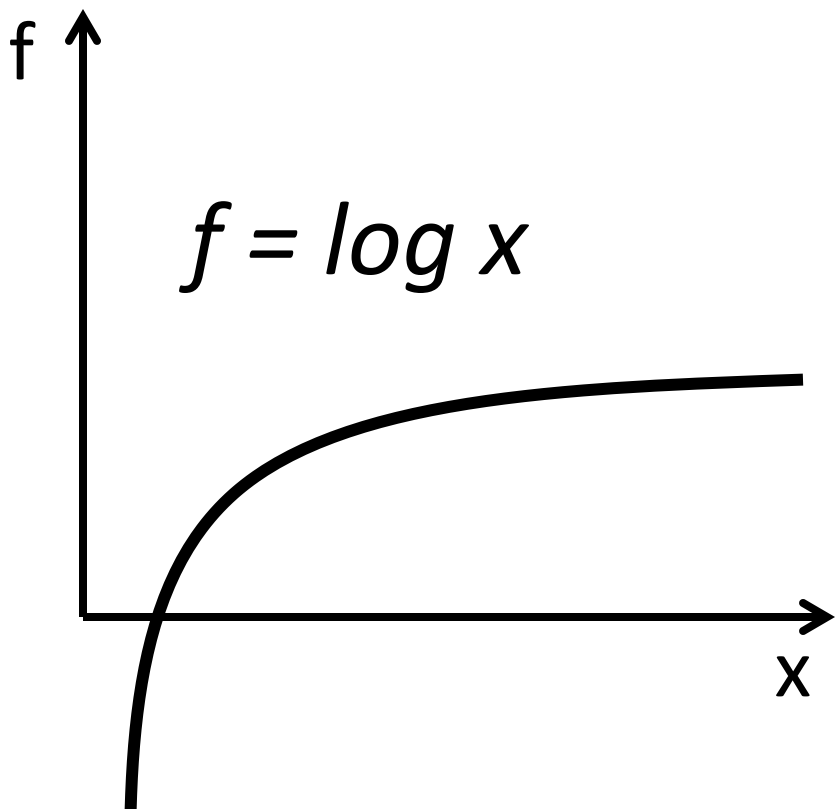
Constraints, Activity and Monotonicity

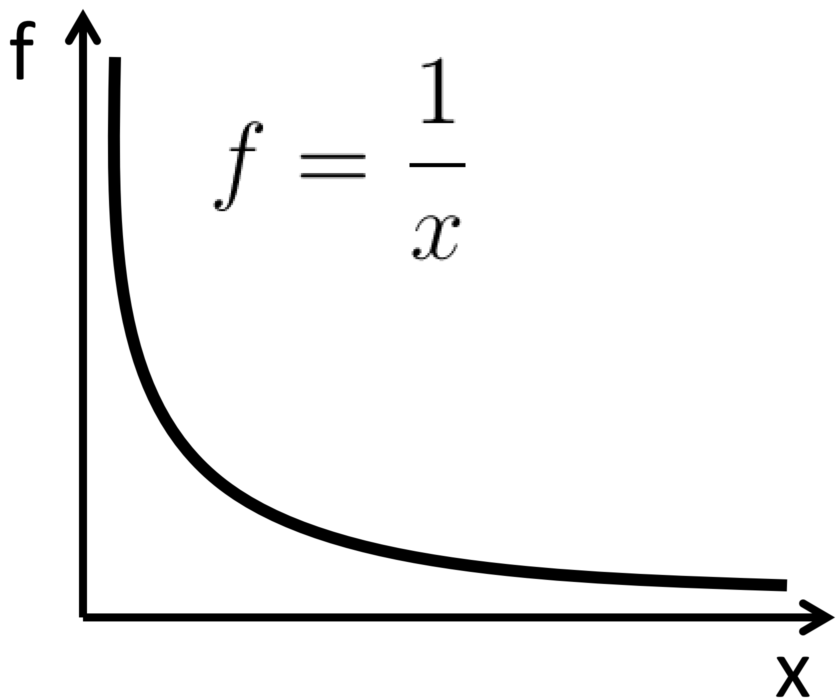
Constraints

In a minimization problem, the objective needs to be *bounded below*.



Constraints

If the objective is bounded below, does it guarantee an optimal solution?



Constraints

An optimization problem should be *well-constrained* in the variable space or otherwise…

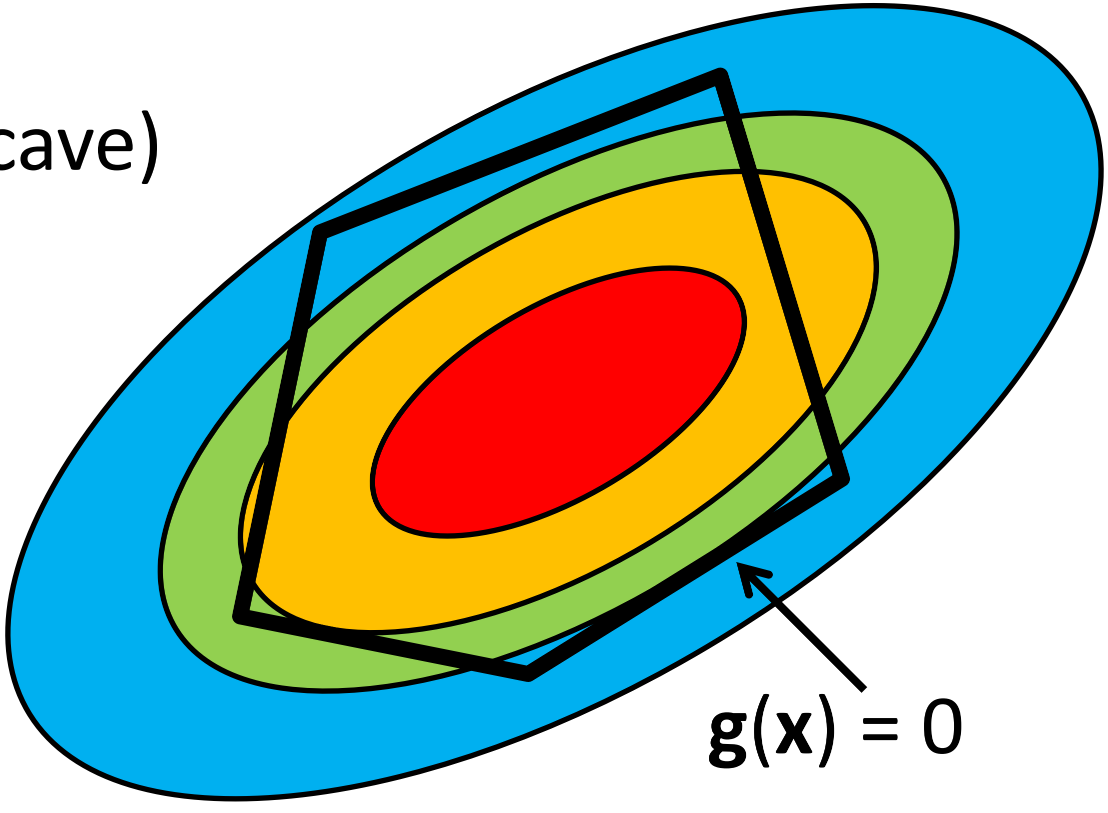
In fmincon: Optimization terminated.
Maximum iteration number reached.

In quadprog: Problem unbounded.

Constraints

If the design space is constrained, does it guarantee an optimal solution?

