Problem 9.6

Let $\textsc{Setmin}(A, B)$ be a function with two input sets A and B and output

$$\textsc{Setmin}(A, B) = \begin{cases} A & \text{if } |A| \le |B| \text{ and } A \ne \emptyset \\ B & \text{otherwise} \end{cases}$$

where $|\cdot|$ denotes the cardinality of a set.

We could write down the following state transition equations:

$$S_{i,v} = \begin{cases} S_{i-1,v} & \text{if } v < v_i \\ \{i\} & \text{if } v = v_i \\ \textsc{Setmin}\big(S_{i-1,v}, S_{i-1,v-v_i} \cup \{i\}\big) & \text{if } v > v_i \end{cases}$$

with initial conditions

$$S_{1,v} = \begin{cases} \{1\} & \text{if } v = v_1 \\ \emptyset & \text{otherwise} \end{cases}$$

There are $l \times l v_{\max} = l^2 v_{\max}$ such $S_{i,v}$ in all and for each one we need constant overhead to calulcate. So the overall time complexity is $O(l^2 v_{\max})$.

One need $\log(v_{\max})$ bits to record the input data, hence the complexity is exponential in the input scale.

Problem 9.7

Let $v_i' = \lfloor v_i / K \rfloor$ be the scaled and truncated value. Note that $K\omega_i \le v_i < (K+1)\omega_i$, which means that picking up an object with value v_i introduces error $\varepsilon < K$ in the overall value.

Let $\boldsymbol{\sigma}^\star \in \{0,1\}^l$ be the optimal solution to the original $\textsc{Knapsack}$ and $v_{\text{opt}} = \sum \sigma_i v_i$ be the associated max value.

We here construct an solution to $\textsc{Approximate-Knapsack}$ and require it to be an $(1-\varepsilon)$-approximation to v_{opt}.

Let $|\boldsymbol{\sigma}^\star|$ be the number of 1's in $\boldsymbol{\sigma}$ and $v_{\text{trunc}} \equiv \max_{w_i \le W} v_i$.

$$\text{total error} \le |\sigma^\star| K \le lK \le \varepsilon v_{\min} \le \varepsilon v_{\text{opt}}$$

For the third inequality to hold, one must have

$$K \le \frac{\varepsilon v_{\text{trunc}}}{l}$$

and we choose

$$K = \frac{\varepsilon v_{\text{trunc}}}{l}$$

where $v_{\text{trunc}} \equiv \max_{w_i \le W} v_i$.

Problem 9.16

To the linear program

$$\max_{\boldsymbol{x}} (\boldsymbol{c}^T \boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \text{ and } \boldsymbol{x} \ge \boldsymbol{0},$$

we associate an auxillary one

$$\min_{\boldsymbol{x},\boldsymbol{s}} \sum s_i \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \text{ and } \boldsymbol{s} \geq \boldsymbol{0}$$

Choose $\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{s} = \boldsymbol{b}$ as the initial solution and try solving the auxillary program.

If the original program has a feasible solution that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$, the auxillary one should reach an optimal solution where $\boldsymbol{s} = \boldsymbol{0}$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. That is exactly an feasible solution of the original program.