Moment (physics)

From Wikipedia, the free encyclopedia

In physics, a **moment** is an expression involving the product of a distance and physical quantity, and in this way it accounts for how the physical quantity is located or arranged.

Moments are usually defined with respect to a fixed reference point; they deal with physical quantities located at some distance relative to that reference point. For example, the moment of force, often called torque, is the product of a force on an object and the distance from the reference point to the object. In principle, any physical quantity can be multiplied by a distance to produce a moment. Commonly used quantities include forces, masses, and electric charge distributions.

Examples [edit]

- The *moment of force*, or *torque*, is a first moment: $\tau = rF$, or, more generally, $\mathbf{r} \times \mathbf{F}$.
- Similarly, *angular momentum* is the 1st moment of momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Note that *momentum* itself is *not* a moment.
- The *electric dipole moment* is also a 1st moment: $\mathbf{p} = q\,\mathbf{d}$ for two opposite point charges or $\int \mathbf{r}\,\rho(\mathbf{r})\,d^3r$ for a distributed charge with charge density $\rho(\mathbf{r})$

Moments of mass:

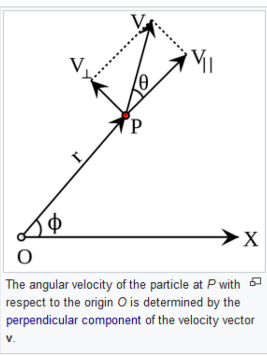
- The *total mass* is the zeroth moment of mass
- The *center of mass* is the 1st moment of mass normalized by total mass: $\mathbf{R} = \frac{1}{M}\sum_i \mathbf{r}_i m_i$ for a collection of point masses, or $\frac{1}{M}\int \mathbf{r}\rho(\mathbf{r})\,d^3r$ for an object with mass distribution $\rho(\mathbf{r})$
- The *moment of inertia* is the 2nd moment of mass: $I = r^2 m$ for a point mass, $\sum_i r_i^2 m_i$ for a collection of point masses, or $\int r^2\rho(\mathbf{r})\,d^3r$ for an object with mass distribution $\rho(\mathbf{r})$. Note that the center of mass is often (but not always) taken as the reference point.

Angular velocity

From Wikipedia, the free encyclopedia

See also: Angular frequency

In physics, **angular velocity** refers to how fast an object rotates or revolves relative to another point, i.e. how fast the angular position or orientation of an object changes with time. There are two types of angular



The angular velocity of the particle at *P* with respect to the origin *O* is determined by the perpendicular component of the velocity vector **v**.

The angular velocity ω is the rate of change of angular position with respect to time, which can be computed from the cross-radial velocity as:

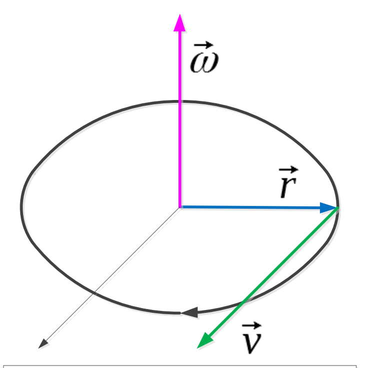
$$\boxed{\omega = \frac{d\phi}{dt} = \frac{v_\perp}{r}}.$$

Here the cross-radial speed $v_\perp$ is the signed magnitude of $\mathbf{v}_\perp$, positive for counter-clockwise motion, negative for clockwise. Taking polar coordinates for the linear velocity $\mathbf{v}$ gives magnitude v (linear speed) and angle θ relative to the radius vector; in these terms, $v_\perp = v\sin(\theta)$, so that

$$\omega = \frac{v\sin(\theta)}{r}.$$

Angular Velocity in 3D

In three dimensions, angular velocity is a pseudovector, with its magnitude measuring the rate at which an object rotates or revolves, and its direction pointing perpendicular to the instantaneous plane of rotation or angular displacement. The orientation of angular velocity is conventionally specified by the right-hand rule.[1]



The orbital angular velocity vector encodes the time rate of change of angular position, as well as the instantaneous plane of angular displacement. In this case (counter-clockwise circular motion) the vector points up.

