ME598/494 Homework 4 Solution

Monotonicity Principle 1 (MP1): If a constrained problem has a solution, and the objective function is monotonically increasing (decreasing), then there exists a non-increasing (non-decreasing) constraint that bound the solution from below (above).

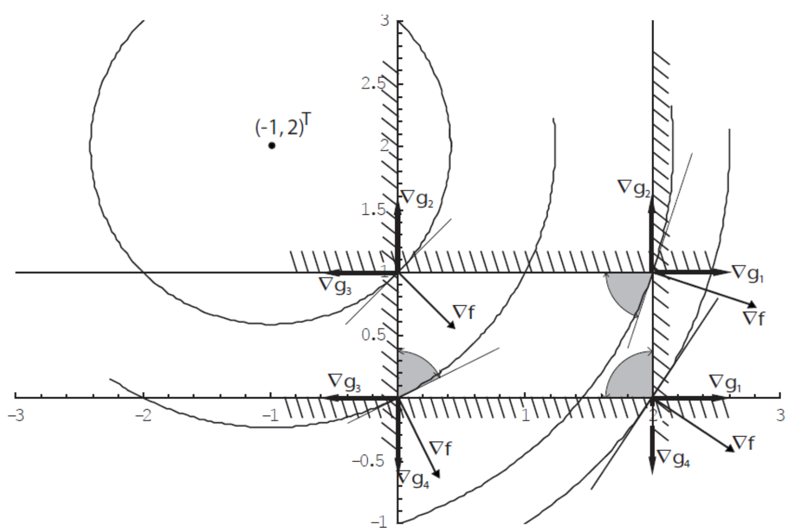
Monotonicity Principle 2 (MP2): If a constrained problem has a solution, and a variable only appears in constraints, then this variable is either irrelevant, or bounded both from above and below.

1

Sketch graphically the problem

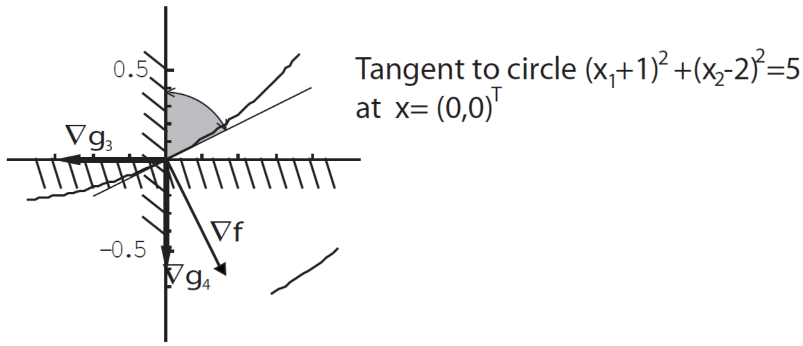
$$\begin{aligned}&\min\ f(\mathbf{x}) = (x_1+1)^2 + (x_2-2)^2\\ &\text{subject to } g_1 = x_1 - 2 \le 0, \quad g_3 = -x_1 \le 0,\\ &g_2 = x_2 - 1 \le 0, \quad g_4 = -x_2 \le 0.\end{aligned}$$

Find the optimum graphically. Determine directions of feasible descent at the corner points of the feasible domain. Show the gradient directions of f and g_is at these points. Verify graphical results analytically using KKT conditions and monotonicity analysis.

