# CSE574 Introduction to Machine Learning

## Support Vector Machine

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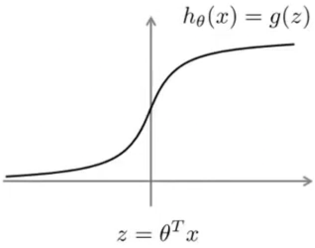
# Outline

# Alternative View of Logistic Regression



A quick review: $h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$

- if $y = 1$, we want $h_\theta(x) \approx 1$, $\theta^T x \gg 0$
- if $y = 0$, we want $h_\theta(x) \approx 0$, $\theta^T x \ll 0$

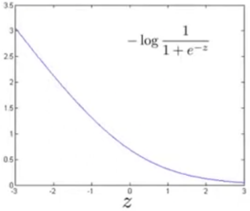



The cost of a single example:

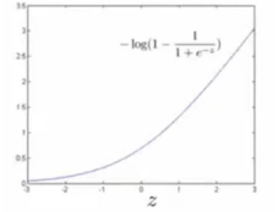
$$
\begin{aligned}
& -\left(y \log h_\theta(x) + (1-y)\log\left(1 - h_\theta(x)\right)\right) \\
= & -y \log \frac{1}{1+e^{-\theta^T x}} - (1-y)\log\left(1 - \frac{1}{1+e^{-\theta^T x}}\right)
\end{aligned}
$$

$$-y \log \frac{1}{1+e^{-\theta^T x}} - (1-y) \log \left(1 - \frac{1}{1+e^{-\theta^T x}}\right)$$

if $y = 1$ (want $\theta^T x \gg 0$)




if $y = 0$ (want $\theta^T x \ll 0$)



# Support Vector Machine

## Cost Function of Logistic Regression

$$\min_{\theta} \frac{1}{m}\left[\sum_{i=1}^{m} y^{(i)}\left(-\log h_{\theta}\left(x^{(i)}\right)\right) + \left(1-y^{(i)}\right)\left(-\log\left(1-h_{\theta}\left(x^{(i)}\right)\right)\right)\right] + \frac{\lambda}{2m}\sum_{j=1}^{n}\theta_j^2$$

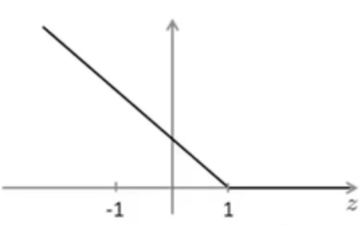
## Cost Function of Support Vector Machine

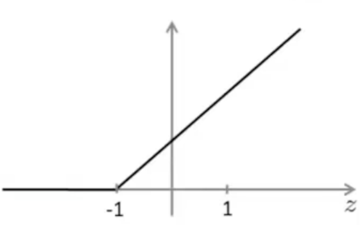
$$\min_{\theta} C\sum_{i=1}^{m}\left[y^{(i)}\operatorname{cost}_1\left(\theta^T x^{(i)}\right) + \left(1-y^{(i)}\right)\operatorname{cost}_0\left(\theta^T x^{(i)}\right)\right] + \frac{1}{2}\sum_{i=1}^{n}\theta_j^2$$

# Large Margin Intuition

**Support Vector Machine**

$$\min_{\theta} C \sum_{i=1}^{m} \left[ y^{(i)} \operatorname{cost}_1 \left( \theta^T x^{(i)} \right) + \left( 1 - y^{(i)} \right) \operatorname{cost}_0 \left( \theta^T x^{(i)} \right) \right] + \frac{1}{2} \sum_{i=1}^{n} \theta_j^2$$




If $y = 1$, we want $\theta^T x \geq 1$ (not just $\geq 0$)

If $y = 0$, we want $\theta^T x \leq -1$ (not just $< 0$)

# A very large $C$



**Support Vector Machine**

$$\min_{\theta} C \sum_{i=1}^{m} \left[ y^{(i)} \operatorname{cost}_1 \left( \theta^T x^{(i)} \right) + \left( 1 - y^{(i)} \right) \operatorname{cost}_0 \left( \theta^T x^{(i)} \right) \right] + \frac{1}{2} \sum_{i=1}^{n} \theta_j^2$$

Given that $C$ is a very large value, we want that the first term to be 0. Let's try to understand the optimization problem in the context of what would it take to make this first term in the objective equal to 0.