Minimize: f(**x**) (concave)
Subject to: g(**x**) < 0



Compact set

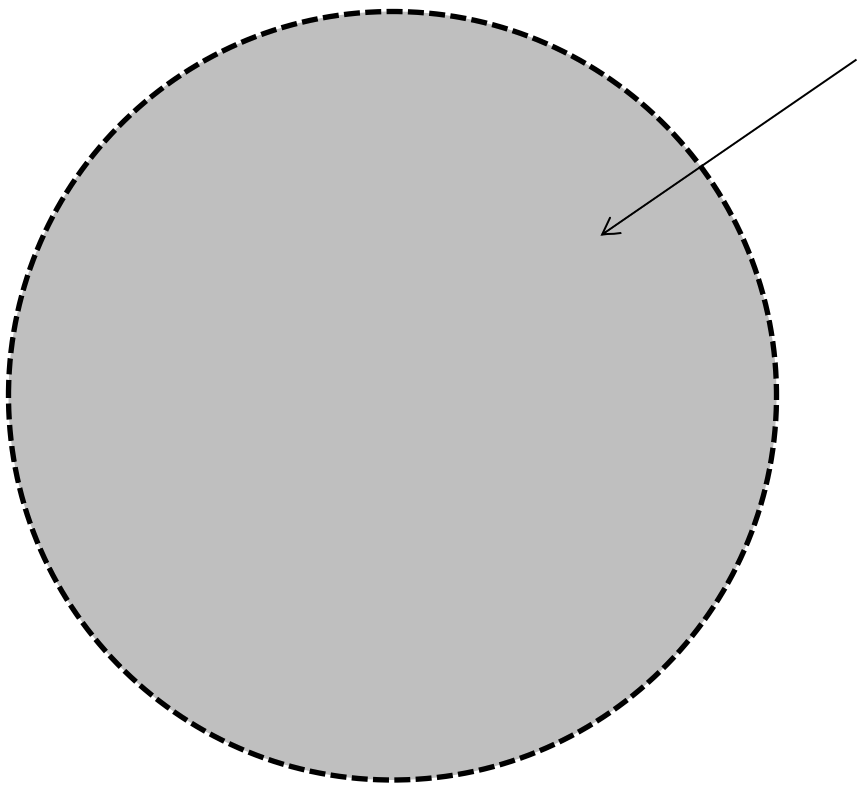
Optimization requires a *compact* variable set.

A *compact set* is *closed* and *bounded*

Closed set: The complement of an *open* set

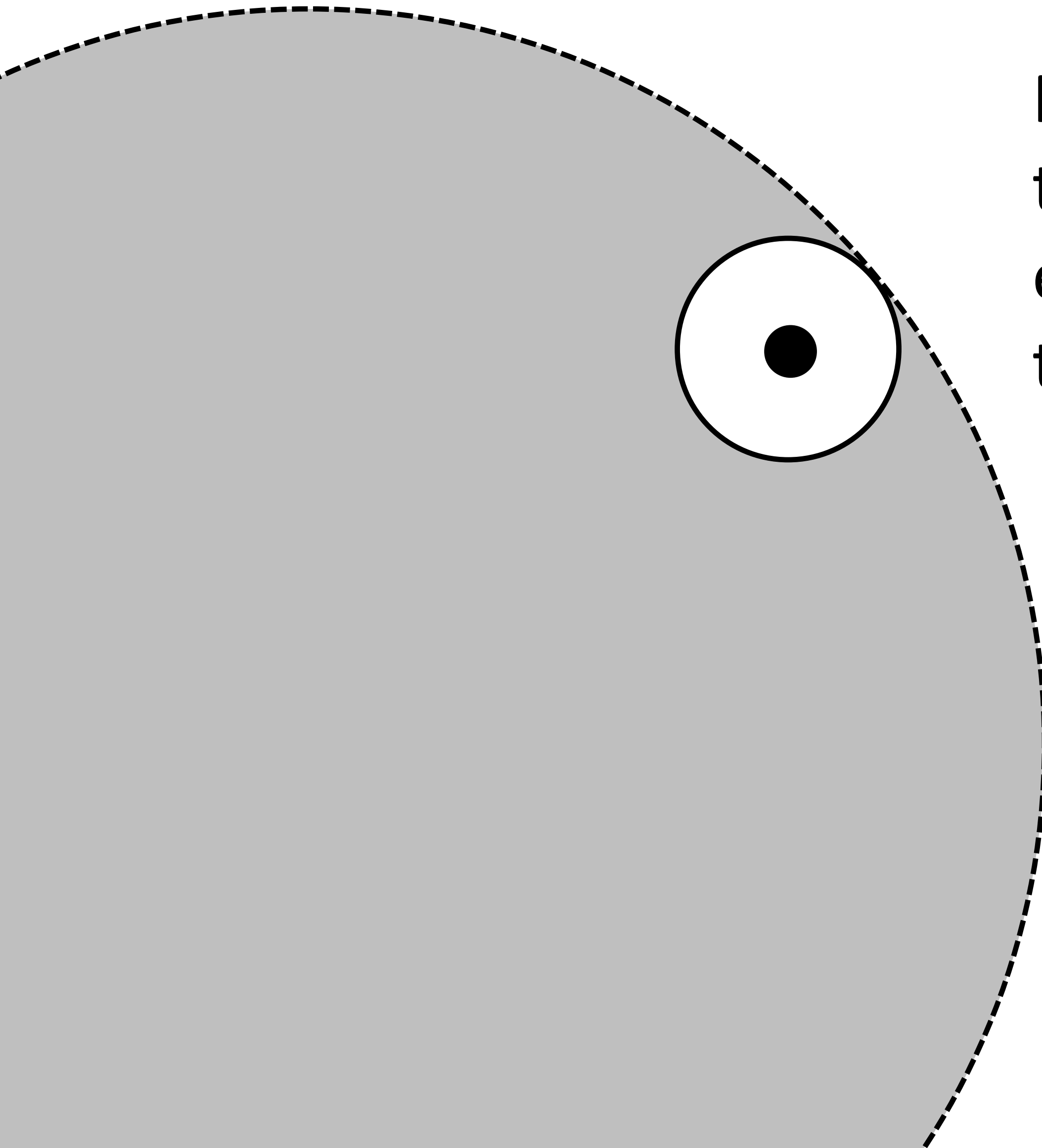
(In Euclidean space) A set is *open* if every point in it has a neighborhood contained in it.

Example of an open set



An open set: All points *in* this circle, but not *on* the circle

Example of an open set

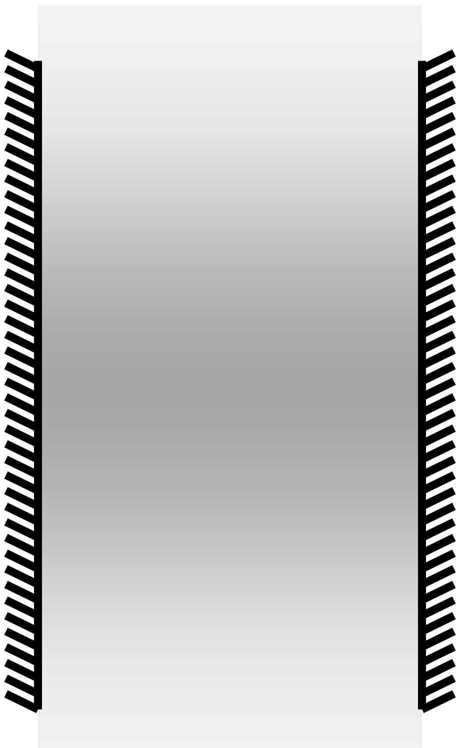


For any point in this set, there always exists a small enough neighborhood in the set

