Fourier Transformation-Lecture 3

Owen Wyn Roberts (owr10) room 2.18

Topics Covered

- A Fourier Analysis

1. A1-Basic Definitions ✓
2. A2-Real/Complex Fourier Series✓
3. A3-Fourier Transforms

- B Diffusion Equation
- C Laplace Equation
- D Wave Equation

Topics Covered

- A Fourier Analysis

1. A1-Basic Definitions ✓ Recap
2. A2-Real/Complex Fourier Series✓ Recap
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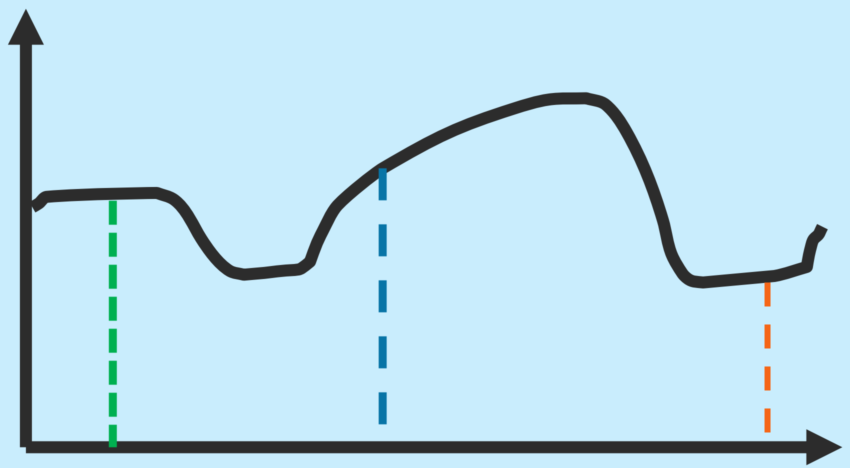
- C Laplace Equation
- D Wave Equation

Review Topic A1/Workshop

- Even functions $f(x) = f(-x)$, Specific example for us $\cos(x)$
- Odd functions $f(-x) = -f(x)$, Specific example for us $\sin(x)$
- $\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$ Partition into two (or more) integrals



Review Topic A1/Workshop Q3

3. $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$

(a)

$$a_0 = \frac{1}{\pi}\int_{x=-\pi}^{x=\pi} f(x)\mathrm{d}x$$

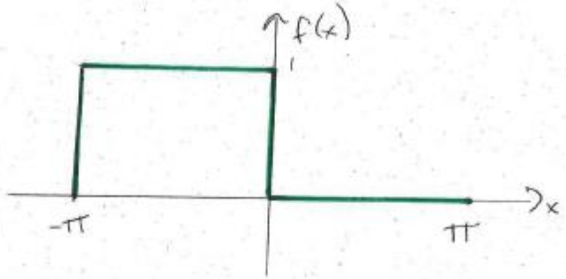
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(1)\mathrm{d}x + \int_{0}^{\pi}(0)\mathrm{d}x$$

$$a_0 = \frac{1}{\pi}[\pi] = 1$$

Partition of the integral

4

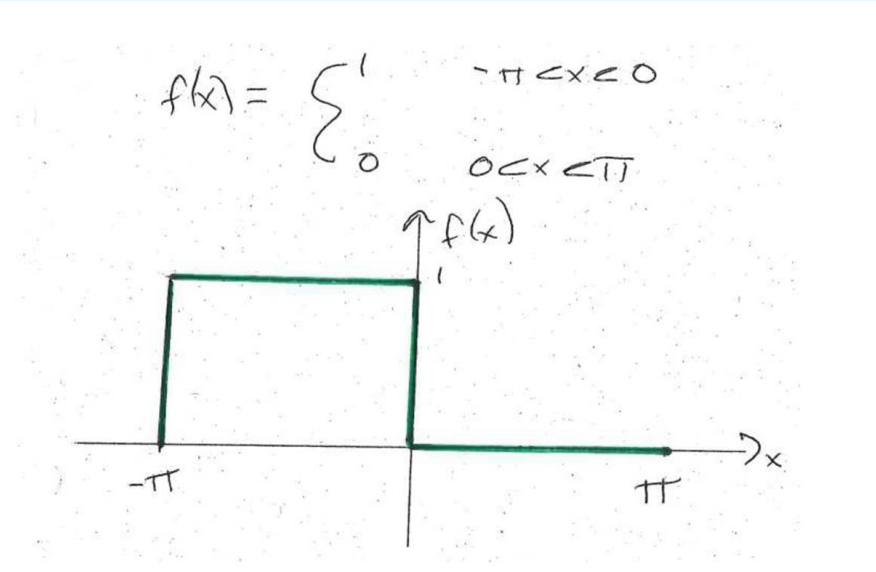
$$f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$$



Review Topic A1

- Even functions $f(x) = f(-x)$, Specific example for us $\cos(x)$
- Odd functions $f(-x) = -f(x)$, Specific example for us $\sin(x)$
- Useful to know when calculating Fourier series (is my function odd or even?)
- Useful to know: $\cos(n\pi) = (-1)^n$ when n is an integer
- Useful to know: $\sin(n\pi) = 0$ when n is an integer

Review Topic A1



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1.\cos(nx)\mathrm{d}x + \int_{0}^{\pi} 0.\cos(nx)\mathrm{d}x\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1.\cos(nx)\mathrm{d}x\right]$$

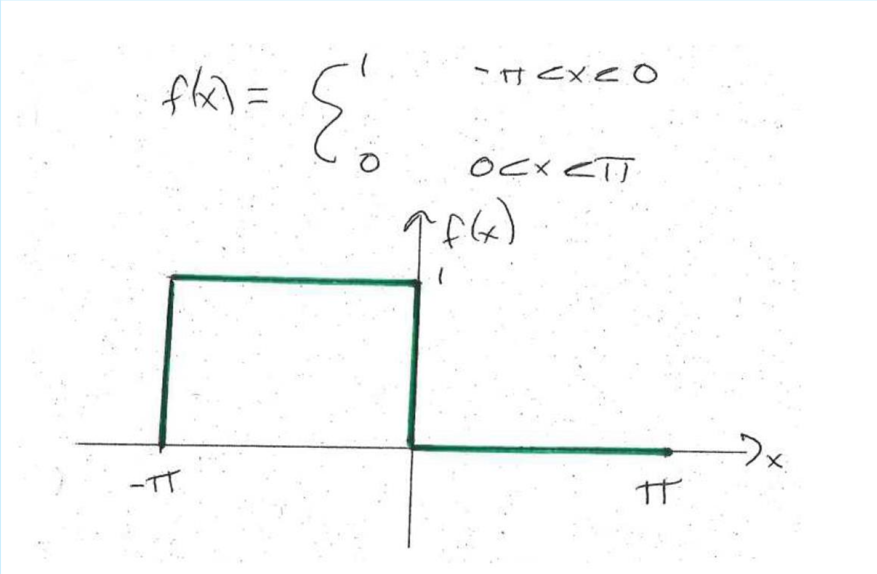
$$a_n = \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]\Bigg|_{x=-\pi}^{x=0}$$

$$a_n = \frac{1}{\pi}\left[\frac{\sin(0)}{n} - \frac{\sin(-n\pi)}{n}\right]$$

$$a_n = 0$$

Could also make a statement here that an should be zero as it is an odd offset function

Review Topic A1



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1.\sin(nx)\mathrm{d}x + \int_{0}^{\pi} 0.\sin(nx)\mathrm{d}x\right]$$

Partition of the integral

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (1)\sin(nx)\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]\Bigg|_{x=-\pi}^{x=0}$$

$$b_n = -\frac{1}{\pi}\left[\frac{\cos(0)}{n} - \frac{\cos(-n\pi)}{n}\right]$$

Use $\cos(n\pi) = (-1)^n$

$$b_n = -\frac{1}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]$$

$$b_n = -\frac{2}{n\pi} \text{ for } n \text{ odd, } 0 \text{ otherwise.}$$

Review Topic A2

- We found that we can estimate any periodic function as a series of sine and cosine functions (Real Fourier Series)
- RFS: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$
- Equivalently we can use complex exponentials and form a Complex Fourier series (CFS)
- CFS: $f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$
- To obtain the coefficients we performed some mathematical tricks and took averages over a period 2π
- This allowed us to isolate each coefficient and obtain a value for them

Review Topic A2

- RFS: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$

- CFS: $f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$

- We can express a function as a superposition of odd (sine) and even (cosine) functions (and an offset term)

A2- Real Fourier Series finding a_n

- RFS: $\boxed{f(x)} = \boxed{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)}$

$$\boxed{LHS = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx}$$

For $n = m$ this is non-zero. It becomes $\cos^2(nx)$

Consider m a constant therefore only one term in the series is important. All other terms when $n \neq m$ vanish so we can get rid of the summation

$$\boxed{RHS = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos(nx)\cos(mx)\,dx}$$

A2- Real Fourier Series finding a_n

$$RHS = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos(nx)\cos(mx)\,dx$$

Prove this is true when $m \neq n$ $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = 0$

Use identity $\cos(nx)\cos(mx) = \frac{1}{2}[\cos((n+m)\,x) + \cos\big((n-m)x\big)]$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}[\cos((n+m)\,x) + \cos\big((n-m)x\big)]dx =$$

$$\frac{1}{4\pi}\left[\frac{sin\big((n+m)x\big)}{n+m} + \frac{sin\big((n-m)x\big)}{n-m}\right]_{-\pi}^{\pi} = 0$$

For $n = m$ this is non-zero. It becomes $\cos^2(nx)$

Consider m a constant therefore only one term in the series is important. All other terms when $n \neq m$ vanish so we can get rid of the summation

You had this question in the workshop to show this

n and m are integers therefore all of these terms are $\sin(a\pi)$ where a is also an integer and these all go to zero

A2- Real Fourier Series putting it all together

- RFS: $\boxed{f(x)} = \boxed{\frac{\boxed{a_0}}{2} + \sum_{n=1}^{\infty} \boxed{a_n} \cos(nx) + \sum_{n=1}^{\infty} \boxed{b_n} \sin(nx)}$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx}$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}$$

Topics Covered

- A Fourier Analysis

1. A1-Basic Definitions ✓ Recap
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Topics Covered

3. A3-Fourier Transforms

- Subtopics:
- Conceptual Understanding
- Doing the integrations
- Convolution theorem

Review Some properties of complex exponentials

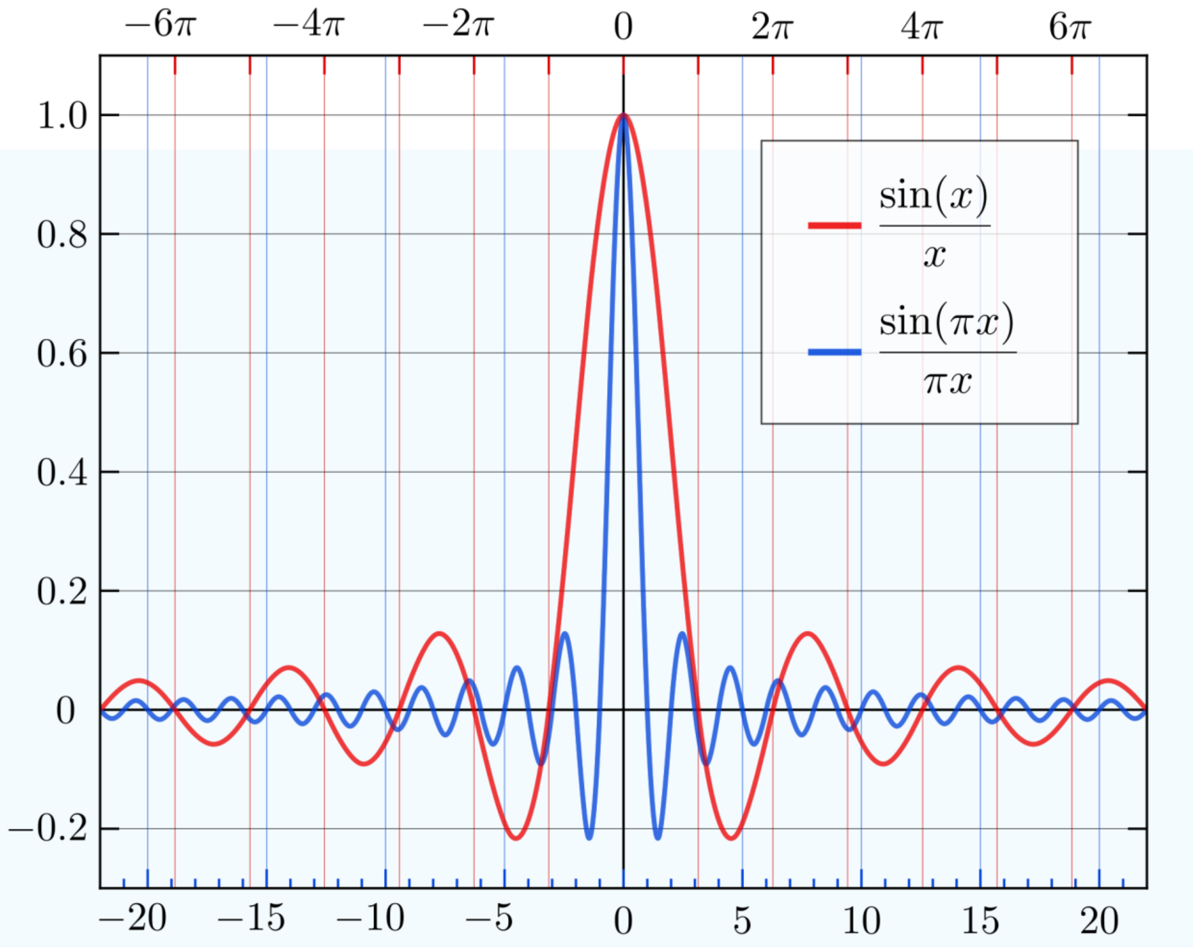
- $e^{ix} = \cos(x) + i\sin(x)$
- $e^{-ix} = \cos(x) - i\sin(x)$
- Re-arrange
- $\sin(nx) = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}nx} - e^{-\mathrm{i}nx})$
- $\cos(nx) = \frac{1}{2}(e^{\mathrm{i}nx} + e^{-\mathrm{i}nx})$
- $e^{0} = 1$
- $e^{a}e^{b} = e^{a+b}$
- $e^{-\infty} = 0$
- $sinc(x) = \frac{\sin(x)}{x}$

$sinc(x)$

$$sinc(x) = \frac{\sin(x)}{x}$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

You can prove this geometrically from the squeeze theorem



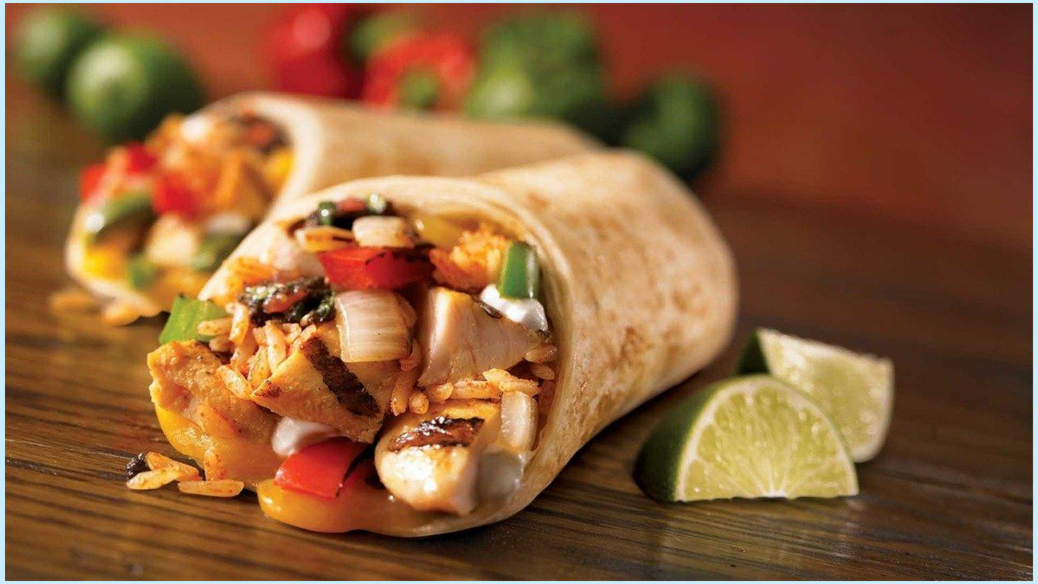
Topics Covered

3. A3-Fourier Transforms

- Subtopics:
- Conceptual Understanding
- Doing the integrations
- Convolution theorem

A3- Fourier Transformation

- You had the best Burrito in your life and you want to reproduce it



https://commons.wikimedia.org/wiki/File:Burrito.JPG

Fourier Transform

Recipe for the best burrito!

