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Miguel Canizares Millán

# CÁLCULO DE LOS SÍMBOLOS DE CHRISTOFEL EN ESPACIO-TIEMPO DE SCHWARZSCHILD

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Métrica  
de Schwarzschild

$$d^2s = -\left(1 - \frac{a}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{a}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

siendo  $a = \frac{2GM}{c^2}$ . Simplificando:

$$d^2s = \underbrace{\left(\frac{a-r}{r}\right)}_{g_{00}} c^2 dt^2 + \underbrace{\left(\frac{r}{r-a}\right)}_{g_{11}} dr^2 + \underbrace{r^2}_{g_{22}} d\theta^2 + \underbrace{r^2 \sin^2 \theta}_{g_{33}} d\phi^2$$

$$\begin{aligned} ct &\equiv x^0 \\ r &\equiv x^1 \\ \theta &\equiv x^2 \\ \phi &\equiv x^3 \end{aligned}$$

## CÁLCULO DE SÍMBOLOS DE CHRISTOFEL (Método derivaciones de productos de bases)

$$\bar{e}_0 \cdot \bar{e}_0 = \frac{a-r}{r}$$

1) -  $\partial_0(\bar{e}_0 \cdot \bar{e}_0) = 0$

$$\begin{aligned} \partial_0(\bar{e}_0 \cdot \bar{e}_0) &= 2 \bar{e}_0 \cdot (\partial_0 \bar{e}_0) = 2 \bar{e}_0 \cdot \Gamma_{00}^m \bar{e}_m = (\text{sólo sobrevive } m=0) = \\ &= 2 \bar{e}_0 \cdot \bar{e}_0 \Gamma_{00}^0 = 2 \frac{a-r}{r} \Gamma_{00}^0 \end{aligned}$$

Sabemos que la derivada es nula  $\Rightarrow 2 \frac{a-r}{r} \Gamma_{00}^0 = 0 \Rightarrow \Gamma_{00}^0 = 0$

2) -  $\partial_1(\bar{e}_0 \cdot \bar{e}_0) = -a/r^2$

$$\begin{aligned} \partial_1(\bar{e}_0 \cdot \bar{e}_0) &= 2 \bar{e}_0 \cdot (\partial_1 \bar{e}_0) = 2 \bar{e}_0 \cdot \Gamma_{10}^m \bar{e}_m = (\text{sólo sobrevive } m=0) = \\ &= 2 \bar{e}_0 \cdot \bar{e}_0 \Gamma_{10}^0 = 2 \frac{a-r}{r} \Gamma_{10}^0 \end{aligned}$$

Sabemos que la derivada vale  $-a/r^2 \Rightarrow 2 \frac{a-r}{r} \Gamma_{10}^0 = -\frac{a}{r^2} \Rightarrow \Gamma_{10}^0 = \frac{a}{2r(r-a)}$

3) -  $\partial_2(\bar{e}_0 \cdot \bar{e}_0) = 0$

$$\partial_2(\bar{e}_0 \cdot \bar{e}_0) = 2 \bar{e}_0 \cdot (\partial_2 \bar{e}_0) = 2 \bar{e}_0 \cdot \Gamma_{20}^m \bar{e}_m = (\text{" "}) = 2 \bar{e}_0 \cdot \bar{e}_0 \Gamma_{20}^0 = 2 \frac{a-r}{r} \Gamma_{20}^0$$

Iguando a cero la derivada  $\Rightarrow 2 \frac{a-r}{r} \Gamma_{20}^0 = 0 \Rightarrow \Gamma_{20}^0 = 0$

4) -  $\partial_3(\bar{e}_0 \cdot \bar{e}_0) = 0$

$$\partial_3(\bar{e}_0 \cdot \bar{e}_0) = 2 \bar{e}_0 \cdot (\partial_3 \bar{e}_0) = 2 \bar{e}_0 \cdot \Gamma_{30}^m \bar{e}_m = (\text{" "}) = 2 \bar{e}_0 \cdot \bar{e}_0 \Gamma_{30}^0 = 2 \frac{a-r}{r} \Gamma_{30}^0$$

Iguando a cero  $\Rightarrow 2 \frac{a-r}{r} \Gamma_{30}^0 = 0 \Rightarrow \Gamma_{30}^0 = 0$



$$\bar{e}_1 \cdot \bar{e}_1 = \frac{r}{r-a}$$

$$5) - \partial_0(\bar{e}_1 \cdot \bar{e}_1) = 0$$

$$\partial_0(\bar{e}_1 \cdot \bar{e}_1) = 2\bar{e}_1 \cdot (\partial_0 \bar{e}_1) = 2\bar{e}_1 \cdot \Gamma_{01}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ \text{para } m=1 \end{smallmatrix} \right) = 2\bar{e}_1 \cdot \bar{e}_1 \Gamma_{01}^1$$

$$\text{Igualando a zero a derivada} \rightarrow 2\frac{r}{r-a} \Gamma_{01}^1 = 0 \rightarrow \Gamma_{01}^1 = 0$$

$$6) - \partial_1(\bar{e}_1 \cdot \bar{e}_1) = -\frac{a}{(r-a)^2}$$

$$\partial_1(\bar{e}_1 \cdot \bar{e}_1) = 2\bar{e}_1 \cdot (\partial_1 \bar{e}_1) = 2\bar{e}_1 \cdot \Gamma_{11}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ m=1 \end{smallmatrix} \right) = 2\bar{e}_1 \cdot \bar{e}_1 \Gamma_{11}^1$$

