Game theory

Competition is a watchword of modern life. A competitive situation exists when two or more opposing parties are making decisions involving conflicting interests and wherein the action of one depends on the action which the opponent takes. Each opponent acts in a rational manner and tries to resolve the conflict in his own favour. Such situations arise in business, military operations, etc. Game theory is used to handle such conflicting situations. It seeks to provide a rational course of action in a conflicting situation.

The term 'game' represents a conflict between two or more parties. There can be several types of games, e.g. two-person and n-person games, zero-sum and non-zero-sum games, constant-sum games, co-operative and non-co-operative games, pure strategy games and mixed strategy games, etc.

When there are two competitors playing a game, it is called a two-person game. If the number of competitors are N (N > 2), it is known as an N person game. When the sum of amounts won by all winners is equal to the sum of the amounts lost by all losers, we call it a zero-sum game. In a non-zero-game there exists a jointly preferred outcome. In other words, in a zero-game or a constant-sum game the sum of gains and losses of the game is zero. As opposed to this, if the sum of gains or losses is not equal to zero, we call it a non-zero-sum game. When the best strategy for each player is to play one particular strategy throughout the game, it is known as a pure strategy game. In case the optimum plan for each player is to employ different strategies at different times, it is called a mixed strategy game. When there is communication between the participants they may reach agreement and increase their pay-off through some forms of co-operative game, otherwise it is a non-co-operative game.

Game theory may be defined as "a body of knowledge that deals with making decisions when two or more intelligent and rational opponents are involved under conditions of conflict and competition". Instead of making inferences from the past behaviour of the opponent, "the approach of game theory is to try and determine a rival's most profitable counter-strategy to one's own 'best' moves and to formulate the appropriate defensive measures". For example, if two firms are locked up in a war to maintain their market share, then a price cut by the first firm will invite reaction from the second firm in the nature of a price cut. This will, in turn, affect the sales and profits of the first firm, which will again have to develop a counter-strategy to meet the challenge from the second firm. The game will thus go on. Game theory helps to determine the best course of action for a firm in view of the expected counter moves from the competitors. Game theory deals with competitive situations of decision-making under uncertainty.

<u>Rules of the Game</u>

Game theory is applicable to situations that satisfy the following conditions:

i. The number of competitors is finite.
ii. The players act rationally and intelligently.
iii. Each player has available to him a finite set of possible courses of action.
iv. There is a conflict of interests between the participants.
v. The players make individual decisions without directly communicating.

vi. The rules governing the choice are specified and known to the players.
vii. The players simultaneously select their respective courses of action.
viii. The payoff (outcome) is fixed and determined in advance.

Basic Terminology

1. Strategy
   A strategy for a player has been defined as a set of rules or alternative courses of action available to him in advance, by which player decides the courses of action that he should adopt. There are two types:
   a. Pure Strategy
      If the player selects the same strategy each time, then it is a pure strategy. In this case each player knows exactly what the other is going to do, i.e. there is a deterministic situation and the objective of the players is to maximize gains or to minimize losses.
   b. Mixed Strategy
      When the players use a combination of strategies and each player is always kept guessing as to which course of action is to be selected by the other, then it is known as a mixed strategy. Thus, there is a probabilistic situation and the objective of the player is to maximize expected gains or to minimize losses. Thus, mixed strategy is a selection among pure strategies with fixed possibilities.

2. Optimal Strategy
   A course of action which puts the player in the most preferred position irrespective of the strategy of his competitors. Any deviation from this strategy results in a decreased pay-off for the player.

3. Value of the Game
   The expected pay-off of the game when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair if it is non-zero.

4. Two-person zero-sum game
   There are two types of Two-person zero-sum games. In one, the most preferred position is achieved by adopting a single strategy and therefore the game is known as the pure strategy game. The second type requires the adoption by both players of a combination of different strategies in order to achieve the most preferred position and is, therefore, referred to as the mixed strategy game.

5. Pay-off matrix
   A two-person zero-sum game is conveniently represented by a matrix. The matrix which shows the outcome of the game as the players select their particular strategies, is known as the pay-off matrix. It is important to assume

that each player knows not only his own list of possible courses of action but also that of his opponent.

