Chapter 4: Time Value of Money

The concept of Time Value of Money:

An amount of money received today is worth more than the same dollar value received a year from now. Why?

Do you prefer a $100 today or a $100 one year from now? why?

- **Consumption forgone has value**
- **Investment lost has opportunity cost**
- **Inflation may increase and purchasing power decrease**

Now,

Do you prefer a $100 today or $110 one year from now? Why?

You will ask yourself one question:
- **Do I have any thing better to do with that $100 than lending it for $10 extra?**
- **What if I take $100 now and invest it, would I make more or less than $110 in one year?**

Note:
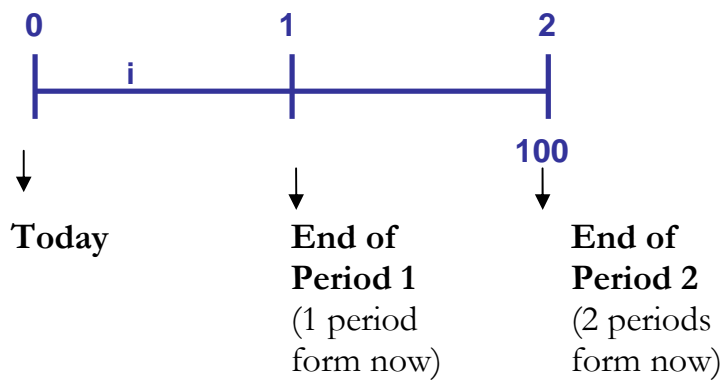**Two elements are important in valuation of cash flows:**
- **What interest rate (opportunity rate, discount rate, required rate of return) do you want to evaluate the cash flow based on?**
- **At what time do these the cash flows occur and at what time do you need to evaluate them?**

Time Lines:

- **Show the timing of cash flows.**
- **Tick marks occur at the end of periods, so Time 0 is today; Time 1 is the end of the first period (year, month, etc.) or the beginning of the second period.**

0	1	2	3
	$i\%$		
CF_0	CF_1	CF_2	CF_3

Example 1 : $100 lump sum due in 2 years

0	1	2
i		
		100
Today	**End of Period 1** (1 period form now)	**End of Period 2** (2 periods form now)

Example 2 : $10 repeated at the end of next three years (ordinary annuity)

| 0	1	2	3
i	**10**	**10**	**10**

Calculations of the value of money problems:

The value of money problems may be solved using

1- Formulas.
2- Interest Factor Tables. (see p.684)
3- Financial Calculators (Basic keys: N, I/Y, PV, PMT, FV).
   I use BAII Plus calculator
4- Spreadsheet Software (Basic functions: PV, FV, PMT, NPER,RATE).
   I use Microsoft Excel.
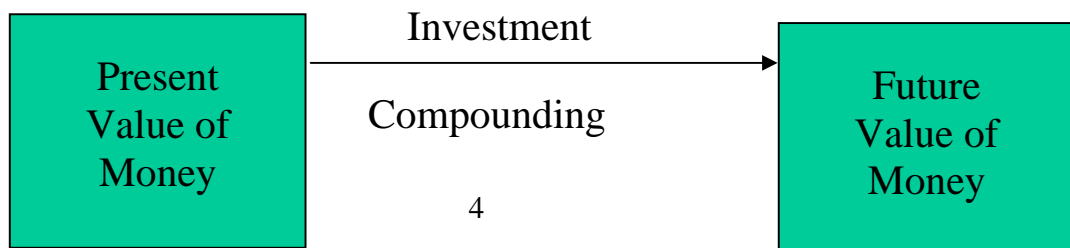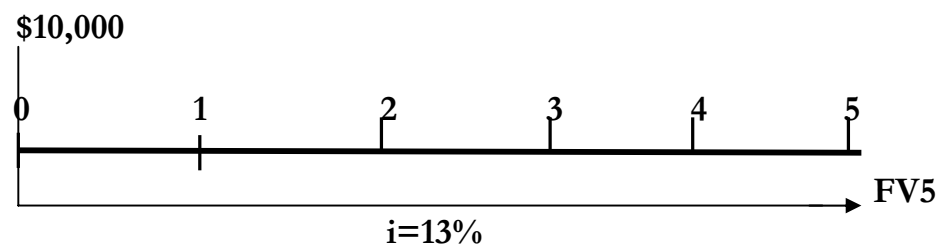
FUTUR VALUE OF A SINGLE CASH FLOW

Examples:

- **You deposited $1000 today in a saving account at BancFirst that pays you 3% interest per year. How much money you will get at the end of the first year ?**

```
        i=3%                    FV1
  ┌─────────────────────────────►
  0                             1
  ├─────────────────────────────┤
  $1000
```
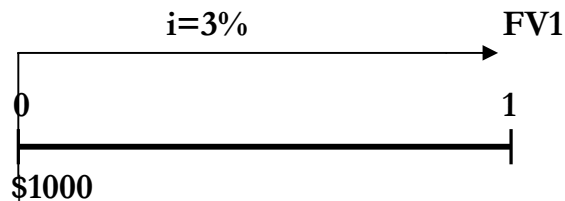
- **You lend your friend $500 at 5% interest provided that she pays you back the $500 dollars plus interest after 2 years. How much she should pay you?**

```
            i=5%                FV2
  ┌─────────────────────────────►
  0           1                 2
  ├───────────┼─────────────────┤
  $500
```
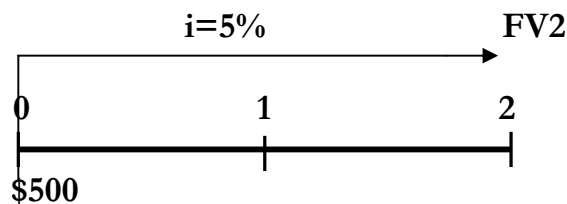
- **You borrowed $10,000 from a bank and you agree to pay off the loan after 5 years from now and during that period you pay 13% interest on loan.**

```
  $10,000
  |
  0      1      2     3      4      5
  ├──────┼──────┼─────┼──────┼──────┤
                                      ► FV5
            i=13%
```

Present Value of Money	Investment	Future Value of Money
	Compounding	

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Detailed calculation:

Simple example:

Invest $100 now at 5%. How much will you have after a year?

FV_1 = PV + INT
        = PV + (PV × i)
        = PV × (1 + i)

FV_1 = $100 + INT
        = $100 + ($100 × .05)
        = $100 + $5
        = $105
Or

FV_1 = $100 × (1+0.05)
        = $100 × (1.05)
        = $105

Another example: Invest $100 at 5% (per year) for 4 years.

0	1	2	3	4
PV = $100	FV_1 = $105	FV_2 = $110.25	FV_3 = $115.76	FV_4 = $121.55
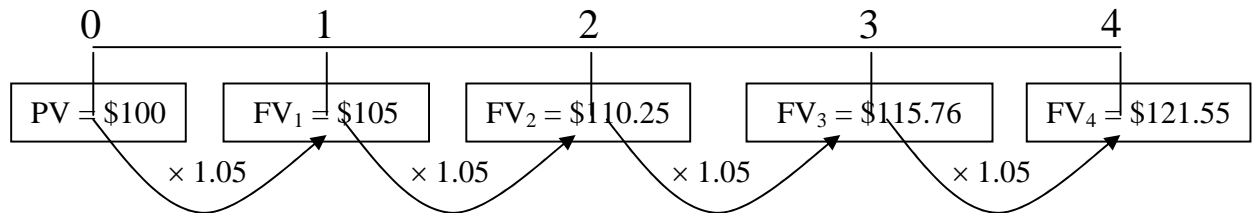
× 1.05   × 1.05   × 1.05   × 1.05

Interest added: + $5.00   + $5.25   + $5.51   + $5.79

$FV_1 = 100 \times (1.05) = \$105$

$FV_2 = 105 \times (1.05) = \$110.25$
$\quad = 100 \times (1.05) \times (1.05) = \$110.25$
$\quad = 100 \times (1.05)^2 = \$110.25$

$FV_3 = 110.25 \times (1.05) = \$115.76$
$\quad = 100 \times (1.05) \times (1.05) \times (1.05) = \$115.76$
$\quad = 100 \times (1.05)^3 = \$115.76$

$FV_4 \quad = \$100 \times (1.05) \times (1.05) \times (1.05) \times (1.05)$
$\qquad\qquad = PV \times (1+i) \times (1+i) \times (1+i) \times (1+i)$
$\qquad\qquad = PV \times (1+i)^4$

In general, the future value of an initial lump sum is: $FV_n = PV \times (1+i)^n$

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To solve for FV, You need
  1- **Present Value (PV)**
  2- **Interest rate per period (i)**
  3- **Number of periods (n)**

Remarks: As PV↑, FV_n↑.
              As i↑, FV_n↑.
              As n↑, FV_n↑.

