Lecture 11:

Even More Curves

Today

1. The main points of curves
2. The Train (to motivate fancier curve things)
3. Arc Length
4. TCB Cardinals
5. B-Splines (C(2) curves)

1. Continuity Conditions

We build curves from pieces

One parameter for curve, or one parameter per piece

We care about **continuity** where the pieces meet

C continuity (derivatives)

G continuity (directions of derivatives)

2. Polynomial Pieces

Canonical Form

$$f(u) = a_0u^0 + a_1u^1 + a_2u^2 + \cdots$$

Basis Function form

$$f(u) = p_1b_1(u) + p_2b_2(u) + p_3b_3(u) + \cdots$$

3. Cubic Forms

Canonical Form

$$f(u) = a_0u^0 + a_1u^1 + a_2u^2 + a_3u^3$$

Hermite Form: begining and end

$$p_0 = f(0)$$
$$p_1 = f(1)$$
$$p'_0 = f'(0)$$
$$p'_1 = f'(1)$$

4. Cardinal Interpolation

$$p'_i = s(p_{i+1} - p_{i-1})$$

Thing we glossed over

5. Bezier Cubic (segment)

4 points

Interpolate endpoints

Tangents = 3 * vector between first (last) points

Convex Hull Property

Variation Diminishing Property

Symmetry

Locality

Cool Algorithms (Geometric)

Things you need to be able to do

6. Geometric Algortihms for Beziers

DeCastlejau construction

1. compute values by repeated interpolation
2. subdivide curves

Useful for deriving properties and basis functions

7. The Generality of Beziers

Works for any degree! ($d = n - 1$ for n points)

2 points (line segment)

3 points (quadratics)

4 points (cubics)

etc.

All properties hold

Scale vector (first points) by degree of curve

Things you should be able to do

8. Convert Between Cubic Forms

APIs generally have Beziers (cubics)

Bezier to Hermite (for cubics)

Cardinal to Hermite

Beziers of different degrees

What can we do with this?

The Train!

This has been part of CS559 since 1999 (the first time I taught it!)

- Good for showing curve ideas!

- 3D Demo (2014)
- 2D Demo (2019)

What's Good About Cardinals?

1. C(1)
2. Local
3. Interpolating

Sketching Cardinals (approximate $s = .5$)

How to draw a cardinal?

1. try a bunch of u values
2. convert to a Bézier (draw with the API)

How far does the cardinal go?

It goes outside of its points.

Convert to Bézier!

Arc Length

If you straighten out the curve - how long is it

Integral of the magnitude of the tangent

Approximate curve with straight lines

Arc Length Parameterization?

- equal steps in u
- equal steps in distance

consider:

$f(u) = u$

$f(u) = u^2$

$f(u) = u^3$

How to implement Arc Length Parameterization?

Hard to do analytically

Approximate *numerically* (make a table, solve, ...)

Fancier Cardinals: TCB curves

- tension
- continuity
- bias

Used by animators to get more control

What's Good About Béziers?

1. Interpolate end-points, stay in Convex Hull
2. Tangents at ends based on points
3. Good algorithms (geometric and algebraic)
4. Variation-Diminishing (limited wiggles), Affine Invariance
5. General (any degree)
6. And other properties…

What's not good about Béziers

1. Not **projective** invariant
2. Polynomial can't represent conics
    - no exact circles!
3. Hard to get better than C[1]/G[1]
4. Sometimes we want **interpolation**

Projective Invariance

Béziers are **affine invariant**

Projection (e.g., general homogeneous coords) requires division

Limited Shapes

Polynomials (like Béziers) can't represent some shapes

Exact circles (and other conic sections)

Rational Polynomial Curves

Represent a curve as the **ratio** of two polynomials

$$f(u) = \frac{\sum a_i u^i}{\sum b_i u^i}$$

Allows for projective invariance

Allows for more shapes (circles, conic sections, ...)

Very complicated - and usually not used in Computer Graphics

Used in mechanical design where exact shapes are required

Better than C(1)/G(1)

requires aligning multiple points

Note:

If we want a very smooth curve, we can make a high degree Bézier

But getting continuity between segments can be tricky

What about Interpolation?

1. Cardinal Splines C(1)
2. High-Order Polynomials (smooth)
3. Natural Splines C(2)

Natural Splines (Cubics)

- Models a "draftsmans spline" (stiff piece of metal)
- C(2)
- second derivative is zero at the ends

- requires solving a linear system to compute coefficients from points
- not local

B-Splines

This should be a big topic - but we will scratch the surface

A general way to think about piecewise polynomials.