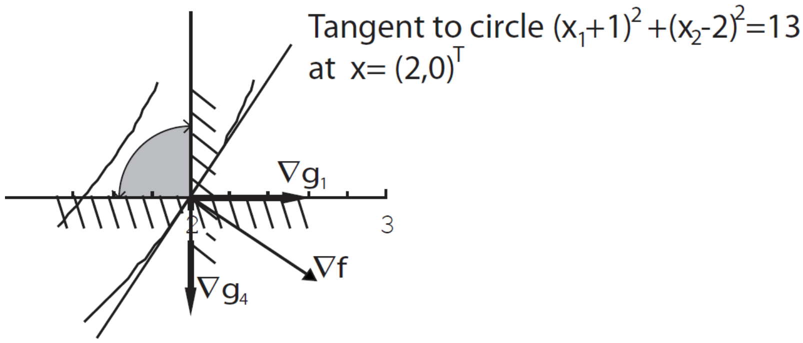


Solution

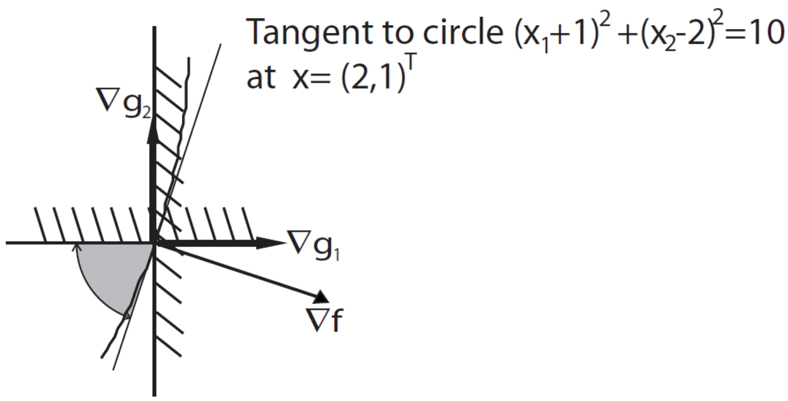
Graphic solution shows the optimum at $\mathbf{x}_* = (0,1)^T$ and $f_* = 2$.



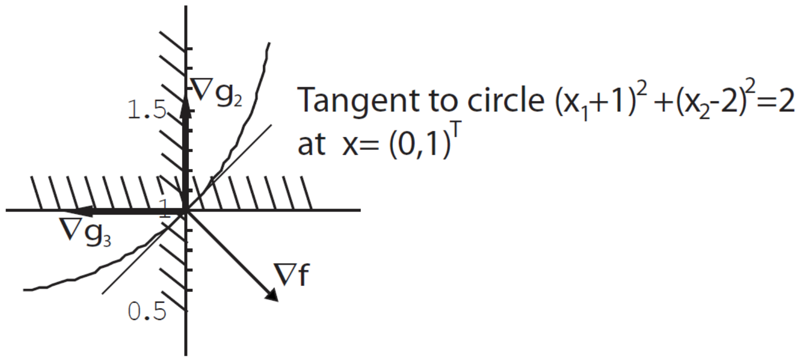
At (0,0): The angle marked includes the directions of feasible direction.



At (2,0): The angle marked includes the directions of feasible direction.



At (2,1): The angle marked includes the directions of feasible direction.



At (0,1): No feasible descent directions.

All feasible directions are ascent directions. This means $\mathbf{x}_* = (0,1)^T$ is the minimizer.

Applying the KKT conditions at (0,1)

- Necessary conditions g_2 and g_3 are active $\Rightarrow$ μ_1 and μ_4 equal 0.

$$
\begin{aligned}
\nabla f - \mu^T \nabla \mathbf{g} &= \mathbf{0}^T \\
&\Rightarrow \begin{pmatrix} 2(x_1+1) \\ 2(x_2-2) \end{pmatrix} + \begin{pmatrix} -\mu_3 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Bigg|_{(x_1,x_2)=(0,1)} \\
&\Rightarrow \begin{pmatrix} 2-\mu_3 \\ -2+\mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mu_2 = 2,\ \mu_3 = 2
\end{aligned}
$$

The KKT necessary conditions are satisfied at (0,1)

- Sufficient conditions The Hessian of Lagrangian is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{, which is positive definite everywhere}$$

Therefore, $\mathbf{x}_* = (0,1)^T$ is the global minimum verifying the graphical solution