1- By Formula $$FV_n = PV_0(1+i)^n$$

2- By Table I $$FV_n = PV_0(FVIF_{i,n})$$

$$\Rightarrow FVIF_{i,n} = (1+i)^n$$

3- By calculator (BAII Plus)

Clean the memory: CLR TVM ➜ CE/C   2nd   FV

INPUTS	3	10	-100	0		
	N	I/Y	PV	PMT		
OUTPUT					CPT FV	133.10

Notes:
- To enter (i) in the calculator, you have to enter it in % form.

- Use +/- **To change the sign of a number.**
For example, to enter -100: 100 +/-

- To solve the problems in the calculator or excel, PV and FV <u>cannot</u> have the same sign. If PV is positive then FV has to be negative.

Example:

Jack deposited $1000 in saving account earning 6% interest rate.
How much will jack money be worth at the end of 3 years?

Time line

0	1	2	3

6%

?

1000

Before solving the problem, List all inputs:
I = 6% or 0.06
N= 3
PV= 1000
PMT= 0

Solution:

By formula: $FV_n = PV \times (1+i)^n$

$$FV_3 = \$1000 \times (1+0.06)^3$$
$$= \$1000 \times (1.06)^3$$
$$= \$1000 \times 1.191$$
$$= \$1,191$$

By Table: $FV_n = PV \times FVIF_{i,n}$

$$FV_3 = \$1000 \times FVIF_{6\%,3}$$
$$= \$1000 \times 1.191$$
$$= \$1,191$$

By calculator:

Clean the memory: CLR TVM ➔ | CE/C | | 2nd | | FV |

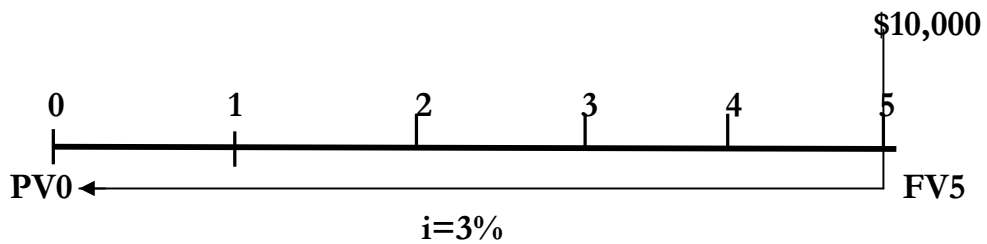| INPUTS	3	6	-1000	0		
	N	I/Y	PV	PMT		
OUTPUT			CPT	FV	1,191.02	

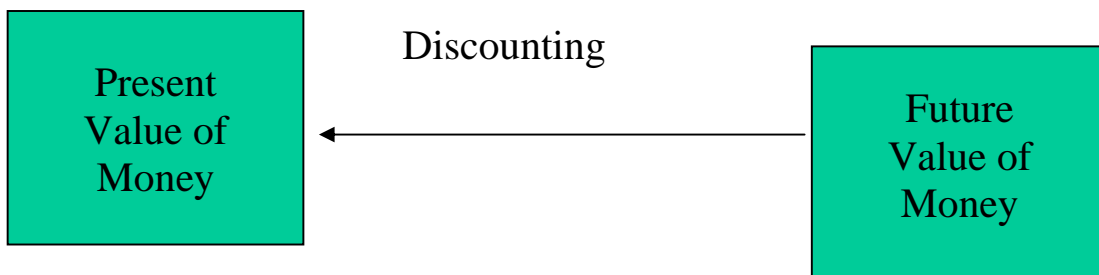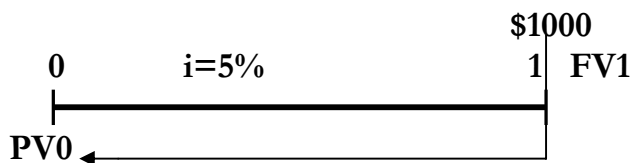**By Excel:**

=FV (0.06, 3, 0,-1000, 0)

PRESENT VALUE OF A SINGLE CASH FLOW

Examples:

- **You need $10,000 for your tuition expenses in 5 years how much should you deposit today in a saving account that pays 3% per year?**

$10,000

```
0        1        2        3        4        5
├────────┼────────┼────────┼────────┼────────┤
PV0 ◄─────────────────────────────────────── FV5
                    i=3%
```

- **One year from now, you agree to receive $1000 for your car that you sold today.**
  How much that $1000 worth today if you use 5% interest rate?

$1000

```
0          i=5%            1   FV1
├───────────────────────────┤
PV0 ◄───────────────────────
```

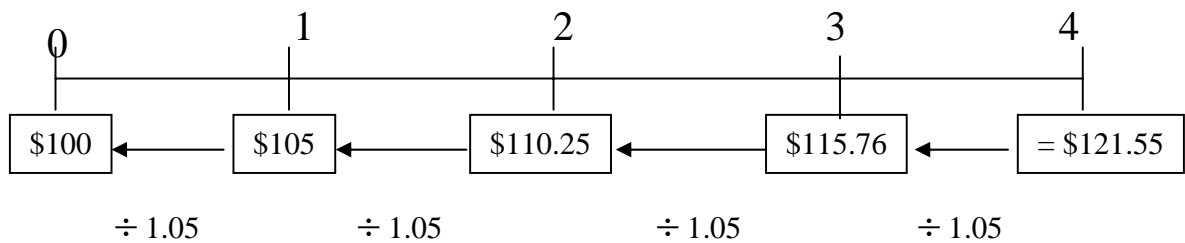|	Discounting	
Present Value of Money	◄───────────	Future Value of Money

Detailed calculation

$$FV_n = PV (1+i)^n$$

$$\Rightarrow PV_0 = \frac{FV_n}{(1+i)^n}$$

$$\Rightarrow PV_0 = FV_n \times \frac{1}{(1+i)^n}$$

Example:

PV$_4$= FV$_4$ = \$121.55

PV$_3$= FV$_4$× [1/(1+i)]
   = \$121.55× [1/(1.05)]
   = \$115.76

PV$_2$= FV$_4$× [1/(1+i)(1+i)]
   = \$121.55× [1/(1.05)(1.05)]
   = \$121.55× [1/(1.05)^2]
   = \$110.25

Or

$PV_2 = FV_3 \times [1/ (1+i)]$

$= \$115.76 \times [1/ (1.05)]$

$= \$110.25$

$PV_1 = FV_4 \times [1/(1+i)(1+i) (1+i)]$

$= \$121.55 \times [1/(1.05)(1.05) (1.05)]$

$= \$121.55 \times [1/(1.05)^3]$

$= \$105$

Or

$PV_1 = FV_2 \times [1/ (1+i)]$

$= \$110.25 \times [1/ (1.05)]$

$= \$105$

$PV_0 = FV_4 \times [1/ (1+i) (1+i) (1+i) (1+i)]$

$= FV_4 \times [1/(1+i)^4]$

$= \$121.55 \times [1/(1.05)(1.05) (1.05) (1.05)]$

$= \$121.55 \times [1/(1.05)^4]$

$= \$100$

In general, the present value of an initial lump sum is: $PV_0 = FV_n \times [1/(1+i)^n]$