Let the pseudovector $\mathbf{u}$ be the unit vector perpendicular to the plane spanned by **r** and **v**, so that the right-hand rule is satisfied (i.e. the instantaneous direction of angular displacement is counter-clockwise looking from the top of $\mathbf{u}$). Taking polar coordinates (r, ϕ) in this plane, as in the two-dimensional case above, one may define the orbital angular velocity vector as:

$$\boldsymbol{\omega} = \omega\mathbf{u} = \frac{d\phi}{dt}\mathbf{u} = \frac{v\sin(\theta)}{r}\mathbf{u},$$

where θ is the angle between **r** and **v**. In terms of the cross product, this is:

$$\boldsymbol{\omega} = \frac{\mathbf{r}\times\mathbf{v}}{r^2}.$$

From the above equation, one can recover the tangential velocity as:

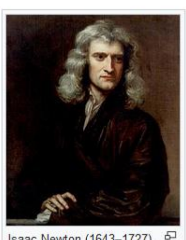
$$\mathbf{v}_\perp = \boldsymbol{\omega}\times\mathbf{r}$$

Momentum

In Newtonian mechanics, **linear momentum**, **translational momentum**, or simply **momentum** (pl. momenta) is the product of the mass and velocity of an object. It is a vector quantity, possessing a magnitude and a direction. If m is an object's mass and $\mathbf{v}$ is its velocity (also a vector quantity), then the object's momentum is:

$$\mathbf{p} = m\mathbf{v}.$$

In SI units, momentum is measured in kilogram meters per second (kg⋅m/s).



Isaac Newton (1643–1727), the physicist who formulated the laws

Newton's second law

The second law states that the rate of change of momentum of a body is directly proportional to the force applied, and this change in momentum takes place in the direction of the applied force.

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}.$$

The second law can also be stated in terms of an object's acceleration. Since Newton's second law is valid only for constant-mass systems,[20][21][22] m can be taken outside the differentiation operator by the constant factor rule in differentiation. Thus,

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a},$$

where $\mathbf{F}$ is the net force applied, m is the mass of the body, and **a** is the body's acceleration. Thus, the net force applied to a body produces a proportional acceleration. In other words, if a body is accelerating, then there is a force on it.

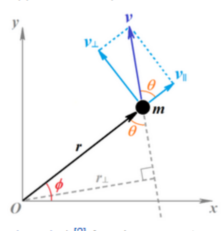
Angular momentum

From Wikipedia, the free encyclopedia

In physics, **angular momentum** (rarely, **moment of momentum** or **rotational momentum**) is the rotational equivalent of linear momentum. It is an important quantity in physics because it is a conserved quantity—the total angular momentum of a closed system remains constant.

In three dimensions, the angular momentum for a point particle is a pseudovector **r** × **p**, the cross product of the particle's position vector **r** (relative to some origin) and its momentum vector; the latter is **p** = *m***v** in Newtonian mechanics. This definition can be applied to each point in

Common symbols	*L*
In SI base units	kg m² s⁻¹
Conserved?	yes
Derivations from other quantities	**L** = **I**ω = **r** × **p**
Dimension	M L²T⁻¹



Velocity of the particle *m* with respect to the origin *O* can be resolved into components parallel to ($v_\parallel$) and perpendicular to ($v_\perp$) the radius vector *r*. The **angular momentum** of *m* is proportional to the perpendicular component $v_\perp$ of the velocity, or equivalently, to the perpendicular distance $r_\perp$ from the origin.

and treat it as a scalar (more precisely, a pseudoscalar).[2] Angular momentum can be considered a rotational analog of linear momentum. Thus, where linear momentum p is proportional to mass m and linear speed v,

$$p = mv,$$

angular momentum L is proportional to moment of inertia I and angular speed ω measured in radians per second.[3]

$$L = I\omega.$$

Because $I = r^2m$ for a single particle and $\omega = \frac{v}{r}$ for circular motion, angular momentum can be expanded, $L = r^2m\cdot\frac{v}{r}$, and reduced to,

$$\boxed{L = rmv,}$$

the product of the radius of rotation r and the linear momentum of the particle $p = mv$, where v in this case is the equivalent linear (tangential) speed at the radius ($= r\omega$).

Differentiation rules

The product rule [edit]

Main article: Product rule

For the functions f and g, the derivative of the function h(x) = f(x) g(x) with respect to x is

$$h'(x) = (fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

In Leibniz's notation this is written

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Proof of the equivalence of definitions [edit]

The definition of angular momentum for a single point particle is:

$$\mathbf{L} = \mathbf{r}\times\boldsymbol{p}$$

where **p** is the particle's linear momentum and **r** is the position vector from the origin. The time-derivative of this is:

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r}\times\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\boldsymbol{p}.$$

This result can easily be proven by splitting the vectors into components and applying the product rule. Now using the definition of force $\mathbf{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$ (whether or not mass is constant) and the definition of velocity $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r}\times\mathbf{F} + \mathbf{v}\times\boldsymbol{p}.$$

The cross product of momentum $\boldsymbol{p}$ with its associated velocity **v** is zero because velocity and momentum are parallel, so the second term vanishes.

