Hugo
Labella

TORO

$$ds^2 = a^2 d\alpha^2 + (6 + a\cos\alpha)^2 d\theta^2$$

Símbolos de Christoffel

$x^1 \equiv \alpha$
$x^2 \equiv \theta$

Γ^1_{11}

$\partial_1 (e_1 \cdot e_1 = a^2)$

$\partial_1(e_1) \cdot e_1 + e_1 \cdot \partial_1(e_1) = 0$

$2\Gamma^{\alpha}_{11} e_\alpha \cdot e_1 = 0$

$2\Gamma^{1}_{11} e_1 \cdot e_1 = 0$

$2\Gamma^1_{11} a^2 = 0$

$\Gamma^1_{11} = 0$

Γ^2_{22}

$\partial_2(e_2 \cdot e_2 = (6 + a\cos\alpha)^2)$

mismo que antes

$2\Gamma^2_{22} e_2 \cdot e_2 = 0$

$2\Gamma^2_{22} (6 + a\cos\alpha)^2 = 0$

$\Gamma^2_{22} = 0$

Γ^2_{12}

$\partial_1((e_2 \cdot e_2) = (6 + a\cos\alpha)^2)$

$2(\partial_1 e_2) \cdot e_2 = 2(6 + a\cos\alpha)(a(-\sin\alpha))$

$2\Gamma^2_{12} h_{22} = 2(6 + a\cos\alpha)(a(-\sin\alpha))$

$\cancel{2}\Gamma^2_{12} (6 + a\cos\alpha)^2 = \cancel{2}(\cancel{6 + a\cos\alpha})(-a\sin\alpha)$

$$\boxed{\Gamma^2_{12} = \Gamma^2_{21} = -\frac{a\sin\alpha}{6 + a\cos\alpha}}$$

Γ^1_{12}

$\partial_2(e_1 \cdot e_1 = a^2)$

$2(\partial_2 e_1) e_1 = 0$

$2\Gamma^1_{12} a^2 = 0$

$\Gamma^1_{12} = 0$

Γ^1_{22}

$\partial_2(e_2 \cdot e_1) = 0$

$(\partial_2 e_2)\cdot e_1 + (\partial_2 e_1)\cdot e_2 = 0$

$\Gamma^1_{22}\, a^2 + \Gamma^2_{12}\,(b + a\cos\alpha) = 0$

$\Gamma^1_{22}\, a^{\cancel{2}} = \frac{\cancel{a}\,\sin\alpha}{\cancel{b + a\cos\alpha}}\,(b + a\cos\alpha)^{\cancel{2}}$

$$\Gamma^1_{22} = \frac{\sin\alpha\,(b + a\cos\alpha)}{a}$$

Γ^2_{11}

$\partial_1(e_2 \cdot e_1) = 0$

$(\partial_1 e_2)\cdot e_1 + (\partial_1 e_1)\cdot e_2 = 0$

$\cancel{\Gamma^1_{21}}^{\to 0}\, a^{2^0} + \Gamma^2_{11}\,(b + a\cos\alpha)^2 = 0$

$\Gamma^2_{11} = 0$

Tensor de Riemann

$$R^{\sigma}{}_{\gamma\delta\beta}\, e_{\sigma} = (\partial_{\delta}\partial_{\beta} - \partial_{\beta}\partial_{\delta})\, e_{\gamma}$$

$$R^{\sigma}{}_{\gamma 11}\, e_{\sigma} = (\partial_1\partial_1 - \partial_1\partial_1)\, e_{\gamma} = R^{\sigma}{}_{\gamma 22} = 0$$

$\delta = 1 \quad \beta = 2 \quad \gamma = 1$

$$R^{\sigma}{}_{112}\, e_{\sigma} = (\partial_1\partial_2 - \partial_2\partial_1)\, e_1 = (\partial_1 \Gamma^{\delta}_{12} e_{\delta}) - \partial_2 (\underbrace{\Gamma^{\delta}_{11}}_{0} e_{\delta}) =$$

$$= (\partial_1 \Gamma^{2}_{12})\, e_2 + \Gamma^{2}_{12}\Gamma^{2}_{12}\, e_2 =$$

$$\partial_1 \Gamma^{2}_{12} = \partial_1\left(-a\frac{\sin\alpha}{b + a\cos\alpha}\right) = -a\,\frac{\cos\alpha(b+a\cos\alpha) + \sin^2\alpha\, a}{(b+a\cos\alpha)^2} = -a\,\frac{b\cos\alpha + a\cos^2\alpha + a\sin^2\alpha}{(b+a\cos\alpha)^2} =$$

$$= -a\,\frac{b\cos\alpha + a\overbrace{(\cos^2\alpha + \sin^2\alpha)}^{1}}{(b+a\cos\alpha)^2} = -a\,\frac{a + b\cos\alpha}{(b+a\cos\alpha)^2}$$

$$\left[-a\,\frac{(a+b\cos\alpha) - (a\sin^2)}{(b+a\cos\alpha)^2}\right] e_2 = -a\,\frac{a(1-\sin^2\alpha) + b\cos\alpha}{(b+a\cos\alpha)^2}\, e_2 = -a\,\frac{a\cos^2\alpha + b\cos\alpha}{(b+a\cos\alpha)^2}\, e_2 =$$