To solve for PV, You need
  4- **Future Value (FV)**
  5- **Interest rate per period (i)**
  6- **Number of periods (n)**

Remarks: **As** $FV_n \uparrow$, **PV**$\uparrow$
        As $i\uparrow$, **PV**$\downarrow$
        As $n\uparrow$, **PV**$\downarrow$

1- By Formula $$PV_0 = FV_n \times \frac{1}{(1+i)^n}$$

2- By Table II $$PV_0 = FV_n (PVIF_{i,n})$$

$$\Rightarrow PVIF_{i,n} = \frac{1}{(1+i)^n}$$

3- By calculator (BAII Plus)

Clean the memory: CLR TVM ➜   [CE/C]   [2nd]   [FV]

INPUTS	3	10	133.10	0	
	N	I/Y	FV	PMT	
OUTPUT			CPT	PV	-100

Example:
Jack needed a $1191 in 3 years to be off some debt. How much should jack put in a saving account that earns 6% today?

Time line

Before solving the problem, List all inputs:
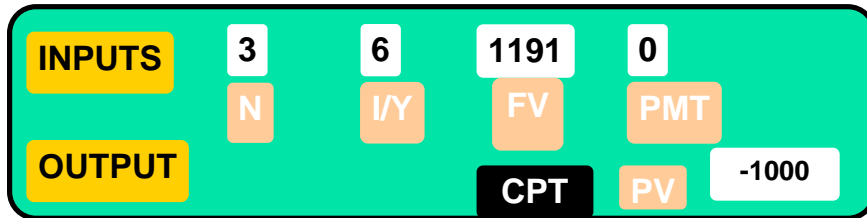I = 6% or 0.06
N= 3
FV= $1191
PMT= 0

Solution:

By formula: $PV_0 = FV_3 \times [1/(1+i)^n]$
$$PV_0 = \$1{,}191 \times [1/(1+0.06)^3]$$
$$= \$1{,}191 \times [1/(1.06)^3]$$
$$= \$1{,}191 \times (1/1.191)$$
$$= \$1{,}191 \times 0.8396$$
$$= \$1000$$

By Table: $= FV_n \times PVIF_{i,n}$
$$PV_0 = \$1{,}191 \times PVIF_{6\%,3}$$
$$= \$1{,}191 \times 0.840$$
$$= \$ 1000$$

By calculator:

Clean the memory: CLR TVM ➔ [CE/C] [2nd] [FV]

INPUTS	3	6	1191	0	
	N	I/Y	FV	PMT	
OUTPUT			CPT	PV	-1000

By Excel:

=PV (0.06, 3, 0, 1191, 0)

Solving for the interest rate i

You can buy a security now for $1000 and it will pay you $1,191 three years from now. What annual rate of return are you earning?

By Formula:
$$i = \left[\frac{FV_n}{PV} \right]^{1/n} - 1$$

$$i = \left[\frac{1191}{1000} \right]^{1/3} - 1 = 0.06$$

By Table:
$$FV_n = PV_0 (FVIF_{i,n})$$

$$\Rightarrow FVIF_{i,n} = \frac{FV_n}{PV_0}$$

$$FVIF_{i,3} = \frac{1191}{1000} = 1.191$$

From the Table I at n=3 we find that the interest rate that yield 1.191 FVIF is 6%

Or
$$PV_0 = FV_n (PVIF_{i,n})$$

$$\Rightarrow PVIF_{i,n} = \frac{PV_0}{FV_n}$$

$$PVIF_{i,3} = \frac{1000}{1191} = 0.8396$$

From the Table II at n=3 we find that the interest rate that yield 0.8396 PVIF is 6%

By calculator:

Clean the memory: CLR TVM ➔ [CE/C] [2nd] [FV]

INPUTS	3	-1000	1191	0	
	N	PV	FV	PMT	
OUTPUT			CPT	I/Y	5.9995

<u>**Solving for _n_:**</u>

Your friend deposits $100,000 into an account paying 8% per year. She wants to know how long it will take before the interest makes her a millionaire.

By Formula: $$n = \frac{\left(Ln\ FV_n\right) - \left(\ln\ PV\right)}{Ln\left(1+i\right)}$$

$$FV_n = \$1,000,000 \quad PV = \$100,000 \quad 1+i = 1.08$$

$$n = \frac{\ln\left(1,000,000\right) - \ln\left(100,000\right)}{\ln(1.08)}$$

$$= \frac{13.82 - 11.51}{0.077} = 30\ years$$

<u>**By Table**</u>: $$FV_n = PV_0\left(FVIF_{i,n}\right)$$

$$\Rightarrow FVIF_{i,n} = \frac{FV_n}{PV_0}$$

$$FVIF_{8,n} = \frac{1,000,000}{100,000} = 10$$

From the Table I at i=8 we find that the number of periods that yield 10 FVIF is 30

Or $$PV_0 = FV_n\left(PVIF_{i,n}\right)$$

$$\Rightarrow PVIF_{i,n} = \frac{PV_0}{FV_n}$$

$$PVIF_{8,n} = \frac{100,000}{1,000,000} = 0.1$$

From the Table II at i=8 we find that the number of periods that yield 0.1 PVIF is 30

By calculator:

Clean the memory: CLR TVM ➔ [CE/C] [2nd] [FV]

| INPUTS	8	-100,000	1,000,000	0
	I/Y	PV	FV	PMT
OUTPUT		CPT N		29.9188

FUTURE VALUE OF ANNUTIES

An annuity is a series of *equal payments* at fixed intervals for a specified number of periods.

PMT = the amount of periodic payment

<u>Ordinary (deferred) annuity</u>: Payments occur at the <u>end</u> of each period.

<u>Annuity due</u>: Payments occur at the *beginning* of each period.

Ordinary

0   1   2   3

i

PM   PM   PM

Due

0   1   2   3

i

PM   PM   PM
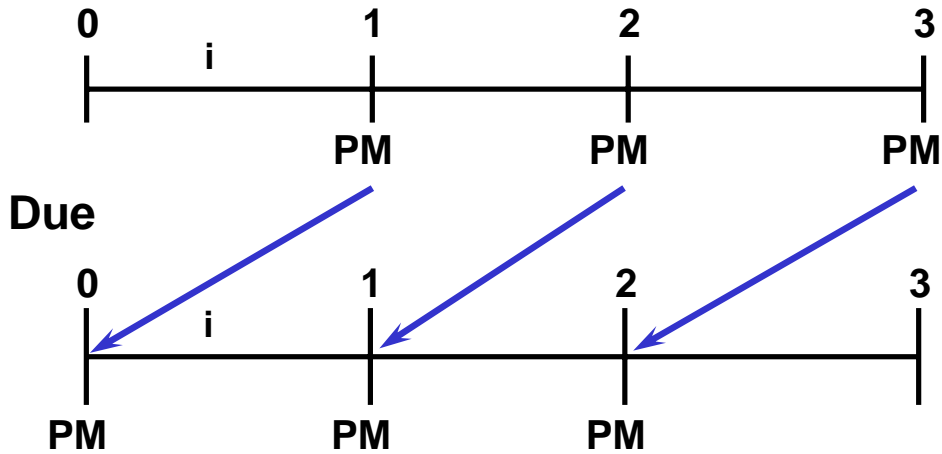
Example: Suppose you deposit $100 at the end of each year into a savings account paying 5% interest for 3 years. How much will you have in the account after 3 years?

