Quaternion

jintaeks@dongseo.ac.kr

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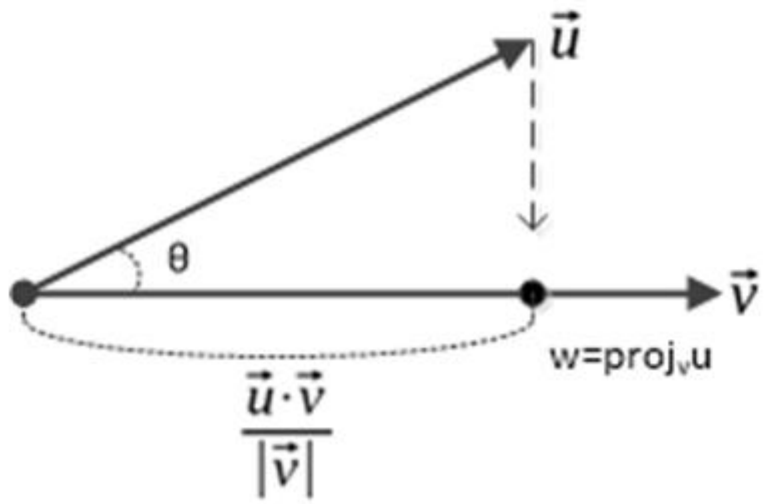
Dongseo University
동서대학교

Vector Decomposition



$$w = proj_v \vec{u} \quad \text{(식6-)}$$

$$|\vec{w}| = |\vec{u}|\cos\theta = \frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{v}|} = \frac{\vec{u} \bullet \vec{v}}{|\vec{v}|} \quad \text{(식6-)}$$

$$proj_v \vec{u} = \vec{w} = \frac{\vec{u} \bullet \vec{v}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\vec{v} \quad \text{(식5-16)}$$

$$perp_v \vec{u} = \vec{u} - proj_v \vec{u} \quad \text{(식6-)}$$

$$\vec{u} = (a, b, c)$$

$$\vec{v} = (d, e, f)$$

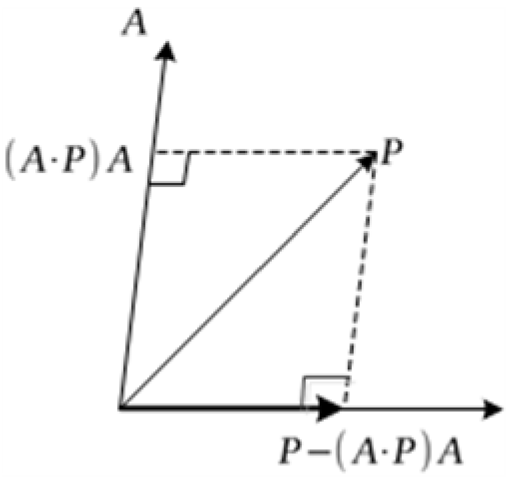
$$proj_v \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{|\vec{v}|^2} (\vec{u} \cdot \vec{v}) \vec{v} \quad \text{(식6-)}$$

$$= \frac{1}{|\vec{v}|^2} \begin{bmatrix} dd & de & df \\ ed & ee & ef \\ fd & fe & ff \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

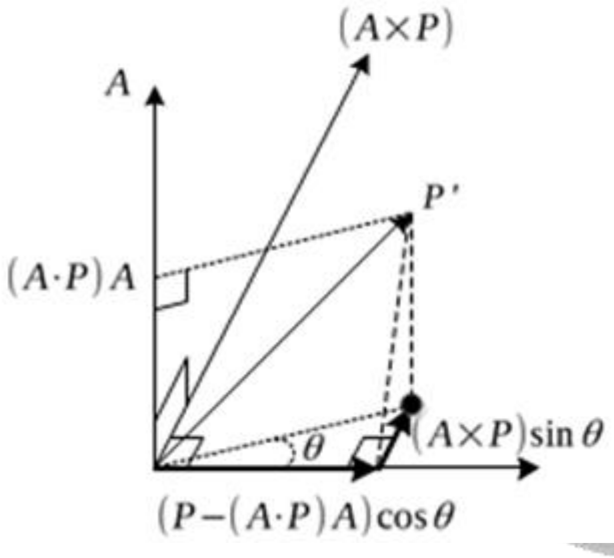
$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