Let player A have m courses of action ($A_1, A_2, A_3, \ldots, A_m$) and player B have n courses of action ($B_1, B_2, B_3, \ldots, B_n$). The numbers m and n need not to be equal. The possible number of outcomes is therefore (mxn). These outcomes are shown in the following matrices:

A's pay-off matrix

		Player B		
		B_1	B_2	B_n
	A_1	a_{11}	a_{12}	a_{1n}
Player A	A_2	a_{21}	a_{22}	a_{2n}
	.	.	.	.
	.	.	.	.
	.	.	.	.
	A_m	a_{m1}	a_{m2}	a_{mn}

B's pay-off matrix

		Player B		
		B_1	B_2	B_n
	A_1	$-a_{11}$	$-a_{12}$	$-a_{1n}$
Player A	A_2	$-a_{21}$	$-a_{22}$	$-a_{2n}$
	.	.	.	.
	.	.	.	.
	.	.	.	.
	A_m	$-a_{m1}$	$-a_{m2}$	$-a_{mn}$

<u>Pure strategies (games with a saddle point)</u>

The minimum value in each row represents the least gain (pay-off) guaranteed to player A, if he plays his particular strategy. These are indicated in the matrix by row-minima. Player A will then select the strategy that maximizes the minimum gains. Player A's selection is called the maximum strategy and his corresponding game is called the maximum value of the game.

Player B on the other hand, would like to minimize his losses. The maximum in each column represents the maximum losses to player B if he plays his particular strategy. These are indicate in the matrix by column maxima. Player B will then select the strategy that minimizes his maximum losses. Player B's selection is called the minimax strategy and his corresponding loss is called the minimax value of the matrix.

If the maximin value equals the minimax value, then the game is said to have saddle point and the corresponding strategies are called optimum strategies. The amount of pay-off at the equilibrium point is known as the value of the game. It may be noted that if player A adopts minimax criterion, then player B has adopted maximin criterion, as it is a two-person zero-sum game.

Note: A game may have more than one saddle point. A game with no saddle point is solved by employing mixed strategies.

Determination of the Saddle Point

Follow the following three steps:

1. Select the minimum element in each row of the pay-off matrix and write them under 'row minima' heading. Then select a largest element among these elements and enclose it in a rectangle.
2. Select the maximum element in each column of the pay-off matrix and write them under 'column maxima' heading. Then select a lowest element among these elements and enclose it in a circle.
3. Find the element(s) which is the same in the circle as well as the rectangle and mark the position of such element(s) in the matrix. This element represent the value of the game and is called the saddle (or equilibrium) point.

Rules of (Principles) Dominance

The concept of dominance can be applied to any two-person zero-sum game with any number of strategies for each player. For a pay-off matrix of large size, the rule of dominance can be used to reduce its size by carefully eliminating certain rows and columns prior to the final analysis to determine the optimum strategy selection for each player. In general the following rules of dominance are used to reduce the size of the matrix.

Rule 1

If all the element in a row (say the i_{th} row) of a pay-off matrix are less than or equal to the corresponding elements of the other row (say the j_{th} row), then the player A will never choose the i_{th} strategy or in other words, the i_{th} strategy is dominated by the j_{th} strategy.

Rule 2

If all the elements in a column (say the r_{th} column) of a pay-off matrix are greater than or equal to the corresponding elements of the other column (say the s_{th} column), then the player B will never choose the r_{th} strategy or in other words, the r_{th} strategy is dominated by the s_{th} strategy.

Rule 3

A pure strategy may be dominated if it is inferior to the average of two or more other pure strategies.

Solution methods of games without saddle points

In certain cases when there is no pure strategy solution for games, i.e. no saddle point exists, both the players must determine an optimum mixture of strategies to find a saddle point. The optimum strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. Since strategies so determined are probabilistic combinations of available choices of strategy, they are mixed strategies. A mixed strategy can be solved by the following methods:

Arithmetic Method

For a zero-sum two-person game in which each of the players, say A and B has strategies A_1 & A_2 and B_1 & B_2 respectively and the pay-offs as given below, then if p_1 is the probability with which A chooses strategy A_1 and if q_1 is the probability that B plays strategy B_1, the pay-off matrix for player A is given by

		Player B's strategy	
		B_1	B_2
Player A's strategy	A_1	a_{11}	a_{12}
	A_2	a_{21}	a_{22}

The following formulae are used to find the value of the game and the optimum strategies.