Monotonicity Analysis In the feasible domain, we have $f(x_1^+, x_2^-)$, i.e., the objective monotonically increases wrt x_1, and decreases wrt x_2. Therefore,

$$
\begin{aligned}
\min \quad & f(x_1^+, x_2^-) = (x_1+1)^2 + (x_2-2)^2 \\
\text{subject to} \quad & g_1(x_1^+) = x_1 - 2 \le 0, \quad g_3(x_1^-) = -x_1 \le 0, \\
& g_2(x_2^+) = x_2 - 1 \le 0, \quad g_4(x_2^-) = -x_2 \le 0.
\end{aligned}
$$

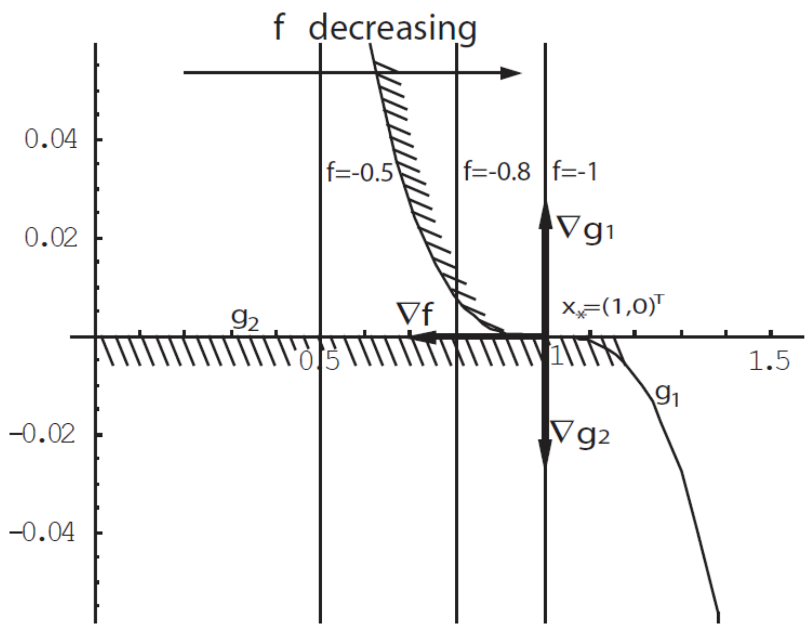
By MP1 w.r.t. x_1, g_3 is active.
By MP1 w.r.t. x_2, g_2 is active.
It gives $x_{1*} = 0$, $x_{2*} = 1$ and $f_* = 2$.

2

Graph the problem

$$\min f = -x_1, \text{subject to}$$
$$g_1 = x_2 - (1 - x_1)^3 \leq 0 \quad \text{and} \quad x_2 \geq 0.$$

Find the solution graphically. Apply the optimality conditions and monotonicity rules. Discuss. (From Kuhn and Tucker, 1951.)



Solution

Form the graph we find the solution is at $(x_{1*}, x_{2*}) = (1, 0)$.

Checking the KKT conditions

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 3(1-x_1^2) \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{aligned} &\mu_1(x_2 - (1-x_1)^3) = 0, \quad && \mu_1 \geq 0 \\ &-\mu_2 x_2 = 0, && \mu_2 \geq 0 \end{aligned}$$

at $(x_{1*}, x_{2*}) = (1, 0)$,

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -1 = 0 : \text{contradict} \\ \mu_1 - \mu_2 = 0 \end{array}$$

Hence, this point is not a KKT point because this is not a regular point.

Using Monotonicity analysis

$$\begin{aligned} \min \quad & f(x_1^-) = -x_1 \\ \text{subject to} \quad & g_1(x_1^+, x_2^+) \leq 0 \\ & g_2(x_2^-) \leq 0 \end{aligned}$$

By MP1 w.r.t. x_1, g_1 is active and x_2 becomes relevant.
By MP2 w.r.t. x_2, g_2 is active.
Solving $g_1 = 0$ and $g_2 = 0$, we get $(x_{1*}, x_{2*}) = (1, 0)$, which is a global minimum.

