Lecture 9: Curves

Today: Curves

- Basics of shape representation
- Basics of curves
- Continuity conditions
- Polynomial pieces
- Cubics

Shapes (informally)

- a set of points (infinite)
- lives in a "space" - dimension of the points
  - a line segment can be in:
    - the plane (2D)
    - space (3D)
    - hyper-space (4D)
    - etc

Curves

- Intuition: set of points drawn with a "pen"
- "Most" points have 2 "neighbors" (next, previous)
  - endpoints
  - crossing
- mapping from time to place
  - $(x, y) = f(t)$ for $t \in [0, 1]$

Curves vs. Areas/Regions/Surfaces

Types of Curve Representations

- Implicit (test function)
  - $f(x, y) = 0$
- Parametric
  - $y = f(x)$
  - $x, y = f(t)$ - for some free parameter t
- Procedural
- Subdivision

Implicit Representations

A function that tests if a point is in the set

- $f(x, y) = 0$

- Easy for geometric tests
- Harder for drawing

Parametric Representations

Index the set with a *free parameter*

- $(x, y) = f(t)$

- easy to generate points - free parameter controls mapping

Same Points, Different Functions

$t \in [0, 1]$

$$f(t) = (t, 0)$$
$$f(t) = (1 - t, 0)$$
$$f(t) = (t^2, 0)$$

- different curves?
- different parameterizations of the same curve?

Mathematics defines curves 2 ways

- the image of a 1D interval
  it's the points!
- the mapping from a 1D interval to a space
  it's the function (mapping)

we'll try to be specific with what we mean if it matters

usually: *curve* is a set of points, *parameterization* is the mapping

A Circle

Implicit

$$x^2 + y^2 - 1 = 0$$

Parametric

$$x = \cos(2\pi t)$$

$$y = \sin(2\pi t)$$

$$t \in [0, 1]$$

Inside the Disc (area - not a curve)

Implicit

$$x^2 + y^2 < 1$$

Parametric

$$x = r\cos(2\pi t)$$

$$y = r\sin(2\pi t)$$

$$t \in [0, 1], r \in [0, 1]$$

Subdivision Representations

- Start with a set of points
- Have a rule that adds new points (possibly moving others)
- Repeat the rule to add more points

- repeat infinitely many times to get the curve
- design rules so it converges
- **limit curve** is what you get after infinite subdivisions

Toy Example

- Rule: insert a new point 1/2 way between
- Limit Curve: line segments

Parametric Forms

Assuming points $\vec{x}$ or $\mathbf{x}$

$$\mathbf{x} = \mathbf{f}(t)$$

For a curve:

- t is a scalar in some range
- $\mathbf{x}$ is a point (in 2D or 3D)
- $\mathbf{f}$ is a function $\mathbb{R} \rightarrow \mathbb{R}^2$ (or $\mathbb{R}^3$)

One "vector" function or functions per dimension

Free Parameters and Shape Parameters

The range of the free parameter

t goes from start to end

can always scale to 0,1

convention: use u for parameter in [0,1]
(unit parameterization)

use t for more general case (which includes unit)