By definition torque **τ** = **r** × **F**. Therefore, torque on a particle is *equal* to the first derivative of its angular momentum with respect to time.

If multiple forces are applied, Newton's second law instead reads $\mathbf{F}_{net} = m\mathbf{a}$, and it follows that

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r}\times\mathbf{F}_{\mathrm{net}} = \boldsymbol{\tau}_{\mathrm{net}}.$$

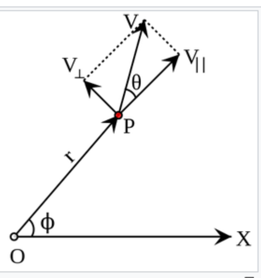
This is a general proof for point particles.

Angular acceleration

From Wikipedia, the free encyclopedia

In physics, **angular acceleration** refers to the time rate of change of angular velocity. As there are two types of angular velocity, namely spin angular velocity and orbital angular velocity, there are naturally also two types of angular acceleration, called spin angular acceleration and orbital angular acceleration respectively. Spin angular acceleration refers to the angular acceleration of a rigid body about its centre of rotation, and orbital angular acceleration refers to the angular acceleration of a point particle about a fixed origin.

Angular acceleration is measured in units of angle per unit time squared (which in SI units is radians per second squared), and is usually represented by the symbol alpha (**α**). In two dimensions, angular acceleration is a pseudoscalar whose sign is taken to be positive if the angular speed increases counterclockwise or decreases clockwise, and is taken to be negative if the angular speed increases clockwise or decreases counterclockwise. In three dimensions, angular acceleration is a pseudovector.[1]



The angular velocity of the particle at *P* with respect to the origin *O* is determined by the perpendicular component of the velocity vector **v**.

Torque

From Wikipedia, the free encyclopedia

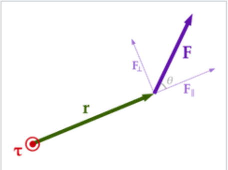
In three dimensions, the torque is a pseudovector; for point particles, it is given by the cross product of the position vector (distance vector) and the force vector. The magnitude of torque of a rigid body depends on three quantities: the force applied, the *lever arm vector*[2] connecting the point about which the torque is being measured to the point of force application, and the angle between the force and lever arm vectors. In symbols:

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$$

$$\tau = \|\mathbf{r}\|\,\|\mathbf{F}\|\sin\theta$$

where

- $\boldsymbol{\tau}$ is the torque vector and τ is the magnitude of the torque,
- r is the position vector (a vector from the point about which the torque is being measured to the point where the force is applied)
- F is the force vector,
- × denotes the cross product, which produces a vector that is perpendicular to both r and F following the right-hand rule,
- θ is the angle between the force vector and the lever arm vector.



A particle is located at position **r** relative to its axis of rotation. When a force **F** is applied to the particle, only the perpendicular component $\mathbf{F}_\perp$ produces a torque. This torque **τ** = **r** × **F** has magnitude $\tau = |\mathbf{r}|\,|\mathbf{F}_\perp| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta$ and is directed outward from the page.

The net torque on a body determines the rate of change of the body's angular momentum,

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$$

where **L** is the angular momentum vector and *t* is time.

For the motion of a point particle,

$$\mathbf{L} = I\boldsymbol{\omega},$$

where I is the moment of inertia and **ω** is the orbital angular velocity pseudovector. It follows that

$$\boldsymbol{\tau}_{\mathrm{net}} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}(I\boldsymbol{\omega})}{\mathrm{d}t} = I\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t}\boldsymbol{\omega} = I\boldsymbol{\alpha} + \frac{\mathrm{d}(mr^2)}{\mathrm{d}t}\boldsymbol{\omega} = I\boldsymbol{\alpha} + 2rp_{||}\boldsymbol{\omega},$$

where **α** is the angular acceleration of the particle, and $p_{||}$ is the radial component of its linear momentum. This equation

Differentiation rules

The product rule [edit]

Main article: Product rule

For the functions *f* and *g*, the derivative of the function *h*(*x*) = *f*(*x*) *g*(*x*) with respect to *x* is

$$h'(x) = (fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

In Leibniz's notation this is written

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}.$$

The chain rule [edit]