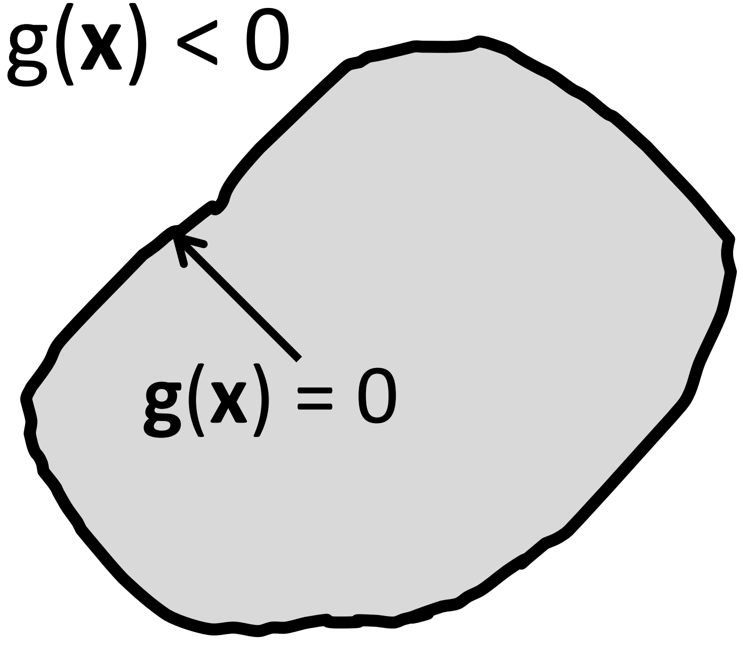
Compact set

- A compact set is (1) closed and (2) bounded

Closed, not bounded



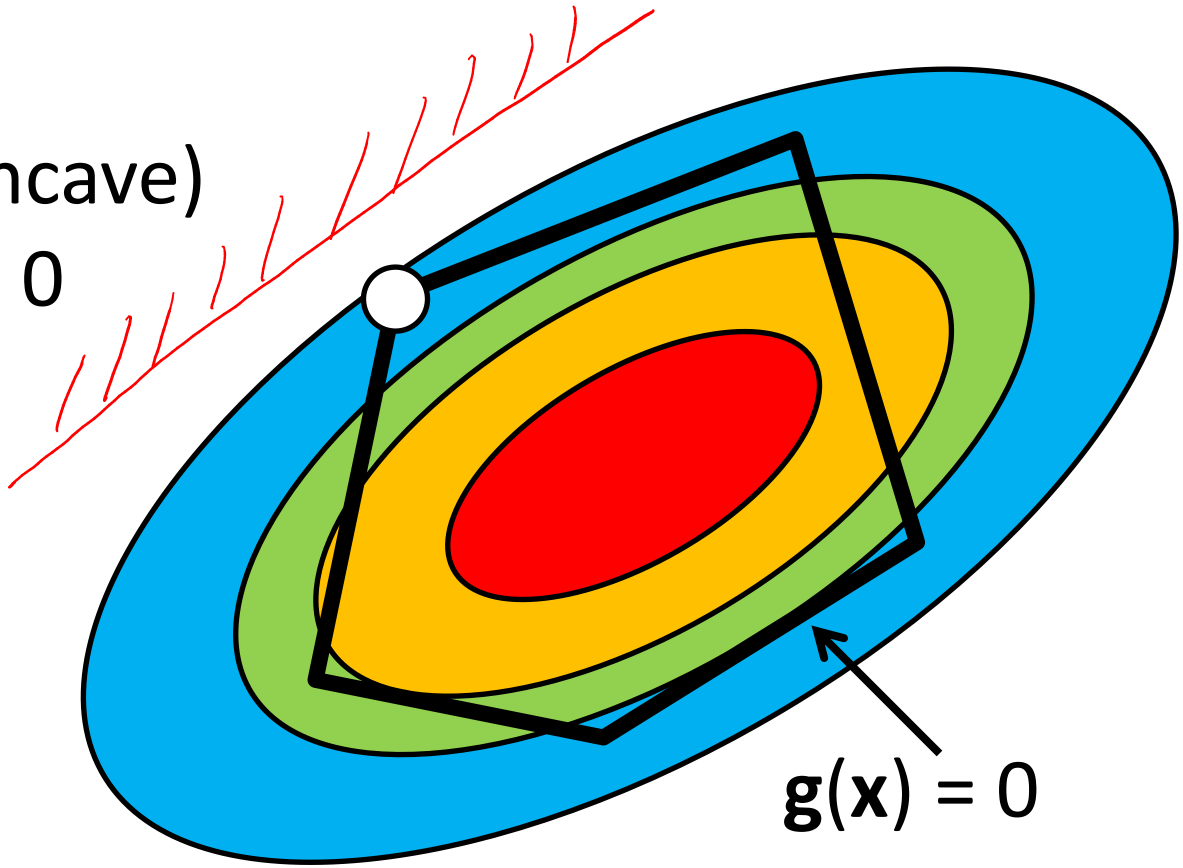
Bounded, not closed



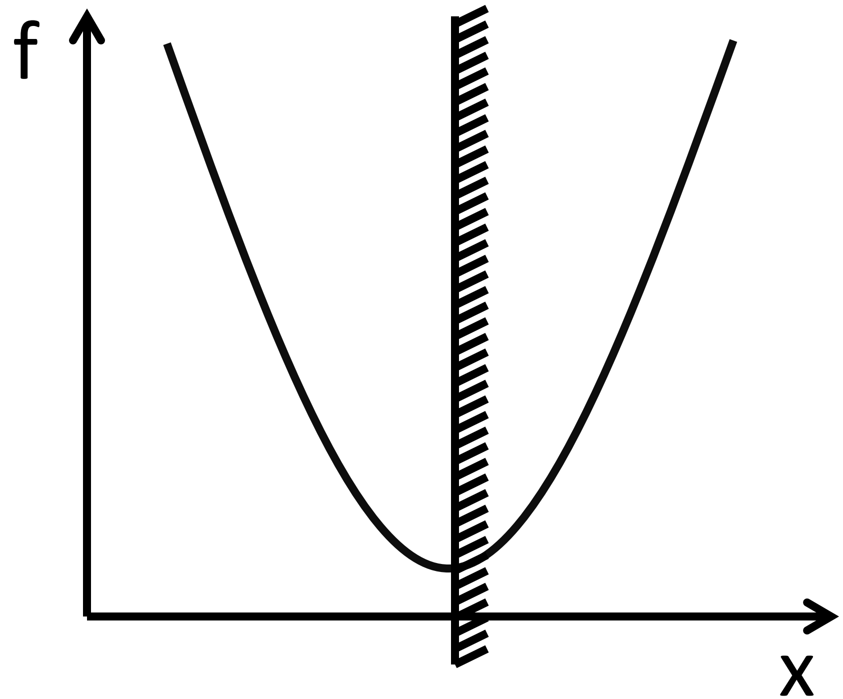
Active constraint

- An inequality constraint is *active* when g(**x**) = 0

Minimize: f(**x**) (concave)
Subject to: g(**x**) <= 0



One exception



Here the constraint happens to be g(x) = 0
However, removing the constraint will not change the solution. So g(x) is not active at the optimal solution.

Active constraints

- Equality constraints can be used to reduce the number of variables (degree of freedom)

- Identifying active inequality constraints may help to reduce the problem (big deal!)

Minimize: $f(\mathbf{x})$

Subject to: $\mathbf{h}(\mathbf{x}) = 0$

$\mathbf{g}(\mathbf{x}) <= 0$

Active constraints

(Exercise 3.4, From W. Braga, Pontificia Universidade Catolica do Rio de Janeiro, Brazil.)

