Lecture 15
Rotations in 3D

Rotations

What is a rotation anyway?

It is a **rigid** transformation

- preserves **distances**
- preserves **handedness**

Two type of rigid transformations

- translations (all points change the same amount)
- rotations (one point - the **center** - doesn't change)

Rotations are Linear Transformations

Orthonormal Matrices

- all rows (columns) have unit length
- all rows (columns) mutually orthogonal

Positive determinant (preserve handedness)

- Not all matrices are rotations

Rotation Facts

- Have an **inverse**
- The inverse is the transpose (only rotations)

- Closed set (composition yields another rotation)
- Associates (do operation in any order)
- Does not commute (in general)

There is a **center** and **axis** of rotation

Rotation vs. Orientation

2D Rotations in 3D

Center of rotation is an axis

Normal 2D rotation is about the Z axis

Center of Rotation

Rotation is about the origin

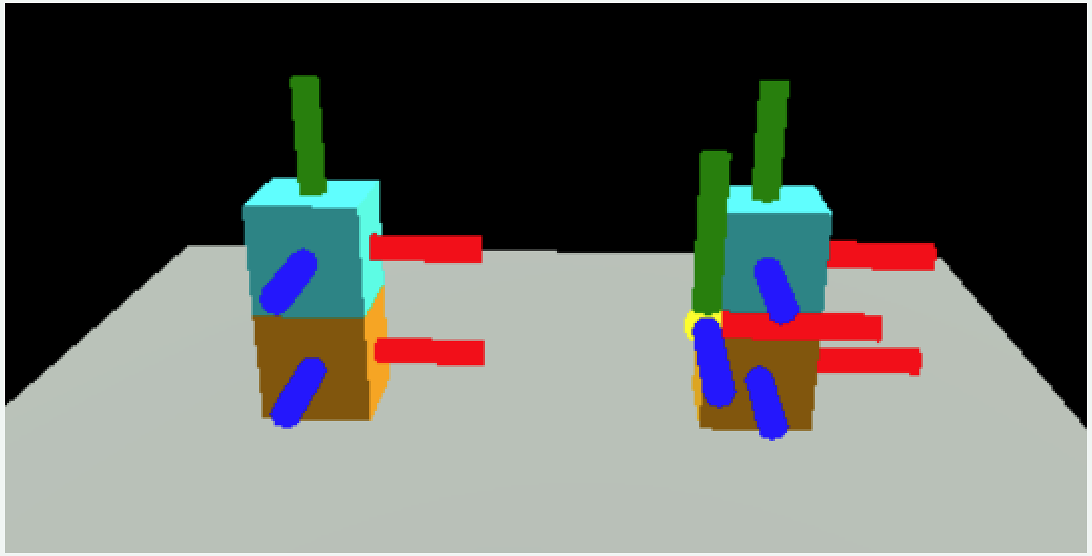
Shift the origin to where you want to rotate

Shift it back after rotation

Center of Rotation in THREE

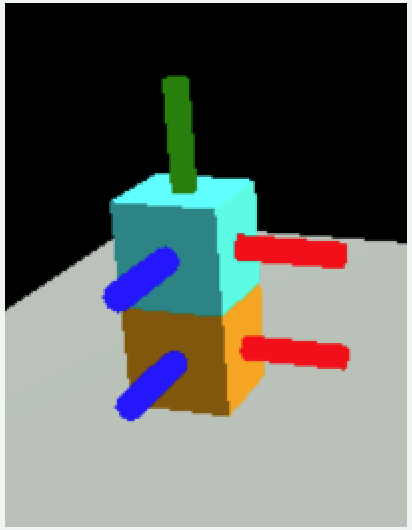
Use a `Group` to put the center in the right place

Put the object in the group (relative to object's center)