0   1   2   3

5%

100   100   100.00

105.00

110.25

$315.25

Time   0   1   2   3   4   ....   n-1   n

PMT PMT PMT PMT   PMT PMT
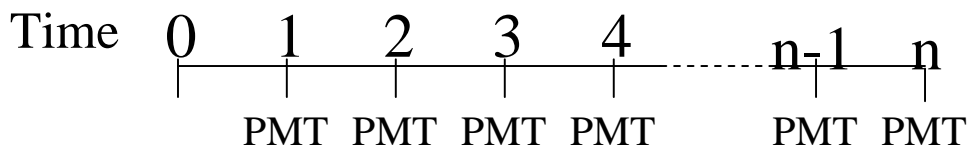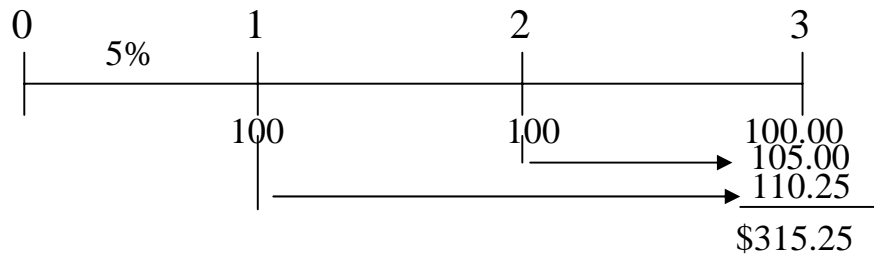
$$FVAN_n = PMT\left(1+i\right)^{n-1} + PMT\left(1+i\right)^{n-2} + ... + PMT$$

(Hard to use this formula)

$$FVAN_n = PMT \left[\frac{(1+i)^n - 1}{i} \right]$$

$$= PMT \,(FVIFA_{i,n})$$

Future Value Interest Factor for an Annuity

Note: **For an *annuity due*, simply multiply the answer above by (1+i).**

So $FVAND_n$ (annuity due) $= PMT\,(FVIFA_{i,n})(1+i)$.

$$= PMT \left[\frac{(1+i)^n - 1}{i} \right](1+i)$$

Annuity:

Years

0	1	2	3

$PMT_1 = \$1,000$ $PMT_2 = \$1,000$ $PMT_3 = \$1,000$

Earns no Interest

Compounded 1 Year
$1,000(1.06)^1$
or $1,000(FVIF_{6,1})$

$\$1,000$

$\$1,060$

Compounded 2 Years
$1,000(1.06)^2$
or $1,000(FVIF_{6,2})$

$\$1,124$

$FVAN_3 = \$3,184$
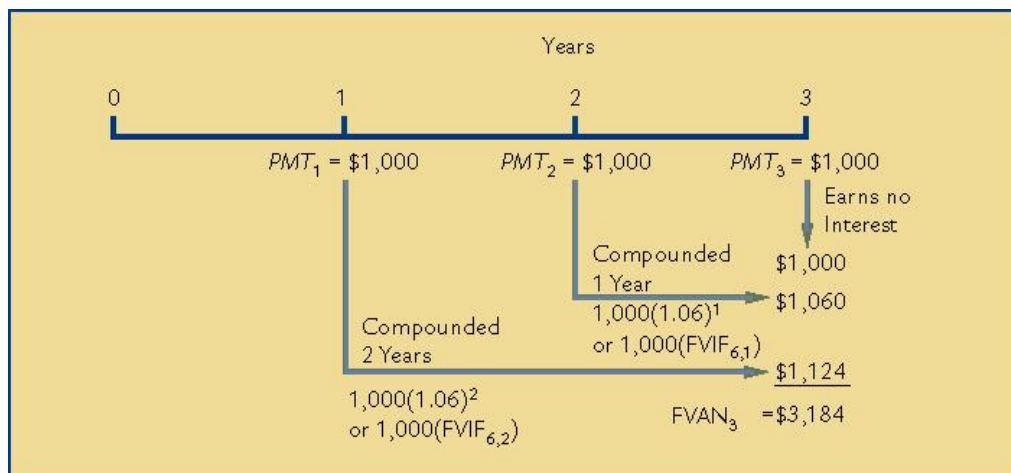
Annuity Due:
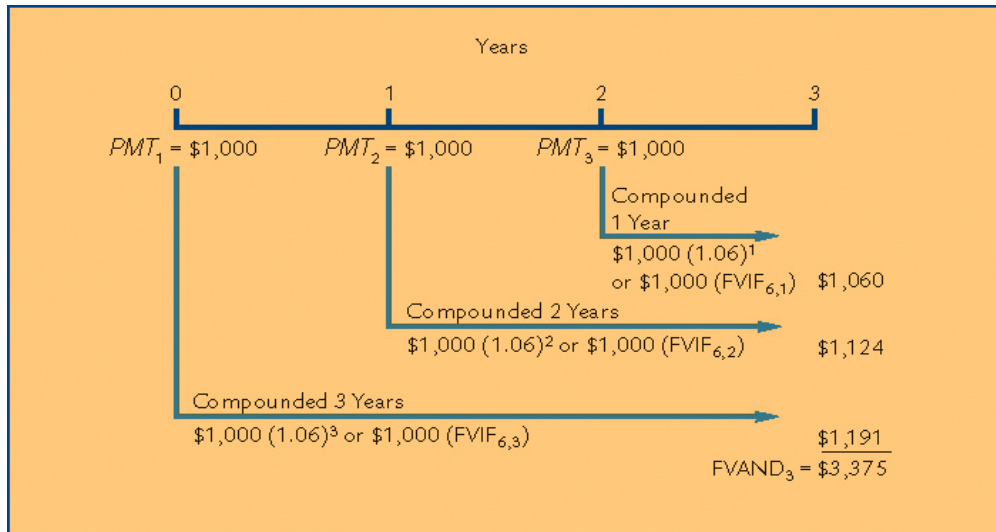
Remark:

$$FVIFA_{i,3} = FVIF_{i,2} + FVIF_{i,1} + FVIF_{i,0}$$

To solve for the future value of Annuities, You need:
  1-Payemnt or annuity amount (PMT)
  2-Interest rate per period (i)
  3-Number of periods (n)

1-BY Formula:

$$FVAN_n = PMT \left[\frac{(1+i)^n - 1}{i} \right] \quad ==\rightarrow \text{Ordinary Annuity}$$

$$FVAND_n = PMT \left[\frac{(1+i)^n - 1}{i} \right] (1+i) \quad ==\rightarrow \text{Annuity Due}$$

$$FVAND_n = FVAN_n (1+i)$$

2- BY Table III:

$$FVAN_n = PMT (FVIFA_{i,n}) \quad ==\rightarrow \text{Ordinary Annuity}$$

$$FVAND_n = PMT (FVIFA_{i,n}) (1+i) \quad ==\rightarrow \text{Annuity Due}$$

3- BY calculator:

Ordinary Annuity:

1- Clean the memory: CLR TVM➔ `CE/C` `2nd` `FV`

2- Set payment mode to END of period: BGN ➔ `2nd` `PMT`
                                              SET ➔ `2nd` `ENTER`

3- Make sure you can see END written on the screen then press `CE/C`

NOTE: If you do not see BGN written on the upper right side of the screen, you can skip
Step 2 and 3.
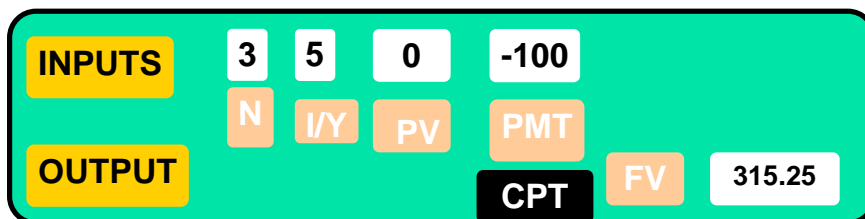
INPUTS	3	5	0	-100		
	N	I/Y	PV	PMT		
OUTPUT				CPT	FV	315.25

Annuity Due:

Clean the memory: CLR TVM ➔ `CE/C` `2nd` `FV`

Set payment mode to BGN of period: BGN ➔ `2nd` `PMT`
SET ➔ `2nd` `ENTER`

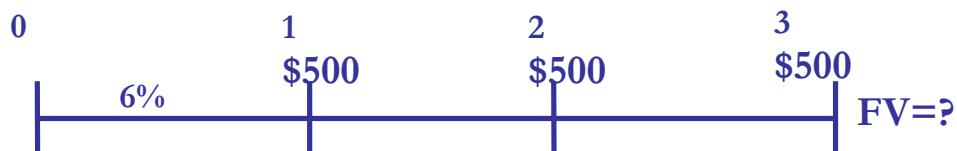**Make sure you can see BGN written on the screen then press** `CE/C`

INPUTS	3	5	0	-100		
	N	I/Y	PV	PMT		
OUTPUT					FV	331.10
				CPT		

Example:

You agree to deposit $500 at the end of every year for 3 years in an investment fund that earns 6%.

