ME 598/494 Exam 1 - September 13, 2018

Problem 1 (20 Points)

Check if the following statements are true or not. Please **explain**.

(a) If x_* is a stationary point of a continuous and differentiable function f, then x_* is a local minimum of f.

(b) The intersection of two convex sets is also convex. If true, please explain; if not, please provide a counter example.

(c) If a stationary point x_* has positive semi definite Hessian, then it is a local solution.

(d) The problem $\min_{x_1,x_2}(x_1-2)^2+(x_2-3)^2$ for $\max\{x_1,x_2\} \leq 1$ (i.e., the larger value of x_1 and x_2 should be no larger than 1) has a unique solution. (Hint: How does the feasible space look like?)

Solutions

(a) False. Could be saddle or local maximum.

(b) True. Consider two convex sets $\mathcal{S}_1$ and $\mathcal{S}_2$ and two elements x_1 and x_2 that both belong to both sets. Since x_1 and x_2 belongs to $\mathcal{S}_1$, we have $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{S}_1$ for any $\lambda \in [0,1]$. Similarly, $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{S}_2$. Therefore, $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{S}_1 \cap \mathcal{S}_2$, i.e., $\mathcal{S}_1 \cap \mathcal{S}_2$ is convex.

(c) False. The Hessian needs to be positive definite for the sufficient condition.

(d) True. $(x_1-2)^2 + (x_2-3)^2$ is a convex function and $\max\{x_1,x_2\} \leq 1$ is a convex set.

Problem 2 (25 Points)

(a) Concisely **explain** the gradient descent algorithm for solving unconstrained optimization problems without line search. (5 Points)

(b) Concisely **explain** the Newton's method for solving unconstrained optimization problems without line search. (5 Points)

(c) Explain why line search is needed for gradient descent and Newton's method. Propose a line search algorithm (can be an existing one). (5 Points)

(d) Discuss briefly the advantages and disadvantages of using Newton's method in solving optimization problems. (10 Points)

Solutions

See notes.

Problem 3 (15 Points)

Consider solving an optimization problem through an iterative algorithm. Let x_k and s_k be the current solution and search direction, respectively; $f(x_k)$ the current objective value; and α the step size.

(a) Derive the second-order Taylor's expansion of $f(x_k + \alpha s_k)$ at $\alpha = 0$. (5 points)

(b) Use Newton's method $s_k = -H_k^{-1} g_k$ with gradient g_k and Hessian H_k. Show that when the expansion in (a) is exact, the optimal step size is $\alpha = 1$. (10 points)

Solutions

Second-order Taylor's expansion of $f(x_k + \alpha s_k)$ at $\alpha = 0$ is

$$f(x_k + \alpha s_k) = f_k + \alpha g_k^T s_k + \frac{1}{2}\alpha^2 s_k^T H_k s_k. \tag{1}$$

Consider $s_k = -H_k^{-1} g_k$, then we have

$$f(x_k + \alpha s_k) = f_k - \alpha g_k^T H_k^{-1} g_k + \frac{1}{2}\alpha^2 g_k^T H_k^{-1} g_k. \tag{2}$$

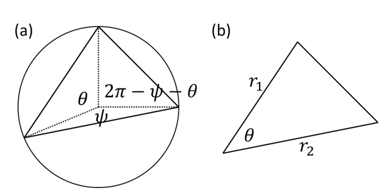
Take derivative to have

$$\frac{df(x_k + \alpha s_k)}{d\alpha} = -g_k^T H_k^{-1} g_k + g_k^T H_k^{-1} g_k \alpha. \tag{3}$$

Set the derivative to zero to have $\alpha^* = 1$.

Problem 4 (20 Points)

Draw a triangle **inside** a circle. Maximize the area of the triangle. (Hint: see figure (a) for one representation of the triangle. The area of the triangle in figure (b) is $r_1 r_2 \sin(\theta)$.)



Solutions

The area of the triangle is proportional to $\sin(\theta) + \sin(\psi) + \sin(2\pi - \theta - \psi)$. The derivatives with respect to θ and ψ are $\cos(\theta) - \cos(2\pi - \theta - \psi)$ and $\cos(\psi) - \cos(2\pi - \theta - \psi)$. Set these to zeros to have $\theta = \psi = 2\pi/3$. The Hessian matrix is $[-\sin(\theta) - \sin(2\pi - \theta - \psi), -\sin(2\pi - \theta - \psi); -\sin(2\pi - \theta - \psi), -\sin(\psi) - \sin(2\pi - \theta - \psi)]$. Take in $\theta = \psi = 2\pi/3$ to have $-\frac{\sqrt{3}}{2}[2, 1; 1, 2]$. The Hessian is negative definite. Therefore the unique stationary point is a global maximum.

Problem 5 (20 Points)

Consider a one-dimensional continuously differentiable function $f(x) = 0$. One way to find its solution is to solve

$$\min_{\mathbf{x}} \quad \frac{1}{2} f(x)^2. \tag{4}$$

(a) In what cases is this problem strictly convex? (5 Points)

(b) Consider solving the problem using Newton's method starting at x_0. Describe the algorithm. (5 Points)

(c) Please explain in what cases you will not be able to use Newton's method, and how you will resolve this issue. (10 Points, optional for MAE494)

Solutions

The gradient of the objective is $f(x)f'(x)$, the Hessian is $f'(x)^2 + f(x)f''(x)$. If the Hessian is positive for all x, then the problem is strictly convex.

input : x_0, $f(x)$
output: x^*
$k = 0$, $\epsilon = 10^{-3}$;
while $|f(x)f'(x)| \geq \epsilon$ **do**
  $s_k = -\frac{f(x)f'(x)}{f'(x)^2 + f(x)f''(x)}$;
  $\alpha_k = \text{lineSearch}(x_k, f(\cdot), s_k)$;
  $x_{k+1} = x_k + \alpha_k s_k$;
  $k = k + 1$;
end

Standard inexact line search applies. If Hessian is not positive definite, we will need to regularize the Hessian, e.g., through trust region.