Take 500g rice

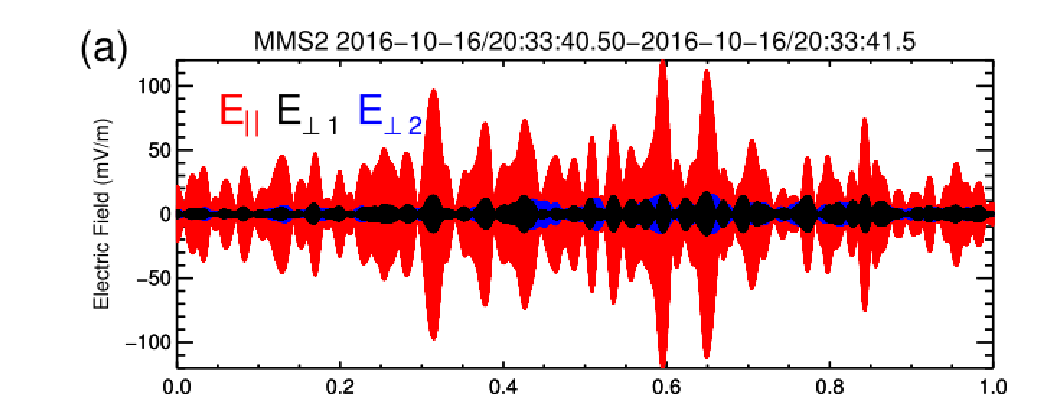
Cut 1 onion

....

.....

A3- Fourier Transformation

- You've measured the best signal in your life and you want to know what frequencies are important!



Roberts et al. (2020) MMS HMFE data

Fourier Transform

Recipe for the signal burrito!

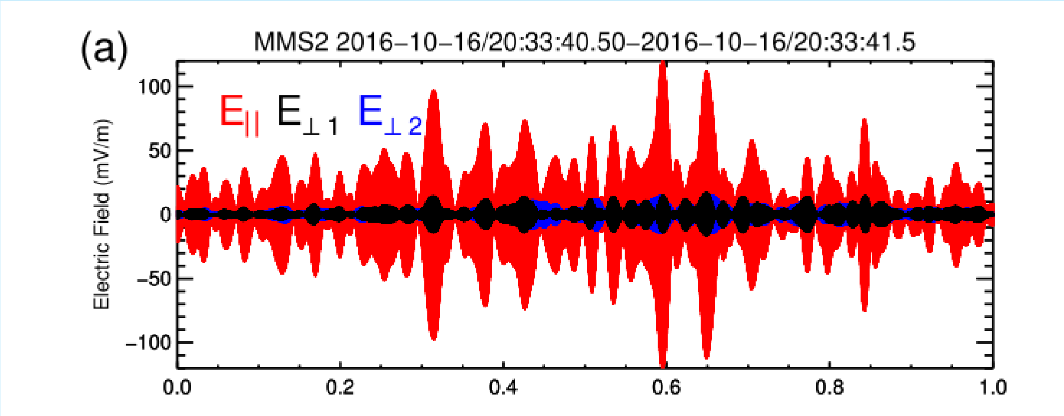
Take 1 sine wave with an angular frequency of 10, an amplitude of 30

Add a cosine wave with a frequency of 30 and an amplitude of 50

.....

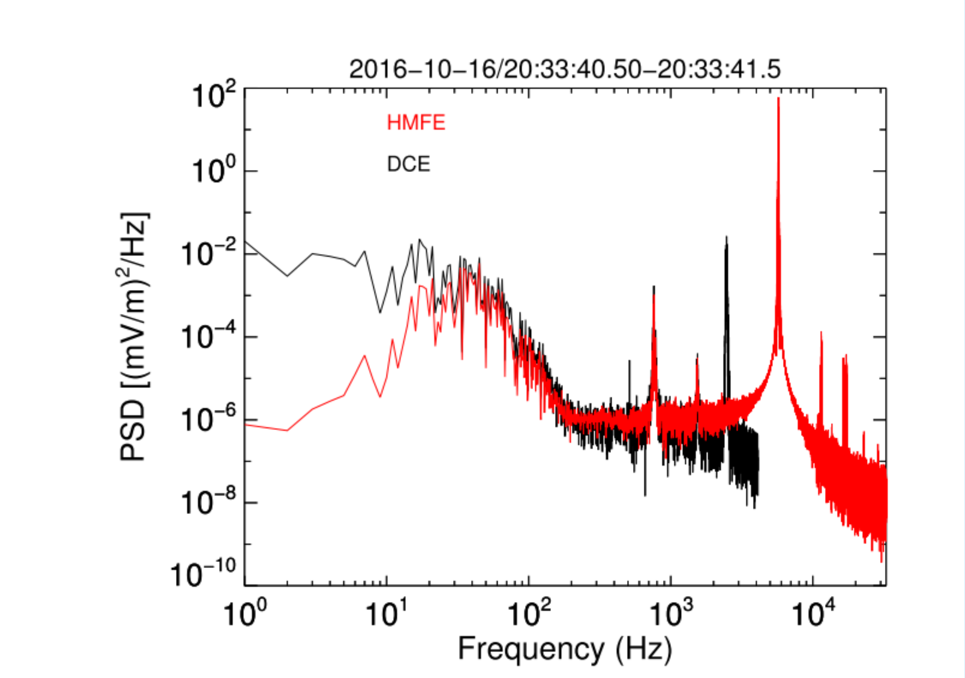
A3- Fourier Transformation

- You've measured the best signal in your life and you want to know what frequencies are important!



Roberts et al. (2020) MMS HMFE data

Fourier Transform



Langmuir wave see also Graham et al (2018)

$$\omega = \sqrt{\frac{n_e q^2}{m_e \epsilon_0}}$$

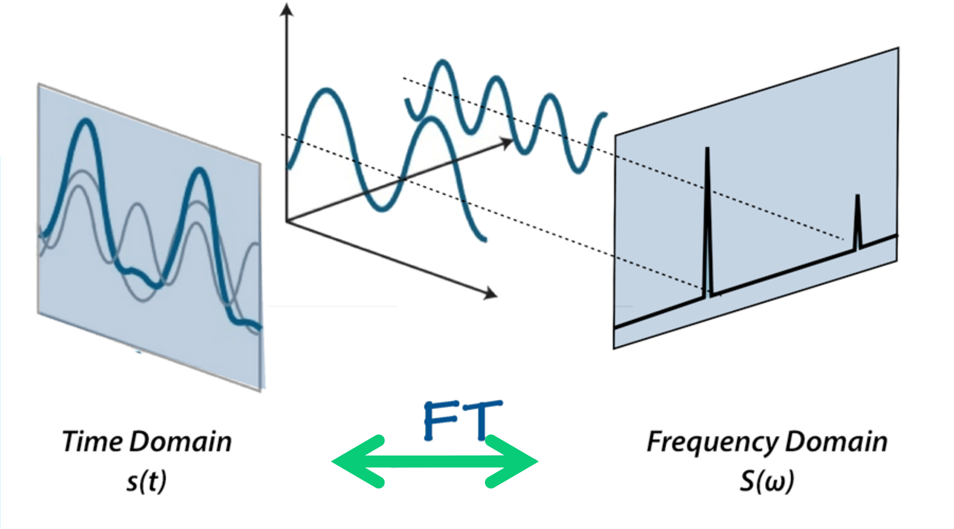
Power spectral density (there is also phase information for each mode).

A3 Fourier Transform

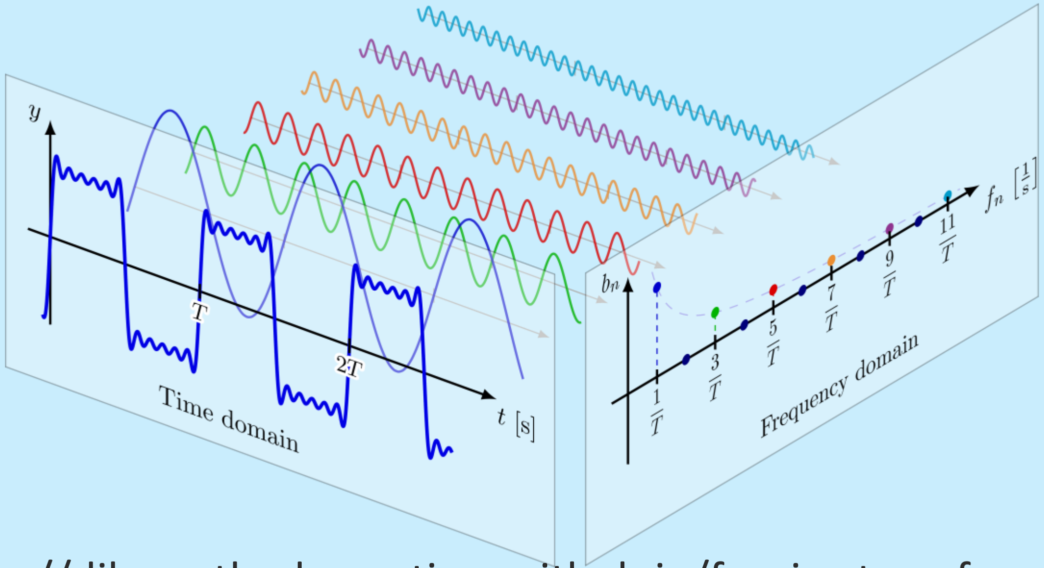
- The Fourier transform is a method of going between the time (or spatial domain) to the frequency (wavenumber domain)

- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

- So, you can see what frequencies are important in your signal



https://mriquestions.com/fourier-transform-ft.html



https://dibsmethodsmeetings.github.io/fourier-transforms/

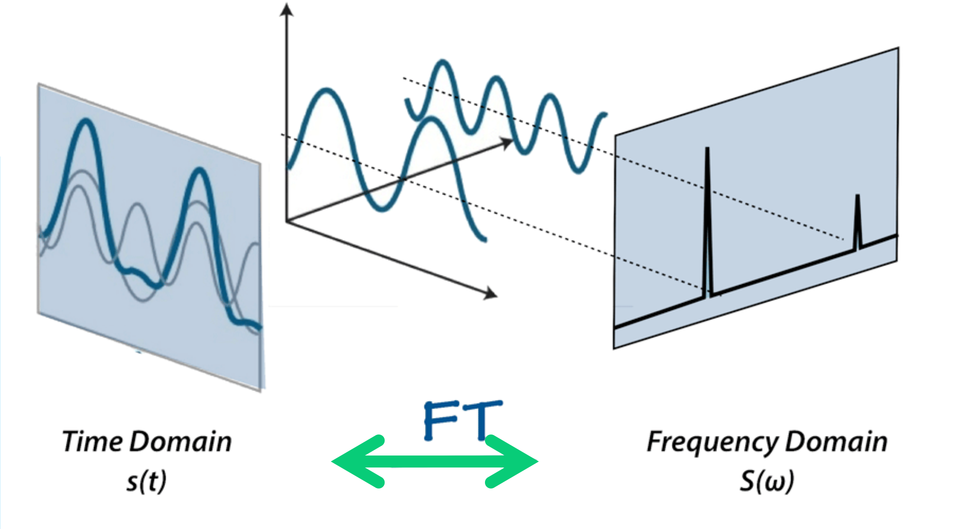
A3 Fourier Transform

- The Fourier transform is a method of going between the time (or spatial domain) to the frequency (wavenumber domain)

You might see different conventions

- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

- So, you can see what frequencies are important in your signal
- Fourier transform has a real and imaginary part