A cubical refrigerated van is to transport fruit between Sao Paulo and Rio. Let n be the number of trips, s the length of a side (cm), a the surface area (cm2), V the volume (cm3), and t the insulation thickness (cm). Transportation and labor cost is $21n$; material cost is $16(1e-4)a$; refrigeration cost is $17(1e-4)an/(t+1.2)$; and insulation cost is $41(1e-5)at$. The total volume of fruit to be transported is $34(1e6)$cm3. Express the problem of designing this van for minimum cost as a constrained optimization problem.

Active constraints

$$\begin{aligned}
\min(\text{cost}) &= (\text{Transportation/Labor}) + (\text{Material}) + (\text{Refrigeration}) + (\text{Insulation}) \\
&= 21\text{n} + 16 \times 10^{-4} a + 17 \times 10^{-4} a \cdot n/(t+1.2) + 41 \times 10^{-5} a \cdot t
\end{aligned}$$

$$\begin{aligned}
\text{subject to } h_1 &= V - (s-2t)^3 = 0 \\
h_2 &= a - 6s^2 = 0 \\
g_1 &= 34 \times 10^6 - nV \leq 0
\end{aligned}$$

How many design variables do you have?

n $\cancel{a}$ t s $\cancel{V}$

?

$$
\begin{aligned}
\min(\text{cost}) &= (\text{Transportation/Labor}) + (\text{Material}) + (\text{Refrigeration}) + (\text{Insulation}) \\
f &= \underline{21\text{n}} + 16 \times 10^{-4} a + 17 \times 10^{-4} a \cdot n/(t + 1.2) + 41 \times 10^{-5} a \cdot t
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to } h_1 &= V - (s - 2t)^3 = 0 \\
h_2 &= a - 6s^2 = 0 \\
g_1 &= 34 \times 10^6 - nV \leq 0
\end{aligned}
$$

Var. $\underline{a}, \underline{t}, n$

$$\frac{\partial f}{\partial n} = 21 + 17 \times 10^{-4} a/(t+1.2) > 0 \quad \forall a, t$$

f is monotonically increasing with $\underline{n}$.

$$\frac{\partial g_1}{\partial n} = -V < 0$$

g_1 is ______ decreasing with n.

g_1 is active. $34 \times 10^6 - nV = 0$

$$n^* = \left\lceil \frac{34 \times 10^6}{V} \right\rceil$$

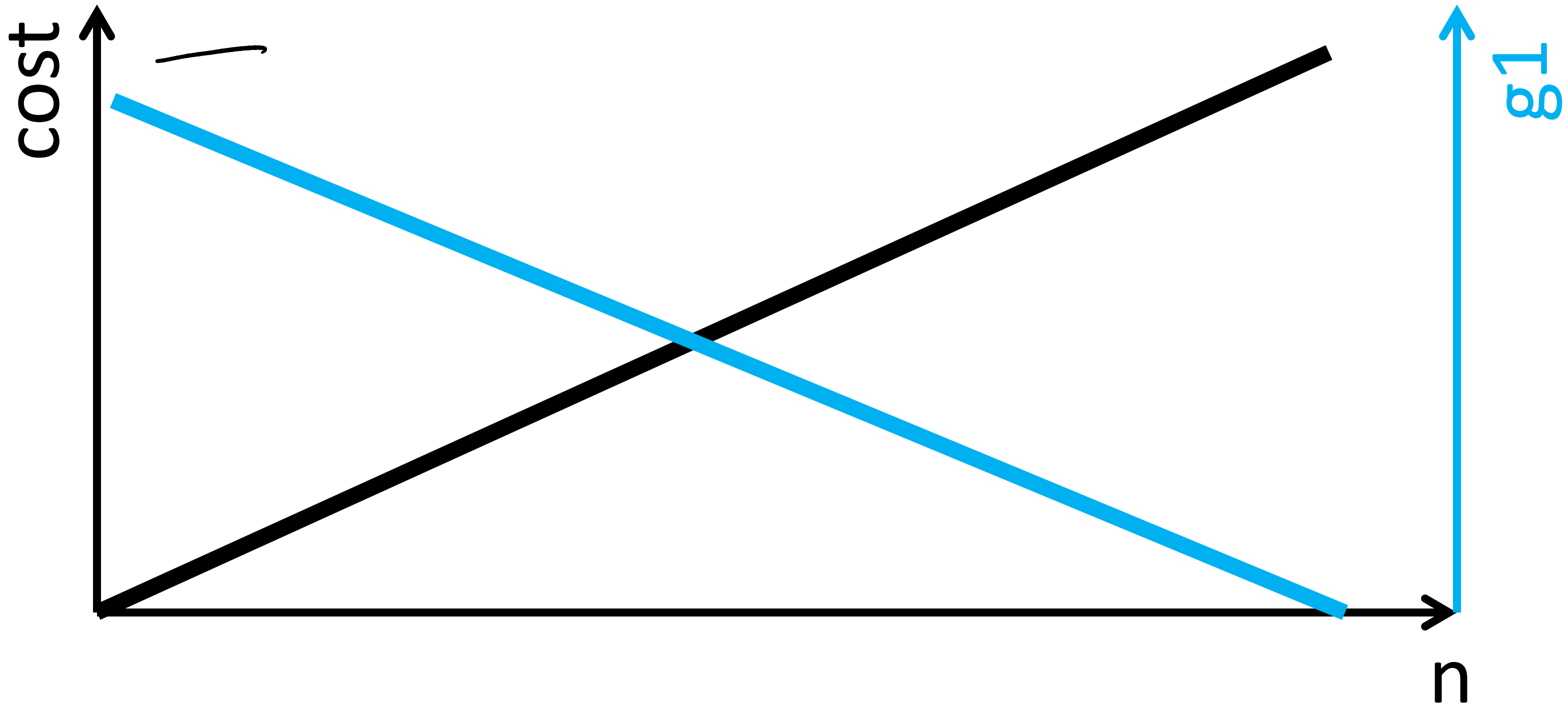
Active constraints

$$\min(\text{cost}) = (\text{Transportation/Labor}) + (\text{Material}) + (\text{Refrigeration}) + (\text{Insulation})$$
$$= \underline{21\text{n} + 16\times10^{-4}a + 17\times10^{-4}a\cdot n/(t+1.2) + 41\times10^{-5}a\cdot t}$$

