http://farside.ph.utexas.edu/teaching/336k/Newtonhtml/node64.html

Moment of Inertia Tensor

Consider a rigid body rotating with fixed angular velocity ω about an axis which passes through the origin--see Figure 28. Let $\mathbf{r}_i$ be the position vector of the ith mass element, whose mass is m_i. We expect this position vector to *precess* about the axis of rotation (which is parallel to ω) with angular velocity ω. It, therefore, follows from Equation (A.1309) that

$$\frac{d\mathbf{r}_i}{dt} = \boldsymbol{\omega} \times \mathbf{r}_i. \qquad (457)$$

Thus, the above equation specifies the velocity, $\mathbf{v}_i = d\mathbf{r}_i/dt$, of each mass element as the body rotates with fixed angular velocity ω about an axis passing through the origin.

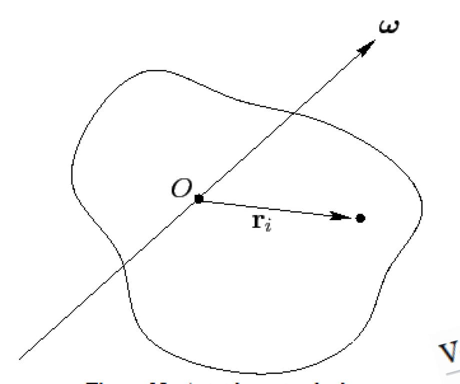


Figure 28: *A rigid rotating body.*

$\mathbf{L} = m\mathbf{r} \times \mathbf{v}$

Vector triple product [edit]

The **vector triple product** is defined as the

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

The total angular momentum of the body (about the origin) is written

$$\mathbf{L} = \sum_{i=1,N} m_i \mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} = \sum_{i=1,N} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \sum_{i=1,N} m_i \left[r_i^2 \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\, \mathbf{r}_i \right], \qquad (458)$$

where use has been made of Equation (457), and some standard vector identities (see Section A.10). The above formula can be written as a matrix equation of the form

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \qquad (459)$$

$L = I\omega$.

where

$$I_{xx} = \sum_{i=1,N} (y_i^2 + z_i^2)\, m_i = \int (y^2 + z^2)\, dm, \qquad (460)$$

$$I_{yy} = \sum_{i=1,N} (x_i^2 + z_i^2)\, m_i = \int (x^2 + z^2)\, dm, \qquad (461)$$

$$I_{zz} = \sum_{i=1,N} (x_i^2 + y_i^2)\, m_i = \int (x^2 + y^2)\, dm, \qquad (462)$$

$$I_{xy} = I_{yx} = -\sum_{i=1,N} x_i\, y_i\, m_i = -\int x\, y\, dm, \qquad (463)$$

$$I_{yz} = I_{zy} = -\sum_{i=1,N} y_i\, z_i\, m_i = -\int y\, z\, dm, \qquad (464)$$

$$I_{xz} = I_{zx} = -\sum_{i=1,N} x_i\, z_i\, m_i = -\int x\, z\, dm. \qquad (465)$$

Here, I_{xx} is called the *moment of inertia* about the x-axis, I_{yy} the moment of inertia about the y-axis, I_{xy} the xy *product of inertia*, I_{yz} the yz product of inertia, *etc*. The matrix of the I_{ij} values is known as the *moment of inertia tensor*. Note that each component of the moment of inertia tensor can be written as either a sum over separate mass elements, or as an integral over infinitesimal mass elements. In the integrals, $dm = \rho\, dV$, where ρ is the mass density, and dV a volume element. Equation (459) can be written more succinctly as

$$\mathbf{L} = \tilde{\mathbf{I}}\, \boldsymbol{\omega}. \qquad (466)$$

Derivation of the tensor components [edit]

The distance r of a particle at $\mathbf{x}$ from the axis of rotation passing through the origin in the $\hat{\mathbf{n}}$ direction is $|\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}|$, where $\hat{\mathbf{n}}$ is unit vector. The moment of inertia on the axis is

$$I = mr^2 = m(\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}})^2 = m(\mathbf{x}^2 - 2\mathbf{x}(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x} \cdot \hat{\mathbf{n}})^2 \hat{\mathbf{n}}^2) = \boxed{m(\mathbf{x}^2 - (\mathbf{x} \cdot \hat{\mathbf{n}})^2)}. \quad \text{(1)}$$

Rewrite the equation using matrix transpose:

$$\text{(2)} \quad I = \boxed{m(\mathbf{x}^T\mathbf{x} - \hat{\mathbf{n}}^T\mathbf{x}\mathbf{x}^T\hat{\mathbf{n}})} = \boxed{m \cdot \hat{\mathbf{n}}^T(\mathbf{x}^T\mathbf{x} \cdot \mathbf{E}_3 - \mathbf{x}\mathbf{x}^T)\hat{\mathbf{n}}}, \quad \text{(3)}$$

where $\mathbf{E}_3$ is the 3 × 3 identity matrix.