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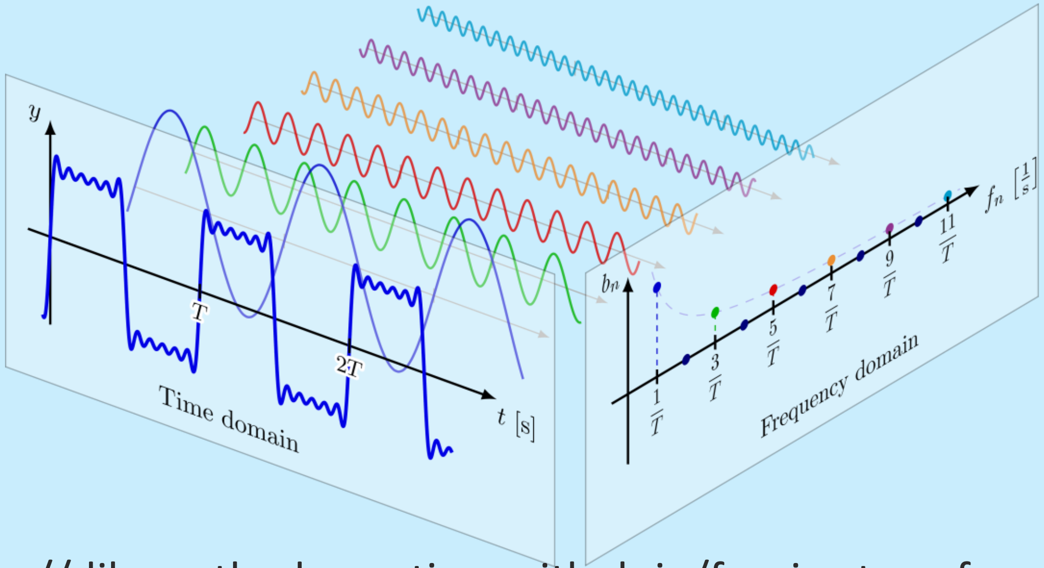
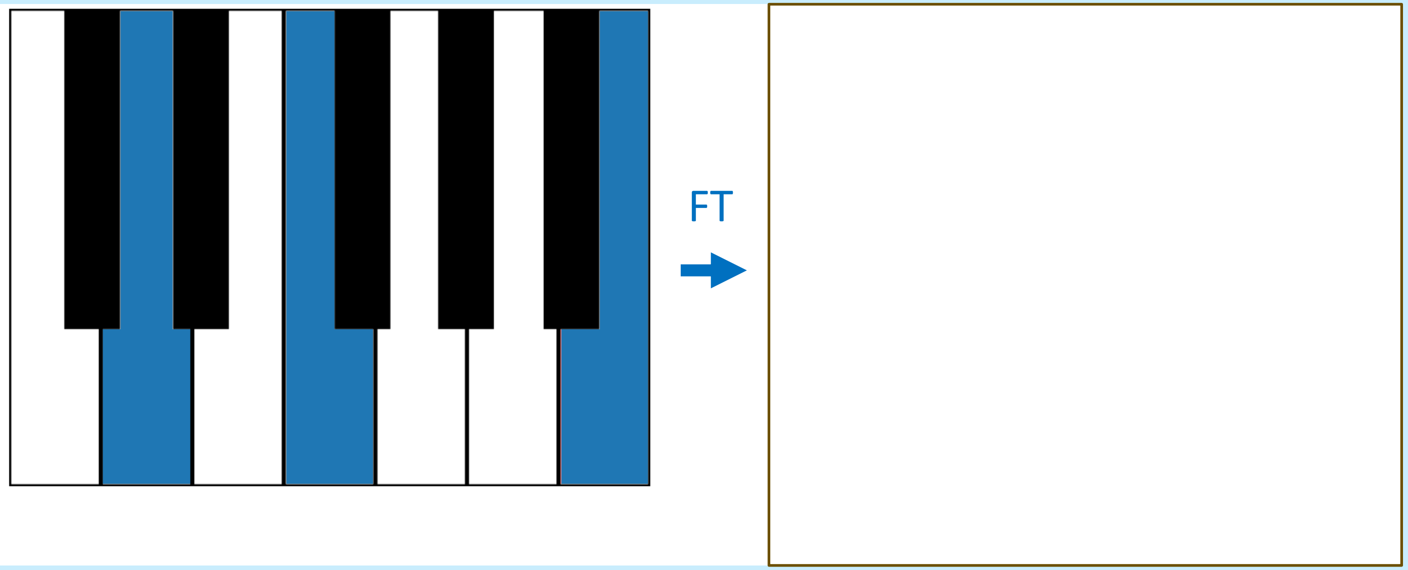


https://dibsmethodsmeetings.github.io/fourier-transforms/

A3 Fourier Transform

What does the Fourier transform of this chord look like?

I'll give you some time to think…



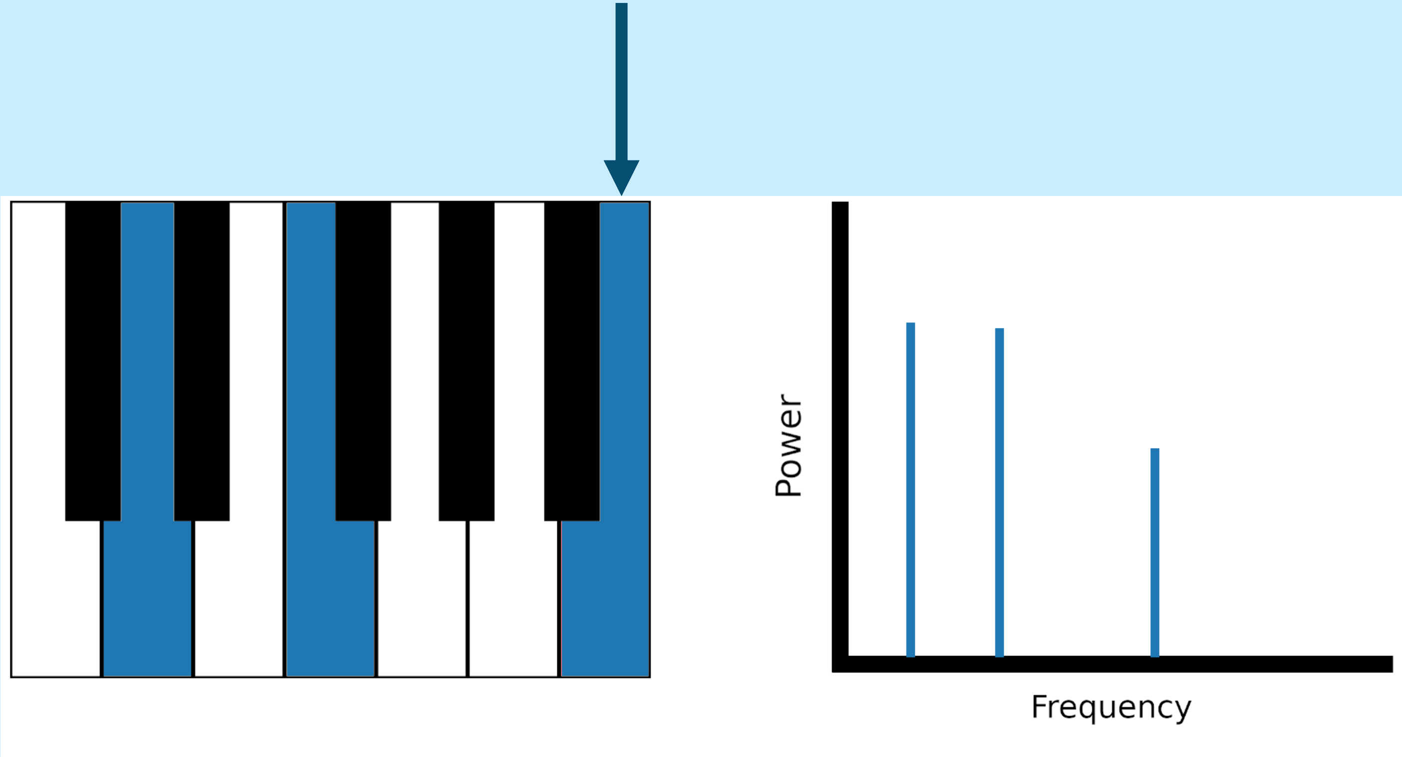
Modified from https://realpython.com/python-scipy-fft/

A3 Fourier Transform

What does the Fourier transform of this chord look like?

I'll give you some time to think…

And what if we play this note quieter?



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A3 Fourier Transform

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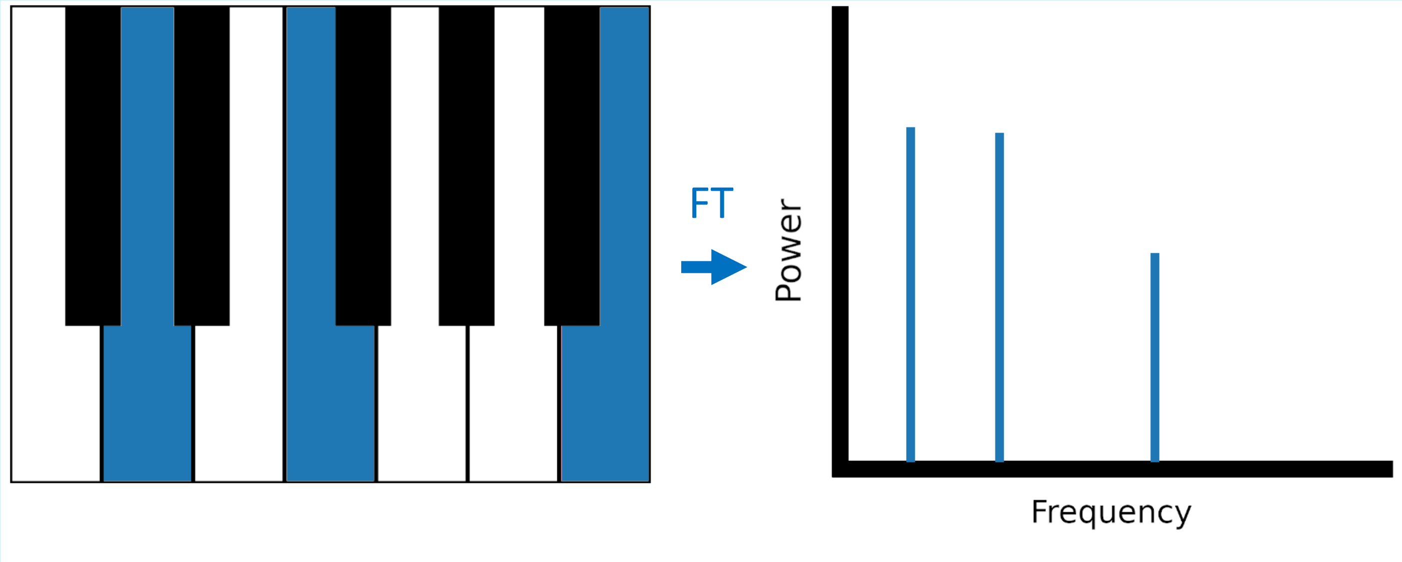
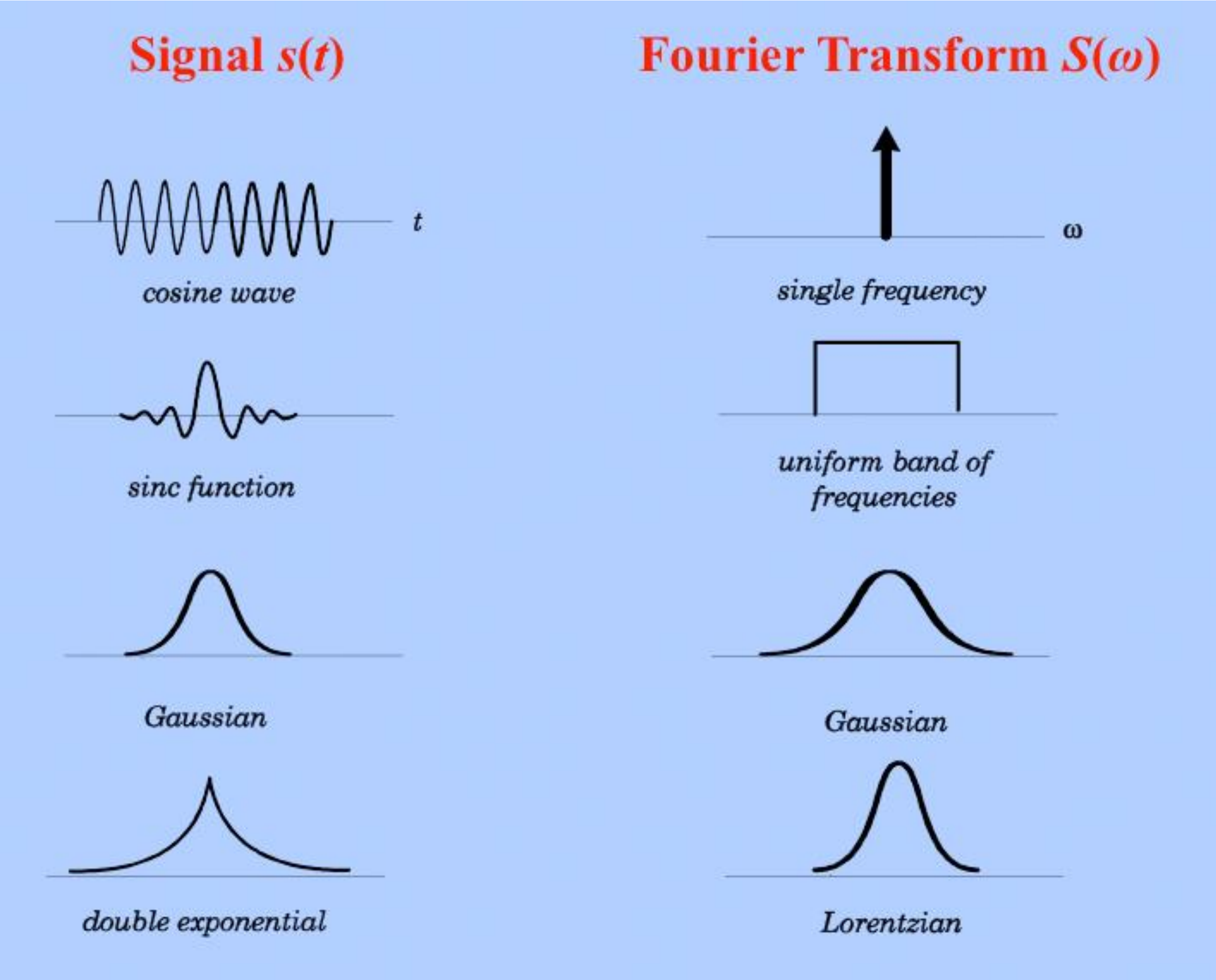


Modified from https://realpython.com/python-scipy-fft/

A3 Fourier Transform pairs

- Figure on the rights shows the signal and the corresponding Fourier transform pairs.
- Just part of the story as it shows only the real part
- Gaussian function is symmetric but
- If you have a narrow Gaussian it transforms to a broad Gaussian (scaling)
- More generally if the energy in time is spread out (cosine wave) it is compact in the Fourier
- Line broadening relevant for spectroscopy