Whenever $y^{(i)} = 1, \theta^\top x^{(i)} \geqslant 1$; Whenever $y^{(i)} = 0, \theta^\top x^{(i)} \leqslant -1$

Now, the optimization problem can be written as:

$$\begin{aligned} &\min C \cdot 0 + \frac{1}{2} \sum_{i=1}^{n} \theta_j^2 \\ \text{s.t. } &\theta^\top x^{(i)} \geqslant 1 \quad \text{if } y^{(i)} = 1 \\ &\theta^T x^{(i)} \leqslant -1 \quad \text{if } y^{(i)} = 0 \end{aligned}$$

## SVM Decision Boundary: Linearly separable case

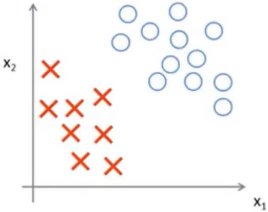
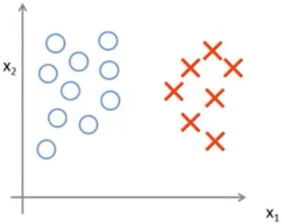
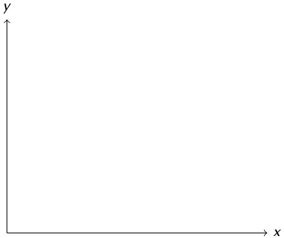


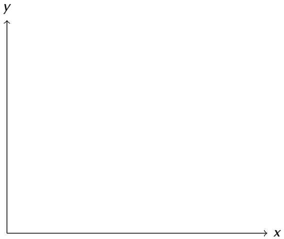

Figure: Linearly Separable Case

## Large margin classifier in presence of outliers



## Vector Inner Product



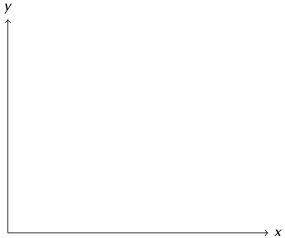

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

# SVM Decision Boundary

$$\min_{\theta} \quad \frac{1}{2}\sum_{j=1}^{n}\theta_j^2$$

$$\text{s.t.} \quad \theta^T x^{(i)} \geq 1 \quad \text{if } y^{(i)} = 1$$

$$\theta^T x^{(i)} \leq -1 \quad \text{if } y^{(i)} = 0$$



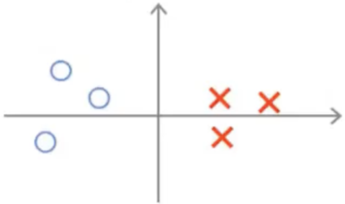
$$\min_{\theta} \quad \frac{1}{2}\sum_{j=1}^{n}\theta_j^2 = \frac{1}{2}\|\theta\|^2$$

$$\text{s.t.} \quad p^{(i)}\cdot\|\theta\| \geq 1 \quad \text{if } y^{(i)} = 1$$

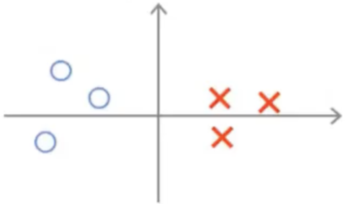
$$p^{(i)}\cdot\|\theta\| \leq -1 \quad \text{if } y^{(i)} = 0$$

where $p^{(i)}$ is the projection of $x^{(i)}$ onto the vector $\theta$. Simplification: $\theta_0 = 0$; this simplification merely makes the decision boundary to pass through (0,0);

**Bad Decision Boundary**



**Good Decision Boundary**

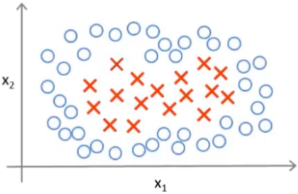


# Reading Assignments



1. Why the parameter vector orthogonal to the decision boundary?
   - Orthogonality in Neural Network
   - SVM

# Non-linear Decision Boundary




Predict $y = 1$ if

$$\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1 x_2 + \theta_4 x_1^2 + \theta_5 x_2^2 + \cdots \geq 0$$

$$h_0(x) = \begin{cases} 1 & \text{if } \theta_0 + \theta_1 x_1 + \cdots \geqslant 0 \\ 0 & \text{otherise.} \end{cases}$$

$$\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 + \cdots$$

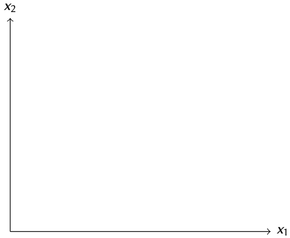
$$f_1 = x_1, \quad f_2 = x_2, \quad f_3 = x_1 x_2,$$

$$f_4 = x_1^2, \quad f_5 = x_2^2, \ldots$$

Is there a different/ better choice of the features $f_1, f_2, f_3, \ldots$?

# Kernel






Given $x$, compute new feature depending on proximity to landmarks $l^{(1)}, l^{(2)}, l^{(3)}$;

Given $x$:

- $f_1 = \text{similarity}(x, l^{(1)}) = \exp(-\frac{\left\|x - l^{(1)}\right\|^2}{2\sigma^2})$
- $f_2 = \text{similarity}(x, l^{(2)}) = \exp(-\frac{\left\|x - l^{(2)}\right\|^2}{2\sigma^2})$
- $f_3 = \text{similarity}(x, l^{(3)}) = \exp(\dots)$

Questions?