RODOLFO GUIDOBONO

CURSO DE RELATIVIDAD GENERAL
J. GARCÍA - EJERCICIOS CAP. 59

HOJA N° 1

FECHA 19/7/19

1) Calcular las componentes del tensor de Ricci para la métrica de un campo gravitatorio uniforme y débil.

Métrica $g_{00} = -1 - 2V/c^2$ $g_{11} = g_{22} = g_{33} = 1$ resto $g_{\alpha\beta} = 0$

$g^{00} \approx -1 + 2V/c^2$ (Taylor) $g^{11} = g^{22} = g^{33} = 1$

Símbolos de Christoffel

$$\Gamma^{\alpha}_{00} = \frac{1}{c^2} g^{\alpha\mu} \partial_\mu V$$

$$\Gamma^{0}_{\beta 0} = \frac{1}{c^2} \frac{1}{1+2V/c^2} \partial_\beta V \approx \frac{1}{c^2}\left(1 - \frac{2V}{c^2}\right) \partial_\beta V$$

Tensor de Ricci (f. 50.1, Crul)

$$R_{\sigma\nu} = \partial_\mu \Gamma^{\mu}_{\nu\sigma} - \partial_\nu \Gamma^{\mu}_{\mu\sigma} + \Gamma^{\mu}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}$$

tener en cuenta $V^2 \to 0$, $V\partial V \to 0$ $\partial V \partial V \to 0$

aproximación campo débil

$$R_{\sigma\nu} \approx \partial_\mu \Gamma^{\mu}_{\nu\sigma} - \partial_\nu \Gamma^{\mu}_{\mu\sigma}$$

Propiedad del Tensor de Ricci f. 16.4 (Crul) $R_{ij} = R_{ji}$

Se calculan entonces las siguientes componentes

R_{00} R_{01} R_{02} R_{03}

R_{11} R_{12} R_{13}

R_{22} R_{23}

R_{33}

(f 59.3 - Crul) $\longrightarrow$ $\boxed{R_{00} \cong \frac{1}{c^2} \nabla^2 V}$

NOTA

HOJA Nº 2

FECHA

$$R_{11} = \partial_\mu \underbrace{\Gamma^\mu_{11}}_{=0} - \partial_1 \Gamma^\mu_{\mu 1} = -\left[\partial_1 \underbrace{\Gamma^0_{01}} + \partial_1 \underbrace{\Gamma^1_{11}}_{=0} + \partial_1 \underbrace{\Gamma^2_{21}}_{=0} + \partial_1 \underbrace{\Gamma^3_{31}}_{=0}\right]$$

$$= -\partial_1 \left(\frac{1}{c^2}\left(1-{}^{2V}/_{c^2}\right)\partial_1 V\right) = -\frac{1}{c^2}\left[-\frac{2}{c^2}\underbrace{\partial_1 V \cdot \partial_1 V}_{\partial V \cdot \partial V \to 0} + \left(1-\frac{2V}{c^2}\right)\partial_1^2 V\right]$$

$$= -\frac{1}{c^2}\partial_1^2 V + \frac{1}{c^4} \underbrace{2V\,\partial_1^2 V}_{V\partial^2 V \to 0}$$

$$\boxed{R_{11} = -\frac{1}{c^2}\partial_1^2 V}$$

$$R_{22} = \partial_\mu \underbrace{\Gamma^\mu_{22}}_{=0} - \partial_2 \Gamma^\mu_{\mu 2} = -\left[\partial_2 \Gamma^0_{02} + \partial_2 \underbrace{\Gamma^1_{12}}_{=0} + \partial_2 \underbrace{\Gamma^2_{22}}_{=0} + \partial_2 \underbrace{\Gamma^3_{32}}_{=0}\right]$$

$$= -\partial_2\left[\frac{1}{c^2}\left(1-{}^{2V}/_{c^2}\right)\partial_2 V\right) = -\frac{1}{c^2}\left[-\frac{2}{c^2}\underbrace{\partial_2 V \cdot \partial_2 V}_{\partial V \cdot \partial V \to 0} + \underbrace{\left(1-\frac{2V}{c^2}\right)\partial_2^2 V}_{V \cdot \partial V \to 0}\right]$$