subject to $h_1 = V - (s-2t)^3 = 0$

$h_2 = a - 6s^2 = 0$

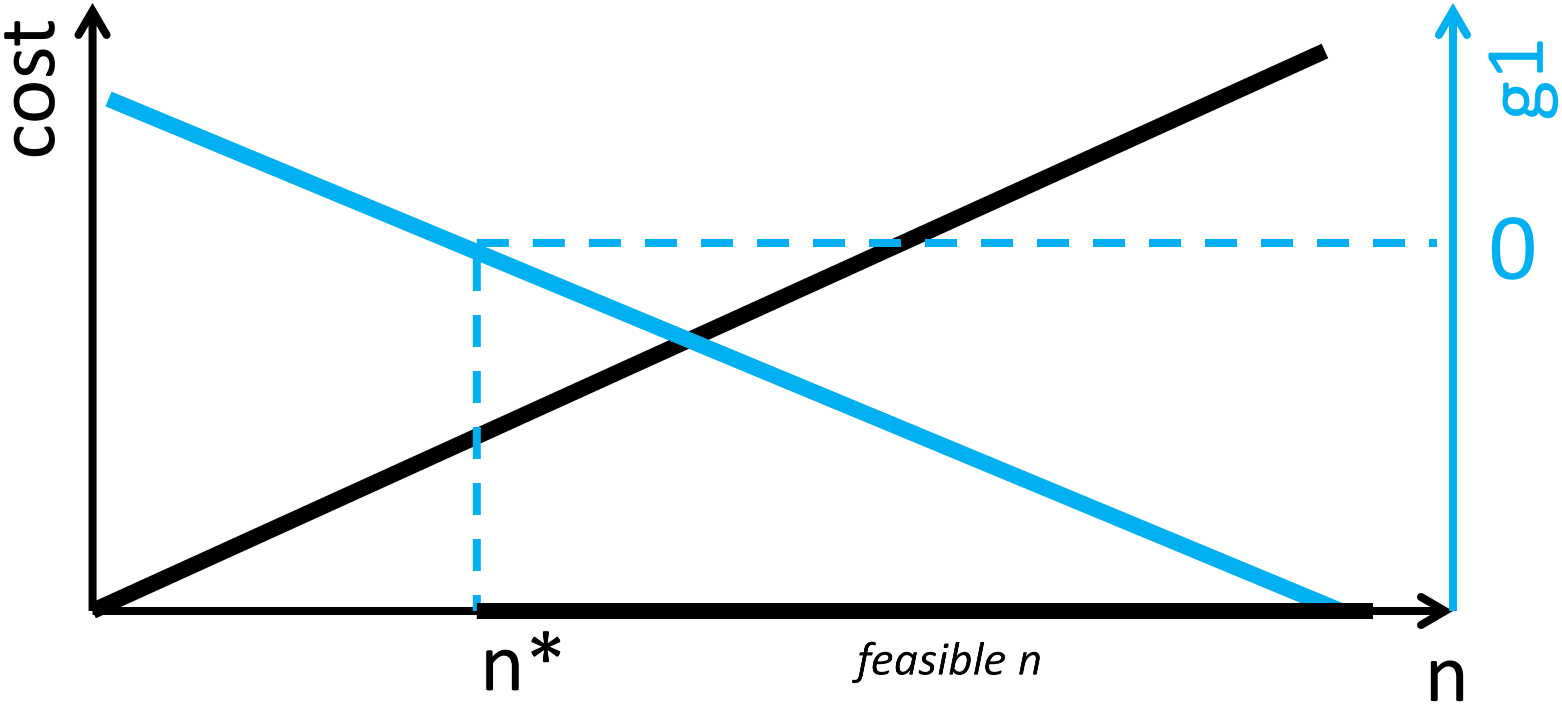
$g_1 = 34\times10^6 - nV \le 0$



Active constraints

$$\begin{aligned}\min(\text{cost}) &= (\text{Transportation/Labor}) + (\text{Material}) + (\text{Refrigeration}) + (\text{Insulation})\\ &= 21\text{n} + 16\times 10^{-4}a + 17\times 10^{-4}a\cdot n/(t+1.2) + 41\times 10^{-5}a\cdot t\end{aligned}$$

$$\begin{aligned}\text{subject to } h_1 &= V-(s-2t)^3 = 0\\ h_2 &= a-6s^2 = 0\\ g_1 &= 34\times 10^6 - nV \le 0\end{aligned}$$



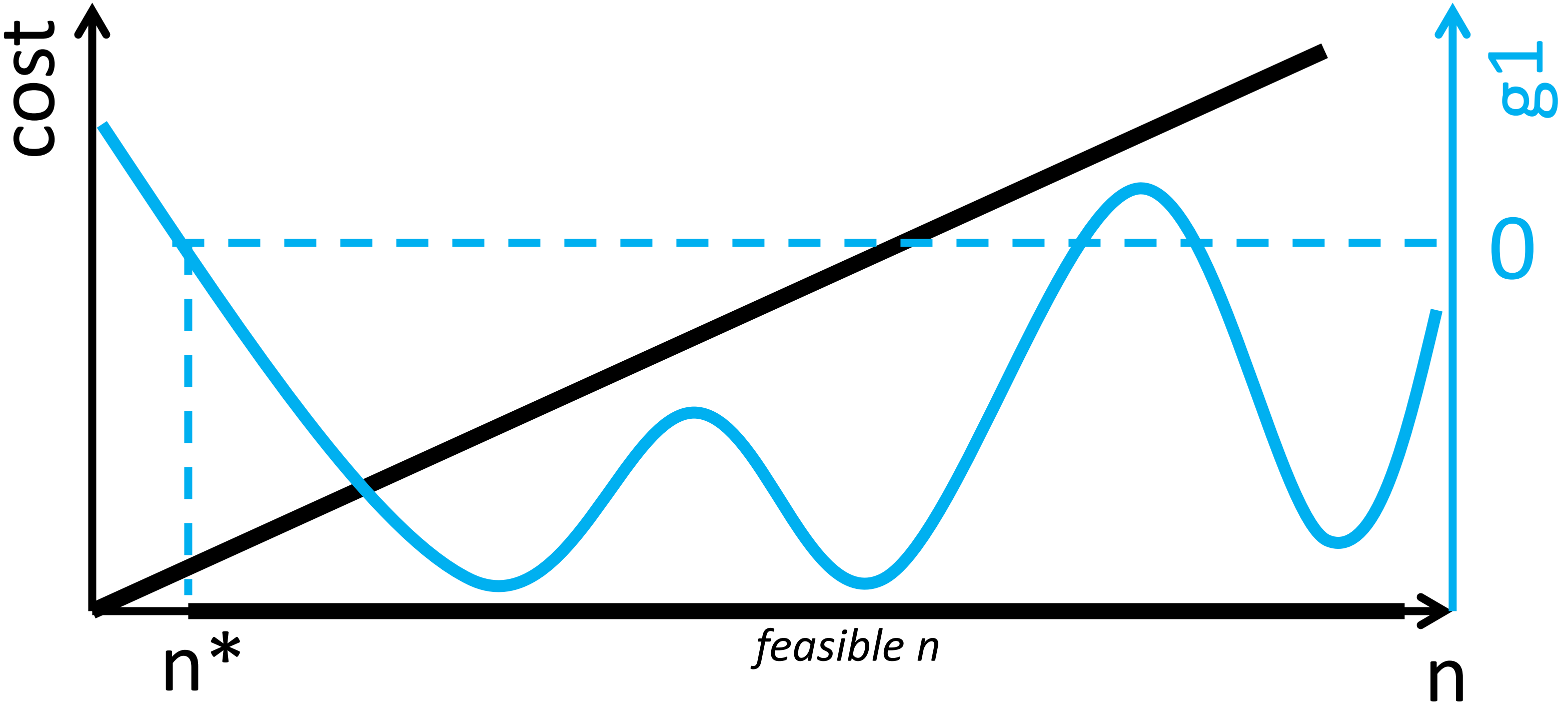
Monotonicity Principle (1)

In a well-constrained minimization problem every increasing variable is bounded below by at least one nonincreasing active constraint;

and every decreasing variable is bounded above by at least one nondecreasing active constraint.

