C|W (CS 2203) (Theory Assignment):- 12/04/23

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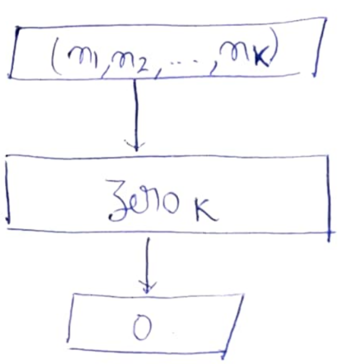
1) (a) Ans → In computability theory, a primitive recursive function takes a fixed number of arguments, each a natural number (non-negative integer) and returns a natural number; if it takes n arguments then it is n-ary primitive recursive function.

The basic primitive recursive function includes:-

(i) Zero Function:-

A K-ary zero function is defined as

$zero_K(n1, n2, \dots, nK) = 0; \; K \geq 0; \; n_1, n_2, \dots, n_K \in N$

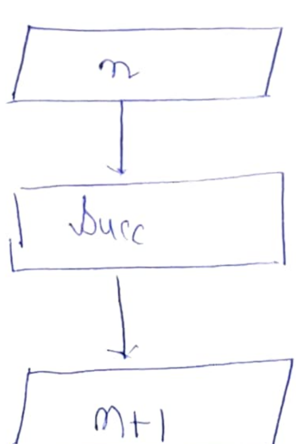


$zero_K : N^K \longrightarrow N$

Eg :- $zero_2(4,5) = 0$

(ii) Successor Function:-

Successor function is defined as $succ(n) = n+1$, $n \in N$.



$succ : N \longrightarrow N$

Eg :- $succ(2) = 3$

(iii) Identity / Projection Function:-

A j^{th} K-ary projection function is defined as,-

$proj_{K,j}(n_1, n_2, \ldots, n_K) = n_j \;\; \forall \; K \geq j > 0; \; n_1, n_2, \ldots, n_K \in N.$

$$\boxed{(n_1, n_2, \ldots, n_K)} \downarrow \boxed{proj_{K,j}} \downarrow \boxed{n_j}$$

$proj_{K,j} : N^K \longrightarrow N$

Eg:-

$proj_{2,8}(9, 72) = 8$

The above three constitute the basic primitive recursive function.

(b) Ans → The rule of composition states that if g is a k-ary function ($g : N^K \longrightarrow N$) and $h_1, h_2, \ldots, h_K$ be I-ary functions ($\forall_{i=1}^{K} h_i : N^I \rightarrow N$), $K \geq 0$, $I \geq 0$,-

Then composition of g with $h_1, h_2, \ldots, h_K$ will be an I-ary function defined as,-

$f : N^I \longrightarrow N$,-

$$f(n_1, n_2, \ldots, n_I) = g(h_1(n_1, n_2, \ldots, n_I), h_2(n_1, n_2, \ldots, n_I), \ldots, h_K(n_1, n_2, \ldots, n_I)).$$

Eg:-

Let $g : N \rightarrow N$,- $g(x) = x + 1$ (g is 1-ary fn.)

$h : N \rightarrow N$,- $h(x) = x + 1$ (h is 1-ary fn.)

