ME 598/494 Exam 1 Solutions - February 16, 2015

Problem 1 (20 Points)

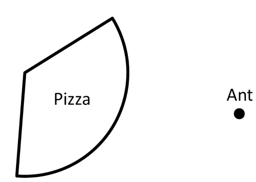
Check if the following statements are true or not. Explain.

(a) If x_* is a stationary point of a continuous and differentiable function f, then x_* is a local minimum of f.

(b) The union of two convex sets is also convex. If true, please explain; if not, please provide a counter example.

(c) The problem $\min_x \frac{1}{x} + x$ for $x \geq 0$ has a minimizer.

(d) An ant is planning a move to the pizza. There is a unique shortest path from this ant to this piece of pizza. (We will talk about how ants do optimization later.)



Solutions

(a) False. Could be saddle or local maximum.

(b) False. Counter example: draw two circles with some overlap.

(c) True. Set derivative to zero to find $x^* = 1$. Also Hessian (second order derivative) is positive.

(d) True. The pizza is a convex shape. The distance function on $\mathbb{R}^2$ is a convex function. Therefore finding a shortest distance within the pizza is a convex problem and has one unique solution.

Problem 2 (20 Points)

Find the minimum of:

$$f(x_1, x_2) = x_1 - 2\log(x_1 + 1) + \exp(x_2) + x_1 x_2, \tag{1}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solutions

Objective is monotonically increasing with respect to x_2. Therefore $x_2^* = 0$. The reduced problem is now to minimize $x_1 - 2\log(x_1 + 1)$ with $x_1 \geq 0$. Set its gradient to zero to have $x_1^* = 1$ which satisfies the constraint. Therefore the solution is $x_1^* = 1, x_2^* = 0$.

Problem 3 (25 Points)

(a) *Describe* the iteration formula for solving n-dimensional unconstrained problems with the gradient method without line search. (5 Points)

(b) *Describe* the iteration formula for solving n-dimensional unconstrained problems with Newton's method without line search. (5 Points)

(c) Explain briefly the meaning of the line search and why it is used to modify (b) and (c) above. (5 Points)

(d) Discuss briefly advantages and disadvantages of using Newton's method in solving optimization problems. (10 Points)

Solutions

Please see POD3 Chapter 4 and lecture notes.

Problem 4 (15 Points)

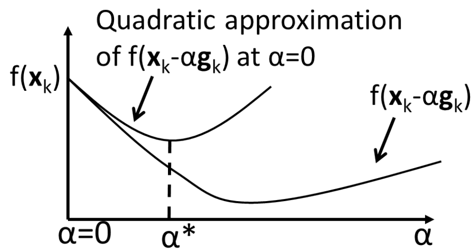
Consider cutting the largest rectangular pizza out of a round pizza. How does the optimal rectangle look like? How do you know it is optimal? (Hint: the optimal solution for a positive function $f(x)$ is also optimal for $f(x)^2$.)

Solutions

Consider the length and width of the rectangle to be a and b, respectively. The optimization problem is to maximize ab, with $a^2 + b^2 = d^2$, where d is the diameter of the pizza. This problem is equivalent to the unconstrained problem of maximizing $(d^2 - b^2)b^2$, which has a solution $b^* = a^* = d/\sqrt{2}$. So the optimal shape should be a square.

Problem 5 (Optional for MAE494, 10 Points)

For an unconstrained function f in $\mathbb{R}$ that is continuous and differentiable, perform the following line search at the current point $\mathbf{x}_k$ with gradient $\mathbf{g}_k$ and Hessian $\mathbf{H}_k$: (1) Consider the one-dimensional function $f(\mathbf{x}_k - \alpha \mathbf{g}_k)$ with respect to α. Derive its second order approximation at $\alpha = 0$. (2) Minimize this approximation to find the optimal step α^*. What could go wrong with this line search method? (Hint: see the figure below)



Solutions

See POD3 Chapter 4 Page 156.

Problem 6 (Optional for MAE494, 10 Points)

(a) Consider the linear programming problem $\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$, where $\mathbf{c}$ is a constant vector. Is this problem well constrained? (2 Points)

(b) Consider the linear programming problem $\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$, subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, where $\mathbf{c}$ is a p-by-1 constant vector, $\mathbf{b}$ is an n-by-1 constant vector, and $\mathbf{A}$ is an n-by-p constant matrix. What can you tell regarding constraint activity if an optimal solution exists (i.e., how many constraints should be active)? (6 Points) (Hint: Use monotonicity analysis.)

(c) Based on (b), briefly describe a strategy for solving a linear programming problem. (2 Points)

Solutions

(a) Not well constrained since the linear function is monotonic.

(b) By MP1, each variable needs to be bounded by a constraint. So in general at least p constraints are active. (Note that there are special cases where this does not hold. For example, when the gradient of a constraint is parallel to that of the objective function.)

(c) From (b), solutions for a linear programming problem are always at vertices of the bounding polytope. Therefore one strategy is to traverse these vertices by adding and deleting active constraints from a set of p presumably active constraints.