3

Find a local solution to the problem

$$\begin{aligned} &\max\ f = x_1 x_2 + x_2 x_3 + x_1 x_3 \\ &\text{subject to } h = x_1 + x_2 + x_3 - 3 = 0. \end{aligned}$$

Use three methods: direct elimination, constrained derivatives, and Lagrange multipliers. Compare. Is the solution global?

Solution

a) Direct Elimination
Substituting $x_3 = 3 - x_1 - x_2$, we obtain

$$\begin{aligned}\max\ f &= x_1x_2 + 3x_2 - x_1x_2 - x_2^2 + 3x_1 - x_1^2 - x_1x_2 \\ &= 3x_1 + x_2 - x_1^2 - x_1x_2 - x_2^2\end{aligned}$$

$\nabla f = \mathbf{0}^T$

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= 3 - 2x_1 - x_2 = 0 \\ \frac{\partial f}{\partial x_2} &= 3 - x_1 - 2x_2 = 0\end{aligned} \tag{1}$$

Solving equation (1), we get

$$x_{1\dagger} = 1, x_{2\dagger} = 1, x_{3\dagger} = 1 \text{ and } f_\dagger = 3.$$

Because $\mathbf{H} = \begin{pmatrix} -2 & -2 \\ -1 & -2 \end{pmatrix}$ is negative definite everywhere, $\mathbf{x}_* = (1,1,1)$ is a global maximum.

b) Constrained Derivatives Let $d_1 = x_1$, $d_2 = x_2$ and $s_1 = x_3$.

$$\begin{aligned}\partial z/\partial \mathbf{d} &= (\partial f/\partial \mathbf{d}) - (\partial f/\partial \mathbf{s})(\partial \mathbf{h}/\partial \mathbf{s})^{-1}(\partial \mathbf{h}/\partial \mathbf{d}) \\ &= \begin{pmatrix} x_2 + x_3 \\ x_1 + x_3 \end{pmatrix} - (x_1 + x_2)(1)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -x_1 + x_3 \\ -x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2 = x_3\end{aligned}$$

Solving $h = x_1 + x_2 + x_3 - 3 = 3x_1 - 3 = 0$, we get

$$x_{1*} = x_{2*} = x_{3*} = 1 \text{ and } f_* = 3$$

c) Lagrange Multiplier

$$\begin{aligned}L &= -f + \lambda h = -x_1x_2 - x_2x_3 - x_3x_1 + \lambda(x_1 + x_2 + x_3 - 3) \\ \nabla L &\Rightarrow \left.\begin{aligned} -x_2 - x_3 + \lambda &= 0 \\ -x_1 - x_2 + \lambda &= 0 \\ -x_1 - x_3 + \lambda &= 0 \end{aligned}\right\} \Rightarrow x_{1*} = x_{2*} = x_{3*} = 1,\ \lambda = 2 \text{ and } f_* = 3\end{aligned}$$

Checking the Second order sufficiency

$$\partial \mathbf{x}^T L_{\mathbf{xx}} \partial \mathbf{x} = \partial x_1 \partial x_2 + \partial x_2 \partial x_3 + \partial x_3 \partial x_1$$

Since $\nabla h \partial \mathbf{x} = \partial x_1 + \partial x_2 + \partial x_3 = 0$,

$$\partial \mathbf{x}^T L_{\mathbf{xx}} \partial \mathbf{x} = -(\partial x_1^2 + \partial x_2^2 + \partial x_1 \partial x_2) = -((\partial x_1 - \partial x_2/2)^2 + \frac{4}{3}\partial x_2^2) \leq 0$$

Therefore, $\mathbf{x}_* = (1, 1, 1)$ is a global maximum.

