(1) Mihalic Pierre - Cancoye - Fargueg . octobre 2018 -

$$ds^2 = a^2 d\alpha^2 + (b + a\cos\alpha)^2 d\theta^2 = a^2 d\alpha^2 + \omega^2_{(\alpha)} d\theta^2$$

avec $\omega(\alpha) = \omega = b + a\cos\alpha$

$\omega'(\alpha) = -a\sin\alpha$

$\omega''(\alpha) = -a\cos\alpha$

avec $\alpha = 1$ $\theta = 2$

$$h_{\mu\nu} \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} a^2 & 0 \\ 0 & \omega^2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \quad h^{\mu\nu} \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 1/a^2 & 0 \\ 0 & 1/\omega^2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

(1) Symbole Christofell $\quad e_1 \cdot e_1 = a^2 \quad e_2 \cdot e_2 = \omega^2 \quad e_1 \cdot e_2 = e_2 e_1 = 0$

$* \ \partial_1(e_2 \cdot e_2) = \partial_1 \omega^2 \iff e_2 \partial_1 e_2 + e_2 \partial_1 e_2 = 2\omega'\omega \iff \cancel{2} e_2 \partial_1 e_2 = \cancel{2}\omega'\omega$

$\Rightarrow e_2 \Gamma^i_{12} e_i = \omega'\omega \qquad e_2 \cdot e_i = \delta^i_2 \iff e_2 \cdot e_i = 0$ sauf si $i = 2$

$\Rightarrow e_2 \cdot e_2 \cdot \Gamma^2_{12} = \omega'\omega \iff \omega^2 \Gamma^2_{12} = \omega'\omega$

$$\boxed{\Gamma^2_{12} = \frac{\omega'}{\omega}} \qquad \text{avec} \quad \boxed{\Gamma^2_{12} = \Gamma^2_{21} = \frac{\omega'}{\omega}}$$

$* \ \partial_2(e_2 e_2) = \partial_2 \omega = 0$ $\quad \omega$ ne dépend pas de θ.

$\iff 2 e_2 e_2 \Gamma^2_{22} = 0 \quad \Rightarrow \boxed{\Gamma^2_{22} = 0}$

$* \ \partial_1(e_1 e_1) = 2 e_1 e_1 \Gamma^1_{11} = 0 \quad \boxed{\Gamma^1_{11} = 0}$

$* \ \partial_2(e_1 e_1) = 2 e_1 e_1 \Gamma^1_{21} = 0 \quad \boxed{\Gamma^1_{21} = \Gamma^1_{12} = 0}$

$* \ \partial_1(e_1 e_2) = e_1 \partial_1 e_2 + e_2 \partial_1 e_1 = e_1 \underset{0}{\Gamma^1_{12}} e_1 + e_2 \Gamma^2_{11} e_2 + \underset{0}{e_1 e_2} \Gamma^2_{11} + \underset{0}{e_2 e_1} \Gamma^1_{11} =$

$\Rightarrow \boxed{\Gamma^2_{11} = 0}$

$* \ \partial_2(e_1 e_2) = e_1 \partial_2 e_2 + e_2 \partial_2 e_1 = e_1 \Gamma^1_{22} e_1 + e_2 \Gamma^2_{21} e_2 = a^2 \Gamma^1_{22} + \omega^2 \Gamma^2_{21}$

$= a^2 \Gamma^1_{22} + \frac{\omega^2 \omega'}{\omega} \Rightarrow \boxed{\Gamma^1_{22} = -\frac{\omega'\omega}{a^2}}$

$$\Gamma^1_{\mu\nu} \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\omega'\omega}{a^2} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \quad \Gamma^2_{\mu\nu} \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 0 & \omega'/\omega \\ \omega'/\omega & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Γ^1_{22} et $\Gamma^2_{21} \neq 0$

autre méthode : $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} h^{\alpha\rho} \left(h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho} \right)$