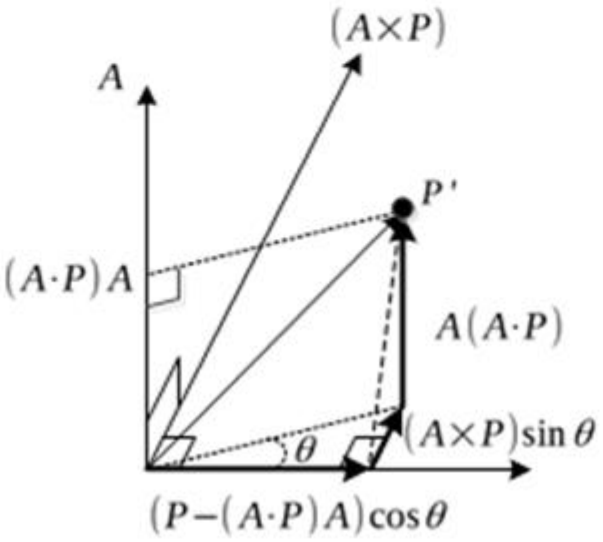
Rotation about Arbitrary Axis

$$proj_A P = (A \cdot P)A$$
$$perp_A P = P - (A \cdot P)A$$



$$P'_{proj} = (P - (A \cdot P)A)\cos\theta + (A \times P)\sin\theta$$





$$
\begin{aligned}
P' &= (P-(A\cdot P)A)\cos\theta + (A\times P)\sin\theta + A(A\cdot P) \quad \text{(식6-)} \\
&= P\cos\theta + (A\times P)\sin\theta + A(A\cdot P)(1-\cos\theta)
\end{aligned}
$$

$$A = (x, y, z)$$

$$c = \cos\theta, \quad s = \sin\theta$$

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P\cos\theta + \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 1 \end{bmatrix} P\sin\theta \quad \text{(식6-)}$$

$$+ \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} P(1-\cos\theta)$$

$$R_A(\theta) = \begin{bmatrix} c+(1-c)x^2 & (1-c)xy-sz & (1-c)xz+sy \\ (1-c)xy+sz & c+(1-c)y^2 & (1-c)yz-sx \\ (1-c)xz-sy & (1-c)yz+sx & c+(1-c)z^2 \end{bmatrix}$$

Quaternion

$$(\cos\beta + i\sin\beta)(\cos\alpha + i\sin\alpha)$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$\cos\theta + i\sin\theta$$

$$\cos\theta + (1,0,0)\sin\theta$$

$$q = a + bi + cj + dk$$

$$q = w + xi + yj + zk$$

$$q = (w, x, y, z)$$

$$ijk = -1$$

$$ij = k, ji = -k$$

$$jk = i, kj = -i$$

$$ki = j, ik = -j$$

×	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

$$\hat{u} = xi + yj + zk = (x, y, z)$$

$$q = (w, \hat{u})$$

Complex number is the special case of Quaternion

$$\cos\theta + isin\theta$$

$$\hat{u} = xi + yj + zk = (1,0,0) = i$$

$$\cos\theta + (1i + 0j + 0k)\sin\theta = \cos\theta + i\sin\theta$$

$$\cos\theta + i\sin\theta$$

$$(\cos\theta, \sin\theta, 0, 0)$$

Pure Quaternion Multiplication

$$q_1 = x_1 i + y_1 j + z_1 k$$

$$q_2 = x_2 i + y_2 j + z_2 k$$

$$\begin{aligned} q_1 q_2 = & x_1 x_2 ii + x_1 y_2 ij + x_1 z_2 ik \\ & + y_1 x_2 ji + y_1 y_2 jj + y_1 z_2 jk \\ & + z_1 x_2 ki + z_1 y_2 kj + z_1 z_2 kk \end{aligned}$$

$$\begin{aligned} q_1 q_2 = & - x_1 x_2 + x_1 y_2 k - x_1 z_2 j \\ & - y_1 x_2 k - y_1 y_2 + y_1 z_2 i \\ & + z_1 x_2 j - z_1 y_2 i - z_1 z_2 kk \end{aligned}$$