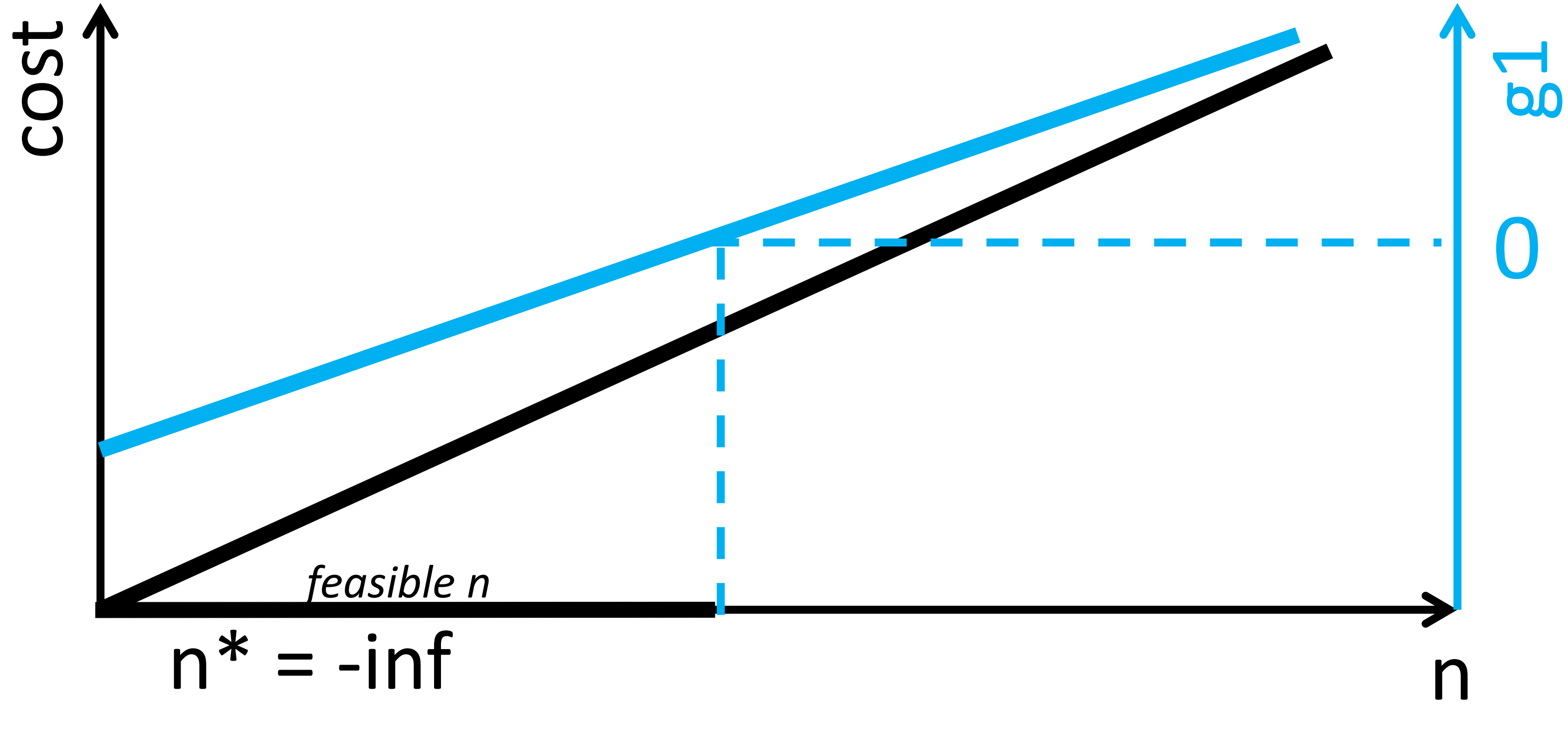
Nonincreasing vs Decreasing

Nonincreasing: Anything but increasing



Unbounded problem

When g1 is increasing, the problem is not bounded



Exercise

Use MP1 to check if the problem is bounded

$$
\begin{aligned}
\min_{x_1, x_2, x_3} \quad & x_1^{-2} + x_2^{-2} + x_3^{-2} \\
\text{subject to} \quad & 1 - x_1 - x_2 - x_3 \leq 0, \\
& x_1^2 + x_2^2 - 2 \leq 0, \\
& 2 - x_1 x_2 x_3 \leq 0, \\
& x_1,\ x_2,\ x_3 \geq 0.
\end{aligned}
$$

$$\min_{x_1,x_2,x_3} f = x_1^{-2} + x_2^{-2} + x_3^{-2}$$

subject to

$$g_1 = 1 - x_1 - x_2 - x_3 \le 0,$$

$$g_2 = x_1^2 + x_2^2 - 2 \le 0,$$

$$g_3 = 2 - x_1 x_2 x_3 \le 0,$$

$$\rightarrow \underline{x_1,\ x_2,\ x_3 \ge 0.}$$

	x_1	x_2	x_3
f	$-$	$-$	$-$
g_1	$-$	$-$	$-$
g_2	$+$	$+$	
g_3	$-$	$-$	$-$
g_4	$-$		
g_5		$-$	
g_6			$-$

$$\frac{\partial f}{\partial x_1} = -2x_1^{-3} \le 0$$

$$\frac{\partial f}{\partial x_2} \le 0, \quad \frac{\partial f}{\partial x_3} \le 0$$

$$\frac{\partial g_1}{\partial x_1} = \frac{\partial g_1}{\partial x_2} = \frac{\partial g_1}{\partial x_3} = -1 \le 0$$

$$\frac{\partial g_2}{\partial x_1} = 2x_1 \ge 0 \qquad \frac{\partial g_2}{\partial x_2} \ge 0$$

$$\frac{\partial g_3}{\partial x_1} \le 0 \qquad \frac{\partial g_3}{\partial x_2} \le 0 \qquad \frac{\partial g_3}{\partial x_3} \le 0$$

$g_4: -x_1 \le 0$ $\qquad \frac{\partial g_4}{\partial x_1} \le 0$

$g_5: -x_2 \le 0$ $\qquad \frac{\partial g_5}{\partial x_2} \le 0$

$g_6: -x_3 \le 0$ $\qquad \frac{\partial g_6}{\partial x_3} \le 0$

Exercise

3.6 Apply monotonicity to the problem

$$
\begin{aligned}
\min \quad & f(x_3^+, x_4^+, x_5^+) = x_3 x_4 + 10 x_5 \\
\text{s.t.} \quad & g_1(x_1^+, x_4^+) = x_1 x_4 - 100 \leq 0 \\
& h_1(x_2^+, x_3^-, x_4^-) = x_2 - x_3 - x_4 = 0 \\
& g_2(x_3^-, x_4^+) = -x_3 + x_4 \leq 0 \\
& h_2(x_1^-, x_4^+, x_5^-) = \frac{1}{x_1} + x_4 - x_5 = 0
\end{aligned}
$$

$$\min \quad f(x_3^+, x_4^+, x_5^+) = x_3 x_4 + 10 x_5$$

$g_3: -x_1 \le 0$, $g_4: -x_3 \le 0$, $g_5: -x_4 \le 0$.

$$\text{s.t.} \quad g_1(x_1^+, x_4^+) = x_1 x_4 - 100 \le 0$$

$$h_1(x_2^+, x_3^-, x_4^-) = x_2 - x_3 - x_4 = 0 \Rightarrow \underline{x_2} = \underline{x_3} + \underline{x_4}$$

$$g_2(x_3^-, x_4^+) = -x_3 + x_4 \le 0$$

$$h_2(x_1^-, x_4^+, x_5^-) = \frac{1}{x_1} + x_4 - x_5 = 0 \Rightarrow x_5 = \frac{1}{x_1} + x_4$$

$$\min \quad f = x_3 x_4 + \frac{10}{x_1} + 10 x_4$$

$$\underline{g_1} = x_1 x_4 - 100 \le 0$$

$$\underline{g_2} = -x_3 + x_4 \le 0$$

$$\frac{\partial f}{\partial x_1} = -\frac{10}{x_1^2} < 0$$

$$\frac{\partial f}{\partial x_3} = \underline{x_4} \qquad \frac{\partial f}{\partial x_4} = \underline{x_3 + 10}$$

$$\frac{\partial g_1}{\partial x_1} = x_4, \quad \frac{\partial g_1}{\partial x_4} = x_1. \quad \frac{\partial g_2}{\partial x_3} = -1. \quad \frac{\partial g_2}{\partial x_4} = 1$$

	x_1	x_3	x_4	
$\underline{f}$	$-$	$+$	$+$	
g_1	$+$		$+$	active
g_2		$-$	$+$	
g_3	$-$			
g_4		$-$		
g_5			$-$	active