$$\text{Igualando: } 2\frac{r}{r-a} \Gamma_{11}^1 = -\frac{a}{(r-a)^2} \rightarrow \Gamma_{11}^1 = -\frac{a}{2r(r-a)}$$

$$7) - \partial_2(\bar{e}_1 \cdot \bar{e}_1) = 0$$

$$\partial_2(\bar{e}_1 \cdot \bar{e}_1) = 2\bar{e}_1 \cdot (\partial_2 \bar{e}_1) = 2\bar{e}_1 \cdot \Gamma_{21}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ m=1 \end{smallmatrix} \right) = 2\bar{e}_1 \cdot \bar{e}_1 \Gamma_{21}^1 = 2\frac{r}{r-a} \Gamma_{21}^1$$

$$\text{Igualando a zero} \rightarrow 2\frac{r}{r-a} \Gamma_{21}^1 = 0 \rightarrow \Gamma_{21}^1 = 0$$

$$8) - \partial_3(\bar{e}_1 \cdot \bar{e}_1) = 0$$

$$\partial_3(\bar{e}_1 \cdot \bar{e}_1) = 2\bar{e}_1 \cdot (\partial_3 \bar{e}_1) = 2\bar{e}_1 \cdot \Gamma_{31}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ m=1 \end{smallmatrix} \right) = 2\bar{e}_1 \cdot \bar{e}_1 \Gamma_{31}^1 = 2\frac{r}{r-a} \Gamma_{31}^1$$

$$\text{Igualando a zero} \rightarrow 2\frac{r}{r-a} \Gamma_{31}^1 = 0 \rightarrow \Gamma_{31}^1 = 0$$

$$\bar{e}_2 \cdot \bar{e}_2 = r^2$$

$$9) - \partial_0(\bar{e}_2 \cdot \bar{e}_2) = 0$$

$$\partial_0(\bar{e}_2 \cdot \bar{e}_2) = 2\bar{e}_2 \cdot (\partial_0 \bar{e}_2) = 2\bar{e}_2 \cdot \Gamma_{02}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ m=2 \end{smallmatrix} \right) = 2\bar{e}_2 \cdot \bar{e}_2 \Gamma_{02}^2 = 2r^2 \Gamma_{02}^2$$

$$\text{Igualando a zero } 2r^2 \Gamma_{02}^2 = 0 \rightarrow \Gamma_{02}^2 = 0$$

$$10) - \partial_1(\bar{e}_2 \cdot \bar{e}_2) = 2r$$

$$\partial_1(\bar{e}_2 \cdot \bar{e}_2) = 2\bar{e}_2 \cdot (\partial_1 \bar{e}_2) = 2\bar{e}_2 \cdot \Gamma_{12}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ m=2 \end{smallmatrix} \right) = 2\bar{e}_2 \cdot \bar{e}_2 \Gamma_{12}^2 = 2r^2 \Gamma_{12}^2$$

$$\text{Igualando a } 2r \rightarrow 2r^2 \Gamma_{12}^2 = 2r \rightarrow \Gamma_{12}^2 = \frac{1}{r}$$

$$11) - \partial_2(\bar{e}_2 \cdot \bar{e}_2) = 0$$

$$\partial_2(\bar{e}_2 \cdot \bar{e}_2) = 2\bar{e}_2 \cdot (\partial_2 \bar{e}_2) = 2\bar{e}_2 \cdot \Gamma_{22}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ m=2 \end{smallmatrix} \right) = 2\bar{e}_2 \cdot \bar{e}_2 \Gamma_{22}^2 = 2r^2 \Gamma_{22}^2$$

$$\text{Igualando a zero} \rightarrow 2r^2 \Gamma_{22}^2 = 0 \rightarrow \Gamma_{22}^2 = 0$$

$$12) - \partial_3(\bar{e}_2 \cdot \bar{e}_2) = 0$$

$$\partial_3(\bar{e}_2 \cdot \bar{e}_2) = 2\bar{e}_2 \cdot (\partial_3 \bar{e}_2) = 2\bar{e}_2 \cdot \Gamma_{32}^m \bar{e}_m = \left( \begin{smallmatrix} \text{só sobrevive} \\ m=2 \end{smallmatrix} \right) = 2\bar{e}_2 \cdot \bar{e}_2 \Gamma_{32}^2 = 2r^2 \Gamma_{32}^2 = 0 \rightarrow \Gamma_{32}^2 = 0$$



$$\bar{e}_3 \cdot \bar{e}_3 = r^2 \sin^2 \theta$$

$$13) - \partial_0(\bar{e}_3 \cdot \bar{e}_3) = 0$$

$$\partial_0(\bar{e}_3 \cdot \bar{e}_3) = 2 \bar{e}_3 (\partial_0 \bar{e}_3) = 2 \bar{e}_3 \Gamma_{03}^m \bar{e}_m = \left( \text{Sobreviver } m=3 \right) = 2 \bar{e}_3 \cdot \bar{e}_3 \Gamma_{03}^3 = 2 r^2 \sin^2 \theta \Gamma_{03}^3$$

$$\text{Ignorando a zero} \rightarrow 2 r^2 \sin^2 \theta \Gamma_{03}^3 = 0 \rightarrow \Gamma_{03}^3 = 0$$

$$14) - \partial_1(\bar{e}_3 \cdot \bar{e}_3) = 2 r \sin^2 \theta$$

$$\partial_1(\bar{e}_3 \cdot \bar{e}_3) = 2 \bar{e}_3 (\partial_1 \bar{e}_3) = 2 \bar{e}_3 \Gamma_{13}^m \bar{e}_m = \left( \text{Sobreviver } m=3 \right) = 2 \bar{e}_3 \cdot \bar{e}_3 \Gamma_{13}^3 = 2 r^2 \sin^2 \theta \Gamma_{13}^3$$