$$\boxed{R_{22} = -\frac{1}{c^2}\partial_2^2 V}$$

$$R_{33} = \partial_\mu \underbrace{\Gamma^\mu_{33}}_{=0} - \partial_3 \Gamma^\mu_{\mu 3} = -\left[\partial_3 \Gamma^0_{03} + \partial_3 \underbrace{\Gamma^1_{13}}_{=0} + \partial_3 \underbrace{\Gamma^2_{23}}_{=0} + \partial_3 \underbrace{\Gamma^3_{33}}_{=0}\right]$$

$$= -\partial_3\left[\frac{1}{c^2}\left(1-{}^{2V}/_{c^2}\right)\partial_3 V\right] = -\frac{1}{c^2}\left[-\frac{2}{c^2}\underbrace{\partial_3 V \cdot \partial_3 V}_{\partial V \cdot \partial V \to 0} + \underbrace{\left(1-\frac{2V}{c^2}\right)\partial_3^2 V}_{V \cdot \partial V \to 0}\right]$$

$$\boxed{R_{33} = -\frac{1}{c^3}\partial_3^2 V}$$

$$R_{10} = \partial_\mu \Gamma^\mu_{10} - \underbrace{\partial_0}_{\partial_0 = 0} \Gamma^\mu_{\mu 1} = \underbrace{\partial_0}_{\partial_0 = 0} \Gamma^0_{10} + \partial_1 \underbrace{\Gamma^1_{10}}_{=0} + \partial_2 \underbrace{\Gamma^2_{10}}_{=0} + \partial_3 \underbrace{\Gamma^3_{10}}_{=0}$$

$$\boxed{R_{10} = 0}$$

puede verse que $R_{01} = -\partial_1 \Gamma^0_{00} = -\partial_1\left[{}^1/_{c^2}\, g^{0\mu}\partial_\mu V\right]$

$$R_{01} = -\frac{1}{c^2}\,\partial_1\left(g^{00}\underbrace{\partial_0 V}_{\partial_0 = 0} + \underbrace{g^{01}}_{=0}\partial_1 V + \underbrace{g^{02}}_{=0}\partial_2 V + \underbrace{g^{03}}_{=0}\partial_3 V\right] = 0 \quad \boxed{R_{0\phi} = 0}$$

NOTA

$$R_{20} = \underbrace{\partial_\mu \Gamma^\mu_{20}}_{\partial_0 \to 0} - \underbrace{\partial_0 \Gamma^\mu_{\mu 2}}_{\partial_0 \to 0} = \partial_0 \Gamma^0_{20} + \partial_1 \underbrace{\Gamma^1_{20}}_{=0} + \partial_2 \underbrace{\Gamma^2_{20}}_{=0} + \partial_3 \underbrace{\Gamma^3_{20}}_{=0}$$

$$\boxed{R_{20} = 0 = R_{02}}$$

$$R_{30} = \partial_\mu \Gamma^\mu_{30} - \underbrace{\partial_0 \Gamma^\mu_{\mu 3} = \partial_0 \Gamma^0_{30}}_{\partial_0 \to 0} + \partial_1 \underbrace{\Gamma^1_{30}}_{=0} + \partial_2 \underbrace{\Gamma^2_{30}}_{=0} + \partial_3 \underbrace{\Gamma^3_{30}}_{=0}$$

$$\boxed{R_{30} = 0 = R_{03}}$$

$$R_{12} = \partial_\mu \underbrace{\Gamma^\mu_{12}}_{=0} - \partial_1 \Gamma^\mu_{\mu 2} = -\left(\partial_1 \Gamma^0_{02} + \partial_1 \underbrace{\Gamma^1_{12}}_{=0} + \partial_1 \underbrace{\Gamma^2_{22}}_{=0} + \partial_1 \underbrace{\Gamma^3_{32}}_{=0}\right)$$

$$= -\partial_1 \left[\frac{1}{c^2}\left(1 - \frac{2V}{c^2}\right)\partial_2 V\right]$$

$$= -\frac{1}{c^2}\left[-\frac{2}{c^2}\underbrace{\partial_1 V \cdot \partial_2 V}_{\to 0} + \underbrace{\left(1 - \frac{2V}{c^2}\right)\partial_{21} V}_{V \cdot \partial V \to 0}\right] = -\frac{1}{c^2}\partial_{21} V$$