Exercise

$$\min \quad f(x_3^+, x_4^+, x_5^+) = x_3 x_4 + 10 x_5$$

$$\begin{aligned}
\text{s.t.} \quad & g_1(x_1^+, x_4^+) = x_1 x_4 - 100 \le 0 \\
& \cancel{h_1(x_2^+, x_3^-, x_4^-) = x_2 - x_3 - x_4 = 0} \\
& g_2(x_3^-, x_4^+) = -x_3 + x_4 \le 0 \\
& h_2(x_1^-, x_4^+, x_5^-) = \frac{1}{x_1} + x_4 - x_5 = 0
\end{aligned}$$

By MP1, g2 is active

Exercise

$$\min \quad x_4^2 + \frac{10}{x_1} + 10x_4$$
$$x_1 x_4 - 100 \leq 0$$

g1 is also active by MP1

Monotonicity Principle (1)

In a ***well-constrained*** minimization problem every increasing variable is bounded below by at least one nonincreasing active constraint;

and every decreasing variable is bounded above by at least one nondecreasing active constraint.

min ~~max~~ : $f = -x_1 + 2x_2$

s.t. $g_1 = x_1 + x_2 - 5 \le 0$

$g_2 = -6x_1 - x_2 \le 0$

	x_1	x_2	
f	$-$	$+$	
g_1	$+$	$+$	active
g_2	$-$	$-$	

$x_1 + x_2 - 5 = 0$ $\quad x_1 = -x_2 + 5$

min : $f = -5 + 3x_2$

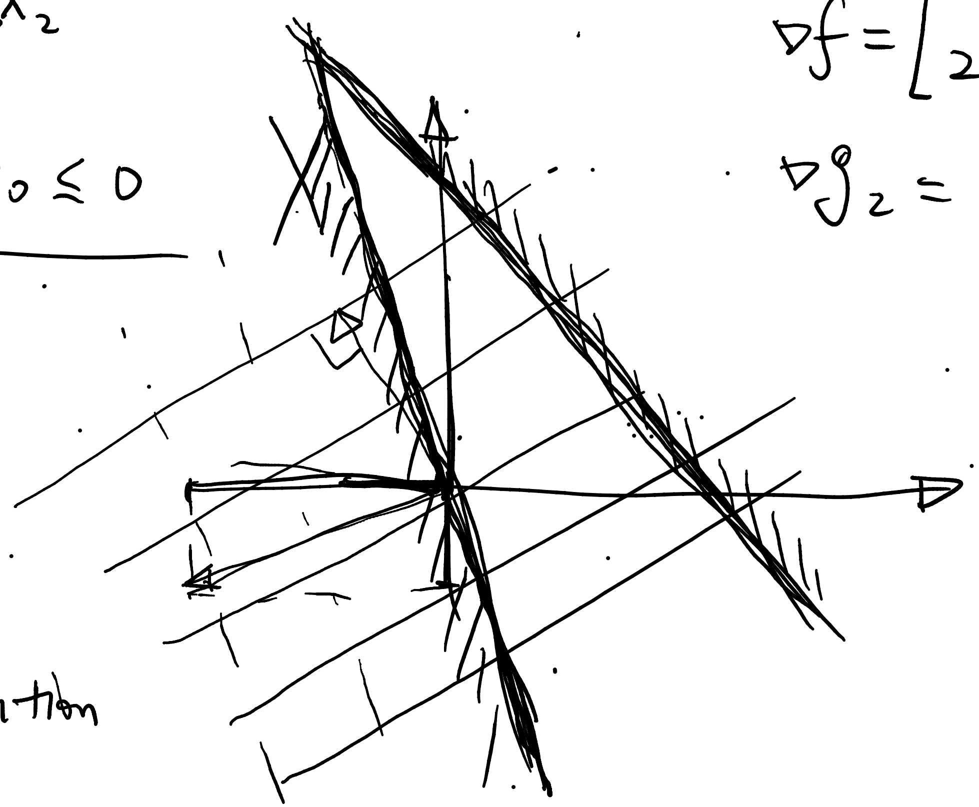
s.t. $g_2 = 5x_2 - 30 \le 0$

$\nabla f = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\nabla g_2 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$

	x_2
f	$+$
g_2	$+$

no solution



Monotonicity Principle 2

In a well-constrained minimization problem every nonobjective variable is either (1) determined by other variables, or (2) bounded both below and above.

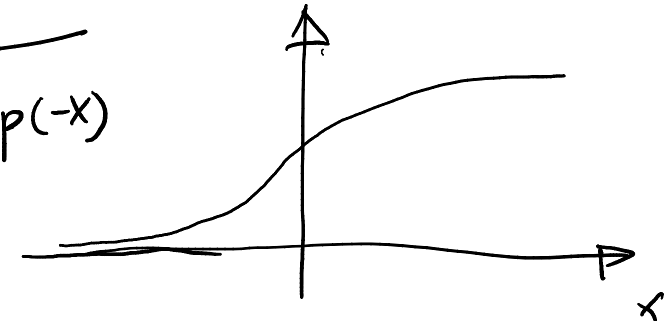
$\min f := -x_1$

s.t. $g_1 := \frac{1}{1+\exp(-x_1)} - x_2 \le 0$

$g_2 := \frac{1}{1+\exp(-x_2)} - x_3 \le 0$

$g_3 := x_3 - 0.1 \le 0$

$\frac{1}{1+\exp(-x)}$



$$\frac{d \frac{1}{1+\exp(-x)}}{dx} = -\frac{\exp(-x)(-1)}{(1+\exp(-x))^2}$$

$$= \frac{\exp(-x)}{(1+\exp(-x))^2} > 0$$

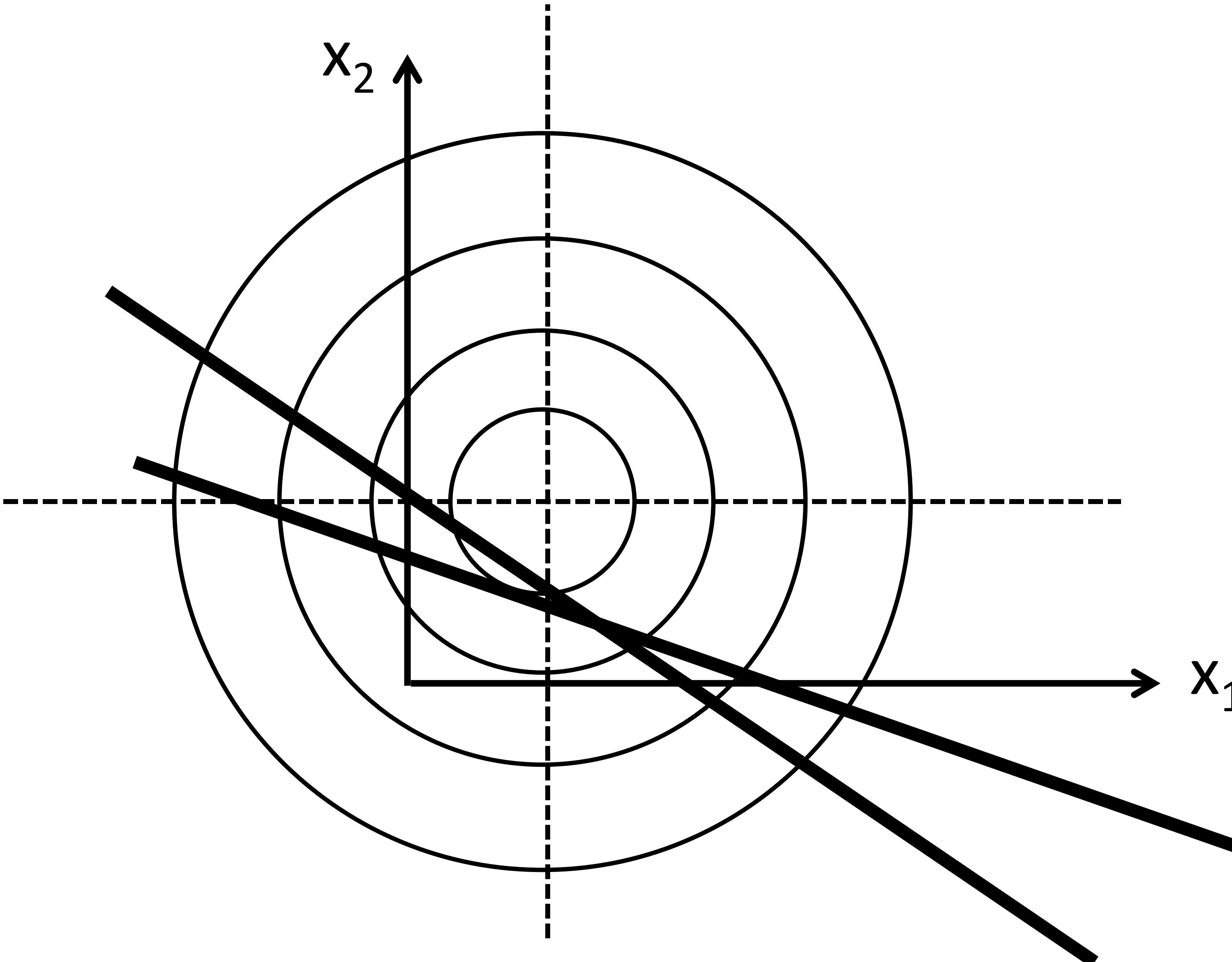
	x_1	x_2	x_3	
f	−			
g_1	+	−		active.
g_2		+	−	
g_3			+	