$h_{11} = \text{cst}$. et si $\mu \neq \nu \Rightarrow h^{\mu\nu} = 0 \Rightarrow$ 3 symb Christoff $\neq 0$ Γ^1_{22} et Γ^2_{12} et Γ^2_{21}

* $\Gamma^1_{22} = \frac{1}{2} h^{11} \left(h_{21,2} + h_{21,2} - h_{22,1} \right) = -\frac{1}{2} h^{11} \left(h_{22,1} \right) = \frac{-1}{2a^2} \cdot 2\omega'\omega = \boxed{-\frac{\omega'\omega}{a^2}}$

* $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} h^{22} \left(\cancel{h_{12,2}} + h_{22,1} - \cancel{h_{12,2}} \right) = \frac{1}{2} \frac{1}{\omega^2} \cdot 2\omega'\omega = \boxed{\frac{\omega'}{\omega}}$

② Tenseur Riemann : $R^{\rho}_{\ \gamma\alpha\beta} = \left(\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha \right) e_\gamma$

$\alpha = \beta \Rightarrow R^{\rho}_{\ \gamma\alpha\alpha} = 0 \quad \forall (\rho, \gamma)$

$\Rightarrow R^1_{\ 111} = R^1_{\ 211} = R^2_{\ 111} = R^2_{\ 211} = R^1_{\ 122} = R^1_{\ 222} = R^2_{\ 122} = R^2_{\ 222} = 0$

ⓐ $\alpha = 1 \quad \beta = 2 \quad \gamma \left.\right\} \begin{matrix} =1 \\ =2 \end{matrix} \quad R^{\rho}_{\ \gamma 12}$???

(a_1) $\alpha = 1 \quad \beta = 2 \quad \gamma = 1$

$R^{\rho}_{\ 112} e_\rho = \left(\partial_1 \partial_2 e_1 - \partial_2 \partial_1 e_1 \right) = \partial_1 \left(\Gamma^i_{21} e_i \right) - \partial_2 \left(\cancel{\Gamma^j_{11} e_j} \right)$ or Γ^2_{12} et $\Gamma^1_{22} \neq 0$

$= \partial_1 \Gamma^2_{21} e_2 = \Gamma^2_{21,1} e_2 + \Gamma^2_{21} \Gamma^j_{21} e_j = \left(\Gamma^2_{21,1} + \Gamma^2_{21} \Gamma^2_{21} \right) e_2$

$R^1_{\ 112} = 0 \quad$ et $\left(R^2_{\ 112} = \Gamma^2_{21,1} + \Gamma^2_{21} \Gamma^2_{21} \right) = \omega''/\omega$

$= \left(\frac{\omega'}{\omega} \right)' + \frac{\omega'^2}{\omega^2} = \frac{\omega''\omega - \omega'^2}{\omega^2} + \frac{\omega'^2}{\omega^2} = \frac{\omega''}{\omega}$

(a_2) $\alpha = 1 \quad \beta = 2 \quad \gamma = 2$

$R^{\rho}_{\ 212} e_\rho = \left(\partial_1 \partial_2 e_2 - \partial_2 \partial_1 e_2 \right) = \partial_1 \left(\Gamma^1_{22} e_1 \right) - \partial_2 \left(\Gamma^2_{12} e_2 \right)$

$= \Gamma^1_{22,1} e_1 + \cancel{\Gamma^j_{11} e_j} - \cancel{\Gamma^2_{12,2} e_2} - \Gamma^2_{12} \Gamma^1_{22} e_1 = e_1 \left(\Gamma^1_{22,1} - \Gamma^2_{12} \Gamma^1_{22} \right)$

$\boxed{R^2_{\ 212} = 0}$

$R^1_{\ 212} = \Gamma^1_{22,1} - \Gamma^2_{12} \Gamma^1_{22} = -\frac{\omega''\omega}{a^2}$

$= -\frac{\omega''\omega - \omega'^2}{a^2} + \frac{\omega'}{\omega} \frac{\omega'\omega}{a^2} = -\frac{\omega''\omega}{a^2}$