Alternative Solution
From the symmetry the problem can be reduced to

$$\begin{aligned} &\max\ f = x_1^2 + 2x_1x_2 \\ &\text{subject to } h = 2x_1 + x_2 - 3 = 0. \end{aligned}$$

a) Direct Elimination

$$\begin{aligned} &\max\ f = -3x_1^2 + 6x_1 \\ &x_{1*} = x_{2*} = x_{3*} = 1 \text{ and } f_* = 3 \end{aligned}$$

This point is indeed a global max since $f''(x_1) = -6$

b) Constrained Derivatives
Let $s = x_1$ and $d = x_2$. From $\partial z/\partial \mathbf{d} = (\partial f/\partial \mathbf{d}) - (\partial f/\partial \mathbf{s})(\partial \mathbf{h}/\partial \mathbf{s})^{-1}(\partial \mathbf{h}/\partial \mathbf{d})$ and $h = 0$, we get

$$\begin{aligned} \partial z/\partial \mathbf{d} &= -s + d = 0 \\ h &= 2s + d - 3 = 0 \end{aligned}$$

Solving gives

$$s = x_{1*} = 1,\ d = x_{2*} = x_{3*} = 1 \text{ and } f_* = 3$$

c)Lagrange Multiplier

$$\begin{aligned} L &= -f + \lambda h = -x_1^2 - 2x_1x_2 + \lambda(2x_1 + x_2 - 3) \\ \nabla L &\Rightarrow \left.\begin{aligned} -x_1 - x_2 + \lambda &= 0 \\ -2x_1 + \lambda &= 0 \\ -2x_1 + x_2 - 3\lambda &= 0 \end{aligned}\right\} \Rightarrow x_{1*} = x_{2*} = x_{3*} = 1,\ \lambda = 2 \text{ and } f_* = 3 \end{aligned}$$

Therefore, $f_* = 3$ at $\mathbf{x}_* = (1, 1, 1)$ is a maximum.

4

Use monotonicity arguments and constrained derivatives to find the value(s) of the parameter b for which the point $x_1 = 1$, $x_2 = 2$ is the solution to the problem

$$\begin{aligned} &\max\ f = 2x_1 + bx_2 \\ &\text{subject to } g_1 = x_1^2 + x_2^2 - 5 \leq 0 \\ &\text{and } g_2 = x_1 - x_2 - 2 \leq 0. \end{aligned}$$

Solution

Rewrite the problem in the negative null form

$$\begin{aligned} \min \quad & f = -2x_1 - bx_2 \\ \text{subject to} \quad & g_1 = x_1^2 + x_2^2 - 5 \leq 0 \\ \text{and} \quad & g_2 = x_1 - x_2 - 2 \leq 0. \end{aligned}$$

Monotonicity Analysis

If the optimum is located at $x_1 = 1, x_2 = 2$,

$$\begin{aligned} g_1 &= 1^2 + 2^2 - 5 = 0 \leftarrow \text{ active} \\ g_2 &= 1 - 2 - 2 = -3 \leftarrow \text{ inactive} \end{aligned}$$

$$\begin{aligned} \min \quad & f(x_1^-, x_2) \\ \text{subject to} \quad & g_1(x_1^+, x_2^+) \leq 0 \\ \text{and} \quad & g_2(x_1^+, x_2^-) \leq 0. \end{aligned}$$

If $f(x_1^-, x_2^+)$, g_2 must be active by MP1 w.r.t. x_2. Therefore, f can't be monotonically increasing w.r.t x_2. That means b¿0

Constrained derivatives

At $x_1 = 1$ and $x_2 = 2$, g_1 is active. Hence, we can consider it as an equality constraint.

let $x_1 = s$ and $x_2 = d$,

$$\begin{aligned} \partial z/\partial \mathbf{d}|_{(1,2)} &= (\partial f/\partial \mathbf{d}) - (\partial f/\partial \mathbf{s})(\partial \mathbf{h}/\partial \mathbf{s})^{-1}(\partial \mathbf{h}/\partial \mathbf{d})|_{(1,2)} = -b - (-2)(2x_1)^{-1}(2x_2)|_{(1,2)} \\ &= -b - (-2)(2)^{-1}(4) = 0 \end{aligned}$$

