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Comprobar que

$$J^\mu = g^{\alpha\beta}\,\delta\Gamma^\mu_{\alpha\beta} - g^{\beta\mu}\,\delta\Gamma^\lambda_{\lambda\beta} = g^{\mu\nu} g^{\alpha\beta}\left(\partial_\alpha \delta g_{\nu\beta} - \partial_\nu \delta g_{\alpha\beta}\right)$$

$$g^{\alpha\beta}\,\delta\Gamma^\mu_{\alpha\beta} = g^{\alpha\beta}\,\frac{1}{2} g^{\mu\nu}\left(\partial_\beta \delta g_{\nu\alpha} + \partial_\alpha \delta g_{\nu\beta} - \partial_\nu \delta g_{\alpha\beta}\right) =$$

(A) $$= \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}\,\partial_\beta \delta g_{\nu\alpha} + \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}\,\partial_\alpha \delta g_{\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}\,\partial_\nu g_{\alpha\beta}$$

$$g^{\beta\mu}\,\delta\Gamma^\lambda_{\lambda\beta} = g^{\beta\mu}\,\frac{1}{2} g^{\lambda\nu}\left(\partial_\beta \delta g_{\nu\lambda} + \partial_\lambda \delta g_{\nu\beta} - \partial_\nu \delta g_{\lambda\beta}\right) =$$

(B) $$= \underbrace{\frac{1}{2} g^{\beta\mu} g^{\lambda\nu}\,\partial_\beta \delta g_{\nu\lambda}}_{T1} + \underbrace{\frac{1}{2} g^{\beta\mu} g^{\lambda\nu}\,\partial_\lambda \delta g_{\nu\beta}}_{T2} - \underbrace{\frac{1}{2} g^{\beta\mu} g^{\lambda\nu}\,\partial_\nu \delta g_{\lambda\beta}}_{T3}$$

$T1 \equiv \frac{1}{2} g^{\beta\mu} g^{\lambda\nu}\,\partial_\beta \delta g_{\nu\lambda}$ índices mudos $\equiv \lambda;\ \nu;\ \beta$

renombramos $\lambda \to \alpha$, $\beta \to \nu$, $\nu \to \beta$

el término T1 queda $\equiv \frac{1}{2} g^{\nu\mu} g^{\alpha\beta}\,\partial_\nu \delta g_{\beta\alpha}$

por simetría $T1 \equiv \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}\,\partial_\nu g_{\alpha\beta}$

$T2 \equiv \frac{1}{2} g^{\beta\mu} g^{\lambda\nu}\,\partial_\lambda \delta g_{\nu\beta}$ índices mudos $\equiv \lambda;\ \nu;\ \beta$

renombramos $\lambda \to \alpha$, $\beta \to \nu$, $\nu \to \beta$

el término T2 queda $\equiv \frac{1}{2} g^{\nu\mu} g^{\alpha\beta}\,\partial_\alpha \delta g_{\beta\nu}$

por simetría $T2 \equiv \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}\,\partial_\alpha g_{\nu\beta}$

$T3 \equiv -\frac{1}{2} g^{\beta\mu} g^{\lambda\nu}\,\partial_\nu \delta g_{\lambda\beta}$ índices mudos $\equiv \lambda;\ \beta;\ \nu$

renombramos $\beta \to \nu$, $\lambda \to \alpha$, $\nu \to \beta$

NOTA

el término T3 queda $\equiv -\frac{1}{2} g^{\nu\mu} g^{\alpha\beta} \partial_\beta g_{\alpha\nu}$

por simetría $T3 \equiv -\frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\beta g_{\nu\alpha}$

Hacemos (A) − (B) reordenando los índices de (B) $\{T_1, T_2, T_3\}$

$$g^{\alpha\beta}\,\delta\Gamma^{\mu}_{\alpha\beta} - g^{\beta\mu}\,\delta\Gamma^{\nu}_{\nu\beta} =$$

$$= \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\beta \delta g_{\nu\alpha} + \cancel{\frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\alpha \delta g_{\nu\beta}} - \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\nu g_{\alpha\beta} -$$

$$-\left(\frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\nu g_{\alpha\beta} + \cancel{\frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\alpha g_{\nu\beta}} - \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\beta g_{\nu\alpha}\right) =$$

$$= \underbrace{g^{\alpha\beta} g^{\mu\nu} \partial_\beta \delta g_{\nu\alpha}} - g^{\alpha\beta} g^{\mu\nu} \partial_\nu g_{\alpha\beta}$$

en este término α y β son mudos, los intercambiamos $\begin{cases} \alpha \to \beta \\ \beta \to \alpha \end{cases}$

$$J^\mu = g^{\alpha\beta} g^{\mu\nu} \partial_\alpha \delta g_{\nu\beta} - g^{\alpha\beta} g^{\mu\nu} \partial_\nu g_{\alpha\beta}$$

$$\boxed{J^\mu = g^{\alpha\beta} g^{\mu\nu} \left(\partial_\alpha \delta g_{\nu\beta} - \partial_\nu g_{\alpha\beta}\right)}$$

Q.E.D.