A lot of history here at Wisconsin

Algebraic and Geometric Constructions

Incredibly General

Thinking in terms of chains of points…

We have points $\mathbf{p_0}, \mathbf{p_1}, \cdots, \mathbf{p_n}$

Each point is the beginning of a segment (except at end)

- (d=1) line segments $[\mathbf{p_0}, \mathbf{p_1}], [\mathbf{p_1}, \mathbf{p_2}], \cdots [\mathbf{p_{n-1}}, \mathbf{p_n}]$

Even with higher order...

We have points $\mathbf{p_0}, \mathbf{p_1}, \cdots, \mathbf{p_n}$

Each point is the beginning of a segment (except at end)

- (d=1) line segments $[\mathbf{p_0}, \mathbf{p_1}], [\mathbf{p_1}, \mathbf{p_2}], \cdots [\mathbf{p_{n-1}}, \mathbf{p_n}]$
- (d=2) quadratics $[\mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2}], [\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}], \cdots [\mathbf{p_{n-2}}, \mathbf{p_{n-1}}, \mathbf{p_n}]$

Each segment is $[\mathbf{p_i}, \mathbf{p_{i+1}}, \cdots, \mathbf{p_{i+d}}]$

The Simplest B-Splines

- Uniform
- Low order Polynomial (d=2 or 3 in graphics)
- Chains of points are chains of polynomials

Cubic B-Splines

- 4 points influence a curve segment
- neighboring segments share 3 points
- each segment is a cubic polynomial

(demo)

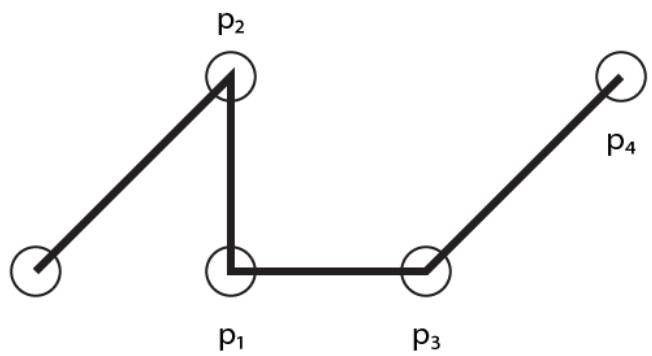
Why B-Splines?

- Curves of any degree d
- Locality: each control point influences $d+1$ segments
- Continuity: curve is $C(d-1)$
- Stays with the Convex Hull
- Symmetric
- Affine Invariant
- Variation Diminishing
- **not interpolating** (except d=1)

