STAT 906 - Assignment #3
Due November 18, 2012

1. Consider the following class of interest rate models

$$dr_t = k(b - r_t) + \sigma r_t^{\gamma} dW_t, \quad t \geq 0.$$

For your selection of the parameters r_0, k, b, σ, and γ (but γ should be different from 1), use the Euler scheme, the Milstein scheme, and the second order approximation scheme to generate paths of this process.

(a) Using the three schemes to simulate paths, propose a Monte Carlo simulation method to find q that satisfies

$$\mathrm{P}\{r_3 > q\} \leq 0.1.$$

By considering the time step Δ equal either to 0.004 or to 0.1, discuss accuracy of each method. In your discussion, you should include the bias of the method and the simulation error.

(b) Suppose now that you want to price bonds with maturities of 3 and 10 years, and for this you are using a Monte Carlo method. By considering time steps equal to $\Delta t = 0.004$ and $\Delta t = 0.1$, discuss accuracy of each approximation method. As in part (a), you should consider the bias and the simulation error.

(c) How do the different approximation methods compare when you consider the computing time? To answer this question, use part (b) above.

The next four questions deal with the problem of finding the integral

$$I = \int_0^2 e^{-x^2} dx. \tag{1}$$

2. Use both crude and antithetic random numbers to estimate the value of the integral (1).

(a) What is the efficiency gain when using antithetic random numbers?

(b) How large a sample size should be, for antithetic and random numbers, in order to estimate correctly I up to three decimal places with probability at least 95%?

3. Propose an importance sampling method for the integral (1). Explain why your selection of the new density should work. What is the efficiency gain?

4. For the problem of finding I in (1), propose a control variate that is a polynomial of order not higher that 3. What is the efficiency gain of your new Monte Carlo estimator?

5. Use a stratified random sample to find the integral (1). What do you recommend for the choice of strata and sample sizes? (consider at most 4 strata.) What is the efficiency gain?

6. We consider the problem of finding the integral

$$\int_0^\infty 1(x \geq q) f(x) dx,$$

where $q = 3.7$ and f is the Rayleigh density function

$$f(x) = 2xe^{-x^2}, \quad x \geq 0.$$

For the Monte Carlo simulation method with importance sampling, obtain the optimal simulation density of the form $f^*(x) = f(x - c),\ c > 0$. You can do this by minimizing the function

$$I(c) := \int_q^\infty \frac{2x^2}{x - c} e^{-(x^2 + 2cx - c^2)} dx \tag{2}$$

over c such that $c < q$. The function in (2) does not have an analytical representation, but you can approximate the optimal c by simulation. For this, draw a graph of Monte Carlo estimates of I as a function of c. Based on this, suggest an estimate of the optimal shift.