$$\text{Ignorando: } 2 r^2 \sin^2 \theta \Gamma_{13}^3 = 2 r \sin^2 \theta \rightarrow \Gamma_{13}^3 = \frac{1}{r}$$

$$15) - \partial_2(\bar{e}_3 \cdot \bar{e}_3) = 2 r^2 \sin \theta \cos \theta$$

$$\partial_2(\bar{e}_3 \cdot \bar{e}_3) = 2 \bar{e}_3 (\partial_2 \bar{e}_3) = 2 \bar{e}_3 \Gamma_{23}^m \bar{e}_m = \left( \text{Sobreviver } m=3 \right) = 2 \bar{e}_3 \cdot \bar{e}_3 \Gamma_{23}^3 = 2 r^2 \sin^2 \theta \Gamma_{23}^3$$

$$\text{Ignorando: } 2 r^2 \sin^2 \theta \Gamma_{23}^3 = 2 r^2 \sin \theta \cos \theta \rightarrow \Gamma_{23}^3 = \cot \theta$$

$$16) - \partial_3(\bar{e}_3 \cdot \bar{e}_3) = 0$$

$$\partial_3(\bar{e}_3 \cdot \bar{e}_3) = 2 \bar{e}_3 (\partial_3 \bar{e}_3) = 2 \bar{e}_3 \Gamma_{33}^m \bar{e}_m = \left( \text{Sobreviver } m=3 \right) = 2 \bar{e}_3 \cdot \bar{e}_3 \Gamma_{33}^3 = 2 r^2 \sin^2 \theta \Gamma_{33}^3$$

$$\text{Ignorando a zero} \rightarrow 2 r^2 \sin^2 \theta \Gamma_{33}^3 = 0 \rightarrow \Gamma_{33}^3 = 0$$

$$\bar{e}_0 \cdot \bar{e}_1 = 0 \quad \text{Qualquer derivada de } (\bar{e}_0 \cdot \bar{e}_1) \text{ será nula}$$

$$17) - \partial_0(\bar{e}_0 \cdot \bar{e}_1) = (\partial_0 \bar{e}_0) \cdot \bar{e}_1 + \bar{e}_0 (\partial_0 \bar{e}_1) = \Gamma_{00}^m \bar{e}_m \cdot \bar{e}_1 + \bar{e}_0 \cdot \Gamma_{01}^n \bar{e}_n = \left( \text{Sobreviver } m=1, n=0 \right) =$$

$$= \Gamma_{00}^1 (\bar{e}_1 \cdot \bar{e}_1) + \Gamma_{01}^0 (\bar{e}_0 \cdot \bar{e}_0) = \left( \text{Conhecemos } \Gamma_{01}^0 = a / 2r(r-a) \right) \left( \text{Sustituirmos} \right)$$

$$= \Gamma_{00}^1 \frac{r}{r-a} + \frac{a}{2r(r-a)} \frac{a-r}{r} = 0 \rightarrow \Gamma_{00}^1 = \frac{a(r-a)}{2r^3}$$

$$18) - \partial_1(\bar{e}_0 \cdot \bar{e}_1) = (\partial_1 \bar{e}_0) \cdot \bar{e}_1 + \bar{e}_0 (\partial_1 \bar{e}_1) = \Gamma_{10}^m \bar{e}_m \cdot \bar{e}_1 + \bar{e}_0 \cdot \Gamma_{11}^n \bar{e}_n = \left( \text{Sobreviver } m=1, n=0 \right) = \Gamma_{10}^1 (\bar{e}_1 \cdot \bar{e}_1) + \Gamma_{11}^0 (\bar{e}_0 \cdot \bar{e}_0)$$

$$\left( \text{Conhecemos (5)} \right) \Gamma_{10}^1 = 0 \text{ y sustituirmos} \rightarrow \Gamma_{11}^0 \frac{a-r}{r} = 0 \rightarrow \Gamma_{11}^0 = 0$$

$$19) - \partial_2(\bar{e}_0 \cdot \bar{e}_1) = (\partial_2 \bar{e}_0) \cdot \bar{e}_1 + \bar{e}_0 (\partial_2 \bar{e}_1) = \Gamma_{20}^m \bar{e}_m \cdot \bar{e}_1 + \bar{e}_0 \cdot \Gamma_{21}^n \bar{e}_n = \left( \text{Sobreviver } m=1, n=0 \right) = \Gamma_{20}^1 (\bar{e}_1 \cdot \bar{e}_1) + \Gamma_{21}^0 (\bar{e}_0 \cdot \bar{e}_0)$$

$$\text{No conocemos ni } \Gamma_{20}^1 \text{ ni } \Gamma_{21}^0 \rightarrow \Gamma_{20}^1 \frac{r}{r-a} + \Gamma_{21}^0 \frac{a-r}{r} = 0$$

$$20) - \partial_3(\bar{e}_0 \cdot \bar{e}_1) = (\partial_3 \bar{e}_0) \cdot \bar{e}_1 + \bar{e}_0 (\partial_3 \bar{e}_1) = \Gamma_{30}^m \bar{e}_m \cdot \bar{e}_1 + \bar{e}_0 \cdot \Gamma_{31}^n \bar{e}_n = \left( \text{Sobreviver } m=1, n=0 \right) =$$

$$= \Gamma_{30}^1 (\bar{e}_1 \cdot \bar{e}_1) + \bar{e}_0 \cdot \bar{e}_0 \Gamma_{31}^0 = \Gamma_{30}^1 \frac{r}{r-a} + \Gamma_{31}^0 \frac{a-r}{r} = 0$$

$$\text{No conocemos } \Gamma_{30}^1 \text{ ni } \Gamma_{31}^0 \rightarrow \Gamma_{30}^1 \frac{r}{r-a} + \Gamma_{31}^0 \frac{a-r}{r} = 0$$