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \qquad p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \qquad q_2 = 1 - q_1$$

and the value of the game is

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

Graphical Method for 2xn or mx2 games

Since the optimum strategies for both the players assign non-zero probabilities to the same number of pure strategies, it is obvious that if one player has only two strategies, the other will also use two strategies. The graphical method is helpful in finding out which of the two strategies can be used. It is also useful if the nature of the game is of the form 2xn or mx2. The graphical method consists of two graphs:

i. The pay-off (gains) available to player A versus his strategy options, and
ii. The pay-off (losses) faced by the player B versus his strategy options.

Consider the following 2xn pay-off matrix of a game without a saddle point

		Player B				Probability
		B_1	B_2	…	B_n	
Player A's strategy	A_1	a_{11}	a_{12}	…	a_{1n}	p_1
	A_2	a_{21}	a_{22}	…	a_{2n}	p_2
Probability		q_1	q_2	…	q_n	

Player A has two strategies, A_1 and A_2 with probabilities of their selection p_1 and p_2 respectively, such that $p_1 + p_2 = 1$ and $p_1, p_2 \geq 0$. Now, for each of the pure strategies available to player B, expected pay-off for player A is given below:

B's pure strategies	A's expected pay-off
B_1	$a_{11} p_1 + a_{21} p_2 = (a_{11} - a_{21}) p_1 + a_{21}$
B_2	$a_{12} p_1 + a_{22} p_2$
B_n	$a_{1n} p_1 + a_{2n} p_2$

Thus, A's expected pay-off varies linearly with p_1.

To plot the expected pay-off lines, we draw two parallel lines one unit apart and make a scale on each of them. These two lines represent the two strategies available to player A. Then we draw lines to represent each of B's strategies. To represent B's first strategy (B_1), we join a_{11} on scale 1 to a_{21} on scale 2. This line will represent the expected pay-off of the line $E_1 = (a_{11} - a_{21}) p_1 + a_{21}$ with p_1 as x-axis and E_1 as y-axis. Similarly, other pay-off lines can be drawn.

The lower boundary of these lines will give minimum expected pay-off and the highest point on this lower boundary will then give the maximum expected pay-off of player A and hence the optimum value of p1. The two optimal strategies for B are then given by the two lines which pass through this maximum point. Thus, the 2xn game is reduced to a 2x2 game which can easily be solved by any of the methods described earlier.

The 2xm games are treated in the same way except that, the minimax point is the lowest point on the upper boundary of the straight lines (instead of the highest point on the lower boundary).

Algebraic Method (Linear Programming)

It is sometimes difficult to solve a game problem with an mxn pay-off matrix having neither a saddle point nor any dominance relation in rows or columns. Game problems in such cases can be formulated as a linear programming problem. Consider a game where the pay-off matrix is $[a_{ij}]_{mxn}$. Let $(p_1, p_2, \ldots, p_m)$ and $(q_1, q_2, \ldots, q_n)$ be the probabilities with which players A and B adopt their mixed strategies $(A_1, A_2, \ldots, A_m)$ and $(B_1, B_2, \ldots, B_n)$ respectively. If V is the value of the game, then expected gain to player A for this game when player B selects strategies $B_1, B_2, \ldots, B_n$ one by one, is given by the left hand side of the simultaneous equations respectively.

Since player A is the gainer player and expects at least V, we must have

$$a_{11}p_1 + a_{21}p_2 + ... + a_{m1}p_m \geq V$$
$$a_{12}p_1 + a_{22}p_2 + ... + a_{m2}p_m \geq V$$

$$a_{1n}p_1 + a_{2n}p_2 + ... + a_{mn}p_m \geq V$$

Where, $p_1 + p_2 + \ldots + p_m = 1$, and $p_i \geq 0$ for all i.

Similarly, the expected loss to player B when player A adopts strategies $A_1, A_2, \ldots, A_m$ can be determined. Since player B is the loser player, he must have

$$a_{11}q_1 + a_{12}q_2 + ... + a_{1n}q_n \leq V$$
$$a_{21}q_1 + a_{22}q_2 + ... + a_{2n}q_n \leq V$$

$$a_{m1}q_1 + a_{m2}q_2 + ... + a_{mn}q_n \leq V$$

Where, $q_1 + q_2 + \ldots + q_n = 1$, and $q_i \geq 0$ for all i.