(b) $\alpha = 2 \quad \beta = 1 \quad \left\{ \begin{array}{l} \gamma = 1 \\ \gamma = 2 \end{array} \right.$ (8) Cauchy - Pauguoy . 2017

(b1) $\alpha = 2 \; \beta = 1 \; \gamma = 1 \quad R^{\rho}_{121} = e_\rho(\partial_2\partial_1 e_1 - \partial_1\partial_2 e_1) = \partial_2 \cancel{\Gamma^j_{11} e_j} - \partial_1 \Gamma^2_{21} e_2$

$= -\Gamma^2_{21,1} e_2 - \Gamma^2_{21}\Gamma^2_{12} e_2 = (-\Gamma^2_{21,1} - \Gamma^2_{21}\Gamma^2_{12}) e_2$ d'où $R^1_{121} = 0$

$R^2_{121} = -\Gamma^2_{21,1} - \Gamma^2_{21}\Gamma^2_{12} = -\left(\frac{\omega'}{\omega}\right)' - \frac{\omega'^2}{\omega^2} = \frac{-\omega''\omega + \omega'^2 - \omega'^2}{\omega^2}$

$= -\frac{\omega''}{\omega}$

(b2) $\alpha = 2 \; \beta = 1 \; \gamma = 2 \quad R^{\rho}_{221} = (\partial_2\partial_1 e_2 - \partial_1\partial_2 e_2) e_\rho$

$= \partial_2 \Gamma^2_{12} e_2 - \partial_1 \Gamma^1_{22} e_1 = \cancel{\Gamma^2_{12,2}} e_2 + \Gamma^2_{12}\Gamma^1_{22} e_1 - \Gamma^1_{22,1} e_1 - \Gamma^1_{21} \cancel{\Gamma^j_{11} e_j}$

$R^{\rho}_{221} = (-\Gamma^1_{22,1} + \Gamma^2_{12}\Gamma^1_{22}) e_1 \quad R^2_{221} = 0$

d $R^1_{221} = -\Gamma^1_{22,1} + \Gamma^2_{12}\Gamma^1_{22} = \frac{\omega''\omega + \omega'^2}{a^2} + \frac{\omega'(-\omega')\cancel{\omega}}{\cancel{\omega} a^2}$

$= \frac{\omega''\omega}{a^2}$

Autre méthode : $R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^i_{kn}\Gamma^n_{lj} - \Gamma^i_{ln}\Gamma^n_{kj}$.

$R^2_{112} = \Gamma^2_{12,1} - \cancel{\Gamma_{11,2}} + \Gamma^2_{12}\Gamma^2_{12} - \cancel{\Gamma\,\Gamma_{11}} = \frac{\omega''}{\omega}$

$R^1_{212} = \Gamma^1_{22,1} - \Gamma^1_{22}\Gamma^2_{21} = -\frac{\omega''\omega}{a^2}$

$R^2_{121} = -\Gamma^2_{12,1} - \Gamma^2_{12}\Gamma^2_{12} = -\frac{\omega''}{\omega}$

$R^1_{221} = -\Gamma^1_{22,1} + \Gamma^2_{12}\Gamma^1_{22} = \frac{\omega''\omega}{a^2}$

(3) Tenseur Ricci : $R_{jl} = R^k_{jkl}$

$R_{11} = R^1_{111} + R^2_{121} = 0 - \frac{\omega''}{\omega} \quad R_{22} = R^1_{212} + R^2_{222} = -\frac{\omega''\omega}{a^2} + 0$

(4) Courbure Scalaire : $R = g^{\mu\nu} R_{\mu\nu} = g^{11} R_{11} + g^{22} R_{22}$

$= \frac{1}{a^2} \cdot \frac{-\omega''}{\omega} + \frac{1}{\omega^2} \cdot \frac{-\omega''\omega}{a^2} = \frac{-2\omega''}{a^2\omega} = \frac{2a\cos\alpha}{a^2(b + a\cos\alpha)} = \boxed{\frac{2\cos\alpha}{a(b + a\cos\alpha)} = R}$