$$\begin{aligned} q_1 q_2 = & (y_1 z_2 - z_1 y_2) i + (z_1 x_2 - x_1 z_2) j \\ & + (x_1 y_2 - y_1 x_2) k - (x_1 x_2 + y_1 y_2 + z_1 z_2) \end{aligned}$$

$$q_1 q_2 = q_1 \times q_2 - q_1 \bullet q_2$$

If q1 and q2 is orthogonal

$$q_1 \cdot q_2 = 0$$

$$q_1 q_2 = q_1 \times q_2$$

$$Q_1 = (w_1, q_1)$$
$$Q_2 = (w_2, q_2)$$

$$Q_1 Q_2 = w_1 w_2 + w_1 q_2 + w_2 q_1 + q_1 \times q_2 - q_1 \bullet q_2$$

$$\begin{aligned} Q_1 Q_2 &= (w_1 + x_1 i + y_1 j + z_1 k)(w_2 + x_2 i + y_2 j + z_2 k) \\ &= (w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2) + \\ &\quad (w_1 x_2 + x_1 w_2 + y_1 z_2 - z_1 y_2) i + \\ &\quad (w_1 y_2 - x_1 z_2 + y_1 w_2 + z_1 x_2) j + \\ &\quad (w_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 w_2) k \end{aligned}$$

Complex Conjugate

$$c = a + bi$$
$$|c| = \sqrt{a^2 + b^2}$$
$$c^* = a - bi$$
$$c^* c = a^2 + b^2 = |c|^2$$
$$c^{-1} = \frac{c^*}{|c|^2} = \frac{c^*}{c^* c} = \frac{1}{c}$$

$$q = (w, x, y, z)$$
$$q^* = (w, -x, -y, -z)$$
$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2}$$
$$q^* q = |q|^2$$
$$q^{-1} = \frac{q^*}{|q|^2} = \frac{q^*}{q^* q} = \frac{1}{q}$$

$$(q_1 q_2)^* = q_2^* q_1^*$$

Vector Quaternion Multiplication

$$v = (x, y, z)$$
$$v_q = (0, x, y, z)$$

Selection Function W

$$v = (x, y, z)$$
$$v_q = (0, x, y, z)$$

$$v' = qvq^*$$

$$W(q) = W(w + xi + yj + zk) = w$$

$$\begin{aligned}
W(v') &= W(qvq^*) \\
&= [(qvq^*) + (qvq^*)^*]/2 \quad \leftarrow (qvq^*) + (qvq^*)^* = 2w \\
&= [qvq^* + qv^*q^*]/2 \quad \leftarrow (ab)^* = b^*a^* \\
&= q[(v + v^*)/2]q^* \\
&= q\,W(v)q^* \quad \leftarrow qrq^* = r|q|^2, where \;\; r \in R \\
&= W(v) \\
&= 0
\end{aligned}$$

$$\cos\theta + \hat{u} sin\theta$$

$$\cos(\theta/2) + \hat{u} sin(\theta/2)$$

$$(P\times Q)\cdot R = (Q\times R)\cdot P = (R\times P)\cdot Q$$ (스칼라 삼중곱)

$$(\boldsymbol{v}\times \boldsymbol{P})\cdot \boldsymbol{v} = (\boldsymbol{v}\times \boldsymbol{v})\cdot \boldsymbol{P} = 0\cdot \boldsymbol{P} = 0$$

$$\vec{P}\times(\vec{Q}\times\vec{R}) = (\vec{P}\cdot\vec{R})\vec{Q} - (\vec{P}\cdot\vec{Q})\vec{R}$$ (벡터 삼중곱)

$$\begin{aligned}\vec{P}\times(\vec{Q}\times\vec{P}) &= \vec{P}\times\vec{Q}\times\vec{P} \\ &= (\vec{P}\cdot\vec{P})\vec{Q} - (\vec{P}\cdot\vec{Q})\vec{P} \\ &= |\vec{P}|^2\vec{Q} - (\vec{P}\cdot\vec{Q})\vec{P}\end{aligned}$$ (삼중곱)

$$vP = v \times P - v \cdot P$$

$$Pv = P \times v - P \cdot v$$

$$(s+v)P = sP + vP = sP + v \times P - v \cdot P$$

$(v \times P)v = (v \times P) \times v - (v \times P) \cdot v$ (순수 쿼터니언 곱)

