ME 598/494 Exam 2 Solution - Nov. 9, 2017

Problem 1 (30 Points)

Check if the following statements are true or not. Explain concisely.

(a) The Lagrangian multipliers for inequality constraints can be negative at the optimal solution.

(b) Line search in GRG is the same as that in the gradient descent or Newton's method.

(c) A KKT point is a solution to a minimization problem, **and** a minimal solution must satisfy the KKT conditions.

(d) GRG can be used when every step of the search is required to be constrained in the feasible domain.

(e) Fitting a nonlinear model to data can always be done through Newton-Ralphson.

(f) Scaling the variables will affect the Lagrangian multipliers at the optimal solution.

Solutions

(a) False. Consider the KKT conditions with inequality constraints: $\nabla_x f + \mu^T \nabla_x g = 0$. Multiply both sides on the right by ∂x to have $\nabla_x f \partial x + \mu^T \nabla_x g \partial x = 0$. At the solution, we have $\nabla_x f \partial x \geq 0$ and $\nabla_x g \partial x \leq 0$. Therefore, if there exists a negative element of μ, say μ_1, then we can pick ∂x such that $\nabla_x g_1 \partial x < 0$ and $\nabla_x g_i \partial x = 0$ for $i \neq 1$ (this is doable since the number of active inequality constraints is smaller than the number of variables). Thus we have $\nabla_x f \partial x + \mu^T \nabla_x g \partial x > 0$. Proved by contradiction. **Note**: No worry. You only need to show that $\nabla_x f \partial x \geq 0$ and $\nabla_x g \partial x \leq 0$.

(b) False. There is a minor difference in that you will need to update state variables along with the decision variables.

(c) False. KKT conditions lead to an optimal solution if the second-order sufficient condition is satisfied. An optimal solution is not necessarily a KKT point when it is irregular.

(d) True. GRG ensures that the solution satisfies equality constraints at every iteration.

(e) False. Newton-Ralphson may diverge.

(f) False. Consider the KKT conditions $\nabla_x f + \mu^T \nabla_x g = 0$. Let $y = kx$. Thus $\nabla_y f = \nabla_x f (dx/dy) = \nabla_x f / k$. Similarly, $\nabla_y g = \nabla_x g / k$. The scaling does not affect the solution.

Problem 2 (30 Points)

Consider a product with price x, market demand $s(x) = 1 - x^2$, and total cost $c(x) = 0.5s(x)$ linearly increasing with the demand.

(a) Formulate an optimization problem for maximizing the profit with respect to the price, with a cost limit of $c_{max} = 9/50$. (5 Points)

(b) Derive the KKT conditions, and show that $x = 4/5$, $\mu = 3/20$ is an optimal solution. (15 Points)

(c) How would increasing the cost limit (from 9/50) affect the optimal profit? (5 Points)

(d) Let c^* be a value such that any cost limit $c_{max} > c^*$ will not change the optimal solution. Discuss how the smallest c^* can be calculated. (5 Points, Optional for MAE494)

Solution

(a) Let the profit be $x(1 - x^2) - 0.5(1 - x^2)$, the problem is:

$$\begin{aligned} \min_{x} \quad & -x(1-x^2) + 0.5(1-x^2) \\ \text{subject to} \quad & 0.5(1-x^2) \leq 9/50 \end{aligned}$$

(b) The KKT conditions are

$$\begin{aligned} -1 + 3x^2 - x - \mu x &= 0 \\ 0.5(1-x^2) < 9/50 \quad & \text{if} \quad \mu = 0 \\ 0.5(1-x^2) = 9/50 \quad & \text{if} \quad \mu > 0 \end{aligned}$$

The solution is $x = 4/5$, $\mu = 3/20$. The Hessian of the Lagrangian is $6x - 1 - \mu > 0$. So the solution is a minimizer (or maximizes the profit). **Note**: If you get $\mu = 3/40$, it is probably because you scaled the inequality constraint by 2. You will get full mark for doing so.

(c) Since $\mu = 3/20$, the profit will be increased by approximately $3/20\Delta c$ when the limit is increased by Δc.

(d) c^* is the limit at which point μ reaches zero. Given that the inequality constraint is active, i.e., $x = \sqrt{1 - 2c^*}$, c^* is the (positive) root of $3x^2 - 1 - x = 0$.

Problem 3 (25 Points)

For the linearly constrained problem (where $\mathbf{A}$ and $\mathbf{b}$ are a matrix and a column vector of parameters, respectively):

$$\min \quad f(x), \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

(a) Derive the reduced gradient. (10 Points)

(b) **Concisely** state all GRG steps for solving **this** problem. (10 Points)

(c) Explain what simplifications occur in GRG due to the linearity of constraints. (5 Points, Optional for MAE494)

Solution

See solutions from previous exams.

Problem 4 (15 Points)

Consider the following problem, with constant m-by-n matrix $\mathbf{A}$ ($m < n$), constant m-by-1 vector $\mathbf{b}$:

$$\min_{\mathbf{x}} \quad \mathbf{x}^T\mathbf{x}, \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

(a) Is this problem convex? Why? (5 Points)

(b) Derive the KKT conditions and find the optimal solution $\mathbf{x}^*$. (10 Points, Optional for MAE494)

Solution

(a) The Hessian of the objective is $2\mathbf{I}$. The linear equality constraints form a convex set since hyperplanes are convex sets, and the intersection of convex sets is convex. Therefore, the problem is convex.

(b) The KKT conditions are $2\mathbf{x} + \mathbf{A}^T\boldsymbol{\lambda} = \mathbf{0}$, $\mathbf{A}\mathbf{x} = \mathbf{b}$, and $\boldsymbol{\lambda} \neq \mathbf{0}$. From the first equality, we have $\mathbf{x} = -0.5\mathbf{A}^T\boldsymbol{\lambda}$. Plug this into the second equality to get $-0.5\mathbf{A}\mathbf{A}^T\boldsymbol{\lambda} = \mathbf{b}$, which leads to $\boldsymbol{\lambda}^* = -2(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{b}$. Then we have $\mathbf{x}^* = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{b}$. Since the problem is convex, this is the solution. Note that we assume $\mathbf{A}$ to have full row-rank. If this is not true, then either $\mathbf{A}\mathbf{x} = \mathbf{b}$ has no solution at all, or there exists a redundant row (or rows) of $[\mathbf{A}|\mathbf{b}]$ that can be removed.

Problem 5 (extra 10 points)

(Principal Component Analysis) Consider the following problem where $\mathbf{A}$ is a symmetric and positive semidefinite matrix:

$$\begin{aligned} \max_{\mathbf{x}} \quad & f = \mathbf{x}^T \mathbf{A} \mathbf{x} \\ \text{subject to} \quad & h = \mathbf{x}^T \mathbf{x} = 1 \end{aligned}$$

Derive the optimal solution. What can you tell about the solution $\mathbf{x}^*$ and λ^*?

Solution

The KKT conditions are:

$$\begin{aligned} \mathbf{A}\mathbf{x} + \lambda \mathbf{x} &= 0 \\ \lambda &\neq 0 \end{aligned}$$

By definition, $\mathbf{x}$ and $-\lambda$ can be any of the eigenvector and eigenvalue pairs of the matrix $\mathbf{A}$. Note that $\mathbf{x}^T \mathbf{A} \mathbf{x} = -\lambda$ at the optimal solution. Therefore, to maximize $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is equivalent to find the maximal eigenvalue of $\mathbf{A}$. The optimal solution for $\mathbf{x}$ is thus the eigenvector corresponding to that maximal eigenvalue.