$\bar{e}_0 \cdot \bar{e}_2 = 0 \rightarrow$  Cualquier derivada de  $(\bar{e}_0 \cdot \bar{e}_2)$  será nula

$$21) - \partial_0(\bar{e}_0 \cdot \bar{e}_2) = (\partial_0 \bar{e}_0) \cdot \bar{e}_2 + \bar{e}_0 \cdot (\partial_0 \bar{e}_2) = \Gamma_{00}^m \bar{e}_m \bar{e}_2 + \bar{e}_0 \Gamma_{02}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=2 \quad n=0 \end{smallmatrix} \right) = \\ = \Gamma_{00}^2 \bar{e}_2 \bar{e}_2 + \bar{e}_0 \bar{e}_0 \Gamma_{02}^0 = \left[ \text{Según (3) conocemos } \Gamma_{02}^0 = 0 \right] = \Gamma_{00}^2 r^2 = 0 \rightarrow \Gamma_{00}^2 = 0$$

$$22) - \partial_1(\bar{e}_0 \cdot \bar{e}_2) = (\partial_1 \bar{e}_0) \cdot \bar{e}_2 + \bar{e}_0 \cdot (\partial_1 \bar{e}_2) = \Gamma_{10}^m \bar{e}_m \bar{e}_2 + \bar{e}_0 \Gamma_{12}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=2 \quad n=0 \end{smallmatrix} \right) = \\ = \Gamma_{10}^2 \bar{e}_2 \bar{e}_2 + \bar{e}_0 \bar{e}_0 \Gamma_{12}^0 \quad \left[ \text{No conocemos ni } \Gamma_{10}^2 \text{ ni } \Gamma_{12}^0 \right] \rightarrow \Gamma_{10}^2 r^2 + \frac{a-r}{r} \Gamma_{12}^0 = 0$$

$$23) - \partial_2(\bar{e}_0 \cdot \bar{e}_2) = (\partial_2 \bar{e}_0) \cdot \bar{e}_2 + \bar{e}_0 \cdot (\partial_2 \bar{e}_2) = \Gamma_{20}^m \bar{e}_m \bar{e}_2 + \bar{e}_0 \Gamma_{22}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=2 \quad n=0 \end{smallmatrix} \right) = \\ = \Gamma_{20}^2 \bar{e}_2 \bar{e}_2 + \bar{e}_0 \bar{e}_0 \Gamma_{22}^0 = \left[ \text{Según (9) conocemos } \Gamma_{20}^2 = 0 \right] = \frac{a-r}{r} \Gamma_{22}^0 = 0 \rightarrow \Gamma_{22}^0 = 0$$

$$24) - \partial_3(\bar{e}_0 \cdot \bar{e}_2) = (\partial_3 \bar{e}_0) \cdot \bar{e}_2 + \bar{e}_0 \cdot (\partial_3 \bar{e}_2) = \Gamma_{30}^m \bar{e}_m \bar{e}_2 + \bar{e}_0 \Gamma_{32}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=2 \quad n=0 \end{smallmatrix} \right) = \\ = \Gamma_{30}^2 \bar{e}_2 \bar{e}_2 + \bar{e}_0 \bar{e}_0 \Gamma_{32}^0 \quad \left[ \text{No conocemos } \Gamma_{30}^2 \text{ ni } \Gamma_{32}^0 \right] \rightarrow \Gamma_{30}^2 r^2 + \frac{a-r}{r} \Gamma_{32}^0 = 0$$

$\bar{e}_0 \cdot \bar{e}_3 = 0 \rightarrow$  Cualquier derivada de  $(\bar{e}_0 \cdot \bar{e}_3)$  será nula.

$$25) - \partial_0(\bar{e}_0 \cdot \bar{e}_3) = (\partial_0 \bar{e}_0) \cdot \bar{e}_3 + \bar{e}_0 \cdot (\partial_0 \bar{e}_3) = \Gamma_{00}^m \bar{e}_m \bar{e}_3 + \bar{e}_0 \Gamma_{03}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=3 \quad n=0 \end{smallmatrix} \right) = \\ = \Gamma_{00}^3 \bar{e}_3 \bar{e}_3 + \bar{e}_0 \bar{e}_0 \Gamma_{03}^0 \quad \left[ \text{Según (4) } \Gamma_{03}^0 = 0 \right] = \Gamma_{00}^3 r^2 \sin^2 \theta = 0 \rightarrow \Gamma_{00}^3 = 0$$

$$26) - \partial_1(\bar{e}_0 \cdot \bar{e}_3) = (\partial_1 \bar{e}_0) \cdot \bar{e}_3 + \bar{e}_0 \cdot (\partial_1 \bar{e}_3) = \Gamma_{10}^m \bar{e}_m \bar{e}_3 + \bar{e}_0 \Gamma_{13}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=3 \quad n=0 \end{smallmatrix} \right) = \Gamma_{10}^3 \bar{e}_3 \bar{e}_3 + \bar{e}_0 \bar{e}_0 \Gamma_{13}^0 \\ = \Gamma_{10}^3 \bar{e}_3 \bar{e}_3 + \bar{e}_0 \bar{e}_0 \Gamma_{13}^0 \quad \left[ \text{No conocemos } \Gamma_{10}^3 \text{ ni } \Gamma_{13}^0 \right] \rightarrow \Gamma_{10}^3 r^2 \sin^2 \theta + \frac{a-r}{r} \Gamma_{13}^0 = 0$$