Main article: Chain rule

The derivative of the function $h(x) = f(g(x))$ is

$$h'(x) = f'(g(x))\cdot g'(x).$$

In Leibniz's notation, this is written as:

$$\frac{d}{dx}h(x) = \frac{d}{dz}f(z)|_{z=g(x)}\cdot\frac{d}{dx}g(x),$$

often abridged to

$$\frac{dh(x)}{dx} = \frac{df(g(x))}{dg(x)}\cdot\frac{dg(x)}{dx}.$$

$$\frac{d}{dt}\left(\frac{v_\perp}{r}\right) = \frac{d}{dt}\left(v_\perp\frac{1}{r}\right)$$

$$\frac{1}{r}\frac{d}{dt}(v_\perp) + v_\perp\frac{d}{dt}\left(\frac{1}{r}\right)$$

$$\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{dt}$$

$$\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{1}{r^2}$$

$$\frac{1}{r}\frac{d}{dt}(v_\perp) - \frac{v_\perp}{r^2}\frac{d}{dt}(r)$$

Particle in two dimensions [edit]

In two dimensions, the orbital angular acceleration is the rate at which the two-dimensional orbital angular velocity of the particle about the origin changes. The instantaneous angular velocity ω at any point in time is given by

$$\omega = \frac{v_\perp}{r},$$

where r is the distance from the origin and $v_\perp$ is the cross-radial component of the instantaneous velocity (i.e. the component perpendicular to the position vector), which by convention is positive for counter-clockwise motion and negative for clockwise motion.

Therefore, the instantaneous angular acceleration α of the particle is given by

$$\alpha = \frac{d}{dt}\left(\frac{v_\perp}{r}\right).$$ [2]

Expanding the right-hand-side using the product rule from differential calculus, this becomes

$$\alpha = \frac{1}{r}\frac{dv_\perp}{dt} - \frac{v_\perp}{r^2}\frac{dr}{dt}.$$

In the special case where the particle undergoes circular motion about the origin, $\frac{dv_\perp}{dt}$ becomes just the tangential acceleration $a_\perp$, and $\frac{dr}{dt}$ vanishes (since the distance from the origin stays constant), so the above equation simplifies to

$$\alpha = \frac{a_\perp}{r}.$$

Particle in three dimensions [edit]

In three dimensions, the orbital angular acceleration is the rate at which three-dimensional orbital angular velocity vector changes with time. The instantaneous angular velocity vector $\boldsymbol{\omega}$ at any point in time is given by

$$\boldsymbol{\omega} = \frac{\mathbf{r}\times\mathbf{v}}{r^2},$$

where $\mathbf{r}$ is the particle's position vector and $\mathbf{v}$ is its velocity vector. [2]

Therefore, the orbital angular acceleration is the vector $\boldsymbol{\alpha}$ defined by

$$\boldsymbol{\alpha} = \frac{d}{dt}\left(\frac{\mathbf{r}\times\mathbf{v}}{r^2}\right).$$

Expanding this derivative using the product rule for cross-products and the ordinary quotient rule, one gets:

$$\boldsymbol{\alpha} = \frac{1}{r^2}\left(\mathbf{r}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt}\times\mathbf{v}\right) - \frac{2}{r^3}\frac{dr}{dt}(\mathbf{r}\times\mathbf{v})$$

$$= \frac{1}{r^2}(\mathbf{r}\times\mathbf{a} + \mathbf{v}\times\mathbf{v}) - \frac{2}{r^3}\frac{dr}{dt}(\mathbf{r}\times\mathbf{v})$$

$$\boxed{= \frac{\mathbf{r}\times\mathbf{a}}{r^2} - \frac{2}{r^3}\frac{dr}{dt}(\mathbf{r}\times\mathbf{v}).}$$

$$\boldsymbol{\alpha} = \frac{\mathbf{r}\times\mathbf{a}}{r^2} - \frac{2}{r}\frac{dr}{dt}\boldsymbol{\omega},$$

Since $\mathbf{r}\times\mathbf{v}$ is just $r^2\boldsymbol{\omega}$, the second term may be rewritten as $-\frac{2}{r}\frac{dr}{dt}\boldsymbol{\omega}$. In the case where the distance r of the particle from the origin does not change with time (which includes circular motion as a subcase), the second term vanishes and the above formula simplifies to

$$\boldsymbol{\alpha} = \frac{\mathbf{r}\times\mathbf{a}}{r^2}.$$

From the above equation, one can recover the cross-radial acceleration in this special case as:

$$\mathbf{a}_\perp = \boldsymbol{\alpha}\times\mathbf{r}.$$

Relation to Torque [edit]

The net *torque* on a point particle is defined to be the pseudovector

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F},$$

where $\mathbf{F}$ is the net force on the particle.[3]