This leads to a tensor formula for the moment of inertia

$$I = m[n_1, n_2, n_3] \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

For multiple particles, we need only recall that the moment of inertia is additive in order to see that this formula is correct.

Example: The Inertia Tensor for a Cube

https://hepweb.ucsd.edu/ph110b/110b_notes/node26.html

We wish to compute the inertia tensor for a uniform density cube of mass M and side s. The density is simply $\rho = \frac{M}{s^3}$.

$$I_{11} = \frac{M}{s^3} \int_{-\frac{s}{2}}^{\frac{s}{2}} \int_{-\frac{s}{2}}^{\frac{s}{2}} \int_{-\frac{s}{2}}^{\frac{s}{2}} (r^2 - x^2)\, dx\, dy\, dz$$

$$I_{11} = \frac{M}{s^3} \int_{-\frac{s}{2}}^{\frac{s}{2}} \int_{-\frac{s}{2}}^{\frac{s}{2}} (y^2 + z^2)\, dy\, dz \int_{-\frac{s}{2}}^{\frac{s}{2}} dx$$

$$I_{11} = \frac{M}{s^2} \int_{-\frac{s}{2}}^{\frac{s}{2}} \int_{-\frac{s}{2}}^{\frac{s}{2}} (y^2 + z^2)\, dy\, dz$$

$$I_{11} = \frac{M}{3s} \left([y^3]_{-\frac{s}{2}}^{\frac{s}{2}} + [z^3]_{-\frac{s}{2}}^{\frac{s}{2}} \right)$$

$$I_{11} = \frac{M}{3s} \frac{s^3}{2} = \frac{M s^2}{6}$$

$$I_{12} = \frac{M}{s^3} \int_{-\frac{s}{2}}^{\frac{s}{2}} \int_{-\frac{s}{2}}^{\frac{s}{2}} \int_{-\frac{s}{2}}^{\frac{s}{2}} (-xy)\, dx\, dy\, dz = 0$$

$$\mathbb{I} = M \frac{s^2}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

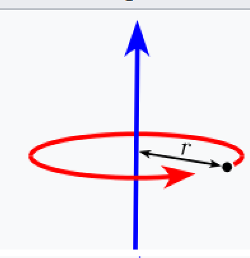
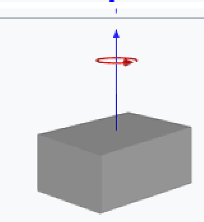
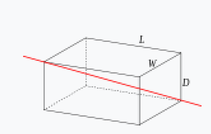
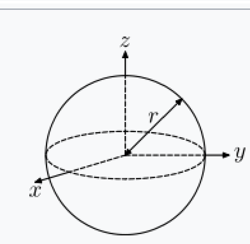
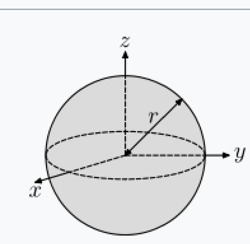
$$r^2 = x^2 + y^2 + z^2$$
$$(r^2 - x^2) = (y^2 + z^2)$$

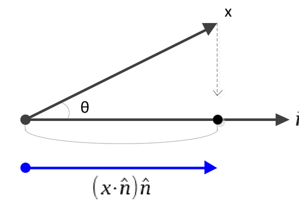
$$\int_{\frac{-s}{2}}^{\frac{s}{2}} 1\, dx = [x]_{\frac{-s}{2}}^{\frac{s}{2}} = s$$

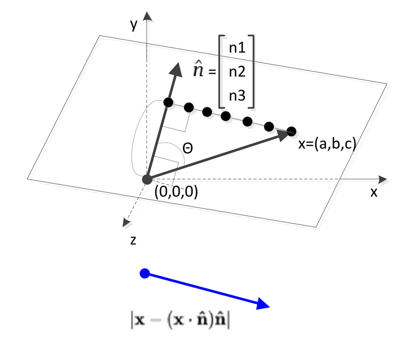
$$\frac{M}{s^3} \int_{\frac{-s}{2}}^{\frac{s}{2}} 1\, dx = \frac{M}{s^3} s = \frac{M}{s^2}$$

Moments of inertia [edit]

Following are scalar moments of inertia. In general, the moment of inertia is a tensor, see below.