Time line

0	1	2	3
6%	$500	$500	$500
			FV=?

Before solving the problem, List all inputs:

I = 6% or 0.06

N= 3

PMT=500

PV= 0

FV=?

Solution:

By formula: $FVAN_n = PMT \left[\dfrac{(1+i)^n - 1}{i} \right]$

$$= 500 \left[\frac{(1+0.06)^3 - 1}{0.06} \right] = 500 \left[\frac{1.191 - 1}{0.06} \right] = 1,591.80$$

By Table: $FVAN_n = PMT \, (FVIFA_{i,n})$

$$FVAN_3 = 500(FVIFA_{6,3})$$

$$= 500(3.184) = 1,592$$

By calculator:

Clean the memory: CLR TVM➔ ▮ CE/C ▮ 2nd ▮ FV

Make sure you do not see BGN written on the upper right side of the screen.

INPUTS	3	6	0	-500		
	N	I/Y	PV	PMT		
OUTPUT					FV	1,591.80
				CPT		

By Excel: =FV (0.06, 3, -500, 0, 0)

Now assume that you deposit the $500 at the beginning of the year not at the end of the year.

Time line

0	1	2	3
$500	$500	$500	FV=?

6%
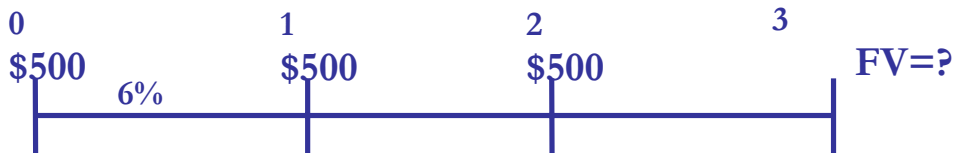
Before solving the problem, List all inputs:
I = 6% or 0.06
N= 3
PMT=500 (beg)
PV= 0
FV=?

Solution:

By formula:

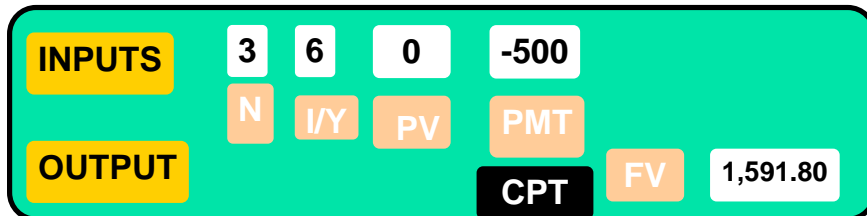$$FVAND_n = PMT \left[\frac{(1+i)^n - 1}{i} \right] (1+i)$$

$$FVAND_3 = 500 \left[\frac{(1+0.06)^n - 1}{0.06} \right] (1+0.06)$$

$$= 500 \left[\frac{0.191}{0.06} \right] (1.06) = 1,687.30$$

By Table:

$$FVAND_n = PMT \, (FVIFA_{i,n})(1+i)$$

$$FVAND_3 = 500(FVIFA_{6,3})(1+0.06)$$

$$= 500(3.184)(1.06) = 1,687.52$$

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By calculator:

Clean the memory: CLR TVM ➔ [CE/C] [2nd] [FV]

Set payment mode to BGN of period: BGN ➔ [2nd] [PMT]
SET ➔
[2nd] [ENTER]

Make sure you can see BGN written on the screen then press [CE/C]

INPUTS	3	6	0	-500		
	N	I/Y	PV	PMT		
OUTPUT					FV	1,687.31
				CPT		
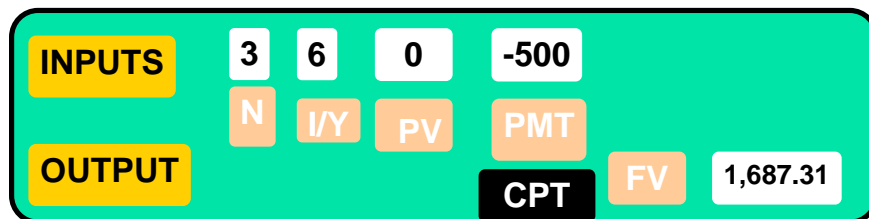
By Excel: =FV (0.06, 3, -500, 0, 1)

PRESENT VALUE OF ANNUTIES

Problem: You have a choice
a) $100 paid to you at the end of each of the next 3 years   or
b) a lump sum today.

i = 5%, since you would invest the money at this rate if you had it.

How big does the lump sum have to be to make the choices equally good?

$$\text{Time}$$

$$95.24 \xleftarrow{\div 1.05}$$
$$90.70 \xleftarrow{\div 1.05^2}$$
$$\underline{86.38} \xleftarrow{\div 1.05^3}$$
$$\text{PVAN}_3 = 272.32$$
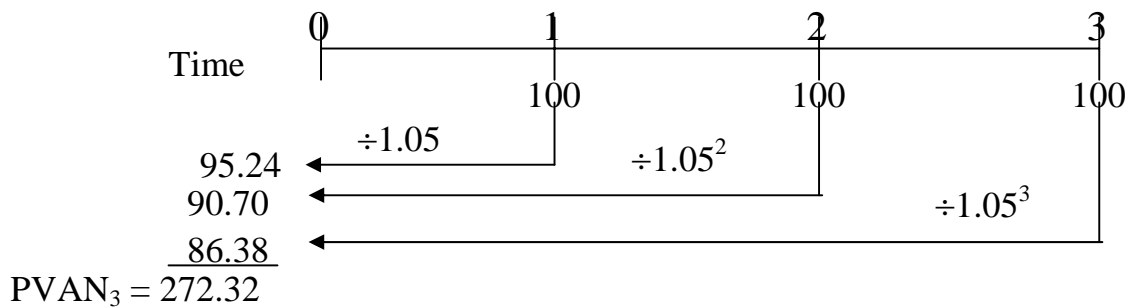
Formula:

$$PVA_n = \frac{PMT}{(1+i)^1} + \frac{PMT}{(1+i)^2} + ... + \frac{PMT}{(1+i)^n}$$

$$= PMT \left[\frac{1 - \dfrac{1}{(1+i)^n}}{i} \right]$$

$$= PMT \left(PVIFA_{i,n} \right) \qquad \boxed{\textit{Present Value Interest Factor}}$$

$$PVA_3 = \$100 \left[\dfrac{1 - \dfrac{1}{1.05^3}}{.05} \right]$$

$$= \$100(2.7232) = \$272.32$$

Note: For *annuities due*, simply multiply the answer above by (1+i)
$PVAND_n$ (annuity due) = PMT ($PVIFA_{i,n}$) (1+i)

To solve for the present value of Annuities, You need:
  1-Payemnt or annuity amount (PMT)
  2-Interest rate per period (i)
  3-Number of periods (n)

  1- BY Formula:

$$PVAN_n = PMT \left[\dfrac{1 - \dfrac{1}{(1+i)^n}}{i} \right] \quad ==\rightarrow \text{Ordinary Annuity}$$

$$PVAND_n = PMT \left[\dfrac{1 - \dfrac{1}{(1+i)^n}}{i} \right](1+i) \quad ==\rightarrow \text{Annuity Due}$$

$$PVAND_n = PVAN_n(1+i)$$

2- BY Table IV:

$$PVAN_n = PMT(PVIFA_{i,n}) \qquad == \rightarrow \text{Ordinary Annuity}$$

$$PVAND_n = PMT(PVIFA_{i,n})(1+i) \qquad == \rightarrow \text{Annuity Due}$$

3- BY calculator:

Ordinary Annuity:

Clean the memory: CLR TVM➔ | CE/C | 2nd | FV |
Make sure you do not see BGN written on the upper right side of the screen.