$\cos^2\alpha + \sin^2\alpha = 1$

$\cos^2\alpha = 1 - \sin^2\alpha$

$$= -a\,\frac{\cos\alpha\, \cancel{(b+a\cos\alpha)}}{(b+a\cos\alpha)^{\cancel{2}}}\, e_2 = -\frac{a\cos\alpha}{b+a\cos\alpha}\, e_2 = R^{1}{}_{112}\, e_1 + R^{2}{}_{112}\, e_2$$

por tanto

$$R^{1}{}_{112} = 0 \qquad R^{2}{}_{112} = -a\,\frac{\cos\alpha}{b + a\cos\alpha}$$

$\gamma=2 \quad \delta=1 \quad \beta=2$

$$R^{\sigma}{}_{212}\, e_\sigma = (\partial_1\partial_2 - \partial_2\partial_1)\, e_2 = \partial_1(\Gamma^{\delta}_{22} e_\delta) - \partial_2(\Gamma^{\delta}_{12} e_\delta) =$$

$$= \partial_1 \Gamma^{1}_{22}\, e_1 + \underset{0}{\underset{\shortparallel}{\Gamma^{\delta}_{11}}}\, \Gamma^{1}_{22}\, e_\delta - \Big(\underset{0}{\underset{\shortparallel}{\underline{\partial_2 \Gamma^{2}_{12}}}}\, e_2 + \Gamma^{2}_{12}\, \Gamma^{1}_{22}\, e_1\Big) =$$

$$= \left[\frac{1}{a}\left(\cos\alpha(b+a\cos\alpha) - \sin^2\alpha(a)\right) - \frac{\sin\alpha \sin\alpha\,(-a)(b+a\cos\alpha)}{a^2(b+a\cos\alpha)}\right] e_1 =$$

$$= \frac{1}{a}\big(\cos\alpha(b+a\cos\alpha) + \underbrace{\sin^2\alpha\, a - \sin^2\alpha\, a}_{0}\big)\, e_1 = \frac{1}{a}\cos\alpha(b+a\cos\alpha)\, e_1$$

$$R^{1}{}_{212}\, e_1 + R^{2}{}_{212}\, e_2 = \frac{1}{a}\cos\alpha(b+a\cos\alpha)\, e_1$$

por tanto

$$\boxed{R^{1}{}_{212} = \frac{1}{a}\cos\alpha\,(b+a\cos\alpha)}$$

$$R^{2}{}_{212} = 0$$

Tensor de Ricci

$$R_{\gamma\beta} = R^{\sigma}{}_{\gamma\sigma\beta}$$

$$\boxed{R_{12}} = \boxed{+R_{21}} = R^{\sigma}{}_{1\sigma 2} = \underset{0}{\underset{\shortparallel}{R^{1}{}_{112}}} + \underset{0}{\underset{\shortparallel}{R^{2}{}_{122}}} = 0$$

$$R_{11} = R^{\sigma}{}_{1\sigma 1} = \underset{0}{\underset{\shortparallel}{R^{1}{}_{111}}} + \underset{-R^{2}{}_{112}}{\underset{\shortparallel}{R^{2}{}_{121}}}$$

$$\boxed{R_{11}} = a\frac{\cos\alpha}{b+a\cos\alpha}$$

$$\boxed{R_{22}} = R^{\sigma}{}_{2\sigma 2} = R^{1}{}_{212} + \underset{0}{\underset{\shortparallel}{R^{2}{}_{222}}} = \frac{\cos\alpha(b+a\cos\alpha)}{a}$$

Escalar de Ricci

$$R = g^{ij} R_{ij}$$

$$g^{ij} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{(b+a\cos\alpha)^2} \end{pmatrix}$$

$$R = g^{11} R_{11} + g^{22} R_{22} + g^{12} \underbrace{R_{12}}_{=0} + g^{21} \underbrace{R_{21}}_{=0} =$$

$$= \frac{1}{a^2} \cancel{a} \frac{\cos\alpha}{b+a\cos\alpha} + \frac{1}{(b+a\cos\alpha)^2} \frac{\cos\alpha}{a} \cancel{(b+a\cos\alpha)} = \boxed{2 \frac{\cos\alpha}{a(b+a\cos\alpha)} = R}$$