stack demo

Default Hierarchy



Cubes have their center in the center

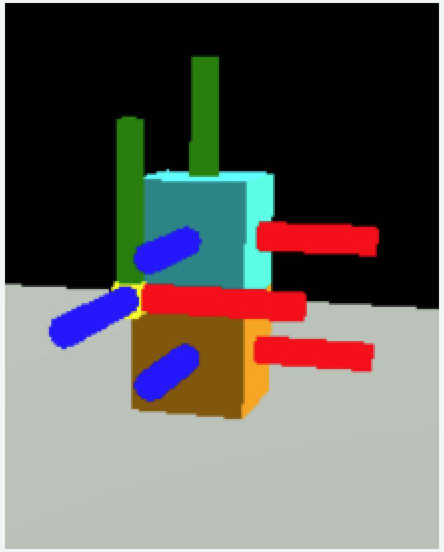
Notice the distance to stack

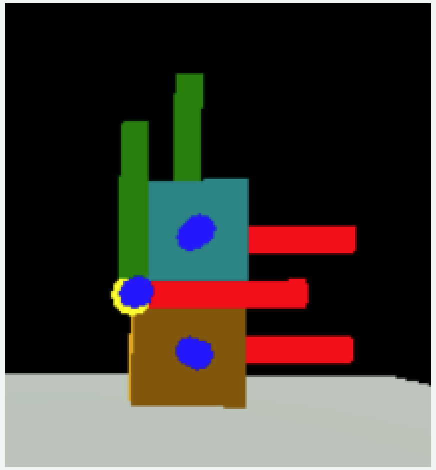
Place one cube in another

```
let cube1 = cube("orange");
cube1.position.set(-2,.5,3);
let cube2 = cube("cyan");
cube1.add(cube2);
cube2.position.set(0,1,0);
scene.add(cube1);
```

```
function cube(color) {
    return new T.Mesh(new T.BoxGeometry(),
                      new T.MeshStandardMaterial({color:color}));
}
```

With a Group





Place the group at the corner of cube

Place the 2nd cube in group

Rotate the Group

```
let cube3 = cube("orange");
cube3.position.set( 2,.5,3);
let group1 = new T.Group();
cube3.add(group1);
group1.position.set(-.5,.5,.5);
let cube4 = cube("cyan");
group1.add(cube4);
cube4.position.set(.5,.5,-.5);
scene.add(cube3);
```

Rotations about Axes

Rotation about X

Rotation about Y

Rotation about Z

demo: EulerToy 1

Axes in the world vs. local axes

demo: EulerToy 1

Sequences of Rotations

demo: Euler Toy 2

Euler's Theorems

1. Any rotation can be represented as a single rotation about some axis
2. Any rotation can be represented as a sequence of three rotations about a fixed axis

Euler Angles

Be careful: Euler invented many different kinds of angles

Rotation around 3 fixed axes

- Could be any order (XYZ, ZYX, ZXY)
- Can repeat (ZXZ)
- Can be local or global
  - easier to think "global to the left"

Play with them

Demo: EulerToy 3

More on Euler Angles

Earlier rotations change the meaning of later ones

Order matters

Local to global (or global to local)

(demo)

(incremental rotations in the workbook)

Composing Rotations

In a single axis (like in 2D):

$$R_z(a) \circ R_z(b) = R_z(a + b)$$

With different axes, this does not hold!

$$R_x(a) \circ R_y(b) = R_?(?)$$

And things in between cause problems

$$R_x(a) \circ R_y(b) \circ R_x(c) \neq R_x(a + c)R_y(b)$$

Getting Stuck

Rotate about X then Y

Rotate about Z is the same as the first rotate about X

Gimbal Lock

No matter what X is, Y=90 aligns Z with it

- There is no way to get the Y axis out of the X=0 plane
- We lost a degree of freedom

(demo EulerToy3)

Two ways to the same place

Rotate about X then Y

Rotate about Y then Z

same! (but different path)

(90,90,0) = (0,90,90) - but can't interpolate!

(demo EulerToy4)

Euler Angles (XYZ)

Good:

Easy for 1 axis

Easy for simple combinations

Bad:

Hard to get what you want (unintuitive combinations)

Can't interpolate

Gimbal lock (can't get there from here)

Axis Angle (Euler's other theorem)

Demo: et-axisangle

Axis Angle

Downsides:

- hard to figure out what axis
- hard to compose

Rotation Vector

Store the angle as the magnitude of the axis

Rotation Matrices

- hard to interpret
- easy to "drift"
- hard to insure it's a rotation
    - Gramm Schmidt Orthonomalization

Unit Quaternions

4 numbers:

- Axis angle: $\theta, \hat{\mathbf{n}}$
- Unit quaternion: $\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\hat{\mathbf{n}}$
- Will have magnitude 1

Why?

What is a Quaternion anyway?

4-dimensional complex number

Consider 2D complex numbers ($a + b\mathbf{i}$)

- we can do arithmetic on them
- multiplication is meaningful

4D Complex Numbers?

Don't worry... you can look up:

- formulas to multiply
- formulas to convert to Matrix form
- formulas to interpolate (and preserve unit-ness)

but you should know...

- these formulas exist
- multiplication preserves unit-ness
- multiplication composes transformations

Why is this better? (or is it?)