Therefore, $b = 4$.
Check 2nd-order sufficiency conditions using reduced gradient.

$$\begin{aligned}
\frac{\partial^2 z}{\partial \mathbf{d}^2} &= (\mathbf{I}, (\partial \mathbf{s}/\partial \mathbf{d})^T) \begin{pmatrix} \frac{\partial^2 f}{\partial \mathbf{d}^2} & \frac{\partial^2 f}{\partial \mathbf{d}\partial \mathbf{s}} \\ \frac{\partial^2 f}{\partial \mathbf{s}\partial \mathbf{d}} & \frac{\partial^2 f}{\partial \mathbf{s}^2} \end{pmatrix} \begin{pmatrix} \mathrm{I} \\ \frac{\partial \mathbf{s}}{\partial \mathbf{d}} \end{pmatrix} + \left(\frac{\partial f}{\partial \mathbf{s}}\right)\left(\frac{\partial^2 \mathbf{s}}{\partial \mathbf{d}^2}\right) \\
&= \left(\frac{\partial f}{\partial \mathbf{s}}\right)\left(\frac{\partial^2 \mathbf{s}}{\partial \mathbf{d}^2}\right) \quad (\text{Since} f_{\mathbf{xx}} = \mathbf{0}) \\
&= -(\partial f/\partial \mathbf{s})(\partial \mathbf{h}/\partial \mathbf{s})^{-1}(\mathbf{I}, (\partial \mathbf{s}/\partial \mathbf{d})^T)\mathbf{h}_{\mathbf{xx}}(\mathbf{I}, (\partial \mathbf{s}/\partial \mathbf{d}))^T\Big|_{(1,2)} \\
&= -(-2)(2x_2)^{-1}(\mathbf{I}, (2x_1)^{-1}(2x_2)^T) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} (\mathbf{I}, (2x_1)^{-1}(2x_2))^T\Bigg|_{(1,2)} \\
&= -(-2)(4)^{-1}(2 + 2(x_2/x_1)^2)|_{(1,2)} \\
&= 5 > 0
\end{aligned}$$

Therefore, $b = 4$ and $x_1 = 1$ and $x_2 = 2$ is a local minimizer.

Alternative Solution
Check 2nd-order sufficiency conditions using KKT. The Lagrangian function is

$$L(x_1, x_2, \mu_1, \mu_2) = -2x_1 - 4x_2 + \mu_1(x_1^2 + x_2^2 - 5) + \mu_2(x_1 - x_2 - 2) \tag{2}$$

Then its gradient and Hessian wrt $\mathbf{x}$ are

$$L_x(x_1, x_2, \mu_1, \mu_2) = [-2 + 2\mu_1 x_1 + \mu_2, -4 + 2\mu_1 x_2 - \mu_2]^T, \tag{3}$$

and

$$L_{xx}(x_1, x_2, \mu_1, \mu_2) = [2\mu_1, 0; 0, 2\mu_1]. \tag{4}$$

Since $L_x(x_1, x_2, \mu_1, \mu_2) = \mathbf{0}$ for $x_1 = 1$, $x_2 = 2$ and $\mu_2 = 0$ (g_2 inactive), we have $\mu_1 = 1$. Thus $L_{xx}(x_1, x_2, \mu_1, \mu_2) = [2, 0; 0, 2]$, p.d. Therefore $x_1 = 1$ and $x_2 = 2$ is a local minimizer.

5

Find the solution for

$$\begin{aligned}
&\min\ f = x_1^2 + x_2^2 + x_3^2 \\
&\text{subject to } h_1 = x_1^2/4 + x_2^2/5 + x_3^2/25 - 1 = 0 \\
&\text{and } h_2 = x_1 + x_2 - x_3 = 0,
\end{aligned}$$

by implementing the generalized reduced gradient method (e.g., using MATLAB).

Solution
Check Matlab code here.