INPUTS	3	5	0	-100		
	N	I/Y	FV	PMT		
OUTPUT				CPT	PV	272.32

Annuity Due:

Clean the memory: CLR TVM ➔ `CE/C` `2nd` `FV`

Set payment mode to BGN of period: BGN ➔ `2nd` `PMT`
SET ➔
`2nd` `ENTER`

Make sure you can see BGN written on the screen then press `CE/C`

INPUTS	3	5	0	-100		
	N	I/Y	FV	PMT		
OUTPUT					PV	285.94
				CPT		
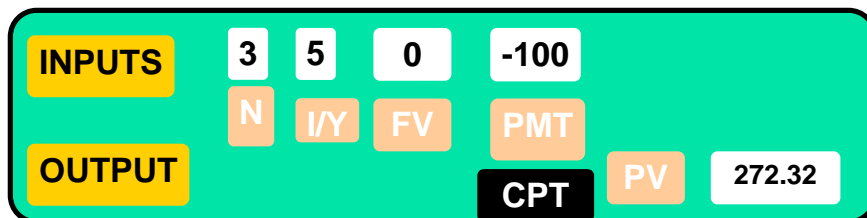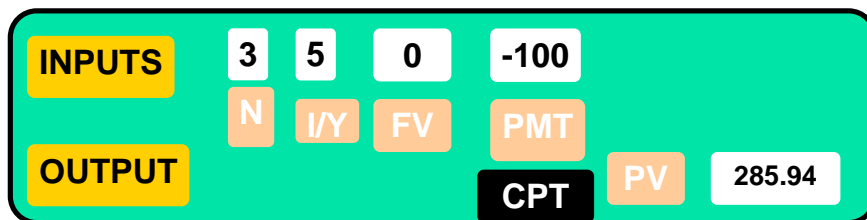
Example:

You agree to receive $500 at the end of every year for 3 years in an investment fund that earns 6%.

Time line

0	1	2	3
PV=?	$500	$500	$500

6%

Before solving the problem, List all inputs:

I = 6% or 0.06

N= 3

PMT=500

FV= 0

PV=?

Solution:

By formula:

$$PVAN_n = PMT \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right]$$

$$PVAN_n = 500 \left[\frac{1 - \frac{1}{(1+0.06)^3}}{0.06} \right] = 500 \left[\frac{1 - \frac{1}{1.191}}{0.06} \right] = \$1,336.51$$

By Table: $PVAN_n = PMT(PVIFA_{i,n})$

$$PVAN_3 = 500(PVIFA_{6,3})$$

$$= 500(2.673) = 1,336.51$$

By calculator:

Clean the memory: CLR TVM➔ CE/C 2nd FV

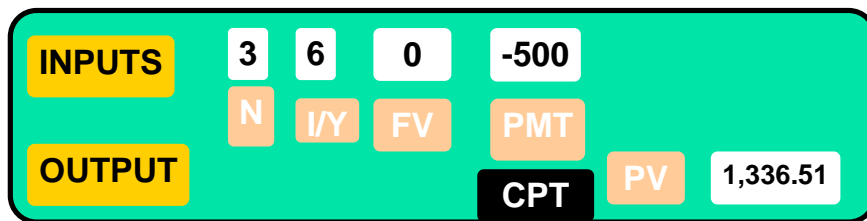**Make sure you do not see BGN written on the upper right side of the screen.**

INPUTS	3	6	0	-500		
	N	I/Y	FV	PMT		
OUTPUT					PV	1,336.51
				CPT		

By Excel: =PV (0.06, 3, -500, 0, 0)

Now assume that you receive the $500 at the beginning of the year not at the end of the year.

Time line

```
0                    1              2              3
$500                $500          $500
      6%
|_____|_____|_____|
PV=?
```

Before solving the problem, List all inputs:
I = 6% or 0.06
N= 3
PMT=500 (beg)
FV= 0
PV=?

Solution

By formula:

$$PVAND_n = PMT \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right] (1+i)$$

$$PVAND_n = 500 \left[\frac{1 - \frac{1}{(1+0.06)^3}}{0.06} \right] (1+0.06) = 500 \left[\frac{1 - \frac{1}{1.191}}{0.06} \right] (1.06)$$

$$= 1,416.70$$

By Table:

$$PVAND_n = PMT(PVIFA_{i,n})(1+i)$$

$$PVAND_3 = 500(PVIFA_{6,3})(1+0.06)$$

$$= 500(2.673)(1.06) = 1,416.69$$

By calculator:

Clean the memory: CLR TVM ➜ [CE/C] [2nd] [FV]

Set payment mode to BGN of period: BGN ➜ [2nd] [PMT]

SET ➜ [2nd] [ENTER]

Make sure you can see BGN written on the screen then press [CE/C]

INPUTS	3	6	0	-500		
	N	I/Y	FV	PMT		
OUTPUT				CPT	PV	1,416.69
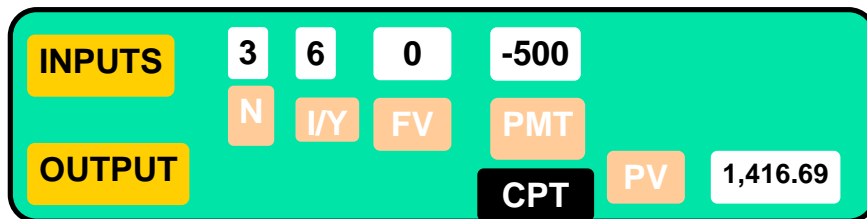
By Excel: =PV (0.06, 3, -500, 0, 1)

Perpetuities

A perpetuity is an annuity that continues forever.

$$PVAN_n = PMT \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right]$$

As n gets very large, $\dfrac{1}{(1+i)^n} \to 0$

$$PVPER_0 (perpetuity) = PMT \times \left[\frac{1-0}{i} \right] = PMT \times \left(\frac{1}{i} \right) = \frac{PMT}{i}$$
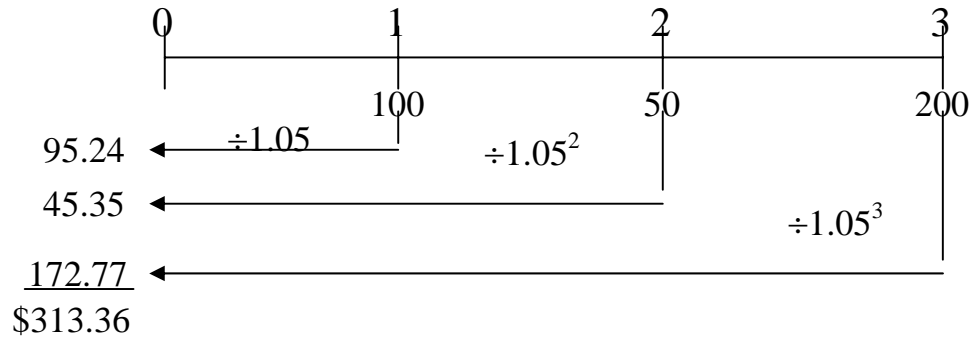
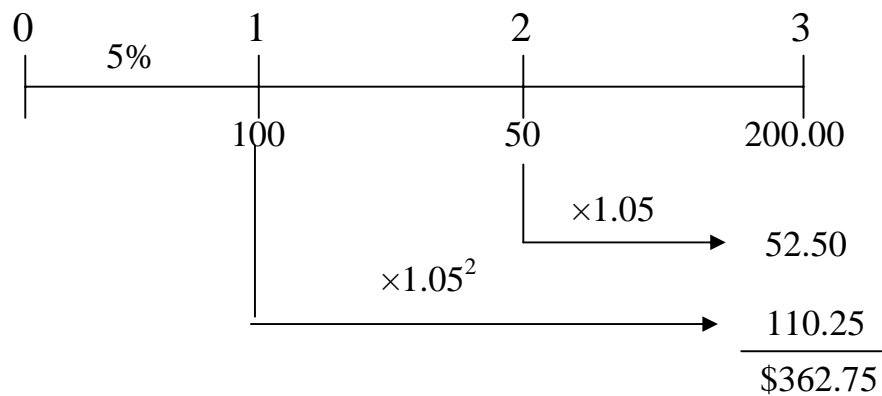**Formula:**