This is **convention** - we can use any variable names we like

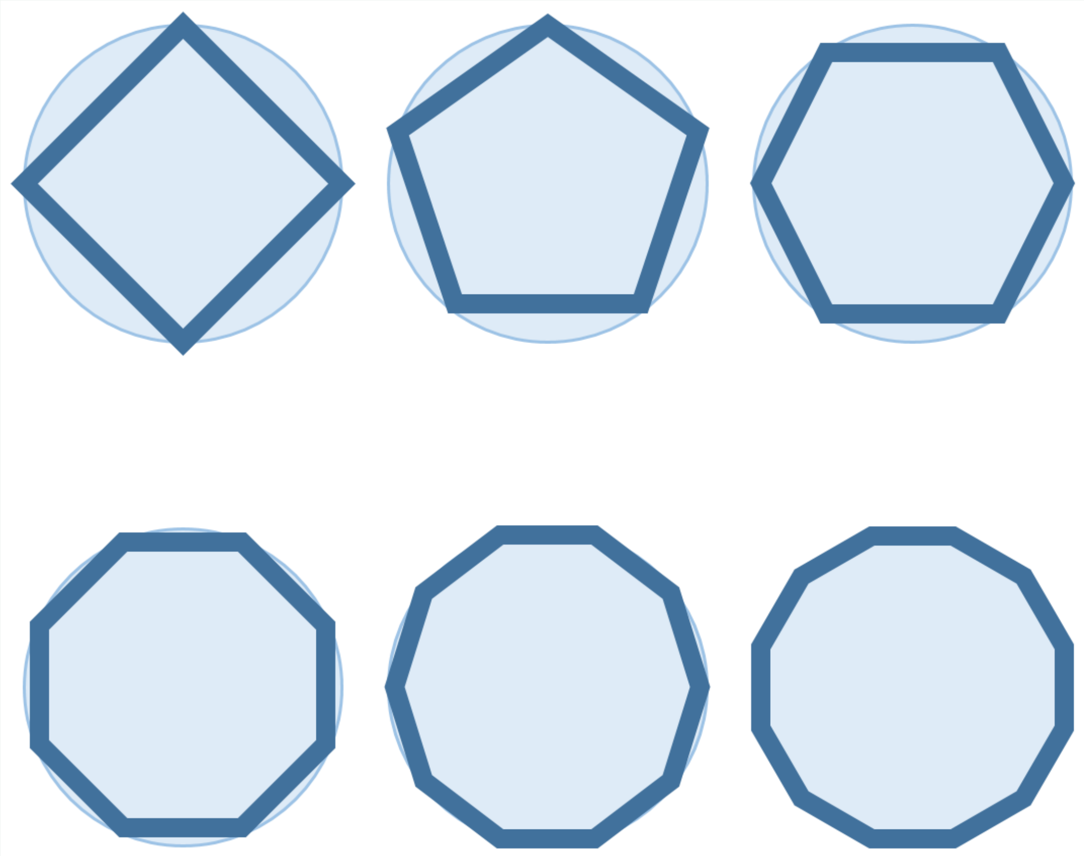
This will keep coming up

Approximation

How many points before it looks "right""?
(smooth)

- Good enough for manufacturing?
  is this round enough to roll?
- What if we zoom in?

Keep "real" curve (infinite...)
Approximate to draw, ...



