ME 555 Exam 1 Solutions - 2013
Closed Book Closed Notes

Problem 1 (20 points)

Check if the following statements are true or not. Explain.

(a) If x_* is a stationary point of a continuous and differentiable function f, then x_* is a local minimum of f.

(b) The intersection of two convex sets is also convex.

(c) A well-constrained linear problem (both the objective and constraints are linear functions of x) can have an interior optimal solution, i.e., with no active constraints.

(d) When using Newton's method, the step size in the line search will be 1 when x_k is close to a local solution.

Problem 1 Solution

(a) False. A stationary point can also be maximum or saddle point.

(b) True. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two convex sets. For $x_1, x_2 \in \mathcal{S}_1 \cap \mathcal{S}_2$, we have $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{S}_1$ and $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{S}_2$, for any $\lambda \in [0, 1]$. Therefore, $\mathcal{S}_1 \cap \mathcal{S}_2$ is convex.

(c) False. Linear problems are monotonic. Therefore by MP1, if the problem is well-constrained, then there must be active constraints, i.e., the unique solution is usually at some vertex of the feasible domain. (Note) In the special case where the objective is constant, any feasible solution will be optimal.

(d) True. When second-order approximation of the objective function is accurate, Newton's method does not require line search. (Note) When x_k is very close to a local solution, we will have g_k close to zero. But this is not the direct reason for $\alpha = 1$.

Problem 2 (20 points)

Solve the problem below using monotonicity analysis. Identify active constraints and prove the global optimum.

$$
\begin{aligned}
\min_{x_1,x_2} \quad & x_1^2 + x_2^2 - 4x_1 + 4 \\
\text{subject to} \quad & g_1 = -x_1 + 1 \leq 0, \\
& g_2 = -x_2 \leq 0, \\
& g_3 = x_2 - (1 - x_1)^3 \leq 0
\end{aligned}
$$

Problem 2 Solution

By MP1, g_2 is active with respect to x_2. Therefore $x_2^* = 0$. Removing x_2 from the problem to have $x_1 \geq 1$ from g_1 and $x_1 \leq 1$ from g_3. Therefore the only feasible solution is $x_1 = 1$. So the optimal solution is $(1, 0)$.

Problem 3 (25 Points)

Consider the problem of finding the minimum of the function

$$f = x_1^2 + x_2^2 - 3x_1x_2$$

(a) Find the stationary point. (5 Points)

(b) Determine what is the nature of the point found above. (5 Points)

(c) Now add the constraints $x_1 \geq 0$ and $x_2 \geq 0$. Can you determine the minimum for this revised problem? Explain. (Hint: Check if there are directions of downslopes away from the stationary point. Also notice that the problem is *symmetric*. If there exists a local solution (x_1^*, x_2^*), what will be the relationship between x_1^* and x_2^*?) (10 Points)

(d) Now add the constraint $g_1 = x_1^2 + x_2^2 - 6 \leq 0$. Can you determine the minimum for this further revised problem? Explain. (Hint: Check if g_1 will be active or not.) (5 Points)

Problem 3 Solution

(a) Set gradient to zero to find the stationary point $x^* = (0, 0)$.

(b) The Hessian is $[2, -3; -3, 2]$. The determinant is negative. Therefore x^* is a saddle point.

(c)
$$\partial f = f(x) - f(x^*) = x_1^2 + x_2^2 - 3x_1x_2 = (x_1 - \frac{3+\sqrt{5}}{2}x_2)(x_1 - \frac{3-\sqrt{5}}{2}x_2).$$

In the region $\left(x_1 - \frac{3+\sqrt{5}}{2}x_2 > 0 \ \& \ x_1 - \frac{3-\sqrt{5}}{2}x_2 < 0\right)$ or $\left(x_1 - \frac{3+\sqrt{5}}{2}x_2 < 0 \ \& \ x_1 - \frac{3-\sqrt{5}}{2}x_2 > 0\right)$, are directions of downslopes away from x^*. For example, along the line $x_1 = x_2$, the function value decreases as x_1, x_2 increase. Therefore $x_1 \geq 0$ and $x_2 \geq 0$ will not bound the problem and the minimum is not determined. (Note) It is not required to find the regions exactly.

(d) Apply MP1 in the region derived from (c) to show that g_1 will be active. Therefore $x_1^2 + x_2^2 = 6$ at the solution. The objective becomes $-x_1x_2$. Notice that $x_1^2 + x_2^2 \geq 2x_1x_2$ and the equality is attained iff $x_1 = x_2$. We then have $x_1^* = x_2^* = \sqrt{3}$.

Problem 4 (35 Points)

(a) State and prove the second order sufficiency condition for a minimum of an unconstrained function f in $\mathbb{R}$ that is continuous and differentiable. (5 Points)

(b) *Derive* the iteration formula for solving n-dimensional unconstrained problems with the gradient method without line search. (5 Points)

(c) *Derive* the iteration formula for solving n-dimensional unconstrained problems with Newton's method without line search. (10 Points)

(d) Explain briefly the meaning of the line search and why it is used to modify (b) and (c) above. (5 Points)

(e) Discuss briefly advantages and disadvantages of using Newton's method in solving optimization problems. (10 Points)

Problem 4 Solution

See textbook.