Since the objective of player A is to maximize the value of the game V, which is equivalent to minimizing $\frac{1}{V}$. The resulting linear programming problem can be stated as:

$$\textit{Minimize } Z_p = \frac{1}{V} = x_1 + x_2 + ... + x_m \qquad \text{where } \frac{p_i}{V} = x_i$$

Subject to constraints

$$a_{11}x_1 + a_{21}x_2 + ... + a_{m1}x_m \geq 1$$
$$a_{12}x_1 + a_{22}x_2 + ... + a_{m2}x_m \geq 1$$

$$a_{1n}x_1 + a_{2n}x_2 + ... + a_{mn}x_m \geq 1$$

where $x_i = \frac{p_i}{V} \geq 0; x_1, x_2, ..., x_m \geq 0$

Similarly, player B has a similar problem with the inequalities of the constraints reversed, i.e. minimizes the expected loss. Since minimizing of V is equivalent to maximizing $\frac{1}{V}$, the resulting linear programming problem can be stated as

$$\textit{Maximize } Z_q = \frac{1}{V} = y_1 + y_2 + ... + y_n$$

Subject to constraints

$$a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n \leq 1$$
$$a_{21}y_1 + a_{22}y_2 + ... + a_{2n}y_n \leq 1$$

$$a_{m1}y_1 + a_{m2}y_2 + ... + a_{mn}y_n \leq 1$$

where $y_j = \frac{q_j}{V} \geq 0; y_1, y_2, ..., y_n \geq 0$

Note: It may be noted that the LPP for player B is the dual of the LPP of player A. Thus the solution of the dual problem can be from the primal simplex table. Since the value of the objective function for both the players is the same, i.e. $z_p = z_q$, the expected gain to player A in the game will be exactly equal to the expected loss of player B.

Examples

Example 1

A company management and the labour union are negotiating a new 3 year settlement. Each of these have 4 strategies:

i. Hard and aggressive bargaining
ii. Reasoning and logical approach
iii. Legalistic strategy
iv. Conciliatory approach

What strategies will the two sides adopt? Also determine the value of the game.

		Company strategy			
		i	ii	iii	iv
Union strategy	*i*	20	15	12	35
	ii	25	14	8	10
	iii	40	2	10	5
	iv	−5	4	11	0

Solution 1

We first determine the saddle point by encircling each row minima and putting squares around each column maxima. Thus, we obtain the saddle point which is enclosed by a circle and a square both:

		i	ii	iii	iv	*Minima*
	i	20	15	12	35	12
	ii	25	14	8	10	8
	iii	40	2	10	5	2
	iv	−5	4	11	0	−5
Maxima		40	15	12	35	

maxima = minimax = V = 12

The company incurs costs and hence its strategy is minimax, i.e. minimizing maximum losses. For the union, negotiations result in gain. Hence, the union strategy will be maximin, i.e. maximizing the minimum gain.

Since there exists a saddle point, strategies are pure and the strategies adopted are:

- The company will always adopt strategy iii (Legalistic strategy)
- The union will always adopt strategy i (hard and aggressive bargaining)

Example 2

Two competitive manufacturers are producing a new toy under license from a patent holder. In order to meet the demand they have the option of running the plant for 8, 16 or 24 hours a day. As the length of production increases so does the cost. One of the manufacturers, say A, has set up the matrix given below, in which he estimates the percentage of the market that he could capture and maintain the different schedules:

s_1 = 8 hours for manufacturer A
s_2 = 16 hours for manufacturer A
s_3 = 24 hours for manufacturer A
c_1 = 8 hours for manufacturer B
c_2 = 16 hours for manufacturer B
c_3 = 24 hours for manufacturer B

		Manufacturer B		
		c_1	c_2	c_3
	s_1	60%	56%	34%
Manufacturer A	s_2	63%	60%	55%
	s_3	83%	72%	60%

i. At which level should each produce?
ii. What percentage of the market will B have?