$$\boxed{R_{12} = -\frac{1}{c^2}\partial_{12} V}$$

$$R_{13} = \partial_\mu \underbrace{\Gamma^\mu_{13}}_{=0} - \partial_1 \Gamma^\mu_{\mu 3} = -\left(\partial_1 \Gamma^0_{03} + \partial_1 \underbrace{\Gamma^1_{13}}_{=0} + \partial_1 \underbrace{\Gamma^2_{23}}_{=0} + \partial_1 \underbrace{\Gamma^3_{33}}_{=0}\right)$$

$$= -\partial_1 \left[\frac{1}{c^2}\left(1 - \frac{2V}{c^2}\right)\partial_3 V\right]$$

$$= -\frac{1}{c^2}\left[-\frac{2}{c^2}\partial_1 V \cdot \partial_3 V + \left(1 - \frac{2V}{c^2}\right)\partial_{13} V\right]$$

$$\boxed{R_{13} = -\frac{1}{c^2}\partial_{13} V}$$

$$R_{23} = \partial_\mu \Gamma^\mu_{23} - \partial_2 \Gamma^\mu_{\mu 3} = -\partial_2 \Gamma^\mu_{\mu 3} = -\partial_2 \Gamma^0_{03}$$

$$= -\partial_2 \left[\frac{1}{c^2}\left(1 - \frac{2V}{c^2}\right)\partial_3 V\right]$$

$$= -\frac{1}{c^2}\left[-\frac{2}{c^2}\partial_2 V \partial_3 V + \left(1 - \frac{2V}{c^2}\right)\partial_{23} V\right]$$

$$\boxed{R_{23} = -\frac{1}{c^2}\partial_{23} V}$$

NOTA

2) Calcular la variación de la acción

$$S_{(g)} = -\frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda) - \frac{c^4}{8\pi G}\int d^3y\sqrt{|h|}\,K + \int d^4x\sqrt{-g}\,\mathcal{L}_m$$

$$S_{(g)} = -\frac{c^4}{16\pi G}\left[\int d^4x\sqrt{-g}\,R + 2\int d^3y\sqrt{|h|}\,K\right] +$$

$$+\frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\,2\Lambda + \int d^4x\sqrt{-g}\,\mathcal{L}_m$$

por 57.2. La variación del primer término es (salvando la constante λ)

$$\delta S^{(1)}_{(g)} = \int d^4x\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\delta g^{\alpha\beta}$$

en el minuto 11 del vídeo de Javier se llega a la variación del tercer término

$$\delta S^{(3)}_{(g)} = \int d^4x\,\frac{1}{2}\sqrt{-g}\,\delta g^{\alpha\beta}\underbrace{\left[-g_{\alpha\beta}\mathcal{L}_m + 2\frac{\delta\mathcal{L}_m}{\delta g^{\alpha\beta}}\right]}_{T_{\alpha\beta}}$$

$$\delta S(g) = \int d^4x\sqrt{-g}\,\delta g^{\alpha\beta}\left[-\frac{c^4}{16\pi G}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right) + \frac{1}{2}T_{\alpha\beta}\right] +$$

$$+\int d^4x\,\frac{c^4}{16\pi G}\cdot 2\Lambda\cdot\delta\sqrt{-g}$$

como $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}$

$$\delta S(g) = \int d^4x\sqrt{-g}\,\delta g^{\alpha\beta}\left[-\frac{c^4}{16\pi G}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right) + \frac{1}{2}T_{\alpha\beta}\right] +$$

$$+\int d^4x\sqrt{-g}\,g^{\alpha\beta}\left(-\frac{1}{2}\right)\frac{c^4}{16\pi G}\,2\Lambda$$

$$\delta S(g) = \int d^4x\sqrt{-g}\,g^{\alpha\beta}\left[\frac{-c^4}{16\pi G}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g^{\alpha\beta}\right) + \frac{1}{2}T_{\alpha\beta}\right]$$

$$\delta S(g) = 0 \Rightarrow -\frac{c^4}{16\pi G}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g^{\alpha\beta}\right) + \frac{1}{2}T_{\alpha\beta} = 0$$

$$\boxed{R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + g_{\alpha\beta}\Lambda = \frac{8\pi G}{c^4}T_{\alpha\beta}}$$

NOTA