$$27) - \partial_2(\bar{e}_0 \cdot \bar{e}_3) = (\partial_2 \bar{e}_0) \cdot \bar{e}_3 + \bar{e}_0 \cdot (\partial_2 \bar{e}_3) = \Gamma_{20}^m \bar{e}_m \bar{e}_3 + \bar{e}_0 \Gamma_{23}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=3 \quad n=0 \end{smallmatrix} \right) = \\ = \Gamma_{20}^3 \bar{e}_3 \bar{e}_3 + \bar{e}_0 \bar{e}_0 \Gamma_{23}^0 \quad \left[ \text{No conocemos } \Gamma_{20}^3 \text{ ni } \Gamma_{23}^0 \right] \rightarrow \Gamma_{20}^3 r^2 \sin^2 \theta + \frac{a-r}{r} \Gamma_{23}^0 = 0$$

$$28) - \partial_3(\bar{e}_0 \cdot \bar{e}_3) = (\partial_3 \bar{e}_0) \cdot \bar{e}_3 + \bar{e}_0 \cdot (\partial_3 \bar{e}_3) = \Gamma_{30}^m \bar{e}_m \bar{e}_3 + \bar{e}_0 \Gamma_{33}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=3 \quad n=0 \end{smallmatrix} \right) = \\ = \Gamma_{30}^3 \bar{e}_3 \bar{e}_3 + \bar{e}_0 \bar{e}_0 \Gamma_{33}^0 \quad \left[ \text{Según (13) } \Gamma_{30}^3 = 0 \right] \rightarrow \frac{a-r}{r} \Gamma_{33}^0 = 0 \rightarrow \Gamma_{33}^0 = 0$$

$\bar{e}_1 \cdot \bar{e}_2 = 0 \rightarrow$  Cualquier derivada de  $(\bar{e}_1 \cdot \bar{e}_2)$  será nula

$$29) - \partial_0(\bar{e}_1 \cdot \bar{e}_2) = (\partial_0 \bar{e}_1) \cdot \bar{e}_2 + \bar{e}_1 \cdot (\partial_0 \bar{e}_2) = \Gamma_{01}^m \bar{e}_m \bar{e}_2 + \bar{e}_1 \Gamma_{02}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=2 \quad n=1 \end{smallmatrix} \right) = \\ = \Gamma_{01}^2 \bar{e}_2 \bar{e}_2 + \bar{e}_1 \bar{e}_1 \Gamma_{02}^1 \quad \left[ \text{No conocemos } \Gamma_{01}^2 \text{ ni } \Gamma_{02}^1 \right] \rightarrow \Gamma_{01}^2 r^2 + \frac{r}{r-a} \Gamma_{02}^1 = 0$$

$$30) - \partial_1(\bar{e}_1 \cdot \bar{e}_2) = (\partial_1 \bar{e}_1) \cdot \bar{e}_2 + \bar{e}_1 \cdot (\partial_1 \bar{e}_2) = \Gamma_{11}^m \bar{e}_m \bar{e}_2 + \bar{e}_1 \Gamma_{12}^n \bar{e}_n = \left( \begin{smallmatrix} \text{Sobreviven} \\ m=2 \quad n=1 \end{smallmatrix} \right) = \\ = \Gamma_{11}^2 \bar{e}_2 \bar{e}_2 + \bar{e}_1 \bar{e}_1 \Gamma_{12}^1 \quad \left[ \text{Según (7) } \Gamma_{12}^1 = 0 \right] \rightarrow \Gamma_{11}^2 r^2 = 0 \rightarrow \Gamma_{11}^2 = 0$$



$$31) - \partial_2(\bar{e}_1 \cdot \bar{e}_2) = (\partial_2 \bar{e}_1) \cdot \bar{e}_2 + \bar{e}_1 \cdot (\partial_2 \bar{e}_2) = \sum_{m=2}^m \bar{e}_m \bar{e}_2 + \bar{e}_1 \sum_{n=2}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=2 & n=1 \end{matrix} \right) = \\ = \sum_{m=2}^2 \bar{e}_m \bar{e}_2 + \bar{e}_1 \sum_{n=2}^1 \bar{e}_n = \left[ \text{Según (10)}: \sum_{m=2}^2 = \frac{1}{r} \right] = \frac{1}{r} \cdot r^2 + \frac{r}{r-a} \sum_{n=2}^1 = 0 \rightarrow \boxed{\sum_{n=2}^1 = a-r}$$

$$32) - \partial_3(\bar{e}_1 \cdot \bar{e}_2) = (\partial_3 \bar{e}_1) \cdot \bar{e}_2 + \bar{e}_1 \cdot (\partial_3 \bar{e}_2) = \sum_{m=3}^m \bar{e}_m \bar{e}_2 + \bar{e}_1 \sum_{n=3}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=2 & n=1 \end{matrix} \right) = \\ = \sum_{m=3}^2 \bar{e}_m \bar{e}_2 + \bar{e}_1 \sum_{n=3}^1 \bar{e}_n \quad [\text{No conocemos } \sum_{m=3}^2 \text{ ni } \sum_{n=3}^1] \rightarrow \boxed{\sum_{m=3}^2 r^2 + \frac{r}{r-a} \sum_{n=3}^1 = 0}$$