Solution 2

	c_1	c_2	c_3	Minima
s_1	60	56	34	34
s_2	63	60	55	55
s_3	83	72	60	60
Maxima	83	72	60	

The saddle point occurs at the s_3/c_3 position. Thus, a maximin/minimax solution is obtained:

- Both A and B should produce at the level of 24 hours per day.
- At the optimum level of 24 hours, B will have 60% of the market and A will have 40%.

Example 3

What is the optimum strategy of the game described by the matrix?

$$\begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix}$$

Solution 3

Suppose the given game is played by players A and B.

		Player B			
		i	ii	iii	iv
	i	-5	3	1	20
Player A	ii	5	5	4	6
	iii	-4	-2	0	-5

The 3rd row dominated by the 2nd row and the reduced matrix looks as follows:

	i	ii	iii	iv
i	-5	3	1	20
ii	5	5	4	6

The 2nd and 4th columns are dominated by the 1st column and the reduced pay-off matrix looks as follows:

	i	iii	Minima
i	-5	1	-5
ii	5	4	4
Maxima	5	4	

From this table, it is apparent that the value of the maximum coincides with the value of the minimax. Therefore a saddle point exist. The solution of the game is:

- The optimum strategy for player A is ii
- The optimum strategy for player B is iii
- The value of the game is 4

Example 4

Explain the principle of dominance in game theory and solve the game

$$\begin{bmatrix} 8 & 10 & 9 & 14 \\ 10 & 11 & 8 & 12 \\ 13 & 12 & 14 & 13 \end{bmatrix}$$

Solution 4

	B_1	B_2	B_3	B_4
A_1	8	10	9	14
A_2	10	11	8	12
A_3	13	12	14	13

In the above game, B_4 is dominated by B_1. Eliminating B_4 the reduced matrix is:

	B_1	B_2	B_3
A_1	8	10	9
A_2	10	11	8
A_3	13	12	14

A_1 is dominated by A_3, so eliminating A_1 reduces the matrix to:

	B_1	B_2	B_3
A_2	10	11	8
A_3	13	12	14

Now we find that the average of B_2 and B_3 dominates B_1.

	B_1	$\frac{B_2 + B_3}{2}$
A_2	10	9.5
A_3	13	13

Eliminating B_1 reduces the matrix to:

	B_2	B_3	Minima
A_2	11	8	8
A_3	12	14	12
Maxima	12	14	

A_3/ B_2 is the saddle point. Hence, the value of the game is 12 as it represents the best pay-off for both players.

Example 5

Given the pay-off matrix for player A, obtain the optimum strategies for both players and determine the value of the game.

$$\begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & 4 \end{bmatrix}$$

Solution 5

Method 1

	B_1	B_2	B_3
A_1	6	-3	7
A_2	-3	0	4

When A chooses strategy A_1 or A_2, B will never go to strategy B_3. Hence, strategy B_3 is redundant.

	B_1	B_2	
A_1	6	-3	-3
A_2	-3	0	-3
	6	0	

Minimax = 0 Maximin = -3
Hence, this is not a pure strategy with a saddle point.

Let the probability of mixed strategies of A for choosing A_1 and A_2 be p_1 and $1 – p_1$ respectively.

$$6p_1 - 3(1 – p_1) = -3p_1 + 0(1 – p_1)$$

or $p_1 = 0.25$

Again, q_1 and $1 – q_1$ being probabilities of strategy B, we get:

$$6q_1 - 3(1 – q_1) = -3q_1 + 0(1 – q_1)$$

or $q_1 = 0.25$

Hence, optimum strategies for player A and B will be as follows:

$$S_A = \left[\frac{1}{4}, \frac{3}{4}\right]; \quad p_2 = 1 - p_1 = \frac{3}{4}$$

$$S_B = \left[\frac{1}{4}, \frac{3}{4}, 0\right]; \quad q_2 = 1 - q_1 = \frac{3}{4}$$

The expected value of the game = $q_1\{6p_1 - 3(1-p_1)\} + (1-q_1)\{-3p_1 + 0(1-p_1)\} = \frac{3}{4}$

Method 2

For player A

$$p_1 = \frac{0 - (-3)}{(6+0) - (-3-3)} = \frac{1}{4} \quad \text{and } p_2 = 1 - p_1 = \frac{3}{4}$$

