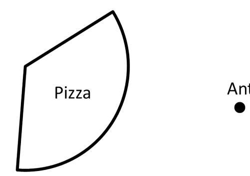
ME 598/494 Exam 1 - September 14, 2017

Problem 1 (20 Points)

Check if the following statements are true or not. Explain.



(a) If x_* is a stationary point of a continuous and differentiable function f, then x_* is a local minimum of f.

(b) The union of two convex sets is also convex. If true, please explain; if not, please provide a counter example.

(c) The problem $\min_x \frac{1}{x} + x$ for $x \geq 0$ has a minimizer.

(d) An ant is planning a move to the pizza (see figure). There is a unique shortest path from this ant to this piece of pizza.

Problem 2 (25 Points)

(a) *Derive* the iteration formula for solving n-dimensional unconstrained problems with the gradient descent method without line search. (5 Points)

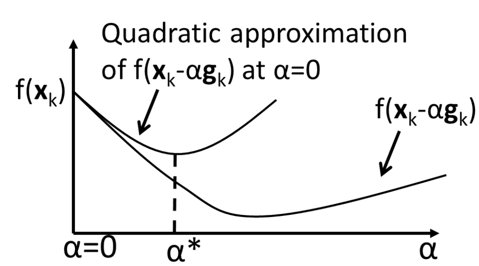
(b) *Derive* the iteration formula for solving n-dimensional unconstrained problems with Newton's method without line search. (5 Points)

(c) Explain why line search is needed for gradient descent and Newton's method. Propose a line search algorithm (can be an existing one). (5 Points)

(d) Discuss briefly advantages and disadvantages of using Newton's method in solving optimization problems. (10 Points)

Problem 3 (10 Points)

For an unconstrained function f in $\mathbb{R}$ that is continuous and differentiable, perform the following line search at the current point $\mathbf{x}_k$ with gradient $\mathbf{g}_k$ and Hessian $\mathbf{H}_k$: (1) Consider the one-dimensional function $f(\mathbf{x}_k - \alpha \mathbf{g}_k)$ with respect to α. Derive its second order approximation at $\alpha = 0$. (2) Minimize this approximation to find the optimal step α^*. What could go wrong with this line search method? (Hint: see figure)



Problem 4 (10 Points)

(a) Consider the linear programming problem $\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$, where $\mathbf{c}$ is a constant vector. Does this problem have a solution? (2 Points)

(b) Consider the linear programming problem $\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$, subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, where $\mathbf{c}$ is a p-by-1 constant vector, $\mathbf{b}$ is an n-by-1 constant vector, and $\mathbf{A}$ is an n-by-p constant matrix. If this problem has solutions (local minima), how many solutions do you expect to have? Please explain. (8 Points) Hint: An optimization problem is convex if the objective is a convex function, and the feasible domain is a convex set. The problem is strictly convex when the objective is strictly convex.

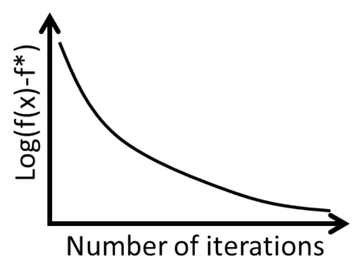
Problem 5 (10 Points)

Consider a triangle with three sides a, b and c. Fix $a = 1$, and $b + c = 2$. How does the triangle look like when it has the largest area (by changing b and c). (Hint: The Heron's formula for triangle area is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$. The optimal solution for a positive function $f(x)$ is also optimal for $f(x)^2$.)

Problem 6 (25 Points)

Consider a linear system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is a n-by-n matrix, $\mathbf{x}$ is n-by-1, and $\mathbf{b}$ is n-by-1. We can find the solution $\mathbf{x}$ by solving

$$\min_{\mathbf{x}} \quad 0.5(\mathbf{A}\mathbf{x} - \mathbf{b})^T(\mathbf{A}\mathbf{x} - \mathbf{b}). \tag{1}$$



(a) Is this problem convex? Please explain. (5 Points)

(b) In what cases is this problem strictly convex? (5 Points, optional for MAE494)

(c) The figure to the right shows the convergence of gradient descent on this problem. Do you trust this result? Please explain. (5 Points)

(d) Consider solving the problem using Newton's method starting at $\mathbf{x}_0$. How many steps will you take to reach the solution? Please explain. (5 Points)

(e) Please explain in what cases you will not be able to use Newton's method, and how you will resolve this issue. (5 Points, optional for MAE494)