Description	Figure	
Point mass M at a distance r from the axis of rotation. A point mass does not have a moment of inertia around its own axis, but using the parallel axis theorem a moment of inertia around a distant axis of rotation is achieved.		$I = Mr^2$
Solid cuboid of height h, width w, and depth d, and mass m. For a similarly oriented cube with sides of length s, $I_{CM} = \frac{1}{6}ms^2$		$I_h = \frac{1}{12}m(w^2 + d^2)$ $I_w = \frac{1}{12}m(d^2 + h^2)$ $I_d = \frac{1}{12}m(w^2 + h^2)$
Solid cuboid of height D, width W, and length L, and mass m, rotating about the longest diagonal. For a cube with sides s, $I = \frac{1}{6}ms^2$.		$I = \frac{1}{6}m\left(\frac{W^2D^2 + D^2L^2 + W^2L^2}{W^2 + D^2 + L^2}\right)$
Hollow sphere of radius r and mass m. A hollow sphere can be taken to be made up of two stacks of infinitesimally thin, circular hoops, where the radius differs from 0 to r (or a single stack, where the radius differs from $-r$ to r).		$I = \frac{2}{3}mr^2$ [1]
Solid sphere (ball) of radius r and mass m. A sphere can be taken to be made up of two stacks of infinitesimally thin, solid discs, where the radius differs from 0 to r (or a single stack, where the radius differs from $-r$ to r).		$I = \frac{2}{5}mr^2$ [1]





$$m(x^2 - 2x(x \cdot \hat{n})\hat{n} + (x \cdot \hat{n})^2 \hat{n}^2)$$

$$(\hat{n})^2 = n \cdot n = |n| = 1$$
$$(x \cdot \hat{n}) \ \ is \ \ scalar, so$$
$$2x(x \cdot \hat{n})\hat{n} = 2(x \cdot \hat{n})^2$$

$$m(x^2 - 2(x \cdot \hat{n})^2 + (x \cdot \hat{n})^2)$$
$$\boxed{m(x^2 - (x \cdot \hat{n})^2)} \quad \text{(1)}$$

$$a \cdot b = a^T b \quad \text{, dot } product\ of\ two\ column\ vectors$$
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$
$$[a\,b\,c] \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$
$$x^2 = x \cdot x = x^T x$$
$$(x \cdot \hat{n})^2 = (x^T \hat{n})^2 = (x^T \hat{n}) \cdot (x^T \hat{n}) = (x^T \hat{n})^T (x^T \hat{n}) = \hat{n}^T x x^T \hat{n}$$

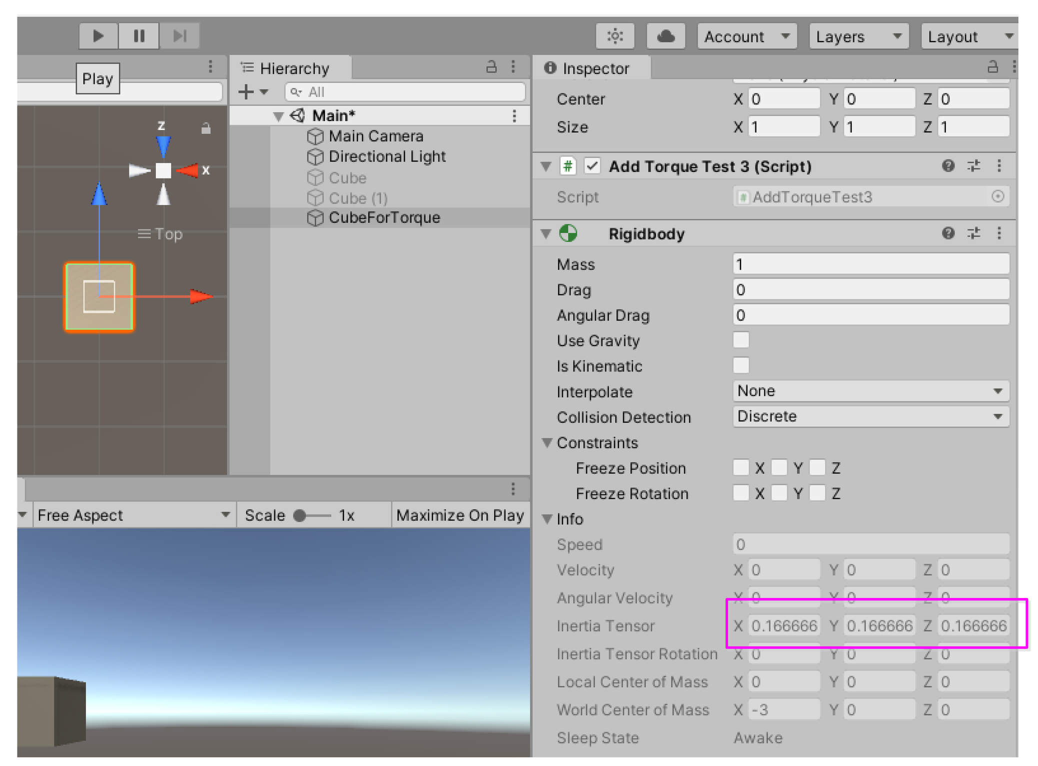
$$\boxed{m(x^T x - \hat{n}^T x x^T \hat{n})} \quad \text{(2)}$$