Thinking in terms of Basis Functions

Consider connecting line segments

$$\mathbf{f}(u) = (1-u)\,\mathbf{p_i} + u\,\mathbf{p_{i+1}}$$


p2
p4
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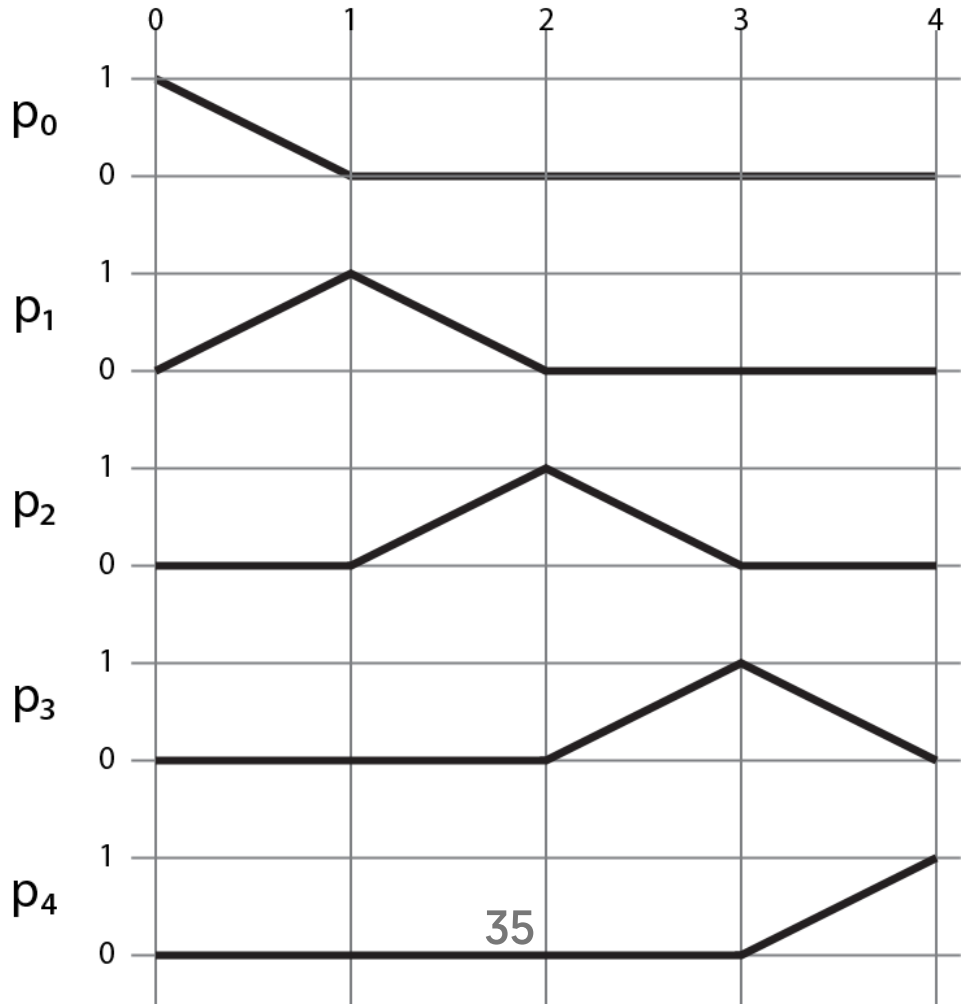
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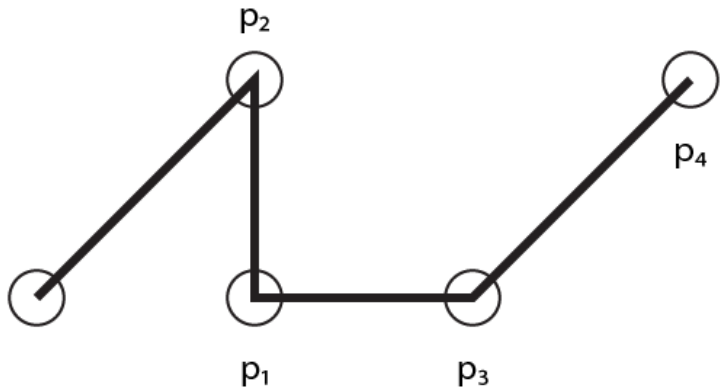
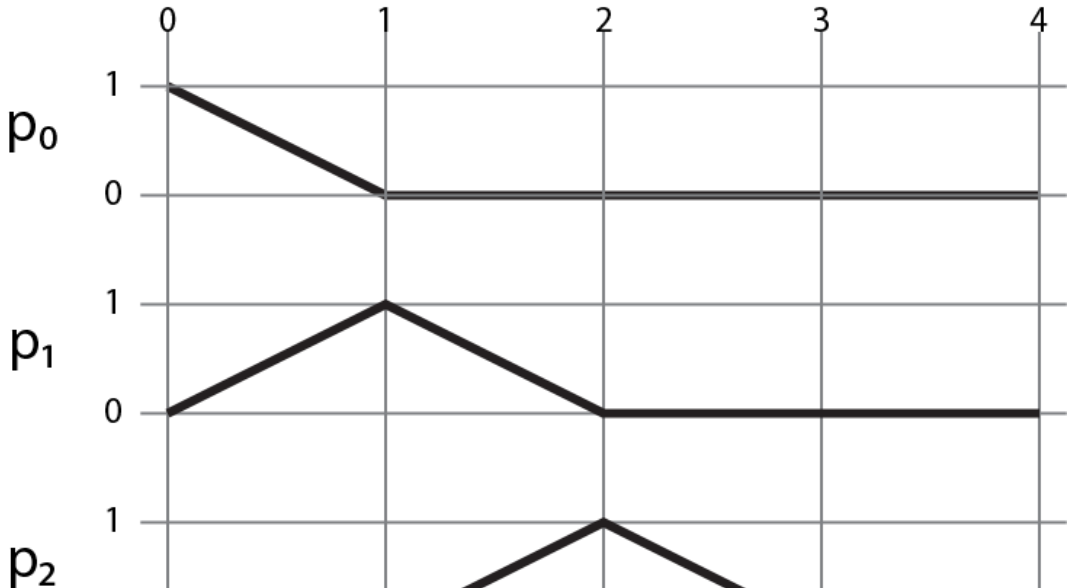
What about these Basis Functions?