$\bar{e}_1 \cdot \bar{e}_3 = 0 \rightarrow$  Cualquier derivada de  $(\bar{e}_1 \cdot \bar{e}_3)$  será nula

$$33) - \partial_0(\bar{e}_1 \cdot \bar{e}_3) = (\partial_0 \bar{e}_1) \cdot \bar{e}_3 + \bar{e}_1 \cdot (\partial_0 \bar{e}_3) = \sum_{m=3}^m \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=3}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=1 \end{matrix} \right) = \\ = \sum_{m=3}^3 \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=3}^1 \bar{e}_n \quad [\text{No conocemos } \sum_{m=3}^3 \text{ ni } \sum_{n=3}^1] \rightarrow \boxed{\sum_{m=3}^3 r^2 \sin^2 \theta + \frac{r}{r-a} \sum_{n=3}^1 = 0}$$

$$34) - \partial_1(\bar{e}_1 \cdot \bar{e}_3) = (\partial_1 \bar{e}_1) \cdot \bar{e}_3 + \bar{e}_1 \cdot (\partial_1 \bar{e}_3) = \sum_{m=1}^m \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=1}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=1 \end{matrix} \right) = \\ = \sum_{m=1}^3 \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=1}^1 \bar{e}_n = [\text{Según (8)}: \sum_{n=1}^1 = 0] = \sum_{m=1}^3 r^2 \sin^2 \theta = 0 \rightarrow \boxed{\sum_{m=1}^3 = 0}$$

$$35) - \partial_2(\bar{e}_1 \cdot \bar{e}_3) = (\partial_2 \bar{e}_1) \cdot \bar{e}_3 + \bar{e}_1 \cdot (\partial_2 \bar{e}_3) = \sum_{m=2}^m \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=2}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=1 \end{matrix} \right) = \\ = \sum_{m=2}^3 \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=2}^1 \bar{e}_n \quad [\text{No conocemos } \sum_{m=2}^3 \text{ ni } \sum_{n=2}^1] \rightarrow \boxed{\sum_{m=2}^3 r^2 \sin^2 \theta + \frac{r}{r-a} \sum_{n=2}^1 = 0}$$

$$36) - \partial_3(\bar{e}_1 \cdot \bar{e}_3) = (\partial_3 \bar{e}_1) \cdot \bar{e}_3 + \bar{e}_1 \cdot (\partial_3 \bar{e}_3) = \sum_{m=3}^m \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=3}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=1 \end{matrix} \right) = \\ = \sum_{m=3}^3 \bar{e}_m \bar{e}_3 + \bar{e}_1 \sum_{n=3}^1 \bar{e}_n = [\text{Según (14)}: \sum_{m=3}^3 = \frac{1}{r}] = \frac{1}{r} r^2 \sin^2 \theta + \frac{r}{r-a} \sum_{n=3}^1 \rightarrow \boxed{\sum_{n=3}^1 = (a-r) \sin^2 \theta}$$

$\bar{e}_2 \cdot \bar{e}_3 = 0 \rightarrow$  Cualquier derivada de  $(\bar{e}_2 \cdot \bar{e}_3)$  será nula

$$37) - \partial_0(\bar{e}_2 \cdot \bar{e}_3) = (\partial_0 \bar{e}_2) \cdot \bar{e}_3 + \bar{e}_2 \cdot (\partial_0 \bar{e}_3) = \sum_{m=3}^m \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=3}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=2 \end{matrix} \right) = \\ = \sum_{m=3}^3 \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=3}^2 \bar{e}_n \quad [\text{No se conoce } \sum_{m=3}^3 \text{ ni } \sum_{n=3}^2] \rightarrow \boxed{\sum_{m=3}^3 r^2 \sin^2 \theta + r^2 \sum_{n=3}^2 = 0}$$

$$38) - \partial_1(\bar{e}_2 \cdot \bar{e}_3) = (\partial_1 \bar{e}_2) \cdot \bar{e}_3 + \bar{e}_2 \cdot (\partial_1 \bar{e}_3) = \sum_{m=1}^m \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=1}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=2 \end{matrix} \right) = \\ = \sum_{m=1}^3 \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=1}^2 \bar{e}_n \quad [\text{No se conoce } \sum_{m=1}^3 \text{ ni } \sum_{n=1}^2] \rightarrow \boxed{\sum_{m=1}^3 r^2 \sin^2 \theta + r^2 \sum_{n=1}^2 = 0}$$

$$39) - \partial_2(\bar{e}_2 \cdot \bar{e}_3) = (\partial_2 \bar{e}_2) \cdot \bar{e}_3 + \bar{e}_2 \cdot (\partial_2 \bar{e}_3) = \sum_{m=2}^m \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=2}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=2 \end{matrix} \right) = \\ = \sum_{m=2}^3 \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=2}^2 \bar{e}_n = [\text{Según (12)}: \sum_{n=2}^2 = 0] = \sum_{m=2}^3 r^2 \sin^2 \theta = 0 \rightarrow \boxed{\sum_{m=2}^3 = 0}$$

$$40) - \partial_3(\bar{e}_2 \cdot \bar{e}_3) = (\partial_3 \bar{e}_2) \cdot \bar{e}_3 + \bar{e}_2 \cdot (\partial_3 \bar{e}_3) = \sum_{m=3}^m \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=3}^n \bar{e}_n = \left( \text{Sobreviven} \begin{matrix} m=3 & n=2 \end{matrix} \right) = \\ = \sum_{m=3}^3 \bar{e}_m \bar{e}_3 + \bar{e}_2 \sum_{n=3}^2 \bar{e}_n = [\text{Según (15)}: \sum_{m=3}^3 = \cot \theta] = \cot \theta \cdot r^2 \sin^2 \theta + r^2 \sum_{n=3}^2 = 0 \\ - \frac{\cos \theta}{\sin \theta} r^2 \sin^2 \theta = r^2 \sum_{n=3}^2 \rightarrow \boxed{\sum_{n=3}^2 = -\cos \theta \cdot \sin \theta}$$