For player B

$$q_1 = \frac{0 - (-3)}{(6+0) - (-3-3)} = \frac{1}{4} \quad \text{and } q_2 = 1 - q_1 = \frac{3}{4}$$

The expected value of the game is:

$$V = \frac{0 - (-3)(-3)}{(6+0) - (-3-3)} = \frac{1}{2}$$

The Graphical Method

Example 6

Solve the following 2x3 game graphically

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{bmatrix} \end{array}$$

Solution 6

Since the problem does not have a saddle point, let player A play the mixed strategy:

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{against player B}, p_2 = 1 - p_1$$

Then A’s expected pay-off against B’s pure moves are given by:

B’s pure move	A’s expected pay-off
B_1	$E_1(p_1) = p_1 + 8(1-p_1) = -7p_1 + 8$
B_2	$E_2(p_1) = 3p_1 + 5(1-p_1) = -2p_1 + 5$
B_3	$E_3(p_1) = 11p_1 + 2(1-p_1) = 9p_1 + 2$

These expected pay-off equations are then plotted as functions of p_1, which shows the pay-offs of each column represented as points on two vertical axes 1 and 2 unit distance apart.

Since player A wishes to maximize his minimum expected pay-off, we consider the highest point of intersection on the lower envelope of A’s expected pay-off equations. This point, H, represents the maximum expected value of the game for A. The lines B_2 and B_3, passing through H, define the only relevant moves B_2 and B_3 that B will play. The solution to the original 2x3 game, therefore reduces to that of the simpler game with the 2x2 pay-off matrix.

	B_2	B_3
A_1	3	11
A_2	5	2

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11}+a_{22}) - (a_{21}+a_{12})} = \frac{3\times 2 - 11\times 5}{(3+2)-(11+5)} = \frac{49}{11}$$

n x n Games

It consists of the following steps:

1. Let A = (a_{ij}) be an nxn pay-off matrix. Obtain a new matrix C whose 1st column is obtained from A by subtracting its 2nd column from 1st, the 2nd column is obtained by subtracting A's 3rd column from the 2nd and so on till the last column of A has been taken care of. This C is an n x (n-1) matrix.
2. Obtain a new matrix R, from A by subtracting its successive rows from preceding ones, in exactly the same manner as was done for columns in step 1. Thus, R is an (n-1) x n matrix.
3. Determine the magnitude of oddments corresponding to each row and each column of A. the oddment to the ith row of A is defined as the determinant $|C_i|$, where C_i is obtained from C by deleting the i^{th} row. Similarly, oddment (j^{th} column of A) = $|R_j|$, where R_j is obtained by deleting its j^{th} column.
4. Write the magnitude of oddments (after ignoring negative signs, if any), against their respective rows and columns.
5. Check whether the sum of row oddments is equal to the sum of column oddments. If so, the oddment expresses as functions of the grand total yield the optimum strategies. If not, the method fails.

Example 7

Use the arithmetic method to solve the following 3x3 game.

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Solution 7

We compute the matrices C and R from the given pay-off matrix, and then obtain the row and column oddments.

$$C = \begin{bmatrix} -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}; \qquad R = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

The oddments are

$$C_1 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \qquad C_2 = \begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} = -3 \qquad C_3 = \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} = 3$$

$$R_1 = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \qquad \mathrm{R}_2 = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3 \qquad \mathrm{R}_3 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3$$

The augmented matrix is

$$\begin{array}{cc} & \text{Row oddments} \\ \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} & \begin{array}{c} 3 \\ 3 \\ 3 \end{array} \\ \textit{Column}\text{ oddments} \quad 3 \quad 3 \quad 3 & 9 \end{array}$$

Since the sum of the row and column oddments are both equal to 9, the optimum strategies are:

Row player: $\left(\frac{1}{3}, \ \frac{1}{3}, \ \frac{1}{3}\right)$

Column player: $\left(\frac{1}{3}, \ \frac{1}{3}, \ \frac{1}{3}\right)$

The value of the game = $\left(\frac{1}{3} \times 0\right) + \left(\frac{1}{3} \times 2\right) + \left(\frac{1}{3} \times 1\right) = 1$