ME 555 Exam 2 Solutions - 2013
Closed Book Closed Notes

Problem 1 (30 Points)

Check if the following statements are true or not. Explain concisely.

(a) The Lagrangian multipliers for an optimal solution represents how fast the optimal objective value changes based on the change in equality and active inequality constraints.

(b) The Lagrangian multipliers should always be non-negative.

(c) In the case we only have equality constraints, we can use the reduced (constrained) gradient method. Here the number of state variables equals the number of equality constraints.

(d) In a constrained minimization problem, a point that satisfies KKT conditions is a local minimum.

(e) For a constrained optimization problem, if there does not exist any KKT point, then the problem does not have an optimal solution.

(f) In each iteration of the generalized reduced gradient method, we first determine a step for the decision variables, and then update the state variables using the first-order approximation of the equality constraints: $(\partial \mathbf{h}/\partial \mathbf{d})\partial \mathbf{d} + (\partial \mathbf{h}/\partial \mathbf{s})\partial \mathbf{s} = 0$.

Problem 1 Solution

(a) True. $\partial f = -\sum \lambda_i \partial h_i - \sum \mu_i \partial g_i$, where ∂h_i are perturbation of the ith equality constraint and ∂g_i is the perturbation of the ith active inequality constraint.

(b) False. Only multipliers for inequality constraints (at KKT points) are non-negative.

(c) True. Let the number of variables be n and the number of equality constraints be m. At any feasible regular point, we can only move in $n - m$ directions, while the rest m are confined by the constraints.

(d) False. It can be a local maximum or a saddle point.

(e) False. There may exist irregular optimal solutions that does not follow KKT conditions.

(f) False. When $\mathbf{h}$ are nonlinear, a nested iteration on s is needed to ensure that the next point is feasible.

Problem 2 (30 Points)

Consider the problem

$$\begin{aligned} \min \quad & f = (x_1 - 1)^2 + x_2^2 \\ \text{subject to} \quad & g_1 = x_1 - 2x_2^2 \le 0 \end{aligned}$$

where the variables are real numbers. (a) Find all KKT points (20 Points); (b) For each solution, check if it is a local minimum or maximum. (10 Points)

Problem 2 Solution

The KKT conditions are

$$\begin{aligned} & 2(x_1 - 1) + \mu = 0 \\ & 2x_2 - 4\mu x_2 = 0 \\ & x_1 - 2x_2^2 \le 0 \\ & \mu(x_1 - 2x_2^2) = 0 \\ & \mu \ge 0 \end{aligned}$$

Case 1: $\mu = 0$. In this case we have $x_1 = 1$ and $x_2 = 0$ which violates the constraint; Case 2: $\mu > 0$. Case 2.1: When $x_2 = 0$, we have $x_1 = 0$ and $\mu = 2$; Case 2.2: When $\mu = 1/2$, we have $x_1 = 3/4$ and $x_2 = \sqrt{3/8}$ or $x_2 = -\sqrt{3/8}$, both are feasible.

In Case 2.1, the Hessian of the Lagrangian is $[2\ 0;\ 0\ 2 - 4\mu]$. Plug in $\mu = 2$ to have $[2\ 0;\ 0\ -6]$. The active inequality constraint requires $\partial x_1 = 0$ at $x_1 = 0$, $x_2 = 0$. Therefore $\partial \mathbf{x}^T[2\ 0;\ 0\ -6]\partial \mathbf{x} = -6\partial x_2^2 < 0$ for any nonzero perturbation in x_2. So $x_1 = 0$, $x_2 = 0$ is a local maximum.

In Case 2.2, the Hessian of the Lagrangian is $[2\ 0;\ 0\ 0]$. The active inequality constraint requires $\partial x_1 - 4\sqrt{3/8}\partial x_2 = 0$ at $x_1 = 3/4$, $x_2 = \sqrt{3/8}$; or $\partial x_1 + 4\sqrt{3/8}\partial x_2 = 0$ at $x_1 = 3/4$, $x_2 = -\sqrt{3/8}$. In either case, the perturbation on neither x_1 or x_2 can be zero, or otherwise both perturbations have to be zeros, which violates the definition of perturbations. Therefore $\partial \mathbf{x}^T[2\ 0;\ 0\ 0]\partial \mathbf{x} = 2\partial x_1^2 > 0$ and both solutions are local minima.

Problem 3 (40 Points)

A company manufactures two types of products: a standard product, A, and a more sophisticated product B. If management charges a price of x_1 for one unit of product A and x_2 for one unit of product B, the company can sell q_A units of A and q_B units of B, where

$$q_A = 400 - 2x_1 + x_2, \qquad q_B = 200 + x_1 - x_2.$$

Manufacturing one unit of product A requires 2 hours of labor and one unit of raw material. For one unit of B, 3 hours of labor and 2 units of raw material are needed. At present, 1000 hours of labor and 200 units of raw material are *freely* available.

(a) Formulate the profit maximization problem and present it in the negative-null form, using x_1 and x_2 for the prices of product A and B. (Hint: You can safely ignore the lower bounds on x_1, x_2, q_A and q_B for this problem.)(5 Points)

(b) Check if the problem is convex or not. (Hint: First check if the Hessian of the objective function is positive definite. Then check if the feasible space is convex.)(5 Points)

(c) List down the KKT conditions for this problem and find its solution(s). Based on (b), are your solution(s) globally optimal? (24 Points)

(d) Roughly, what is the maximum the company would be willing to pay for

   (i) another hour of labor? (3 Points)

   (ii) another unit of raw material? (3 Points)

Problem 3 Solution

(a) The problem can be formulated as

$$\begin{aligned}
\min \quad & -400x_1 - 200x_2 + 2x_1^2 + x_2^2 - 2x_1x_2 \\
\text{s.t.} \quad & -x_1 - x_2 + 400 \leq 0 \\
& -x_2 + 600 \leq 0
\end{aligned}$$

(b) The problem is convex since the Hessian of the objective function is positive definite and constraints are linear.

(c) The KKT conditions are

$$\begin{aligned}
& -400 + 4x_1 - 2x_2 - \mu_1 = 0 \\
& -200 + 2x_2 - 2x_1 - \mu_1 - \mu_2 = 0 \\
& -x_1 - x_2 + 400 \leq 0 \\
& -x_2 + 600 \leq 0 \\
& \mu_1(-x_1 - x_2 + 400) = 0 \\
& \mu_2(-x_2 + 600) = 0 \\
& \mu_1 \geq 0 \\
& \mu_2 \geq 0
\end{aligned}$$

The unique solution is $x_1 = 400$, $x_2 = 600$, $\mu_1 = 0$ and $\mu_2 = 200$. The profit at this solution is 80000 unit price.

(d) The company will not be willing to pay for extra hours of labor since this constraint is not active. The company will be will to pay 200 unit price for one extra unit of raw material.

One can verify this result by setting the available material at 201 unit, which results in the optimal solution of $x_1 = 399.5$ and $x_2 = 599$. The profit at this new solution is 80199.5. Therefore the Lagrangian multiplier provides a close estimate of the increase in profit.