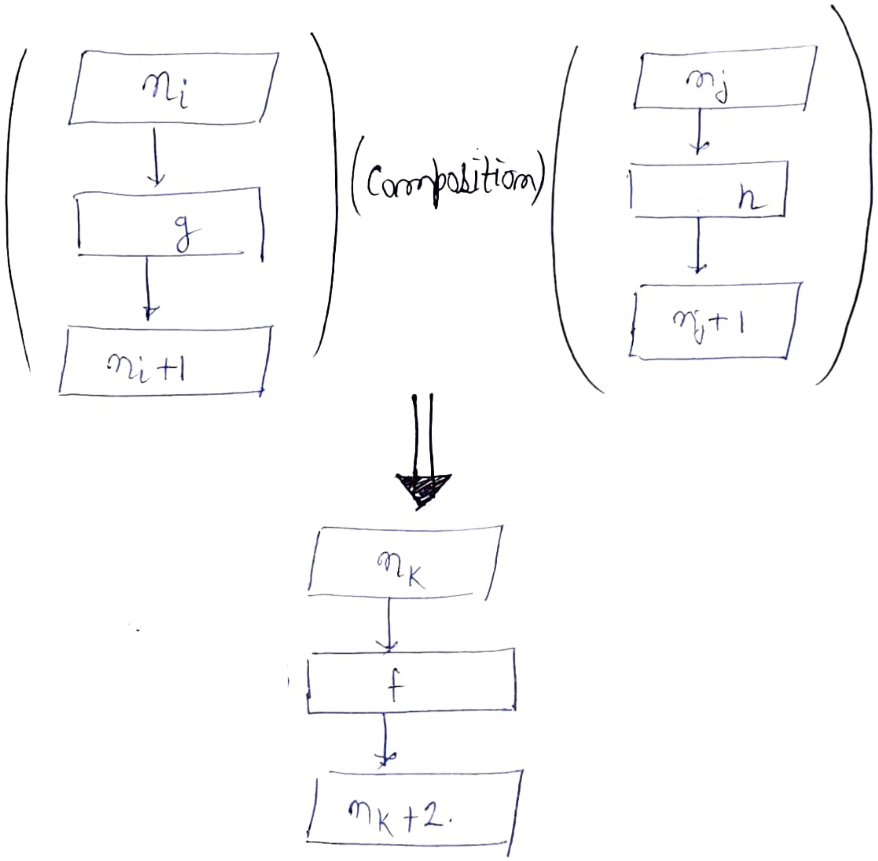
Then by rule of composition,–

$f : N \longrightarrow N$,–

$f(x) = g(h(x)) = h(x)+1$

$= x+2.$

$\because$ Both g and h are basic functions (successor),– by rule of composition f will also be primitive recursive function.



; $n_i, n_j \in N$

Rule of primitive recursion states that if,–

g is a K-ary function ($g: N^K \longrightarrow N$)

h is a $(K+2)$-ary function ($h: N^{K+2} \longrightarrow N$)

Then,–

Function f is a $(K+1)$-ary function ($f: N^{K+1} \longrightarrow N$),– defined recursively by g and h as;–

$$f(n_1, n_2, \ldots, n_K, 0) = g(n_1, n_2, \ldots, n_K)$$

$$f(n_1, n_2, \ldots, n_K, m+1) = h(n_1, n_2, \ldots, n_K, m, f(n_1, n_2, \ldots, m))$$

Eg :-

$plus(m, 0) = m \quad [f(n, 0) = g(n)]$

$plus(m, n+1) = succ(plus(m, n)) \quad [f(n, m+1) = h(n, m, f(n, m))]$

(c) <u>Exponent</u> :-

Exp (M, N) is defined as,-

Exp: $N \to N$, - $\quad Exp(M, N) = M^N$.

We will prove exponent as primitive recursive function by using rule of primitive recursion.

<u>Base Case</u> :-

zero fn. & Successor fn. are primitive recursive fn.

so,- by rule of composition,-

succ (zero (x)) will also be primitive recursive.

$Exp(M, 0) = succ(zero(M)) = succ(0) = 1.$

$\therefore Exp(M, 0)$ is primitive recursive $\forall M \in N$.

Now, –

$$\begin{aligned} Exp(M, N+1) &= M^{N+1} \\ &= M^N . M \\ &= Exp(M, N). M. \\ &= Mult(Exp(M, N), M). \\ &= id_{3,3}(M, N, Mult(Exp(M, N), M)). \end{aligned}$$

Since, –

Add (M, N) and Mult (M, N) are primitive recursive function, –

By, – rule of primitive recursion, –

Exp (M, N+1) is primitive recursive function.

<u>Non - Negative Subtraction</u> :-

NNSub (M, N) is defined as; –

NNSub : $N \rightarrow N$; NNSub (M, N) = M – N

We will prove non - negative subtraction as primitive recursive function using primitive recursion.

<u>Base Case</u> :-

$$NNSub(M, 0) = M = proj_{1,M}(M)$$

Since $proj_{1,M}$ is basic function, –

NNSub (M, 0) is primitive recursive function.

Now, –

Let, – $Pred(0) = 0$;

$Pred(N+1) = N$. ; $Pred : N \to N$ is a primitive recursive fn.

$\therefore NNsub(M, N+1) = M - (N+1)$

$= M - N - 1$

$= NNsub(M, N) - 1$

$= Pred(NNsub(M, N))$

$= id_{3,3}(M, N, Pred(NNsub(M, N)))$, –

$\therefore$ By principle of primitive recursion, –

$NNsub(M, N)$ is primitive recursive function.

(d) Ans → μ - Recursive functions consists of all basic functions alongwith functions obtained by successive application of composition, recursion and minimization of minimizible function.

Minimization of a (K+1)-ary function 'g' into a (k)-ary function is defined as :-

$$f(n_1, n_2, \ldots, n_K) = \begin{cases} \text{(i) min. value of } n \text{ such that, } g(n_1, n_2, \ldots, n_{K+1}) = 1 \text{ if } n \text{ exists,} \\ \text{(ii) } 0 \text{ otherwise.} \end{cases}$$

Eg :-

Let, -

$$g(a, b, p) = 1, \text{ if } a^p \geq b$$
$$= 0, \text{ otherwise.}$$

; - ∴ g is primitive recursive function

∴ Minimization of g, -

$$f(a, b) = \begin{cases} \text{Min. value of } p, \text{ - such that } g(a, b, p) = 1, \\ 0, \text{ otherwise.} \end{cases}$$

∴ f being a minimization of g, - f is primitive recursive function, - implying that f is μ - recursive function.

Q2) Ans → The class diagram for Assignment 04, - Problem No. 2 (College Program) is :-