Fourier Transform Pairs

Superposition of Cosine waves

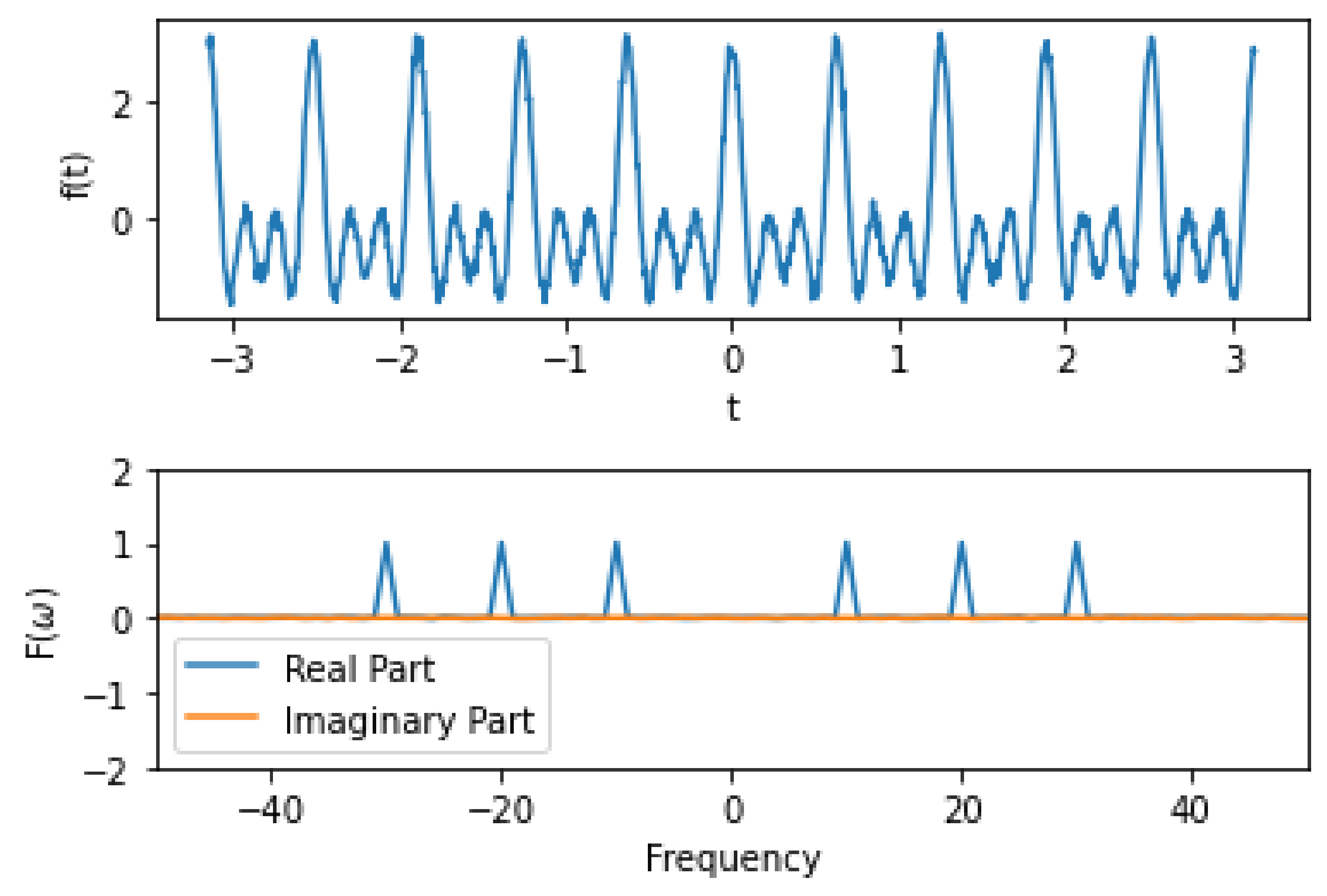
$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

No imaginary part!

FT of an even function is even

Two peaks at positive and negative frequency. A consequence of complex exponentials:

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$$



Fourier Transform Pairs

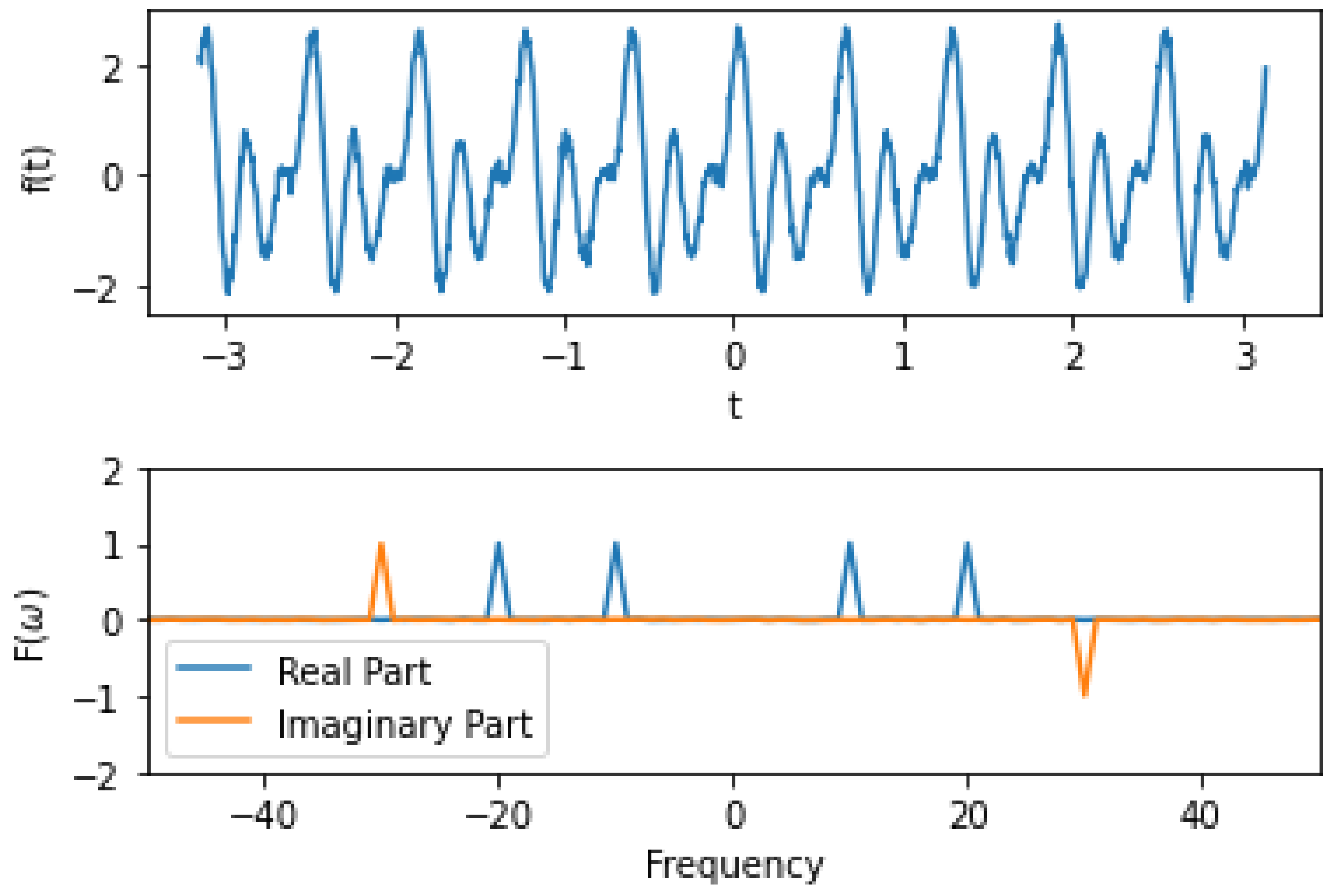
Change one wave to a sine wave

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

No real part!

FT of an odd function is odd

Usually, we look at magnitudes and phases.

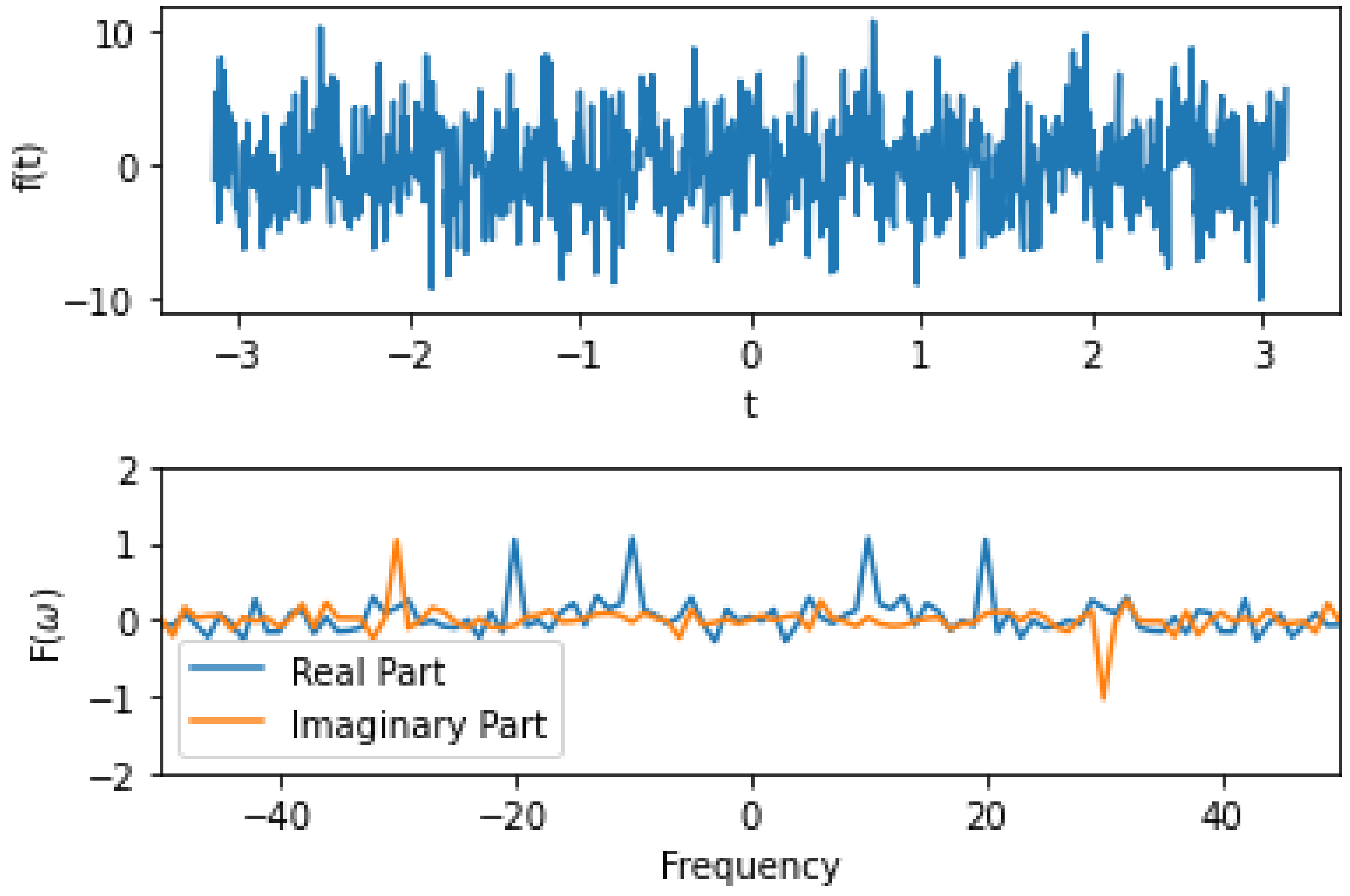


Fourier Transform Pairs

Add more noise!

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

FT of noise is noise



Fourier Transform Pairs

Superposition of Sine waves

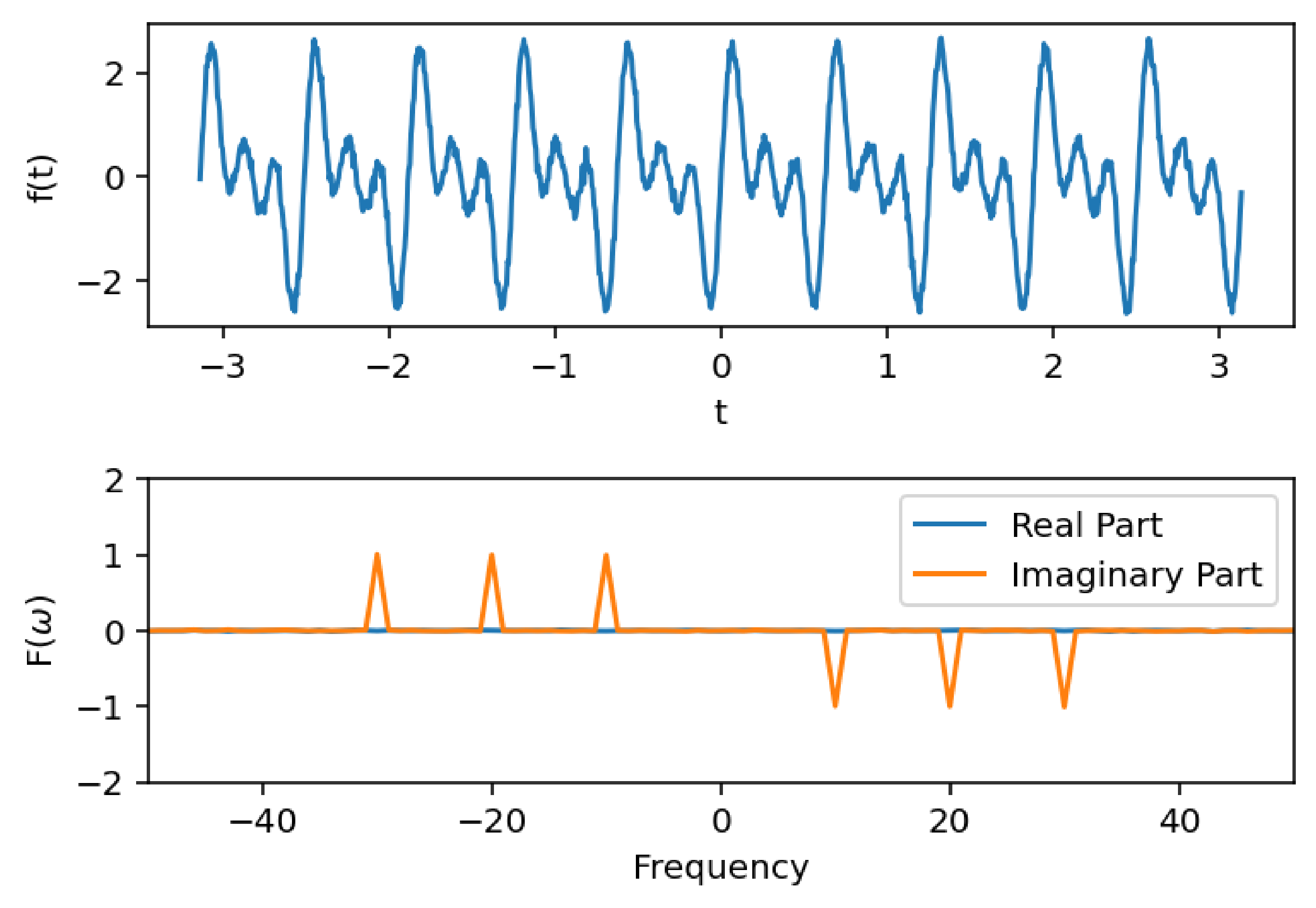
$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

No real part!