- No Gimbal Lock (but antipodes)
- Represents orientations
- Close things are close (except for sign flips)

But Really:

- Easy to compose
- Easy to interpolate (not linear interpolation)
- Other nice math (interpolation)
- 3x3 rotation matrices are a pain
- Easy to fix drift

Convert to Quaternions

(Other direction is MUCH harder)

Axis angle $(\theta, \hat{\mathbf{v}})$ -> $(\cos(\frac{\theta}{2}), sin(\frac{\theta}{2})\hat{\mathbf{v}}))$

Euler Angles XYZ (x,y,z)

- make a quaterion for each $(\cos(\frac{x}{2}), sin(\frac{x}{2}[1, 0, 0])))$
- multiply the quaternions together

THREE.js and rotations

Internally, stores quaternions

- it provides all conversions
- it does conversions automatically (beware errors!)
- it provides good quaternion functions
- it gives you operations using other forms
  - axis angle, euler angle, matrix,

You never **need** to see the quaternions... unless you want to

THREE and Rotations

State (variables / orientation)

matrix (normalMatrix, ...)

position

scale

quaternion

rotation

Transforms (motions / rotations)

applyMatrix4

translate (x,y,z, onAxis, ...)

applyQuaternion

rotate (x,y,z, onAxis, ...)

`lookAt`, `setFrom` are special (a method that sets) an absolute orientation

Internally...

The **quaternion** is used for everything

If you do something else, it is converted to the quaternion

If you apply a matrix it must be **decomposed** into rotate, translate, scale

```
	applyMatrix: function ( matrix ) {
		this.matrix.multiplyMatrices( matrix, this.matrix );
		this.matrix.decompose( this.position, this.quaternion, this.scale );
	}, // in Object3D.js
```

Internally

```
translateX: function () {
        var v1 = new Vector3( 1, 0, 0 );
        return function translateX( distance ) {
                return this.translateOnAxis( v1, distance );
        };
}(),
translateOnAxis: function () {
        // translate object by distance along axis in object space
        // axis is assumed to be normalized
        var v1 = new Vector3();
        return function translateOnAxis( axis, distance ) {
                v1.copy( axis ).applyQuaternion( this.quaternion );
                this.position.add( v1.multiplyScalar( distance ) );
                return this;
        };
}(),
```

Old School JavaScript hidden constant

```
translateX: function () {
        var v1 = new Vector3( 1, 0, 0 );
        return function translateX( distance ) {
                return this.translateOnAxis( v1, distance );
        };
}(),
```

A Special Rotation: LookAt

Point the Z axis towards a point

- Useful for cameras
- Useful for other objects

Note this is not unique

- Only specifies 2 dergees of freedom

Up Vector!

Lookfrom / Lookat / Up

- In Three
  - `position` of object center
  - `lookat` method
  - `up` vector (object property)

Internally, it will convert to quaternion

Geometric Derivation

1. Point z at target

   normalize(at - from) $(\widehat{\mathrm{at} - \mathrm{from}})$

2. Find x (right) as $\widehat{up} \times z$

3. Find y (local up) as $z \times x$

Notice: we have built a rotation matrix!

It has all the right properties

We never figured out angles

Rotations Summary: What you need to know

1. Basic facts (rigid, orthonormal, composition, ...)
2. Single Axis Rotations
3. Euler Angles - be able to think about them
    - local vs. global
    - how things compose (and complexities)
4. Axis Angle forms - understand what they are
5. Quaternions
    - basic facts - and know they are inside THREE
6. Lookfrom/Lookat/VUp
7. Use in THREE (including centers)

JavaScript Tip

Inheritance is important for Workbook 7

You will make your own **subclasses** of the framework class

(there is a tutorial on the course web, will post to Piazza)

Classes in Javascript

```
class Parent {
    constructor(a,b) {
        this.a = a;
        this.b = b;
        this.c = 10;
    }
    method() {
        console.log(this.a,this.c)
    }
};
```

```
let thing1 = new Parent(1,2);
thing1.method(); // prints 1,10
```

SubClasses in Javascript

```
class Parent {
    constructor(a,b) {
        this.a = a;
        this.b = b;
        this.c = 10;
    }
    method() {
        console.log(this.a,this.c);
    }
};
```

```
class Child extends Parent {
    constructor(b) {
        super(3,b);
        this.c = 20;
    }
}
```

```
let thing1 = new Parent(1,2);
thing1.method(); // prints 1,10
```

```
let thing2 = new Child(5);
thing1.method(); // prints 3,20
```

SubClasses in Javascript

Child class **extends** parent class

```
class Child extends Parent {
    constructor(b) {
        super(3,b);
        this.c = 20;
    }
}
```

Child class has its own constructor

Child constructor calls parent

`super()` - takes parent arguments

`this` doesn't exist until `super()`

Child class uses parent methods

(unless it overrides them)

```
let thing2 = new Child(5);
thing1.method(); // prints 3,20
```

Why do you need to know this?

The CS559 Software Framework uses this!

You define types (subclasses) of `GrObject`

`GrObject` has a list of THREE `Object3D`

You pass the `GrObject` constructor the `Object3D` it should contain

```
class BasicSphere extends GrObject {
    constructor() {
        let geom = new T.SphereGeometry();
        let mat = new T.BasicMaterial({color:"green"});
        let mesh = new T.Mesh(geom,mat);
        super("Basic Sphere", mesh);
        this.mesh = mesh;
    }
}
```