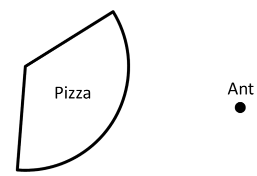
ME 598/494 Exam 1 - September 14, 2017

Problem 1 (20 Points)

Check if the following statements are true or not. Explain.



(a) If x_* is a stationary point of a continuous and differentiable function f, then x_* is a local minimum of f.

(b) The union of two convex sets is also convex. If true, please explain; if not, please provide a counter example.

(c) The problem $\min_x \frac{1}{x} + x$ for $x \geq 0$ has a minimizer.

(d) An ant is planning a move to the pizza (see figure). There is a unique shortest path from this ant to this piece of pizza.

Solutions

(a) False. Could be saddle or local maximum.

(b) False. Counter example: draw two circles with some overlap.

(c) True. Set derivative to zero to find $x^* = 1$. Also Hessian (second order derivative) is positive.

(d) True. The pizza is convex. The distance function on $\mathbb{R}^2$ is a convex function. Therefore finding a shortest distance within the pizza is a convex problem and has one unique solution.

Problem 2 (25 Points)

(a) *Derive* the iteration formula for solving n-dimensional unconstrained problems with the gradient descent method without line search. (5 Points)

(b) *Derive* the iteration formula for solving n-dimensional unconstrained problems with Newton's method without line search. (5 Points)

(c) Explain why line search is needed for gradient descent and Newton's method. Propose a line search algorithm (can be an existing one). (5 Points)

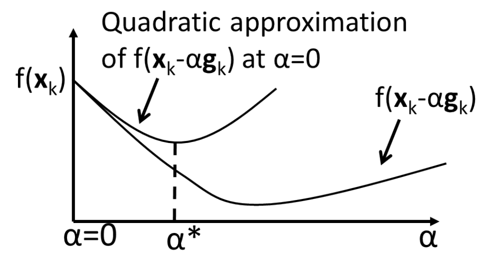
(d) Discuss briefly advantages and disadvantages of using Newton's method in solving optimization problems. (10 Points)

Solutions

See notes.

Problem 3 (10 Points)

For an unconstrained function f in $\mathbb{R}$ that is continuous and differentiable, perform the following line search at the current point $\mathbf{x}_k$ with gradient $\mathbf{g}_k$ and Hessian $\mathbf{H}_k$: (1) Consider the one-dimensional function $f(\mathbf{x}_k - \alpha\mathbf{g}_k)$ with respect to α. Derive its second order approximation at $\alpha = 0$. (2) Minimize this approximation to find the optimal step α^*. What could go wrong with this line search method? (Hint: see figure)



Solutions

See notes.

Problem 4 (10 Points)

(a) Consider the linear programming problem $\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$, where $\mathbf{c}$ is a constant vector. Does this problem have a solution? (2 Points)

(b) Consider the linear programming problem $\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$, subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, where $\mathbf{c}$ is a p-by-1 constant vector, $\mathbf{b}$ is an n-by-1 constant vector, and $\mathbf{A}$ is an n-by-p constant matrix. If this problem has solutions (local minima), how many solutions do you expect to have? Please explain. (8 Points) Hint: An optimization problem is convex if the objective is a convex function, and the feasible domain is a convex set. The problem is strictly convex when the objective is strictly convex.

Solutions

(a) No. Let $\mathbf{x} = -t\mathbf{c}$, where $t > 0$. Then the objective is $-t||\mathbf{c}||^2 < 0$. This value can go to $-\infty$ as $t \to \infty$.

(b) Notice that the objective is a convex function, and the feasible domain of $\mathbf{x}$ is a convex set, the problem has a global solution, but this solution is not necessarily unique. See hrefftp://ftp.cs.wisc.edu/pub/techreports/1978/TR316.pdfthe uniqueness proof. **Note**: You get full points if you show convexity of the objective and the contraints.

Problem 5 (10 Points)

Consider a triangle with three sides a, b and c. Fix $a = 1$, and $b + c = 2$. How does the triangle look like when it has the largest area (by changing b and c). (Hint: The Heron's formula for triangle area is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$. The optimal solution for a positive function $f(x)$ is also optimal for $f(x)^2$.)

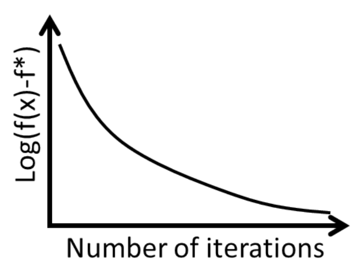
Solutions

The objective is proportional to the negative squared area: $f(b,c) = -(3-2b)(3-2c)$, with the constraint $b+c = 2$. Convert this to an unconstrained problem: $\bar{f}(b) = -(-1+2b)(3-2b)$. The derivative is $df/db = -2(3-2b)+2(-1+2b) = 0$. Thus $b^* = 1$ and $c^* = 1$. The second-order derivative is $d^2f/db^2 = 2 > 0$. Therefore the stationary point is a global minimizer.

Problem 6 (25 Points)

Consider a linear system of equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a n-by-n matrix, $\mathbf{x}$ is n-by-1, and $\mathbf{b}$ is n-by-1. We can find the solution $\mathbf{x}$ by solving

$$\min_{\mathbf{x}} \quad 0.5(\mathbf{Ax}-\mathbf{b})^T(\mathbf{Ax}-\mathbf{b}). \tag{1}$$



(a) Is this problem convex? Please explain. (5 Points)

(b) In what cases is this problem strictly convex? (5 Points, optional for MAE494)

(c) The figure to the right shows the convergence of gradient descent on this problem. Do you trust this result? Please explain. (5 Points)

(d) Consider solving the problem using Newton's method starting at $\mathbf{x}_0$. How many steps will you take to reach the solution? Please explain. (5 Points)

(e) Please explain in what cases you will not be able to use Newton's method, and how you will resolve this issue. (5 Points, optional for MAE494)

Solutions

(a) The Hessian $\mathbf{A}^T\mathbf{A}$ is psd. To show this, let $\mathbf{x}$ be an arbitrary non-zero vector. Then $\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = ||\mathbf{Ax}||^2 \geq 0$. Therefore the problem is convex.

(b) It is strictly convex when all eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive, or equivalently $\mathbf{A}$ is non-singular.

(c) No. Since this is a convex problem, the convergence should be linear.

(d) One step. The problem is quadratic and convex.

(e) When $\mathbf{A}$ is singular. We can use a modified Hessian $\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I}$ with $\lambda > 0$ to ensure the convergence.