FT of an odd function is odd

Two peaks at positive and negative frequency. A consequence of complex exponentials:

$$\sin(\omega t) = \frac{1}{2i}\left(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}\right)$$



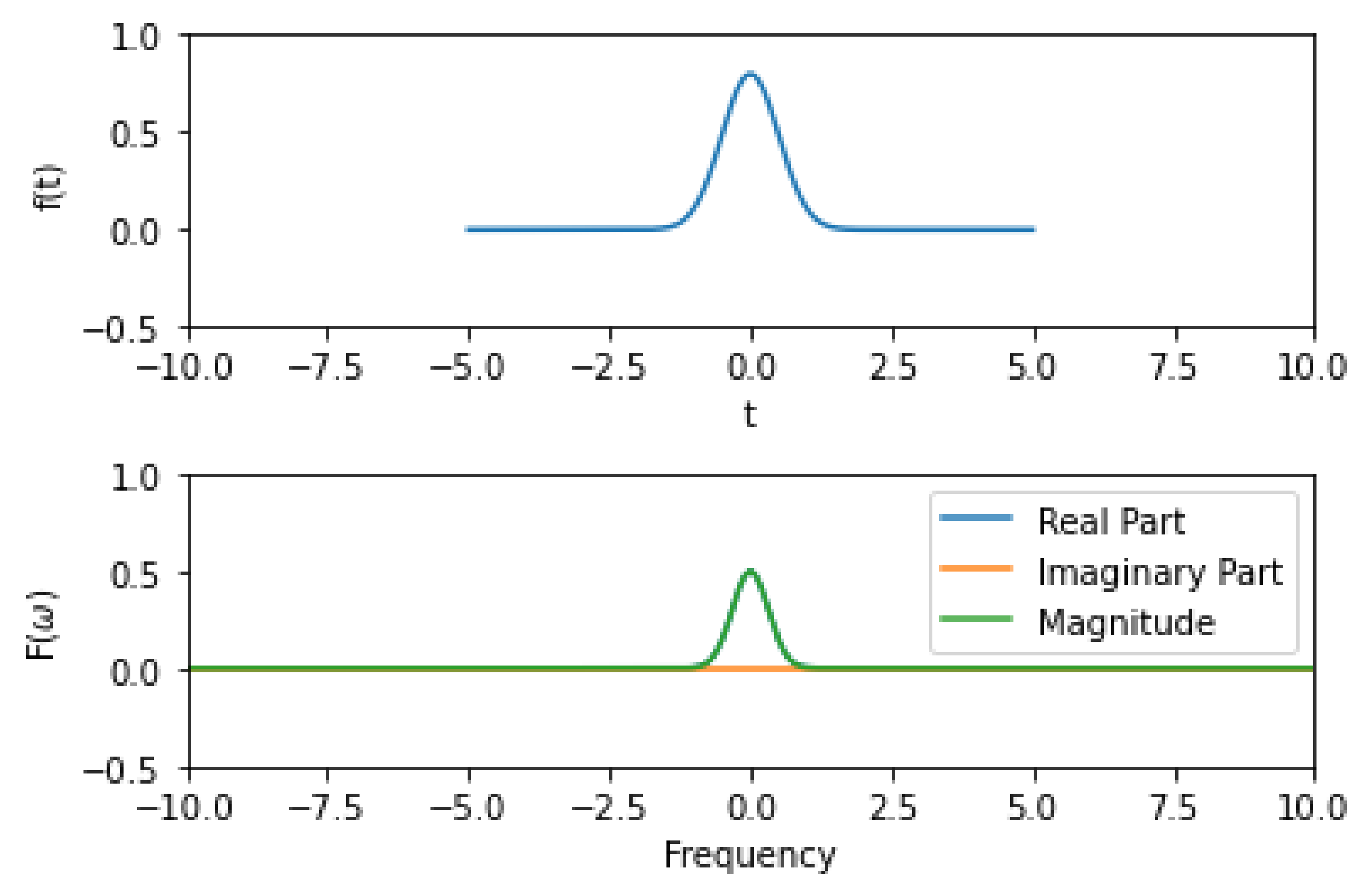
Fourier Transform Pairs

Gaussian

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

If it is a real even function then it will give a real even FT

(Gaussian centered at zero is even)



Fourier Transform Pairs

Gaussian

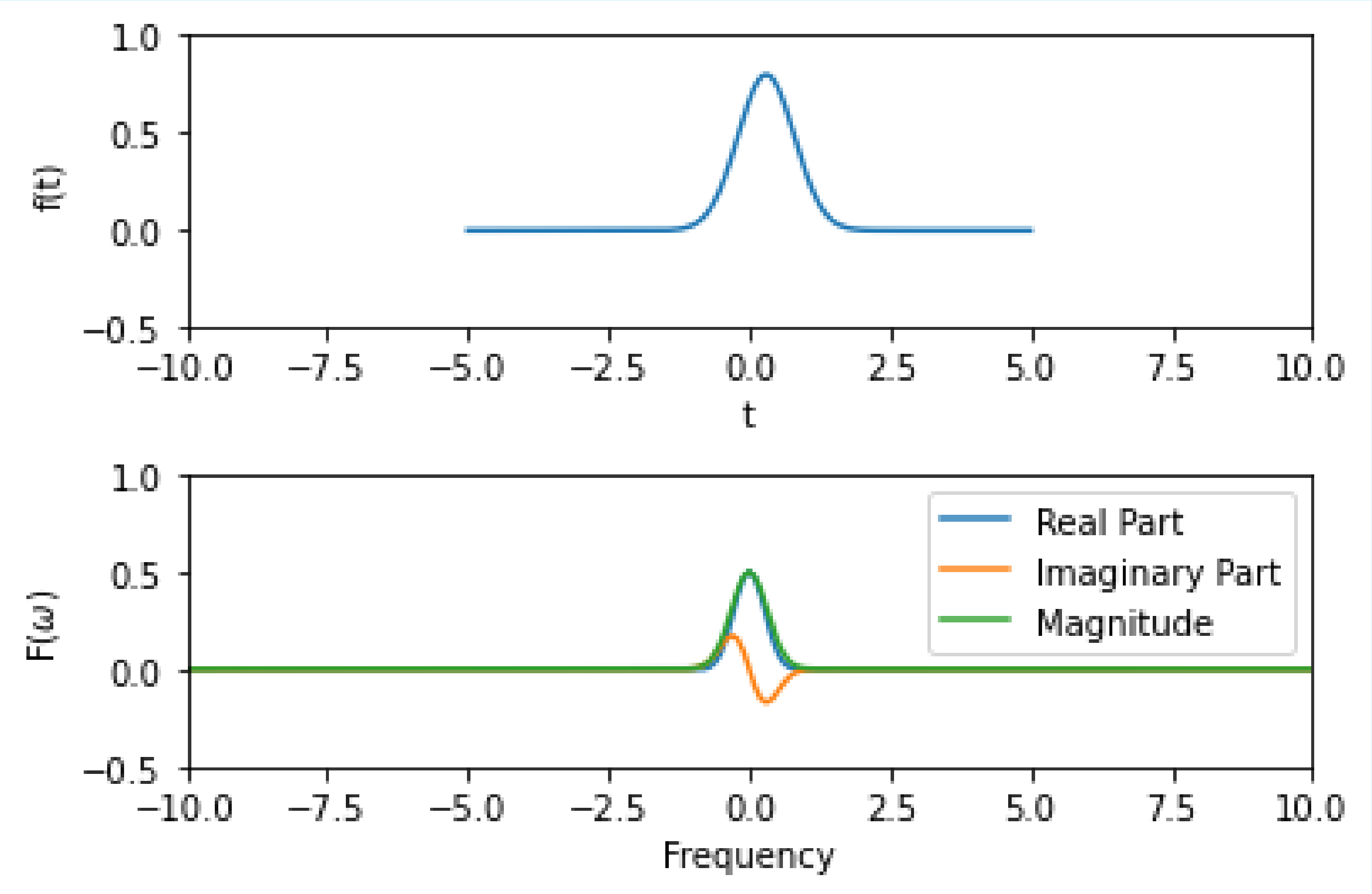
$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

If it is a real even function then it will give a real even FT

(Gaussian centered at zero is even)

Gaussian which is not centered at zero will not be even and will then have some imaginary component.

However, magnitude remains the same



Fourier Transform Pairs

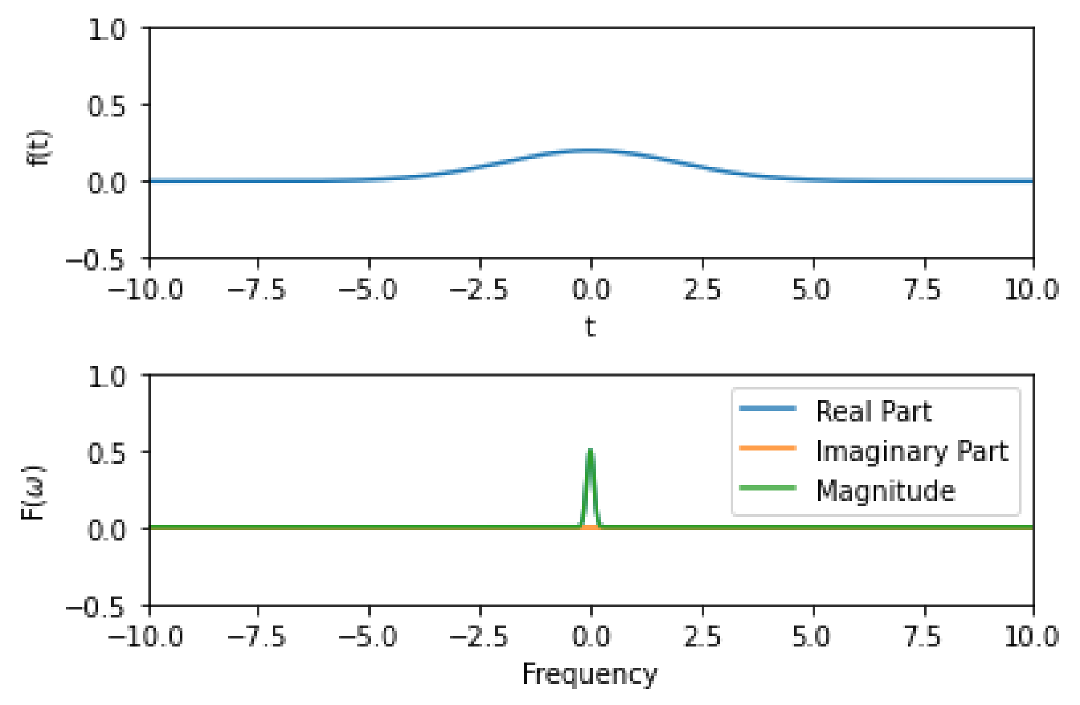
Gaussian

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

Narrow gaussian FT-> Broad Gaussian

Broad Gaussian FT-> Narrow Gaussian

Python script available to play in ‘Bonus content’