Quaternion Multiplication

$$q = s + v,\quad q^{-1} = s - v$$

$$\begin{aligned} qPq^* &= (s+v)P(s-v) \\ &= (sP + vP)(s-v) \end{aligned}$$

$$vP = v \times P - v \cdot P$$

$$\begin{aligned} qPq^* &= (s+v)P(s-v) \\ &= (sP + vP)(s-v) \\ &= (sP + v \times P - v \cdot P)(s-v) \\ &= s^2P + s(v \times P) - s(v \cdot P) - sPv - (v \times P)v + (v \cdot P)v \end{aligned}$$

$$Pv = P \times v - P \cdot v$$

$$(v \times P)v = (v \times P) \times v - (v \times P) \cdot v$$

$$\begin{aligned}
qPq^* &= \ldots \\
&= s^2P + s(v \times P) - s(v \cdot P) - sPv - (v \times P)v + (v \cdot P)v \\
&= s^2P + s(v \times P) - s(v \cdot P) - s(P \times v - P \cdot v) \\
&\quad - (v \times P)v + (v \cdot P)v \\
&= s^2P + s(v \times P) - s(v \cdot P) - s(P \times v - P \cdot v) \\
&\quad - ((v \times P) \times v - (v \times P) \cdot v) + (v \cdot P)v \\
&= s^2P + s(v \times P) - s(v \cdot P) - s(P \times v) + s(P \cdot v) \\
&\quad - (v \times P) \times v + (v \times P) \cdot v + (v \cdot P)v
\end{aligned}$$

$$
\begin{aligned}
qPq^* = &\ \ldots \\
= &\ s^2P + s(v\times P) - s(v\cdot P) - s(P\times v) + s(P\cdot v) \\
&- (v\times P)\times v + (v\times P)\cdot v + (v\cdot P)v \\
= &\ s^2P + s(v\times P) - s(v\cdot P) + s(v\times P) + s(v\cdot P) \\
&- (v\times P\times v) + (v\times P)\cdot v + (v\cdot P)v
\end{aligned}
$$

$$
(v\times P)\cdot v = (v\times v)\cdot P = 0\cdot P = 0
$$

$$
\begin{aligned}
qPq^* = &\ \ldots \\
= &\ s^2P + s(v\times P) - s(v\cdot P) + s(v\times P) + s(v\cdot P) \\
&- (v\times P\times v) + (v\times P)\cdot v + (v\cdot P)v \\
= &\ s^2P + s(v\times P) + s(v\times P) \\
&- (v\times P\times v) + 0 + (v\cdot P)v \\
= &\ s^2P + 2s(v\times P) + (v\cdot P)v - (v\times P\times v)
\end{aligned}
$$

$$v \times P \times v = v^2 P - (v \cdot P)v$$

$$\begin{aligned} qPq^* \\ &= s^2 P + 2s(v \times P) + (v \cdot P)v - (v \times P \times v) \\ &= s^2 P + 2s(v \times P) + (v \cdot P)v - (v^2 P - (v \cdot P)v) \\ &= s^2 P + 2s(v \times P) + (v \cdot P)v - v^2 P + (v \cdot P)v \\ &= (s^2 - v^2)P + 2s(v \times P) + 2(v \cdot P)v \end{aligned}$$

$$qPq^* = (s^2 - v^2)P + 2s(v \times P) + 2(v \cdot P)v \quad \text{(식6-)}$$

$$v = tA$$

$$A^2 = A \cdot A = |A|^2 = 1$$

$$v^2 = t^2 A^2 = t^2 1 = t^2$$

$$\begin{aligned} qPq^* &= (s^2 - v^2)P + 2s(v \times P) + 2(v \cdot P)v \\ &= (s^2 - t^2)P + 2s((tA) \times P) + 2((tA) \cdot P)(tA) \\ &= (s^2 - t^2)P + 2st(A \times P) + 2t^2(A \cdot P)A \end{aligned}$$

$$\begin{aligned} P' &= (P - (A \cdot P)A)cos\theta + (A \times P)\sin\theta + A(A \cdot P) \\ &= Pcos\theta + (A \times P)\sin\theta + A(A \cdot P)(1 - \cos\theta) \end{aligned}$$

