligned}$$

BASIC PROBABILITY: THEORY

Master of Logic, University of Amsterdam, 2018 — LICENSE CC BY-NC-SA 4.0

TEACHERS Alexandre Cremers and Jakub Dotlačil TA Bas Cornelissen

Board questions set 7

Problem 1: The normal distribution

Recall that the normal distribution with parameters μ and σ has support over $\mathbb{R}$ and the pdf $f(x) = K \exp(\frac{-(x-\mu)^2}{2\sigma^2})$, where $K = (\sqrt{2\pi\sigma^2})^{-1}$ is the normalization constant. In this exercise we look at a normal random variable X for which $\sigma = 1$, so its density is

$$f_X(x) = K \cdot \exp\Big(-\frac{1}{2}(x-\mu)^2\Big), \qquad \text{where } K = \frac{1}{\sqrt{2\pi}}$$

(a) Show that when $\mu = 0$, then $\mathrm{E}[X] = 0$.

(b) Show that in general, $\mathrm{E}[X] = \mu$.
Hint: You can show that $\mathrm{E}[X-\mu] = 0$ *and use the linearity of* E.

In general, if we have n independent samples from a continuous random variable, we define the likelihood as:

$$f(x_1, \dots, x_n | \theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

(c) Now consider n independent Normal$(\mu, 1)$-distributed random variables. Show that the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a sufficient statistics for μ
Hints: The factorization theorem applies to the pdf. The following decomposition might help: $\sum(x_i - \mu)^2 = \sum\left[(x_i - \bar{x}) + (\bar{x} - \mu)\right]^2$

(d) Show that the MLE for μ is $\bar{x}$.

Cheat sheet:

$$\begin{aligned}
\mathrm{e}^{x+y} &= \mathrm{e}^x \mathrm{e}^y \\
\ln(\mathrm{e}^x) &= x \\
[(x-\mu)^2]' &= 2(x-\mu) \\
\int_a^b g'(x) \cdot \mathrm{e}^{g(x)}\, dx \;&=\; \left[\mathrm{e}^{g(x)}\right]_a^b \;=\; \mathrm{e}^{g(b)} - \mathrm{e}^{g(a)}
\end{aligned}$$