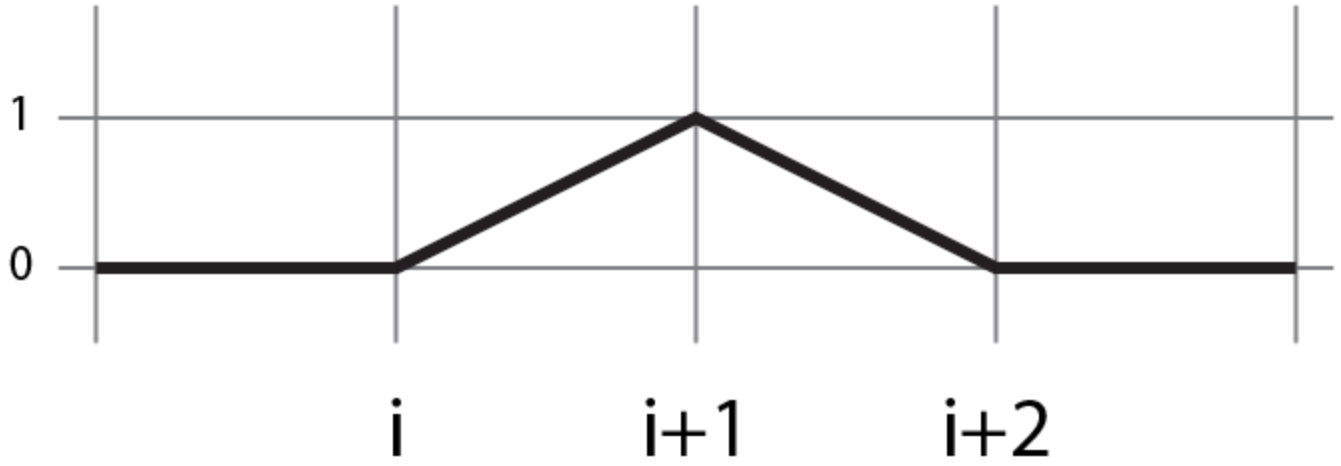




- They are piecewise polynomials
- They have the same continuity
- They repeat (except ends)

- If you have one, you'd have them all

The Linear Basis Functions



$$b_i(t) = \begin{array}{l} \text{if } t < i \text{ then } 0 \\ \text{elif } t < i+1 \text{ then } t - i \\ \text{elif } t < i+1 \text{ then } i + 2 - t \\ \text{else } 0 \end{array}$$

On a chain of points? (lines)

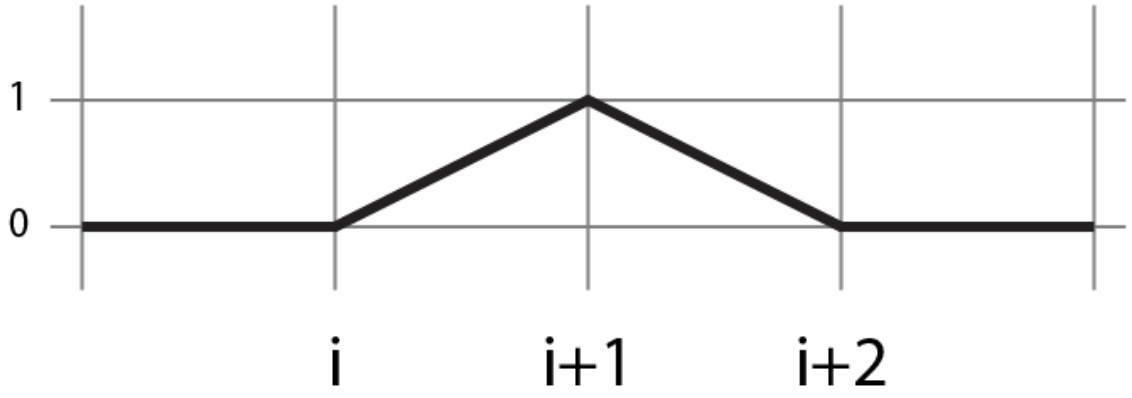
At any parameter value, 2 control points are active

One in each "phase"