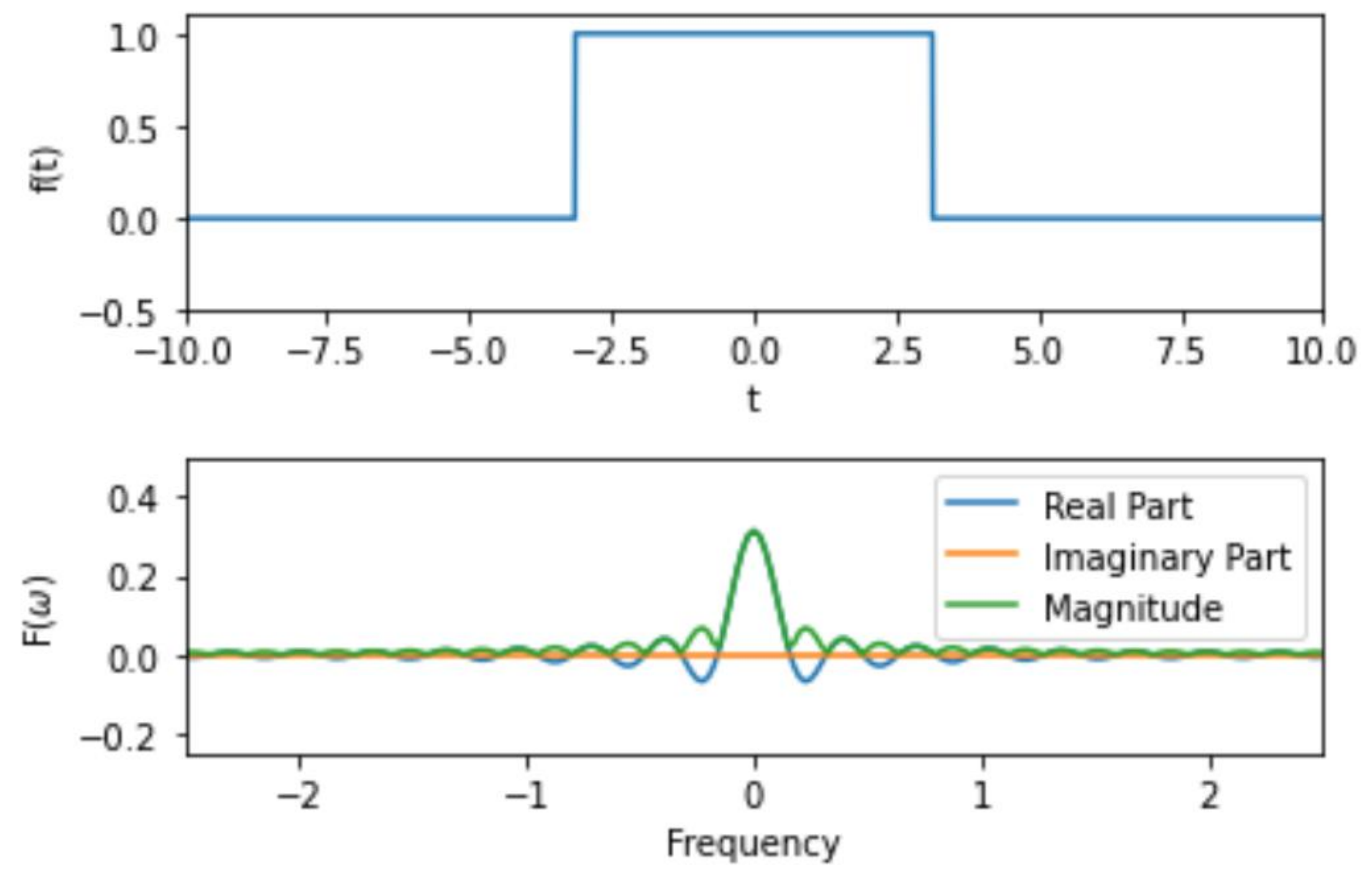


Fourier Transform Pairs

Top hat/Boxcar

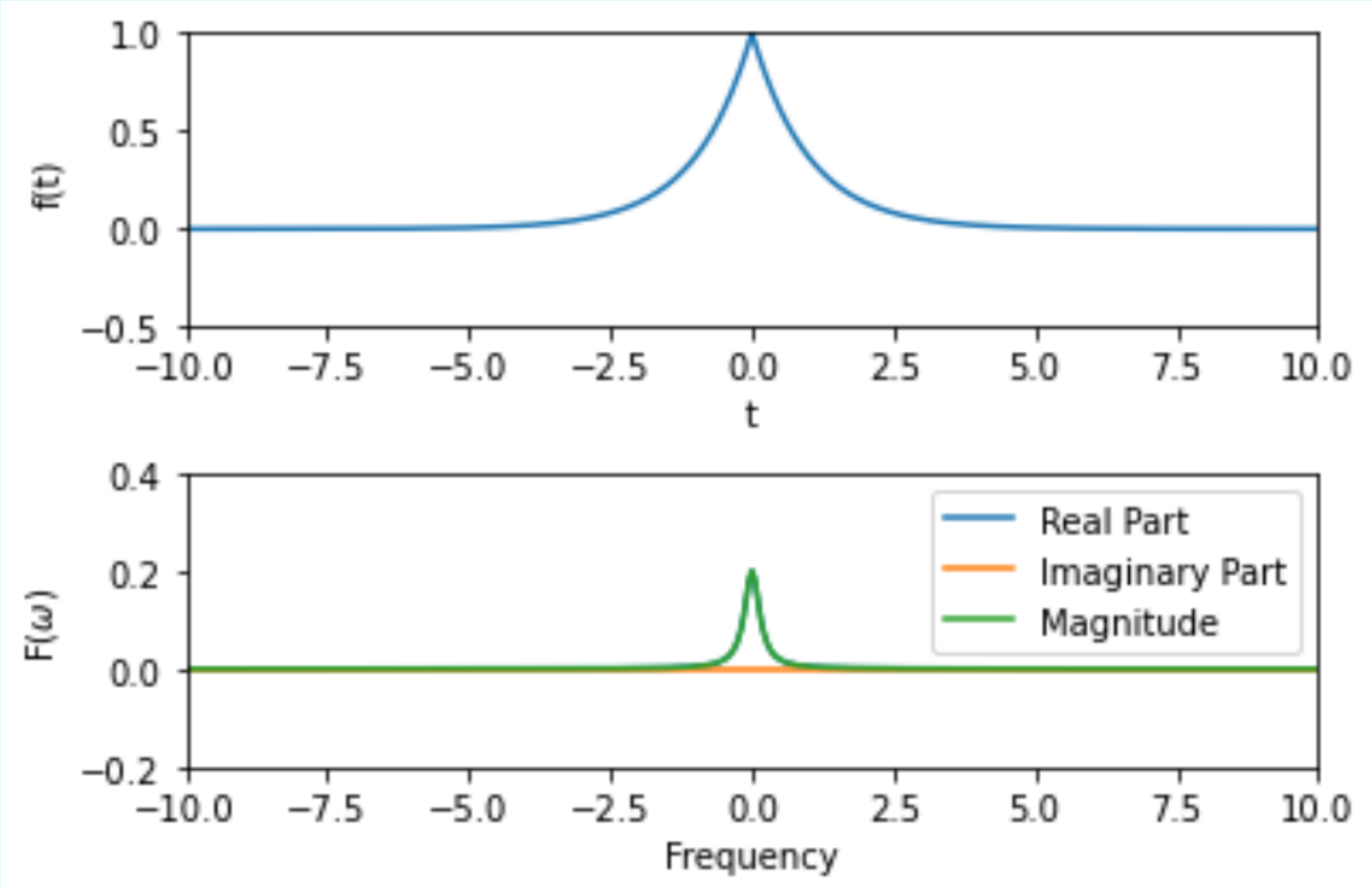
Python scripts available to play in ‘Bonus content’

Gives a Sinc function. You will show this in the workshop questions.



Fourier Transform Pairs

Double Exponential



Fourier Transform Pairs

Play with changing these parameters

```
# -*- coding: utf-8 -*-
"""
Created on Fri Nov  8 13:16:25 2024
@author: owr10
"""
import numpy as np
import matplotlib.pyplot as plt
sigma= 2.0#change variance
shift=0.0# change value of the midpoint of gaussian
#
#
NP=4096
fs=NP/(20*sigma)  #sampling frequency scaled to sigma
tmax = 10*sigma
t = np.arange(-tmax, tmax, 1/fs) # time domain
variance=sigma**2
x=1/(np.sqrt(2*np.pi*variance))*(np.exp(-(t-shift)**2/(2*variance))) # gaussian pulse
L=len(x)
NFFT = NP # length of FFT
f = (fs/NFFT)*np.arange(-NFFT/2, NFFT/2) # frequency domain
#here we need to shift the function because of how the fft works
#i.e. it starts counting from 0 element x=0 in real space
#should be the zeroeth element
X = 10 * sigma / L * np.fft.fftshift(np.fft.fft(np.fft.ifftshift(x),NFFT)) # FFT of the gaussian
fig, (ax1, ax2) = plt.subplots(2, 1)
ax1.plot(t,x)
ax1.set(xlabel='t', ylabel='f(t)')
ax1.axis([-10, 10, -0.5, 1.0])
ax2.set(xlabel='Frequency', ylabel='F($\omega$)')
ax2.plot(f, X.real,f,X.imag,f,abs(X))
ax2.axis([-10, 10, -0.5, 1.0])
ax2.legend(["Real Part","Imaginary Part","Magnitude"],loc="best")
plt.subplots_adjust(hspace = 0.5)
plt.show()
```

Topics Covered

3. A3-Fourier Transforms

- Subtopics:
- Conceptual Understanding ✓
- Doing the integrations
- Convolution theorem

A3 Doing the integrations

- Fourier Transform
- $F(\omega)= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
- Inverse Fourier Transform
- $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}d\omega$

A3 Doing the integrations

- Fourier Transform
- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$
- Inverse Fourier Transform
- $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$

Different conventions but forward needs to be the opposite of the inverse! Also, (electrical) engineers prefer $j = \sqrt{-1}$. To avoid confusion with i for current!

A3 Doing the integrations

- Fourier Transform

- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

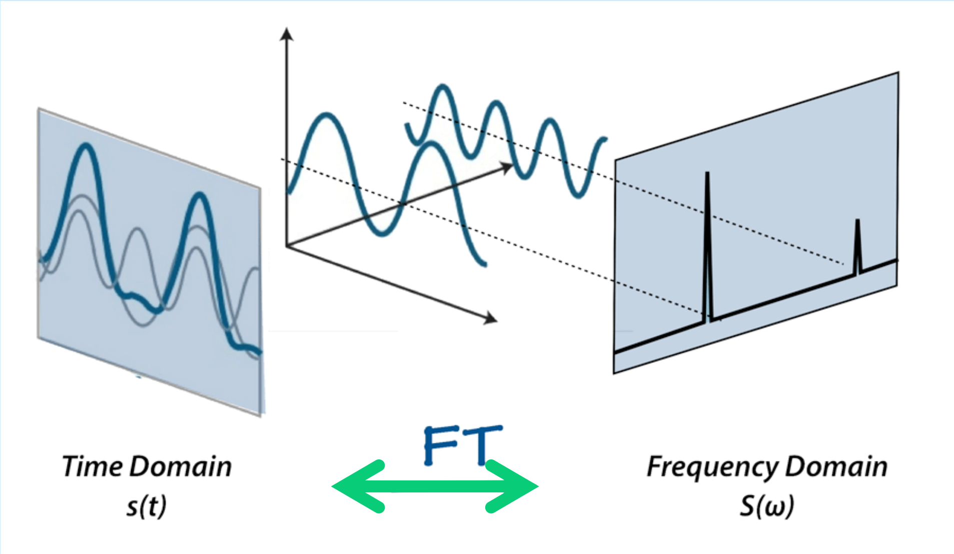
- Inverse Fourier Transform

- $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$

You might encounter different numbers here (different normalizations)

A3 Application of inverse FT

- Example if we have an audio signal and we know there is high frequency noise in the signal.
- We also know which frequencies we expect the signal
- We can apply the FT to the signal, then set all the high frequency coefficients (where the noise is) to zero
- Then to the inverse FT



Example

Given the following function $f(t)$, calculate the Fourier transform $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \begin{cases} 0, t \leq 0 \\ e^{-4t}, t \geq 0 \end{cases}$$

Example

Given the following function $f(t)$, calculate the Fourier transform $F(\omega)$

$$F(\omega)= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$f(t) = \begin{cases} 0, t \leq 0 \\ e^{-4t}, t \geq 0 \end{cases}$$

It is (just) doing integrations:

Example

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$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \begin{cases} 0, t \leq 0 \\ e^{-4t}, t \geq 0 \end{cases}$$

It is (just) doing integrations:

$$F(\omega) = \int_{-\infty}^{0} 0 \, e^{-i\omega t} dt + \int_{0}^{\infty} e^{-4t} e^{-i\omega t} dt$$

Example

Given the following function $f(t)$, calculate the Fourier transform $F(\omega)$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$f(t)=\begin{cases} 0, t \leq 0 \\ e^{-4t}, t \geq 0 \end{cases}$$

It is (just) doing integrations:

$$F(\omega)=\cancelto{0}{\int_{-\infty}^{0} 0\, e^{-i\omega t}dt} + \int_{0}^{\infty} e^{-4t}e^{-i\omega t}dt$$

Example

Given the following function $f(t)$, calculate the Fourier transform $F(\omega)$

$$F(\omega)= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

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Recall: $e^{a}e^{b} = e^{a+b}$

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0

$$F(\omega) = \int_{-\infty}^{0} 0\, e^{-i\omega t} dt + \int_{0}^{\infty} e^{-4t} e^{-i\omega t} dt$$

$$F(\omega) = \int_{0}^{\infty} e^{-4t-i\omega t} dt$$

Example

$$F(\omega)=\int_0^{\infty} e^{-4t-i\omega t}dt$$

$$F(\omega)=\int_0^{\infty} e^{-(4+i\omega)t}dt$$

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Example

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$$F(\omega) = \frac{1}{-(4+i\omega)}[e^{-\infty} - e^{0}]$$

Recall: $e^{-\infty} = 0$

and $e^{0} = 1$

$$F(\omega) = \frac{1}{-(4+i\omega)}[0-1]$$

Example

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Recall: $e^{-\infty} = 0$

$$F(\omega)=\int_0^{\infty} e^{-(4+i\omega)t}dt$$

and $e^{0} = 1$

$$F(\omega) = \frac{1}{-(4+i\omega)}\left[e^{-(4+i\omega)t}\right]_0^{\infty}$$

$$F(\omega) = \frac{1}{-(4+i\omega)}[0-1]$$

$$F(\omega) = \frac{1}{-(4+i\omega)}[e^{-\infty} - e^{0}]$$

$$\boxed{F(\omega) = \frac{1}{(4+i\omega)}}$$

Summary of this example FT

- Let’s take a moment to reflect on that!
- What did we need to do?
- Split up the integral using the limits
- Recall the properties of indices
- Integrate exponential functions
- Recall identities for exponential functions

Properties of the Fourier Transform

- Linearity
- $f(t) \leftrightarrow F(\omega)$ and $h(t) \leftrightarrow H(\omega)$ then $\mathrm{z}(t) = \alpha f(t) + \beta h(t) \leftrightarrow \alpha F(\omega) + \beta H(\omega) = Z(\omega)$
- Show that the linearity property is valid:
- $F(\omega)= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ $H(\omega)= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$
- $Z(\omega)= \int_{-\infty}^{\infty} z(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} (\alpha f(t) + \beta h(t))\, e^{-i\omega t}\, dt$
- $Z(\omega)=\alpha \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}dt + \beta \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}dt$
- $Z(\omega)=\alpha F(\omega)+\beta H(\omega)$

Properties of the Fourier Transform

- Scaling
- $f(t) \leftrightarrow F(\omega)$ and $h(t) = f\left(\frac{t}{\alpha}\right)$
- $H(\omega) = \alpha F(\alpha\omega)$
- $F(\omega)= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ $H(\omega)= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f\left(\frac{t}{\alpha}\right) e^{-i\omega t}dt$
- Let $t' = t/\alpha$ where α is a constant $t = \alpha t' \Rightarrow$ $dt = \alpha dt'$
- $H(\omega)= \int_{-\infty}^{\infty} f(t')e^{-i\omega(\alpha t')}\alpha dt')= \alpha \int_{-\infty}^{\infty} f(t')e^{-i\omega(\alpha t')}dt'$
- $F(\omega)= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
- $H(\omega)=\alpha F(\alpha\omega)$

Properties of the Fourier Transform

- Scaling
- $f(t) \leftrightarrow F(\omega)$ and $h(t) = f\left(\frac{t}{\alpha}\right)$
- $H(\omega) = \alpha F(\alpha\omega)$
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- Let $t' = t/\alpha$ where α is a constant $t = \alpha t' \Rightarrow dt = \alpha dt'$
- $H(\omega) = \int_{-\infty}^{\infty} f(t') e^{-i\omega(\alpha t')} \alpha dt') = \boxed{\alpha \int_{-\infty}^{\infty} f(t') e^{-i\omega(\alpha t')} dt'}$
- $F(\omega) = \boxed{\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt}$

Compare

- $H(\omega) = \alpha F(\alpha\omega)$

Properties of the Fourier Transform

- Scaling- Implications
- $h(t) = f\left(\frac{t}{\alpha}\right)$
- $H(\omega) = \alpha F(\alpha\omega)$
- Compression in time $\frac{t}{\alpha}$ expands in the frequency domain $\alpha\omega$ and vice versa!

Properties of the Fourier Transform Even/Odd

- Fourier transform of an even function is even

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

- Proof:

Properties of the Fourier Transform Even/Odd

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- Proof:

- $f(t) = f(-t)$ for an even function!

Properties of the Fourier Transform Even/Odd

- Fourier transform of an even function is even
- Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

- $f(t) = f(-t)$ for an even function!
- $F(\omega) = \int_{t=-\infty}^{t=\infty} f(t) e^{-i\omega t} dt$
- $F(\omega) = \int_{t=-\infty}^{t=\infty} f(-t) e^{-i\omega t} dt$
- Make a substitution $u = -t, \quad t = -u$ so $\frac{du}{dt} = -1$

Properties of the Fourier Transform Even/Odd

$$F(\omega) = \int_{t=-\infty}^{t=\infty} f(-t)e^{-i\omega t}dt$$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$u = -t, \quad t = -u \qquad \text{so } \frac{du}{dt} = -1$$

$$\int_{u=\infty}^{u=-\infty} -f(u)e^{-i\omega(-u)}du$$

Change limits around recall: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Properties of the Fourier Transform Even/Odd

$$\int_{u=-\infty}^{u=+\infty} f(u)e^{-i\omega(-u)}du$$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Change where the negative sign is:

$$\int_{u=-\infty}^{u=\infty} f(u)e^{-i(-\omega)u}du = F(-\omega)$$

Exercise Prove that the FT of an odd function is Odd

If odd then $f(-t) = -f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Or $-f(-t) = f(t)$

Start with substituting $f(t) = -f(-t)$

Let $u = -t$, $\quad \frac{du}{dt} = -1$

Is left as an exercise to the reader…

Exercise: Prove that FT of a Real even function is real

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)(\cos(\omega t)-\mathrm{i}\sin(\omega t))\,dt$$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt-\mathrm{i}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

Integral of an odd function (odd times even function is odd)
over a symmetric interval is zero

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt$$

Which is real and even

Summary: Properties of the Fourier Transform

You can prove these properties with a change of variable.