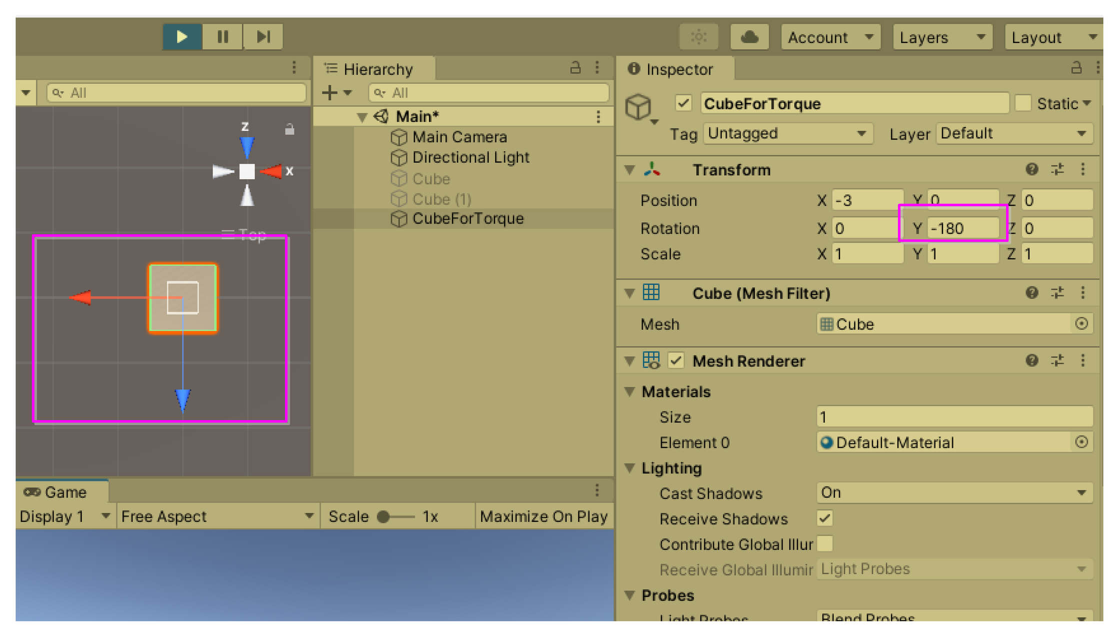
$$x^T x = \hat{n}^T (x^T x \cdot E_3) \hat{n}$$
$$1 = n \cdot n = \hat{n}^T E_3 \hat{n}$$
$$n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \hat{n}^T E_3 = (a, b, c) \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix} = [a\,b\,c]$$
$$\hat{n}^T E_3 \hat{n} = [a\,b\,c] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = n \cdot n = 1$$

$$\boxed{m \cdot \hat{n}^T (x^T x \cdot E_3 - x x^T) \hat{n}} \quad \text{(3)}$$

$$x^T x \cdot E_3 = \begin{bmatrix} x^2 + y^2 + z^2 & 0 & 0 \\ 0 & x^2 + y^2 + z^2 & 0 \\ 0 & 0 & x^2 + y^2 + z^2 \end{bmatrix}$$
$$x x^T = \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}$$

Inertia Tensor in Unity






```
float theta = Mathf.PI;
//Vector3 w = Vector3.down * theta / Time.fixedDeltaTime;
Vector3 w = Vector3.down * theta * magnitude;
Quaternion q = transform.rotation * _rb.inertiaTensorRotation;
Vector3 torque = q * Vector3.Scale(_rb.inertiaTensor, (Quaternion.Inverse(q) * w));
_rb.AddTorque(torque);
_timer = 5.0f;
```