Exercise

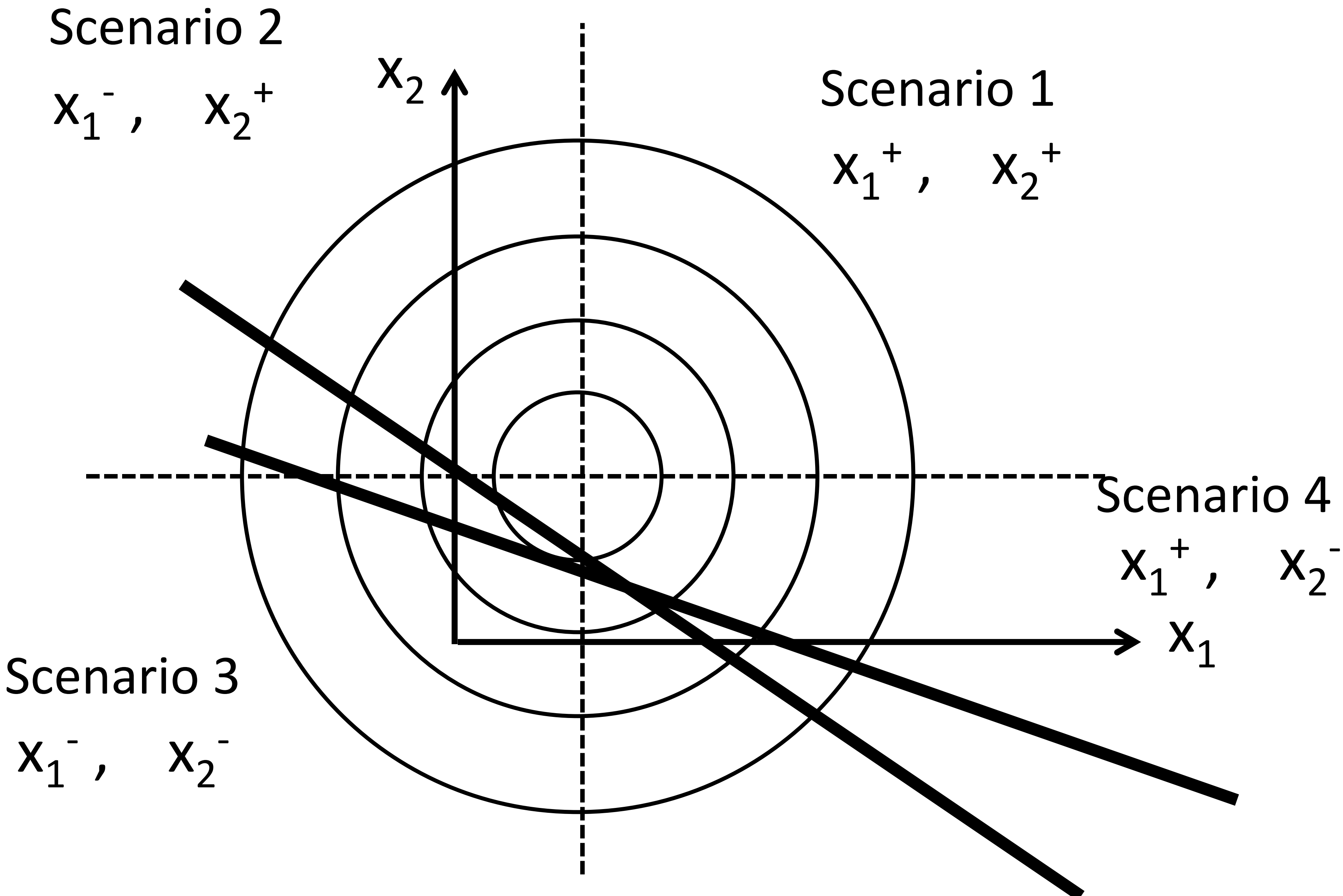
3.14 Apply monotonicity to the problem

$$
\begin{aligned}
\min_{x_1, x_2} \quad & x_1^2 + x_2^2 - 2x_1 - 4x_2 \\
\text{subject to} \quad & x_1 + 4x_2 - 5 \leq 0, \\
& 2x_1 + 3x_2 - 6 \leq 0, \\
& x_1,\ x_2 \geq 0.
\end{aligned}
$$

Exercise



Exercise


Scenario 2
x_1^- , x_2^+
x_2
Scenario 1
x_1^+ , x_2^+
Scenario 4
x_1^+ , x_2^-
x_1
Scenario 3
x_1^- , x_2^-

Scenario 3

$$
\begin{aligned}
\min_{x_1, x_2} \quad & x_1^2 + x_2^2 - 2x_1 - 4x_2 \\
\text{subject to} \quad & x_1 + 4x_2 - 5 \leq 0, \\
& 2x_1 + 3x_2 - 6 \leq 0, \\
& x_1 - 1 \leq 0, \\
& -x_1 \leq 0, \\
& x_2 - 2 \leq 0, \\
& -x_2 \leq 0.
\end{aligned}
$$

	x_1	x_2
f	-	-
g1	+	+
g2	+	+
g3	+	
g4	-	
g5		+
g6		-

g2 dominated by g1, so either g3 and g5 are both active, or g1 must be active.

g3 & g5 active → Infeasible;

g1 active → x_1 = 13/17, x_2 = 18/17, feasible

Scenario 4

$$\min_{x_1, x_2} \quad x_1^2 + x_2^2 - 2x_1 - 4x_2$$
$$\text{subject to} \quad x_1 + 4x_2 - 5 \leq 0,$$
$$2x_1 + 3x_2 - 6 \leq 0,$$
$$-x_1 + 1 \leq 0,$$
$$x_2 - 2 \leq 0,$$
$$-x_2 \leq 0.$$

	x_1	x_2
f	+	-
g1	+	+
g2	+	+
g3	-	
g5		+
g6		-

g3 active → $x_1 = 1$, $x_2 = 1$, feasible