Property	Relation
Linearity	$h(t) = \alpha f(t) \quad + \quad \beta g(t)$ $H(\omega) = \alpha F(\omega) \quad + \quad \beta G(\omega)$
Time-shifting	$h(t) = f(t - t_0)$ $H(\omega) = e^{-i\omega t_0} F(\omega)$
Scaling	$h(t) = f(\frac{t}{\alpha})$ $H(\omega) = \alpha F(\alpha\omega)$

Properties of the Fourier Transform

- Multiplication
- $FG(\omega)= \int_{-\infty}^{\infty} f(t)g(t)e^{-i\omega t}dt \neq F(\omega)G(\omega)$
- Caution!! The Fourier transform of the product of functions $f(t)$ and $g(t)$ is not the product of the Fourier transforms $F(\omega), G(\omega)$
- Rather there is a different operation called the convolution
- $F(\omega)G(\omega) = f(t) * g(t)$
- Where $f(t) * g(t) = \int_{-\infty}^{\infty} f(t)g(t_0 - t)dt$
- What does that mean?

Topics Covered

3. A3-Fourier Transforms

- Subtopics:
- Conceptual Understanding ✓
- Doing the integrations ✓
- Convolution theorem

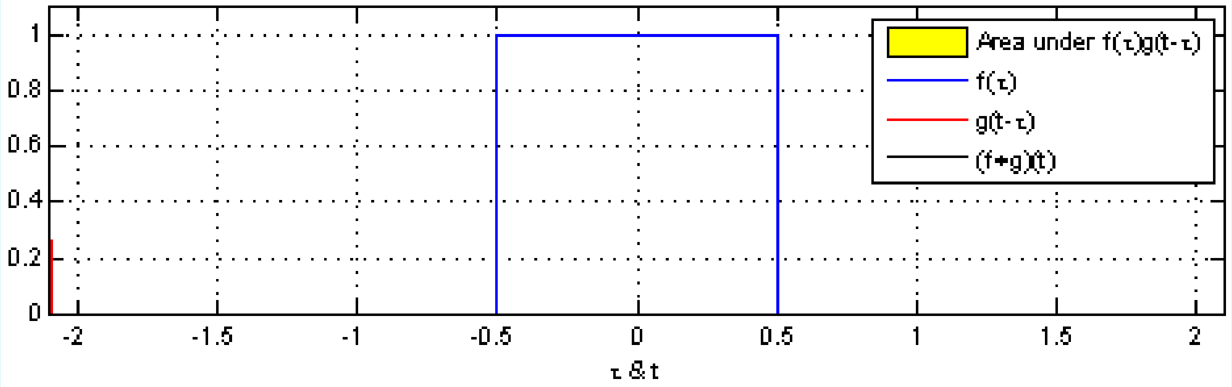
Convolution

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t)g(t_0 - t)dt$$

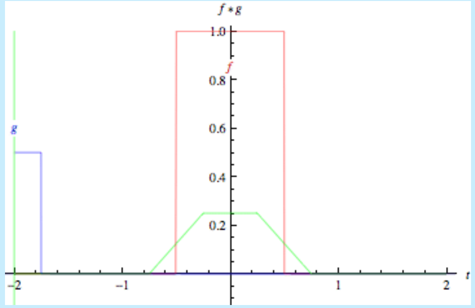
We take our function $g(t)$, reverse it (example on the right is symmetric) and we shift it in time and find the overlapping area.

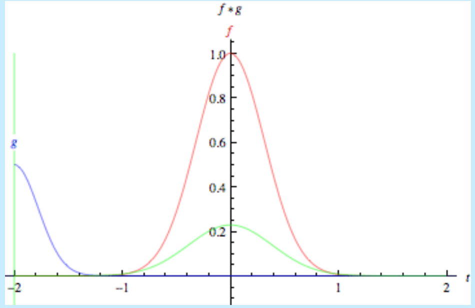
We see that for two square waves (box cars) the shape of the convolution is triangular

Different convolutions can come from different amplitudes etc.



https://commons.wikimedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif author: Brian Amberg





https://mathworld.wolfram.com/Convolution.html

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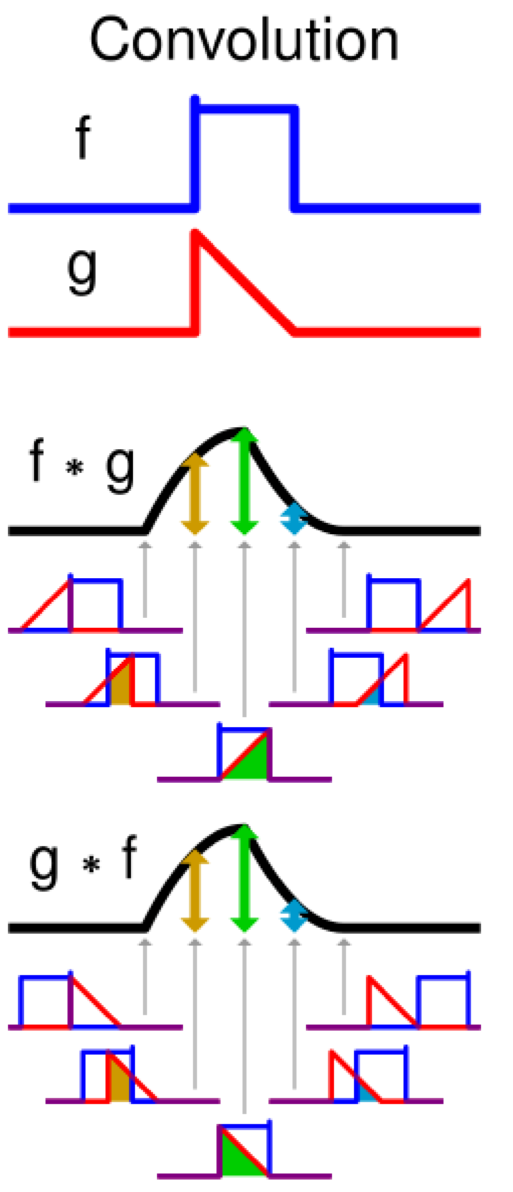
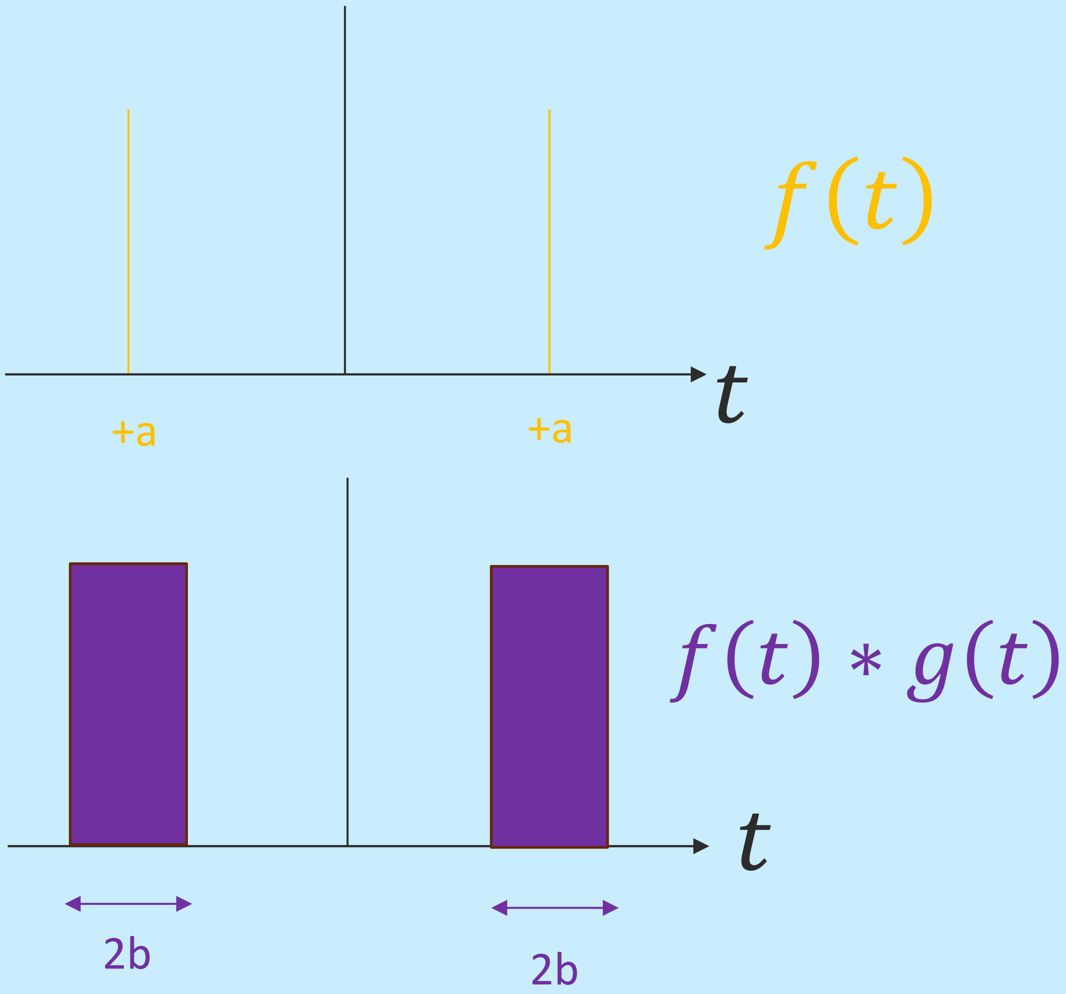
Non-symmetric example

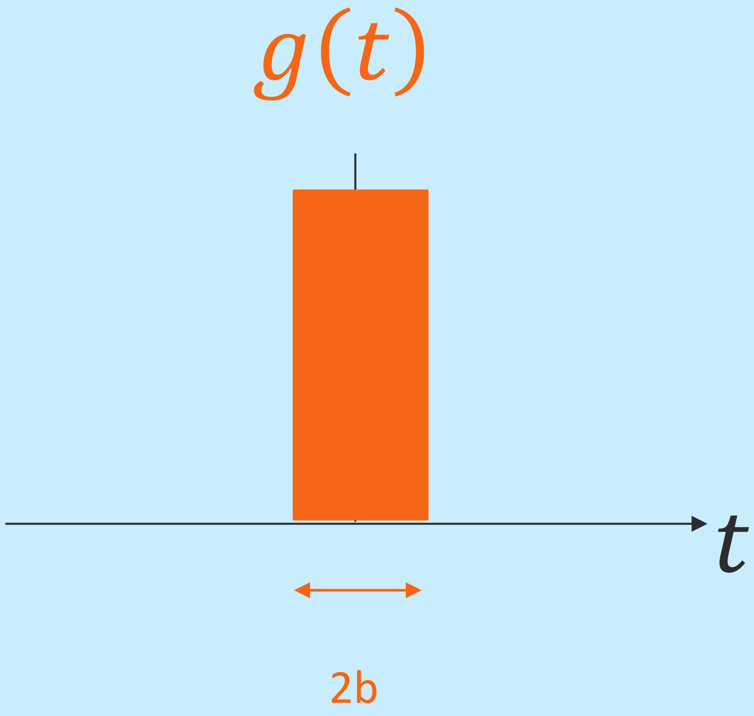
You can flip either f or g (commutative)



https://commons.wikimedia.org/wiki/File:Comparison_convolution_correlation.svg by cmglee

Convolution





Review

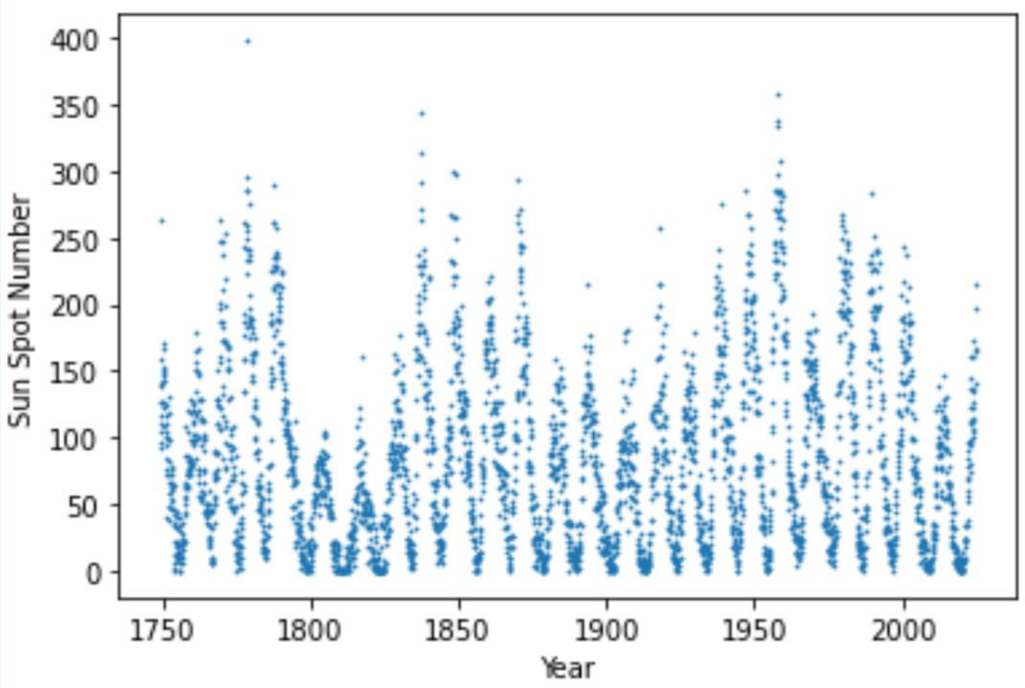
The Fourier transform is a method to go from time (real space) to frequency (inverse space). From burrito to recipe space.

It is (or the fast Fourier transform algorithm is) everywhere!

So many applications! Signal processing, data compression, reducing noise, image processing…

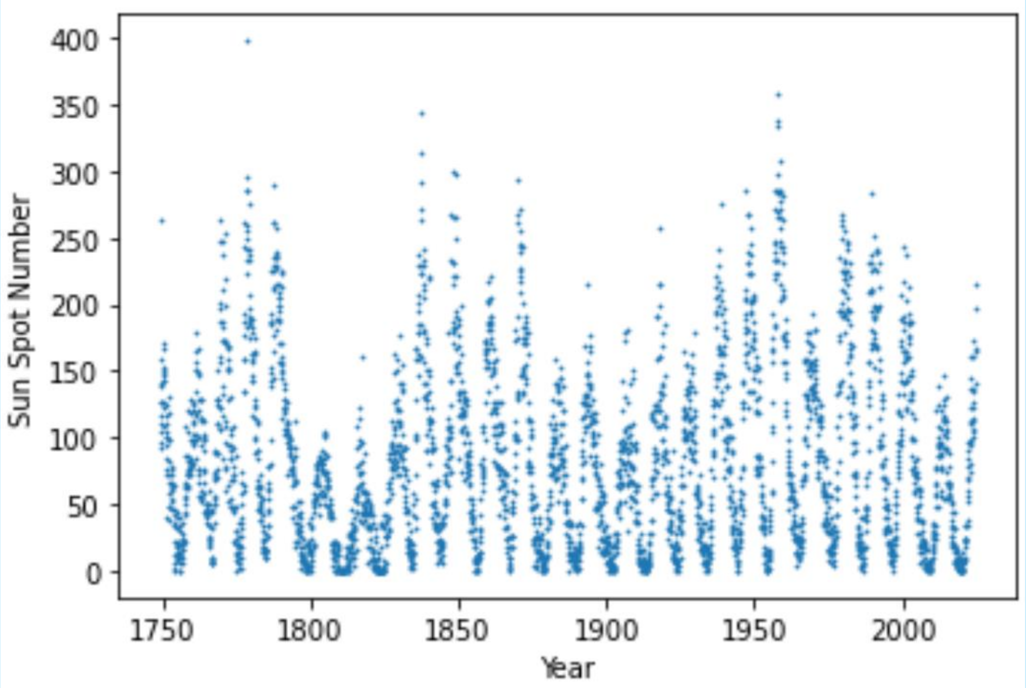
Key skills: integration, manipulating indices, recognising identities etc.