$$PVPER_0 = \frac{PMT}{i}$$

UNEVEN CASH FLOWS

How do we get PV and FV when the periodic payments are unequal?

Present Value

```
        0            1            2            3
        |            |            |            |
                    100           50          200
95.24  ←──÷1.05──────┘
45.35  ←────────÷1.05²───────────┘
172.77 ←──────────────÷1.05³──────────────────┘
$313.36
```

$$PV = CF_0 + \frac{CF_1}{1+i} + \frac{CF_2}{(1+i)^2} + ... + \frac{CF_n}{(1+i)^n}$$

Future Value

```
        0            1            2            3
        |     5%     |            |            |
                    100           50         200.00
                                  └──×1.05──→  52.50
                     └─────×1.05²────────────→ 110.25
                                              $362.75
```

$$FV = CF_0(1+i)^n + CF_1(1+i)^{n-1} + ... + CF_n(1+i)^0$$

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Example:

<u>Present Value of Uneven Cash Flows</u>

Years

0	1	2	3
	$100	$200	$300

$90.9 ← Discounted 1 Year
$100/(1.10)^1
or $100 (PVIF$_{10,1}$)

$165.2 ← Discounted 2 Years
$200/(1.10)^2 or $200 (PVIF$_{10,2}$)

$225.3 ← Discounted 3 Years
$300/(1.10)^3 or $300 (PVIF$_{10,3}$)
_____
$481.4
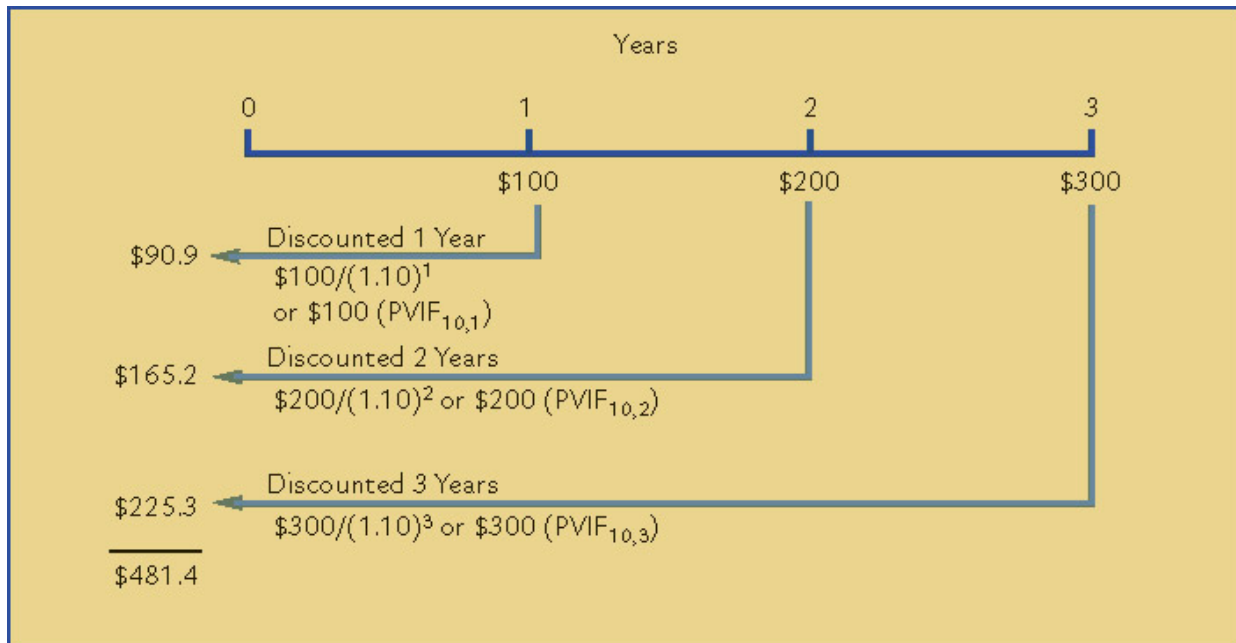
By Calculator:

Clean the memory: **CF** **2nd** **CE/C**

Input cash flows in the calculator's CF register:
CF0 = 0 ➜ 0 **ENTER**

CF1 = 100 ➜ **↓** C01 100 **ENTER** **↓** F01 1 **ENTER**

CF2 = 200 ➜ **↓** C02 200 **ENTER** **↓** F02 1 **ENTER**

CF3 = 300 ➜ **↓** C03 300 **ENTER** **↓** F03 1 **ENTER**

Press **NPV** , then the it will ask you to enter the Interest rate (I)
Enter I = 10 ➜ 10 **ENTER**

Use **↓** to get to the NPV on the screen
When you read NPV on the screen, press **CPT**
You will get NPV = $481.59 (Here NPV = PV.)

NOTE:

To calculate the future value of uneven cash flows, it is much easier to start by calculating the Present value of the cash flows using NPV function then calculate the future value using the future value of a single cash flow rules. The single cash flow in this case will be the present value.

Simple and Compound Interest

- **Simple Interest**
  - **Interest paid on the principal sum only**

- **Compound Interest**
  - **Interest paid on the principal and on interest**

Example:

Calculate the future value of $1000 deposited in a saving account for 3 years earning 6% . Also, calculate the simple interest, the interest on interest, and the compound interest.

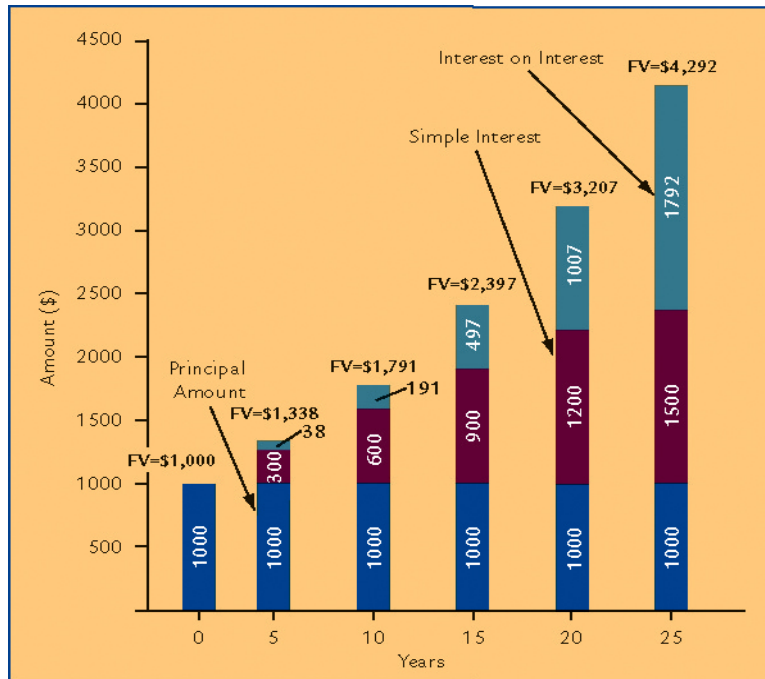**$FV3 = 1000\ (1.06)^3 = \$1{,}191.02$**