Torque is the rotational analogue of force: it induces change in the rotational state of a system, just as force induces change in the translational state of a system. Since the net force on a particle may be connected to the acceleration of the particle by the equation $\mathbf{F} = m\mathbf{a}$, one may hope to construct a similar relation connecting the net torque on a particle to the angular acceleration of the particle. That may be done as follows:

First, substituting $\mathbf{F} = m\mathbf{a}$ into the above equation for torque, one gets

$$\boldsymbol{\tau} = m(\mathbf{r}\times\mathbf{a}) = mr^2\left(\frac{\mathbf{r}\times\mathbf{a}}{r^2}\right).$$

But from the previous section, it was derived that

$$\boldsymbol{\alpha} = \frac{\mathbf{r}\times\mathbf{a}}{r^2} - \frac{2}{r}\frac{dr}{dt}\boldsymbol{\omega},$$

where $\boldsymbol{\alpha}$ is the orbital angular acceleration of the particle and $\boldsymbol{\omega}$ is the orbital angular velocity of the particle. Therefore, it follows that

$$\boldsymbol{\tau} = mr^2\left(\boldsymbol{\alpha} + \frac{2}{r}\frac{dr}{dt}\boldsymbol{\omega}\right)$$

$$= mr^2\boldsymbol{\alpha} + 2mr\frac{dr}{dt}\boldsymbol{\omega}.$$

In the special case where the distance r of the particle from the origin does not change with time, the second term in the above equation vanishes and the above equation simplifies to

$$\boldsymbol{\tau} = mr^2\boldsymbol{\alpha},$$

Moment of inertia

- The *moment of inertia* is the 2nd moment of mass: $I = r^2m$ for a point mass,

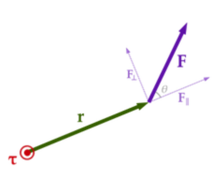
If the shape of the body does not change, then its moment of inertia appears in Newton's law of motion as the ratio of an applied torque τ on a body to the angular acceleration α around a principal axis, that is

$$\boxed{\tau = I\alpha.}$$

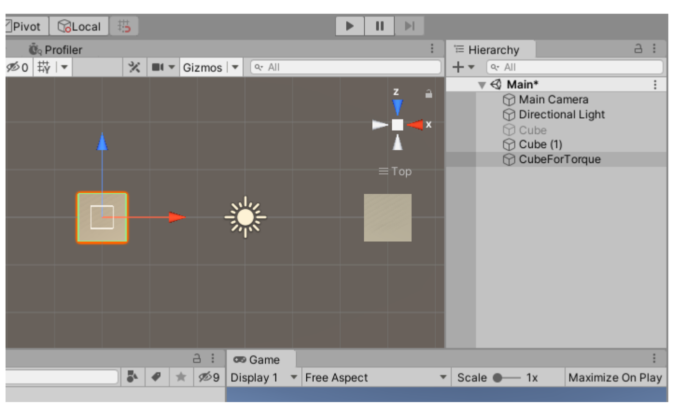
Torque

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$$

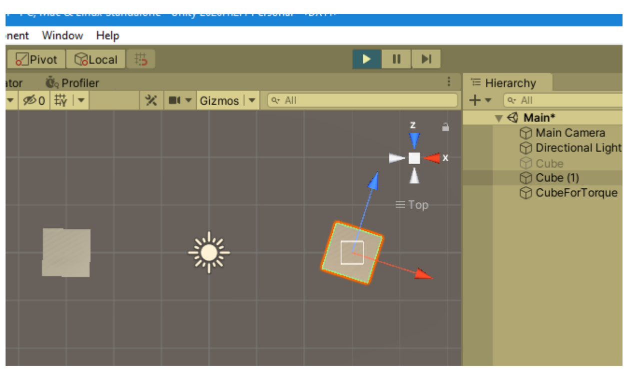
$$\tau = \|\mathbf{r}\|\,\|\mathbf{F}\|\sin\theta$$



Demo in Unity




```
// test 01
Vector3 force = new Vector3(0, 0, 1) * magnitude;
Vector3 torque = Vector3.Cross(Vector3.right, force);
_rb.AddTorque(torque);
_timer = 5.0f;
```




```
// test 03: approximate 180 degree rotation per 5 second
float theta = Mathf.PI;
Vector3 w = Vector3.down * theta * magnitude;
Vector3 torque = Vector3.Scale(_rb.inertiaTensor, w);
_rb.AddTorque(torque);
_timer = 5.0f;
```