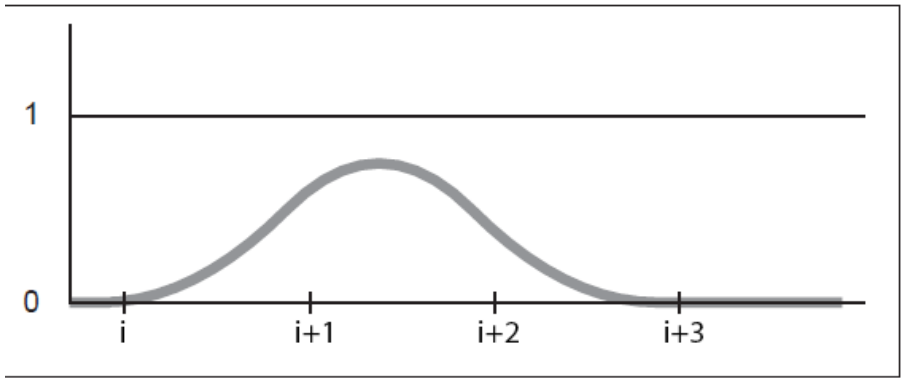
Higher-degree basis functions

Only linear interpolates (like Beziers)

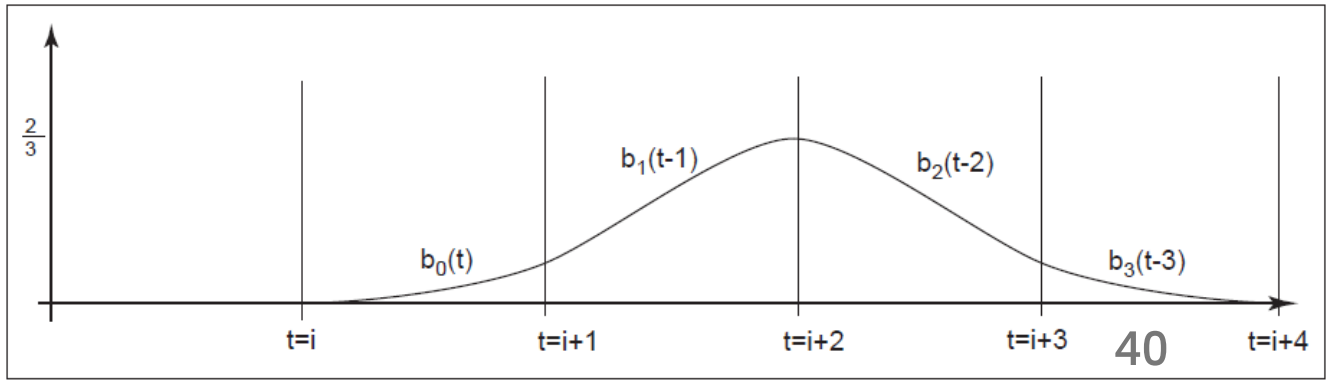
Degree d has $d+1$ segments, meet with $C(d-1)$ continuity



Linear



Quadratic



Cubic

Geometric Intuition for B-Splines

or

A More Interesting Subdivision Curve

A Geometric Approach: Chakin Corner Cutting

Subdivide each line segment:

- Mark 1/4 and 3/4
- Cut it there

Each iteration gets smoother

- In the limit it will be smooth

Notice:

- does not interpolate its "control points"

What to learn from that?

- Practical? (might need a lot of iterations)
- Subdivision: in the limit, it is a piecewise quadratic
- It converges to a B-Spline
- Approximating curve: it does not interpolate its (initial) points

In practice…

1. Know that it converges to piecewise quadratics
2. Look up the basis functions

What to do with these?

Use cubic B-Splines if you want C(2) curves

Even though the do not interpolate, they "behave nicely"

Look up the blending functions if you need them

Train Demo

See how nice B-Splines Are?

Are Curves different in 3d?

Curves are not Surfaces!

Dimensions are independent (just 3 numbers per vector)

Equations are the same

Tangent vector - normal plane (perpendicular to vector)

Curve Summary

- Use **parametric** curves
- Use **piecewise** representations, connect with **continuity**
- Use **polynomial** segments, usually **quadratics** or **cubics**
- Use **Hemite** or **Cardinal** interpolation (if you need interpolation)
- Use **Bézier** curves for good control (and API compatibility)
- Use **B-Splines** to get very smooth curves

Get ready for 3D!