Aside: Drawing Curves

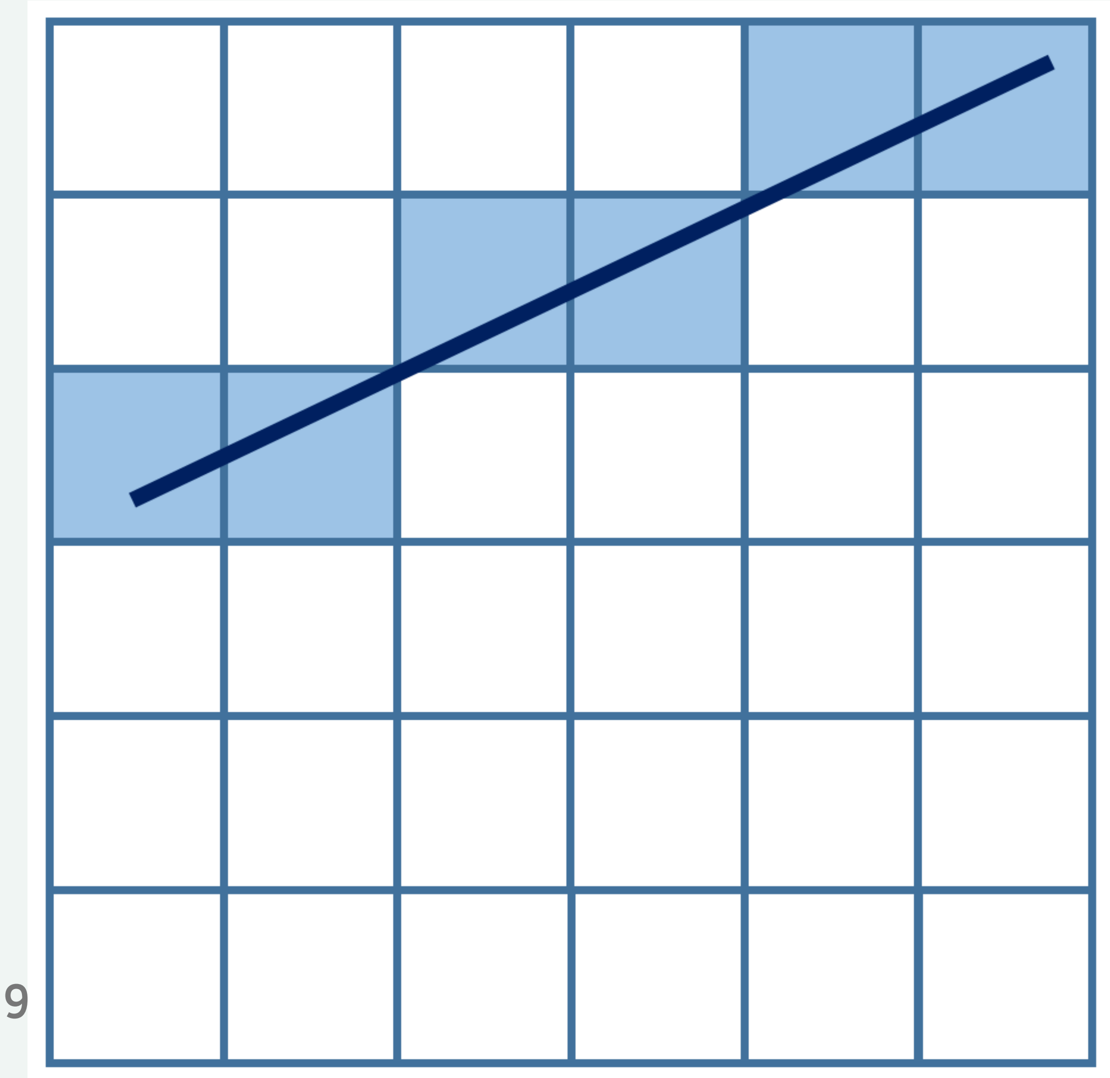
Ultimately approximate with pixels

Good algorithms for basic shapes

- lines, circles
- bezier curves
- later in class

Raster Algorithms

- in the library/API
- (often) in hardware



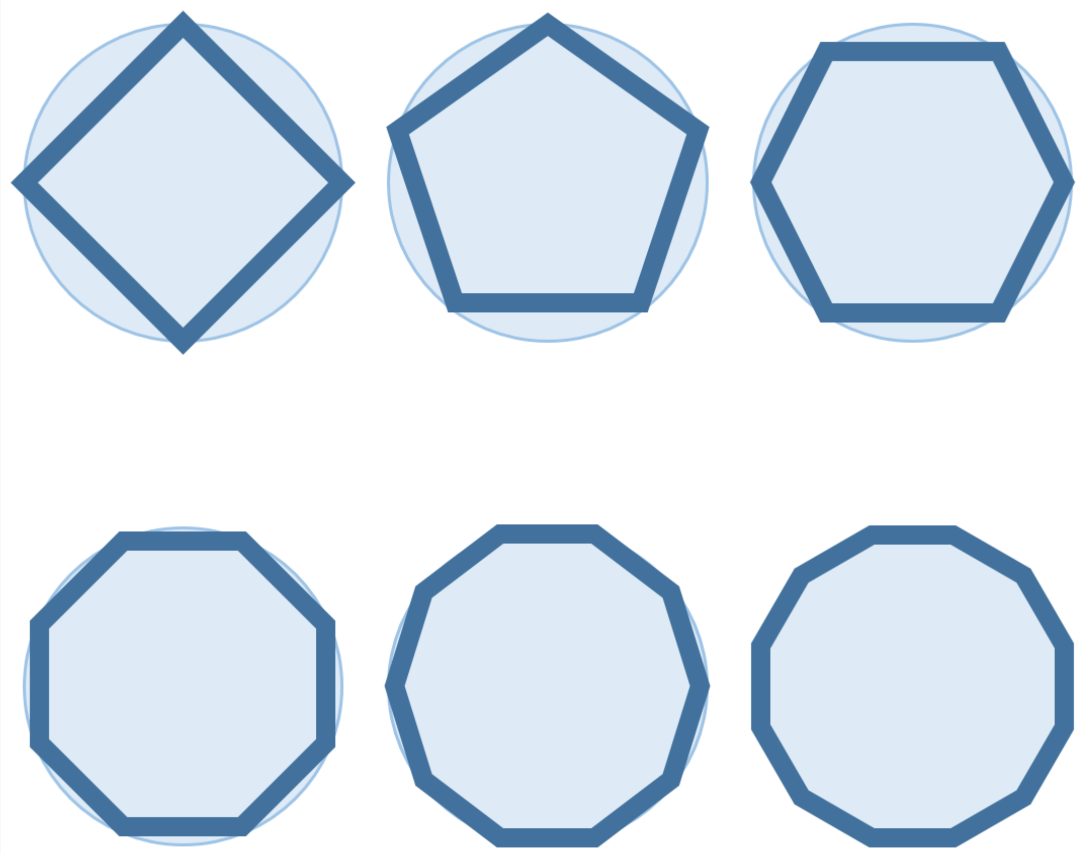
Defining Smoothness

We will actually define

continuity

Does it have abrupt changes?

- breaks / gaps
- corners
- changes in higher derivatives



Continuity vs. Other Smoothness

Continuity defined

Are the points next to each other?

Can we draw without lifting the pen?

At a parameter value u

$f(u^-) = f(u^+)$

This is continuity in **value**

Continuity in Direction

Does the curve change direction suddenly?

Tangent Vectors

Line that touches the curve at the point

Velocity (vector) of the pen's travel

Derivative of position with respect to free parameter

$\mathbf{x} = \mathbf{f}(t)$

$\dot{\mathbf{x}} = \mathbf{f}'(t)$, where $\mathbf{f}' = \frac{\partial \mathbf{f}}{\partial t}$

Tangent/velocity is a **vector**

It is a **function** of the free parameter

Discontinuity Example

Piecewise line segments:

```
f(u) = if u<.5 then (u,0) else (u,1)
or
f(u) = (u<.5) ? (u,0) : (u,1)
```

Position discontinuity at `u=.5`

Discontinuity Example

Piecewise line segments:

```
f(u) = if u<.5 then (u,u) else (u,.5)
```

Tangent (first derivative) discontinuity at `u=.5`

Note: discontinuities happen when we switch

Continuity Conditions

We say a curve is $C(n)$ continuous

If all its derivatives up to (and including) n are continuous

$C(0)$ - positions
$C(1)$ - positions and tangents (1st derivatives)
$C(2)$ - positions and tangents and 2nd derivatives

How much continuity do we need?

$C(0)$ - no gaps

$C(1)$ - no corners

$C(2)$ - looks smooth

Higher...

Important for airflow (airplane, car, boat design)