# RECAPITULACIÓN

Nº del cálculo donde aparece el Christofel

- Se destacan en rojo los distintos de cero (No se repiten los de subíndices Simétricos)
- Se destacan en verde los indeterminados

1) $\begin{bmatrix} 0 \\ 00 \end{bmatrix}$ 0	17) $\begin{bmatrix} 1 \\ 00 \end{bmatrix}$ $\frac{a(r-a)}{2r^3}$	21) $\begin{bmatrix} 2 \\ 00 \end{bmatrix}$ 0	25) $\begin{bmatrix} 3 \\ 00 \end{bmatrix}$ 0
2) $\begin{bmatrix} 0 \\ 01 \end{bmatrix}$ $\frac{a}{2r(r-a)}$	5) $\begin{bmatrix} 1 \\ 01 \end{bmatrix}$ 0	22) $\begin{bmatrix} 2 \\ 01 \end{bmatrix}$	26) $\begin{bmatrix} 3 \\ 01 \end{bmatrix}$
3) $\begin{bmatrix} 0 \\ 02 \end{bmatrix}$ 0	19) $\begin{bmatrix} 1 \\ 02 \end{bmatrix}$	9) $\begin{bmatrix} 2 \\ 02 \end{bmatrix}$ 0	27) $\begin{bmatrix} 3 \\ 02 \end{bmatrix}$
4) $\begin{bmatrix} 0 \\ 03 \end{bmatrix}$ 0	20) $\begin{bmatrix} 1 \\ 03 \end{bmatrix}$	24) $\begin{bmatrix} 2 \\ 03 \end{bmatrix}$	13) $\begin{bmatrix} 3 \\ 03 \end{bmatrix}$ 0
18) $\begin{bmatrix} 0 \\ 11 \end{bmatrix}$ 0	6) $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$ $-\frac{a}{2r(r-a)}$	30) $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$ 0	34) $\begin{bmatrix} 3 \\ 11 \end{bmatrix}$ 0
19) $\begin{bmatrix} 0 \\ 12 \end{bmatrix}$	7) $\begin{bmatrix} 1 \\ 12 \end{bmatrix}$ 0	10) $\begin{bmatrix} 2 \\ 12 \end{bmatrix}$ $\frac{1}{r}$	35) $\begin{bmatrix} 3 \\ 12 \end{bmatrix}$
20) $\begin{bmatrix} 0 \\ 13 \end{bmatrix}$	8) $\begin{bmatrix} 1 \\ 13 \end{bmatrix}$ 0	32) $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$	14) $\begin{bmatrix} 3 \\ 13 \end{bmatrix}$ $\frac{1}{r}$
23) $\begin{bmatrix} 0 \\ 22 \end{bmatrix}$ 0	31) $\begin{bmatrix} 1 \\ 22 \end{bmatrix}$ $a-r$	11) $\begin{bmatrix} 2 \\ 22 \end{bmatrix}$ 0	39) $\begin{bmatrix} 3 \\ 22 \end{bmatrix}$ 0
24) $\begin{bmatrix} 0 \\ 23 \end{bmatrix}$	32) $\begin{bmatrix} 1 \\ 23 \end{bmatrix}$	12) $\begin{bmatrix} 2 \\ 23 \end{bmatrix}$ 0	15) $\begin{bmatrix} 3 \\ 23 \end{bmatrix}$ $\cotg \theta$
28) $\begin{bmatrix} 0 \\ 33 \end{bmatrix}$ 0	36) $\begin{bmatrix} 1 \\ 33 \end{bmatrix}$ $(a-r)\sin^2 \theta$	40) $\begin{bmatrix} 2 \\ 33 \end{bmatrix}$ $-\cos \theta \sin \theta$	16) $\begin{bmatrix} 3 \\ 33 \end{bmatrix}$ 0

Los 12 Christofels que aún están indeterminados, están relacionados por 12 ecuaciones que reescribo a continuación:

$$\left. \begin{aligned} 19) \begin{bmatrix} 1 \\ 20 \end{bmatrix} \frac{r}{r-a} + \begin{bmatrix} 0 \\ 21 \end{bmatrix} \frac{a-r}{r} &= 0 \\ 22) \begin{bmatrix} 2 \\ 10 \end{bmatrix} r^2 + \begin{bmatrix} 0 \\ 21 \end{bmatrix} \frac{a-r}{r} &= 0 \\ 29) \begin{bmatrix} 2 \\ 10 \end{bmatrix} r^2 + \begin{bmatrix} 1 \\ 20 \end{bmatrix} \frac{r}{r-a} &= 0 \end{aligned} \right\} \begin{aligned} &\begin{bmatrix} 1 \\ 20 \end{bmatrix} \\ &\begin{bmatrix} 0 \\ 21 \end{bmatrix} \\ &\begin{bmatrix} 2 \\ 10 \end{bmatrix} \end{aligned}$$