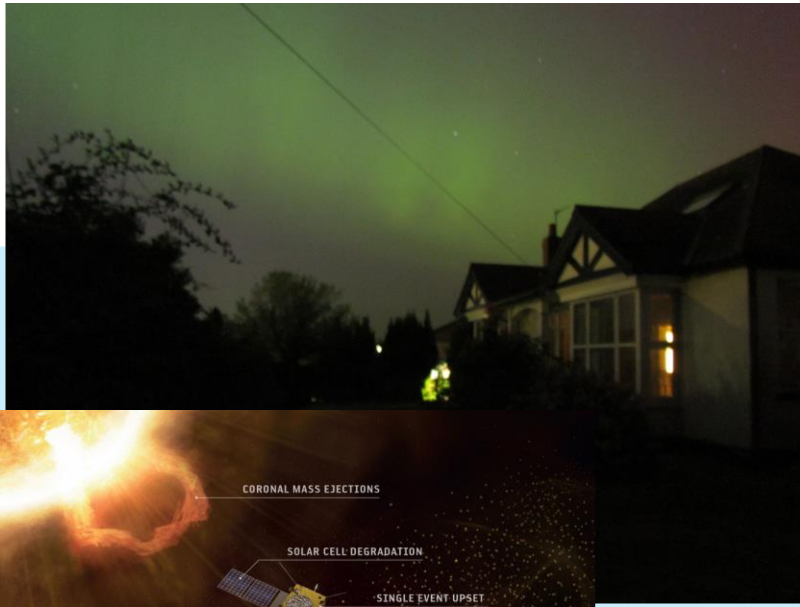
Bonus Example - Sunspot number

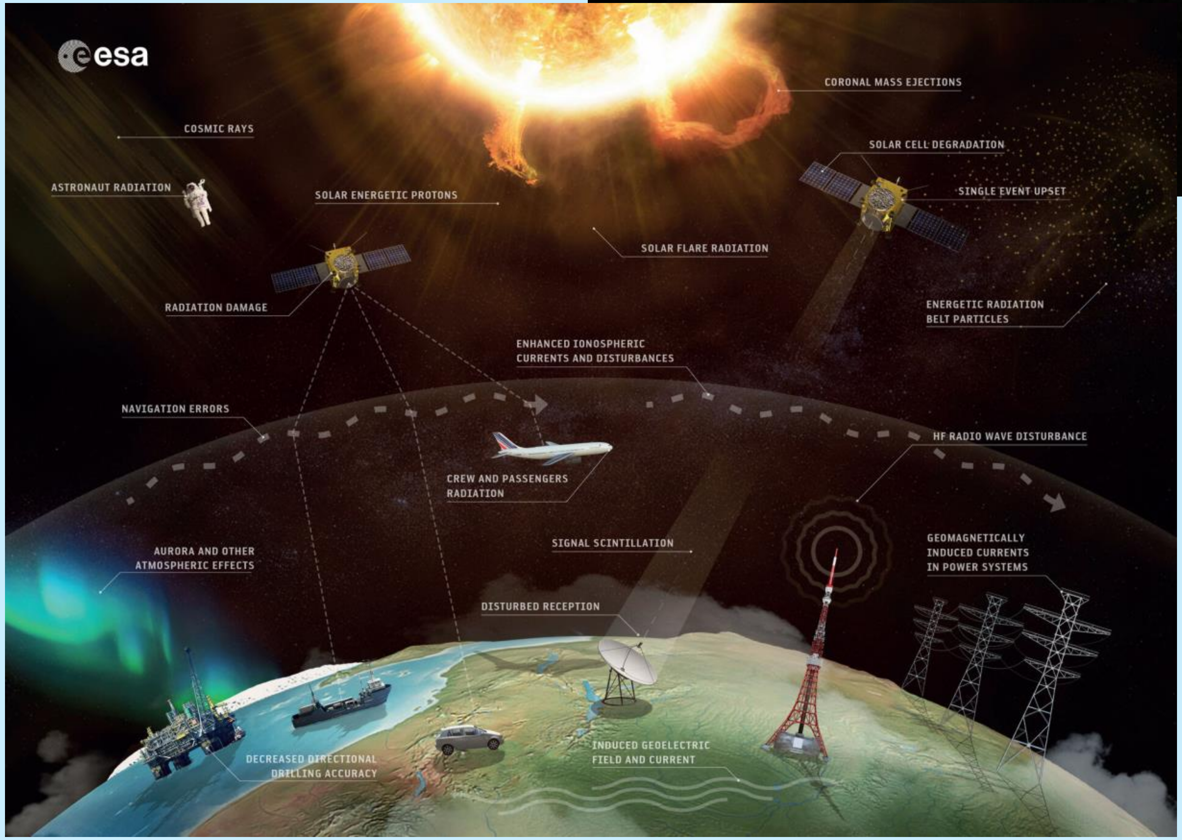


The number of sunspots on the surface of the sun exhibits some periodicity. This is monthly data from SIDC. What is the period??

Bonus Example - Sunspot number







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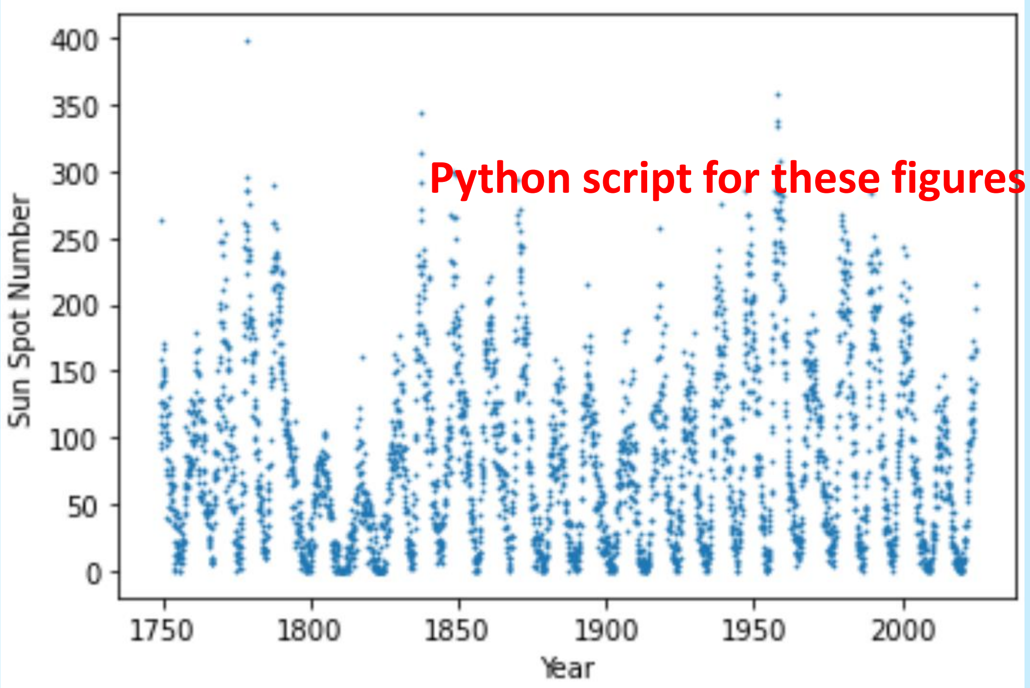
Python script for these figures and the data available in ‘Bonus Content on’ Blackboard!



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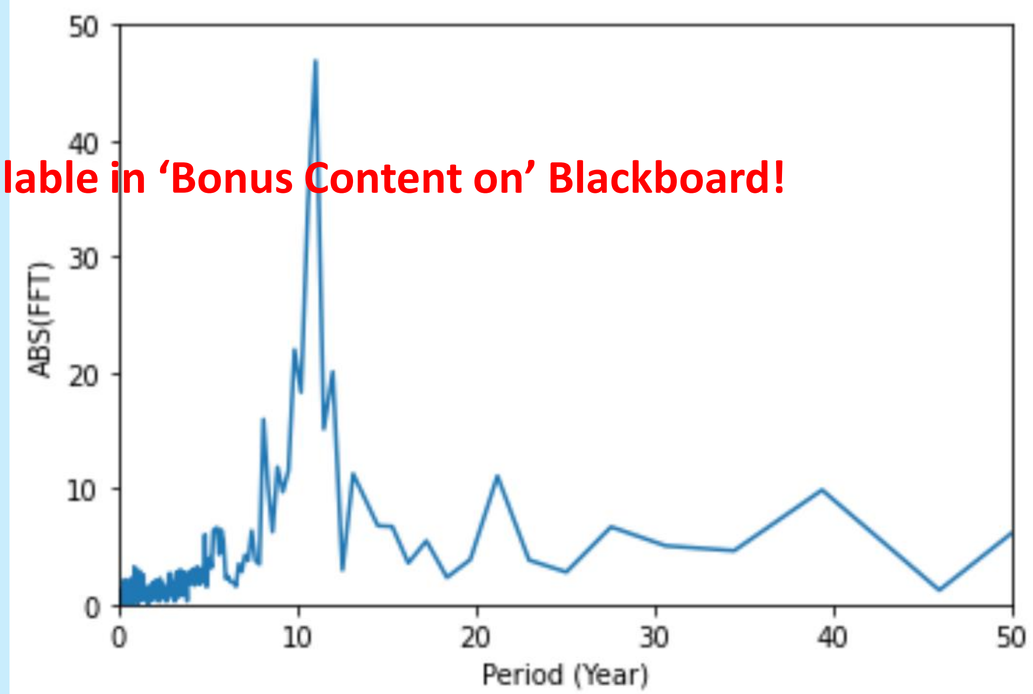
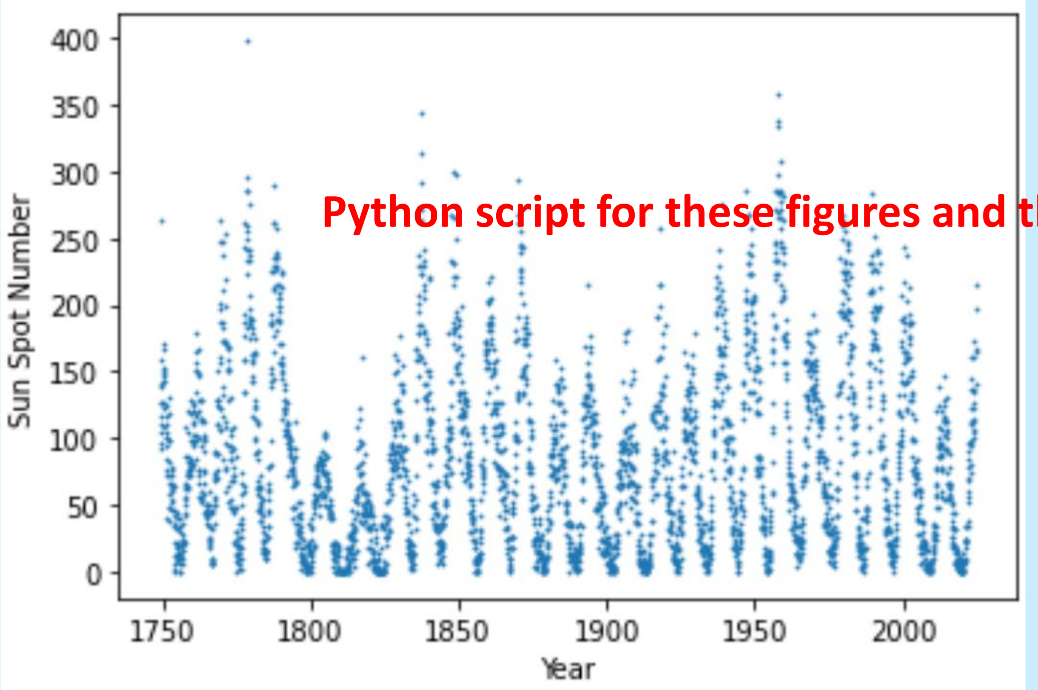
FT



We do the Fourier transform of the data and plot the absolute values of the data. Put it in terms of Period rather than frequency (I don’t want to think in frequencies of 1/Years!) We have a strong peak at 11.49 years! Another peak at 22 years.

Bonus Example - Sunspot number



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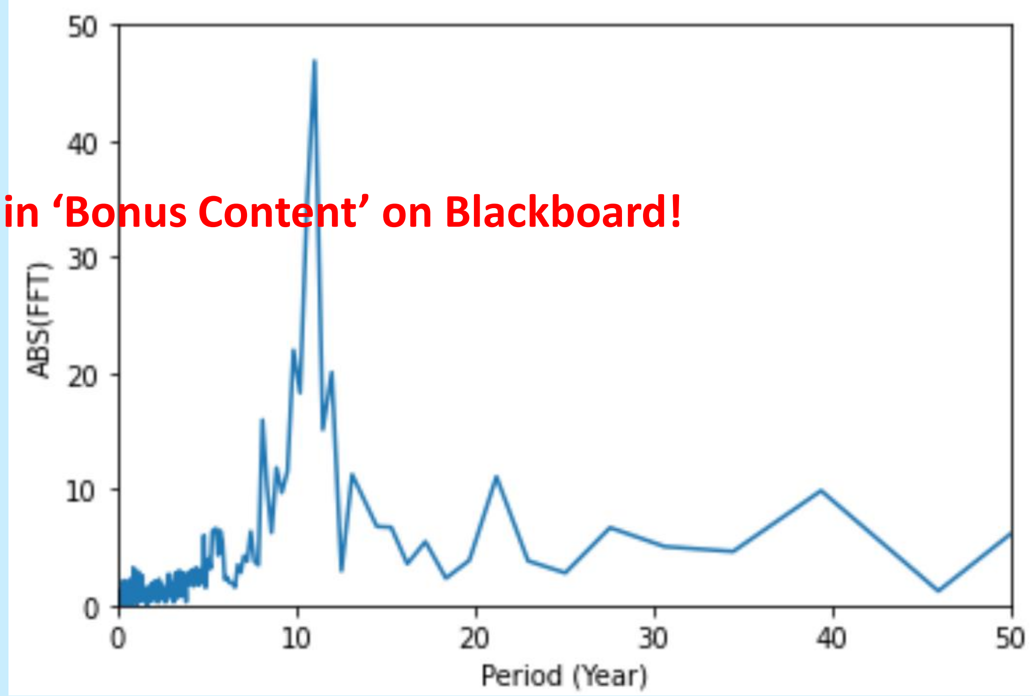

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