Important for reflections

Speed Matters?

```
f(u) = if u<0.5 then (u,0) else (2u-0.5,0)
```

It's a horizontal line

The pen doesn't change direction

It does change "speed" at the point

C and G continuity

$C(n)$ continuity - all derivatives up to n match

$G(n)$ continuity - the directions of the derivatives match

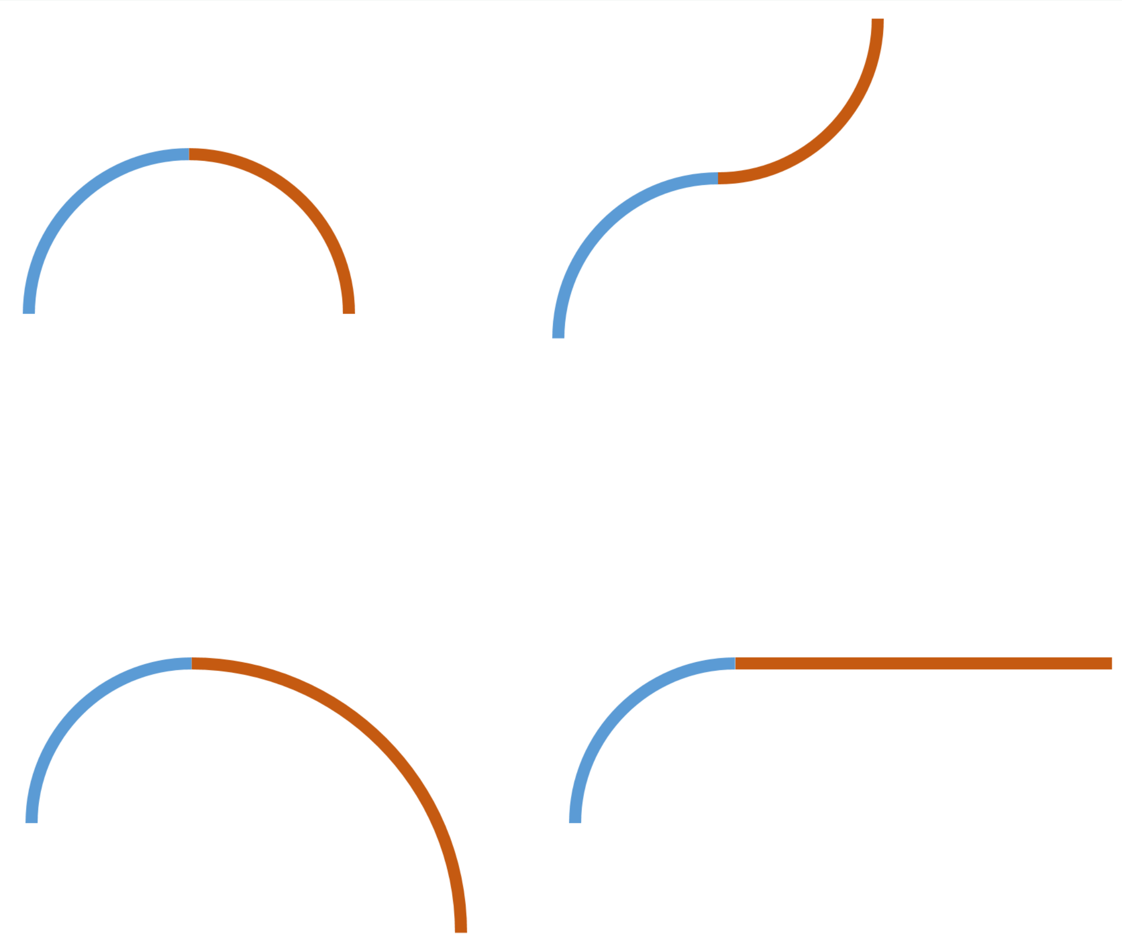
Technically: requires some terms we haven't learned yet

Consider continuity where segments come together

Better pieces than line segments

Circular arcs?

C and G continuity with arcs



Piecewise Polynomials

Chains of low-degree polynomials

- line segment chains (1st degree)
- chains of 2nd or 3rd degree (or more)

Why not pieces of higher degree?

Given n points, you can make an $n-1$ degree polynomial

- hard to compute
- hard to control
- unwanted wiggles

Come back to this later

Piecewise Parameterizations

Overall parameterization (t)

Per-piece parameterization (u)

General Polynomials

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

for 2D, we need:

$$f_x(t) = a_{0x} + a_{1x} t + a_{2x} t^2 + \cdots + a_{nx} t^n$$
$$f_y(t) = a_{0y} + a_{1y} t + a_{2y} t^2 + \cdots + a_{ny} t^n$$

or use vector notation

$$\mathbf{f}(t) = \mathbf{a_0} + \mathbf{a_1} t + \mathbf{a_2} t^2 + \cdots + \mathbf{a_n} t^n$$

Note: the dimensions are independent

General Polynomials

$$\mathbf{f}(t) = \mathbf{a_0} + \mathbf{a_1} t + \mathbf{a_2} t^2 + \cdots + \mathbf{a_n} t^n$$

$$\mathbf{f}(t) = \sum_{i=0}^{n} \mathbf{a_i} t^i$$

Polynomials

Linear in the coefficients (given u)

Polynomial Forms: Line Segment

$\mathbf{a}_0$ and $\mathbf{a}_1$

$\mathbf{f}(\mathbf{u}) = \mathbf{a_0} + \mathbf{a_1} u$

is this convenient?