Principal = PV = \$1000
Compound interest = FV – PV = 1191.02 – 1000 = 191.02
Simple Interest = PV * i * n =1000 * 0.06 * 3 = \$180
Interest on interest = Compound interest - Simple Interest = 191.02 – 180 = 11.02

Effect of Compounding over Time

Other Compounding Periods

So far, our problems have used <u>annual compounding.</u> In practice, interest is usually compounded more frequently.
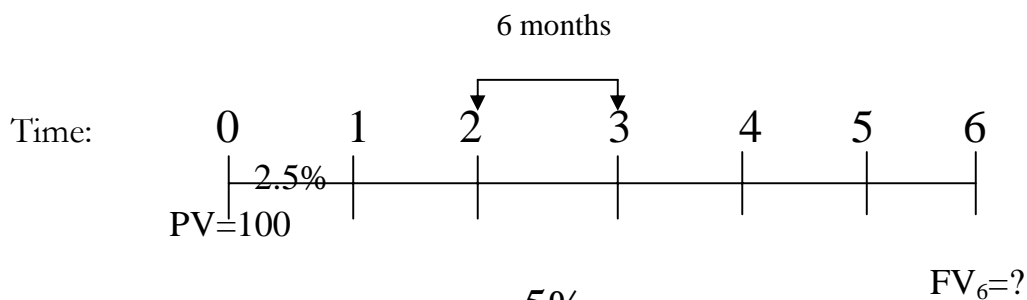
Example: You invest $100 today at 5% interest for 3 years.

Under annual compounding, the future value is:

$$FV_3 = PV(1+i)^3$$
$$= \$100(1.05)^3$$
$$= \$100(1.1576)$$
$$= \$115.76$$

What if interest is compounded <u>semi-annually</u> (twice a year)?

Then the periods on the time line are no longer years, but half-years!

$$i = \text{Periodic interest rate} = \frac{5\%}{2} = 2.5\%$$
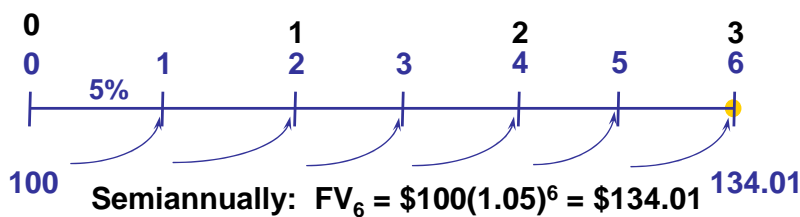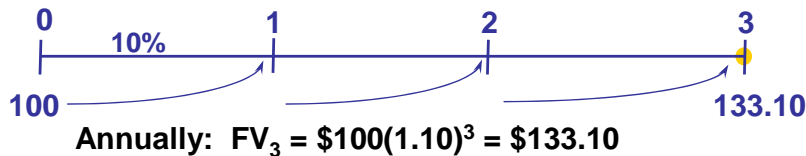
$$n = \text{No. of periods} = 3 \times 2 = 6$$

$$FV_n = PV(1+i)^n$$

$$FV_6 = \$100(1.025)^6$$
$$= \$100(1.1597)$$
$$= \$115.97$$

Note: the final value is slightly higher due to more frequent compounding.

Will the FV of a lump sum be larger or smaller if compounded more often, holding the stated I% constant?

- LARGER, as the more frequently compounding occurs, interest is earned on interest more often.

```
0              1              2              3
|    10%       |              |              |
100 ———————————————————————————————————> 133.10
        Annually:  FV₃ = $100(1.10)³ = $133.10
```

Annually: $FV_3 = \$100(1.10)^3 = \$133.10$

```
0                1              2              3
0        1       2       3      4      5      6
|   5%   |       |       |      |      |      |
100 ——————————————————————————————————————> 134.01
        Semiannually:  FV₆ = $100(1.05)⁶ = $134.01
```

Semiannually: $FV_6 = \$100(1.05)^6 = \$134.01$

Important: When working *any* time value problem, make sure you keep straight what the relevant periods are!
n = the number of periods
i = the periodic interest rate

<u>From now on:</u>
n = m*n
i = i/m

Where m = 1   for annual compounding
    m = 2   for semiannual compounding
    m = 4   for quarterly compounding
    m = 12   for monthly compounding
    m = 52   for weekly compounding
    m = 365   for daily compounding

For continuously compounding: $(1+i)^n = e^{in}$
   =➜ FVn = PV $(e)^{in}$
   =➜ PV = FVn $(e)^{-}$

EFFECTIVE INTREST RATE

You have two choices:
1- 11% annual compounded rate of return on CD
2- 10% monthly compounded rate of return on CD

How can you compare these two nominal rates?

A nominal interest rate is just a stated (quoted) rate. An APR (annual percentage rate) is a nominal rate.

For every nominal interest rate, there is an effective rate.

The effective annual rate is the interest rate actually being earned per year.

To compare among different nominal rates or to know what is the actual rate that you're getting on any investment you have to use the Effective annual interest rate.

Effective Annual Rate: $i_{eff} = \left(1 + \dfrac{i}{m}\right)^m - 1$

To compare the two rates in the example,

1- $i_{eff} = \left(1 + \dfrac{0.11}{1}\right)^1 - 1 = 0.11$ or 11% **(Nominal and Effective rates are equal in annual compounding)**

2- $i_{eff} = \left(1 + \dfrac{0.10}{12}\right)^{12} - 1 = 0.1047$ or 10.47 %

You should choose the first investment.

To compute effective rate using calculator:

ICONV ➔ [2nd] [2]

Enter Nominal Rate ➔ NOM   10   [ENTER]

Enter compounding frequency per year (m) ➔ [↑] C/Y   12   [ENTER]

Compute the Effective rate ➔ [↑] EFF [CPT]

Nominal Versus Real Interest Rate

Nominal rate r_f is a function of:

➢ Inflation premium i_n :compensation for inflation and lower purchasing power.

➢ Real risk-free rate r_f' : compensation for postponing consumption.

$$(1 + r_f) = (1 + r_f')(1 + i_n)$$

$$r_f = r_f' + i_n + r_f' i_n$$

$$r_f \approx r_f' + i_n$$

Amortized Loans

An amortized loan is repaid in equal payments over its life.

Example: You borrow $10,000 today and will repay the loan in equal installments at the end of the next 4 years. How much is your annual payment if the interest rate is 9%?

Time | 0 9% 1   2   3   4

PVA N= $10.000   PMT   PMT   PMT   PMT

Inputs:

The periods are years. (m=1)
n = 4
i = 9%
PVAN$_4$ = $10,000
FV =0
PMT = ?

$$PVAN_4 = PMT\ (PVIFA_{9\%,4})$$

$$\$10,000 = PMT\ (3.240)$$

$$PMT = \frac{\$10,000}{3.240} = \$3,087$$

Table 5.7 Loan Amortization Table

Year	Beginning Balance	Annual Payment	Interest Amount	Principal Reduction	Ending Balance
1	$10,000	$3,087	$900	$2,187	$7,813
2	7,813	3,087	703	2,384	5,429
3	5,429	3,087	489	2,598	2,831
4	2,831	3,087	255	2,832	0

Interest amount = Beginning balance * i
Principal reduction = annual payment - Interest amount
Ending balance = Beginning balance - Principal reduction

Beginning balance: Start with principal amount and then equal to previous year's ending balance.

As a loan is paid off:

- at the beginning, much of each payment is for interest.

- later on, less of each payment is used for interest, and more of it is applied to paying off the principal.