$$\left. \begin{aligned} 20) \begin{bmatrix} 1 \\ 30 \end{bmatrix} \frac{r}{r-a} + \begin{bmatrix} 0 \\ 31 \end{bmatrix} \frac{a-r}{r} &= 0 \\ 26) \begin{bmatrix} 3 \\ 10 \end{bmatrix} r^2 \sin^2 \theta + \begin{bmatrix} 0 \\ 31 \end{bmatrix} \frac{a-r}{r} &= 0 \\ 33) \begin{bmatrix} 3 \\ 10 \end{bmatrix} r^2 \sin^2 \theta + \begin{bmatrix} 1 \\ 30 \end{bmatrix} \frac{r}{r-a} &= 0 \end{aligned} \right\} \begin{aligned} &\begin{bmatrix} 1 \\ 30 \end{bmatrix} \\ &\begin{bmatrix} 0 \\ 31 \end{bmatrix} \\ &\begin{bmatrix} 3 \\ 10 \end{bmatrix} \end{aligned}$$

$$\left. \begin{aligned} 24) \begin{bmatrix} 2 \\ 30 \end{bmatrix} r^2 + \begin{bmatrix} 0 \\ 23 \end{bmatrix} \frac{a-r}{r} &= 0 \\ 27) \begin{bmatrix} 3 \\ 20 \end{bmatrix} r^2 \sin^2 \theta + \begin{bmatrix} 0 \\ 23 \end{bmatrix} \frac{a-r}{r} &= 0 \\ 37) \begin{bmatrix} 3 \\ 20 \end{bmatrix} r^2 \sin^2 \theta + \begin{bmatrix} 2 \\ 30 \end{bmatrix} r^2 &= 0 \end{aligned} \right\} \begin{aligned} &\begin{bmatrix} 2 \\ 30 \end{bmatrix} \\ &\begin{bmatrix} 0 \\ 23 \end{bmatrix} \\ &\begin{bmatrix} 3 \\ 20 \end{bmatrix} \end{aligned}$$

$$\left. \begin{aligned} 32) \begin{bmatrix} 2 \\ 31 \end{bmatrix} r^2 + \begin{bmatrix} 1 \\ 32 \end{bmatrix} \frac{r}{r-a} &= 0 \\ 35) \begin{bmatrix} 3 \\ 21 \end{bmatrix} r^2 \sin^2 \theta + \begin{bmatrix} 1 \\ 32 \end{bmatrix} \frac{r}{r-a} &= 0 \\ 38) \begin{bmatrix} 3 \\ 21 \end{bmatrix} r^2 \sin^2 \theta + \begin{bmatrix} 2 \\ 31 \end{bmatrix} r^2 &= 0 \end{aligned} \right\} \begin{aligned} &\begin{bmatrix} 2 \\ 31 \end{bmatrix} \\ &\begin{bmatrix} 1 \\ 32 \end{bmatrix} \\ &\begin{bmatrix} 3 \\ 21 \end{bmatrix} \end{aligned}$$

Vemos que están relacionados de 3 en 3, con cuatro sistemas de ecuaciones.

Cada "sistema de ecuaciones" tiene 3 ecuaciones y 3 incógnitas.



Resolviendo los sistemas de ecuaciones que nos relacionan los Christofels, aun indeterminados, fácilmente se llega a la conclusión que todos ellos deben ser nulos

Por ejemplo, resolvemos el primer sistema:

$$\left. \begin{array}{l} 19) \quad \Gamma_{20}^1 \frac{r}{r-a} + \Gamma_{21}^0 \frac{a-r}{r} = 0 \\ 22) \quad \Gamma_{10}^2 r^2 + \Gamma_{21}^0 \frac{a-r}{r} = 0 \end{array} \right\} \rightarrow \left. \begin{array}{l} \Gamma_{20}^1 \frac{r}{r-a} = -\Gamma_{21}^0 \frac{a-r}{r} \\ \Gamma_{10}^2 r^2 = -\Gamma_{21}^0 \frac{a-r}{r} \end{array} \right\} \Rightarrow \Gamma_{20}^1 \frac{r}{r-a} = \Gamma_{10}^2 r^2$$

$$29) \quad \Gamma_{10}^2 r^2 + \Gamma_{20}^1 \frac{r}{r-a} = 0 \rightarrow \text{Sustituyendo: } 2 \Gamma_{10}^2 r^2 = 0 \Rightarrow \Gamma_{10}^2 = 0$$

$$\text{Sustituyendo } \Gamma_{10}^2 = 0 \text{ en (22)} \rightarrow \Gamma_{21}^0 \frac{a-r}{r} = 0 \Rightarrow \Gamma_{21}^0 = 0$$

$$\text{Sustituyendo } \Gamma_{21}^0 = 0 \text{ en (19)} \rightarrow \Gamma_{20}^1 \frac{r}{r-a} = 0 \Rightarrow \Gamma_{20}^1 = 0$$

Los otros tres sistemas de ecuaciones tienen la misma estructura, por lo que se llegaría a:

$\Gamma_{30}^1 = 0$	$\Gamma_{31}^0 = 0$	$\Gamma_{10}^3 = 0$
$\Gamma_{30}^2 = 0$	$\Gamma_{23}^0 = 0$	$\Gamma_{20}^3 = 0$
$\Gamma_{31}^2 = 0$	$\Gamma_{32}^1 = 0$	$\Gamma_{21}^3 = 0$

Por lo tanto, en el espacio-tiempo de Schwarzschild, los únicos símbolos de Christofel distintos de cero son los 9 que están expuestos en la tabla de la página anterior.

(Se consideran como un único Christofel a cada pareja con índices simétricos, p.ej:  $\Gamma_{01}^0 = \Gamma_{10}^0 = \frac{a}{2r(r-a)}$ )