Polynomial Forms: Line Segment

$\mathbf{a_0}$ and $\mathbf{a_1}$

$$\mathbf{f(u)} = \mathbf{a_0} + \mathbf{a_1}u$$

$\mathbf{p_0}$ and $\mathbf{p_1}$

$$\mathbf{f(u)} = (1-u)\mathbf{p_0} + u\mathbf{p_1}$$

easy to specify

easy to check continuity between segments

easy to convert between forms

Polynomial Forms: Line Segment

$\mathbf{a_0}$ and $\mathbf{a_1}$

$$\mathbf{f(u)} = \mathbf{a_0} + \mathbf{a_1}u$$

$\mathbf{p_0}$ and $\mathbf{p_1}$

$$\mathbf{f(u)} = (1-u)\mathbf{p_0} + u\mathbf{p_1}$$

$\mathbf{c}$ and $\mathbf{d}$ (center and displacement)

$$\mathbf{f(u)} = \mathbf{c} + 2 * (u - .5) * \mathbf{d}$$

and many others

Change of parameters

$\mathbf{a_0}$ and $\mathbf{a_1}$

$\mathbf{f(u)} = \mathbf{a_0} + \mathbf{a_1}u$

$\mathbf{p_0}$ and $\mathbf{p_1}$

$\mathbf{f(u)} = (1 - u)\mathbf{p_0} + u\mathbf{p_1}$

easy to compute $\mathbf{a_i}$ from other parameters

Beyond a line...

We need curved segments to get better continuity

Quadratic (2nd degree) Segments

$\mathbf{a_0}$, $\mathbf{a_1}$, and $\mathbf{a_2}$

$\mathbf{f(u)} = \mathbf{a_0} + \mathbf{a_1}u + \mathbf{a_2}u^2$

what can we do with this?

note:

- $\mathbf{f}(0)$
- $\mathbf{f}'(0)$
- $\mathbf{f}(1) = \mathbf{a_0} + \mathbf{a_1} + \mathbf{a_2}$
  - if you want to specify where the curve ends, you can compute $\mathbf{a_2}$
  - are $\mathbf{a_1}$ and $\mathbf{a_2}$ convenient?

Quadratic (2nd degree) Segments

$\mathbf{a_0}$, $\mathbf{a_1}$, and $\mathbf{a_2}$

$$\mathbf{f(u)} = \mathbf{a_0} + \mathbf{a_1}u + \mathbf{a_2}u^2$$

$\mathbf{p_0}$, $\mathbf{p_1}$, and **??**

- interpolate $\mathbf{p}_{\frac{1}{2}}$
- stay inside triangle (influence)
- specify derivatives (to help match neighbors)

Cubics

The most popular choice in computer graphics

- specify position and 1st derivative at the ends
- C(1), interpolation, local control
- 4x4 matrices (just like 3D transformations)

Cubics

$$\mathbf{f}(\mathbf{u}) = \mathbf{a_0} + \mathbf{a_1}u + \mathbf{a_2}u^2 + \mathbf{a_3}u^3$$

coefficient form is not convenient

Hermite Form

specify position and 1st derivative at ends

$\mathbf{p_0}, \mathbf{p_1}$ as well as $\mathbf{p'_0}, \mathbf{p'_1}$

need to compute $\mathbf{a_i}$ from these

derivation in the book (or old versions of the class)

Hermite Equations

$$\begin{aligned} f(u) = p_0 \; u^0 + \\ p'_0 \; u^1 + \\ (-3p_0 - 2p'_0 + 3p_1 - p'_1) \; u^2 + \\ (2p_0 + p'_0 - 2p_1 - p'_1) \; u^3 \end{aligned}$$

so...

$\mathbf{a_0} = \mathbf{p_0}$ and so on...

A more useful form

$$
\begin{aligned}
f(t) = \left(1 - 3u^2 + 2u^3\right) p_0 + \\
\left(u - 2u^2 + 1\right) p_0' + \\
\left(3u^2 - 2u^3\right) p_1 + \\
\left(-u^2 + u^3\right) p_1'
\end{aligned}
$$

functions of u for each "control point"

$$f(t) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_0' + b_3(u)p_1'$$

$$b_0(u) = 1 - 3u^2 + 2u^3$$

basis functions

Interpolation

Given a set of points, make a curve through them

But which one?

- shortest? (line segments)
- smooth?

what happens in between points?

Designing with Hermite Curves

We can make C(1) shapes easily

Control "in-between" with derivatives

Avoid specifying derivatives?

Compute derivatives based on neighbor points

Cardinal Splines

Catmul-Rom Splines

Tension Parameter

$f_i' = s(f_{i+1} - f_{i-1})$

$s = \frac{1-t}{2}$

$t = 0, s = \frac{1}{2}$

Cardinal Interpolation