$s^2 - t^2 = \cos\theta$ (식6-)

$$2st = \sin\theta$$

$2t^2 = 1 - \cos\theta$ (식6-)

$$s^2 - t^2 = \cos\theta \quad \text{(식6-)}$$

$$2st = \sin\theta$$

$$2t^2 = 1 - \cos\theta \quad \text{(식6-)}$$

$$t = \sqrt{\frac{1-\cos\theta}{2}} = \sin\frac{\theta}{2}$$

$$(s^2 - t^2) + 2t^2 = \cos\theta + (1 - \cos\theta)$$

$$s^2 + t^2 = 1 \quad \text{(식6-)}$$

$$s = \cos\frac{\theta}{2}$$

$$\boldsymbol{q} = \cos\frac{\theta}{2} + \boldsymbol{A}\,sin\frac{\theta}{2} \quad \text{(식6-)}$$

Polar Coordinate

$$q = a + ix + jy + kz = a + \boldsymbol{v}$$

$$q = |q|(\cos\theta + \boldsymbol{n}\sin\theta) = |z|e^{n\theta} = e^{\ln|z|}e^{n\theta} = e^{\ln|z| + n\theta}$$

$$|q| = \sqrt{a^2 + x^2 + y^2 + z^2}$$

$$|\boldsymbol{v}| = \sqrt{x^2 + y^2 + z^2}$$

$$\cos\theta = \frac{a}{|z|}$$

$$\sin\theta = \frac{|\boldsymbol{v}|}{|z|}$$

$$\boldsymbol{n} = \frac{\boldsymbol{v}}{|z|\sin\theta} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

$$b = e^{\ln b} \quad (\text{식}5\text{-}98\text{b})$$

$$be^{i\alpha} = e^{\ln b}e^{i\alpha} = e^{\ln b + i\alpha} \quad (\text{식}5\text{-}98\text{c})$$

$$e^{a+bi} = e^{a}e^{bi}$$

$$\ln a + i\theta$$

$$e^{\ln a + i\theta} = e^{\ln a}e^{i\theta} = ae^{i\theta}$$

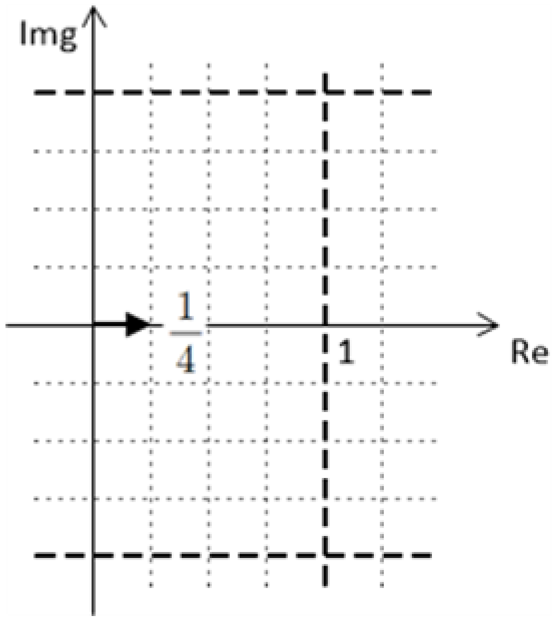
$$\left(\frac{1}{2}\right)^{2+i}$$

$$\ln\frac{1}{2} = \ln 2^{-1} = -\ln(2) \quad \text{(식5-98f)}$$

$$\frac{1}{2} = e^{\ln\frac{1}{2}} = e^{-\ln 2} \quad \text{(식5-98g)}$$

$$\frac{1}{2}^{2+i} = \frac{1}{2}^{2}\left(\frac{1}{2}\right)^{i} = \frac{1}{4}\left(\frac{1}{2}\right)^{i} = \left(\frac{1}{4}\right)\left(e^{-\ln 2}\right)^{i} = \frac{1}{4}e^{(-\ln 2)i}$$

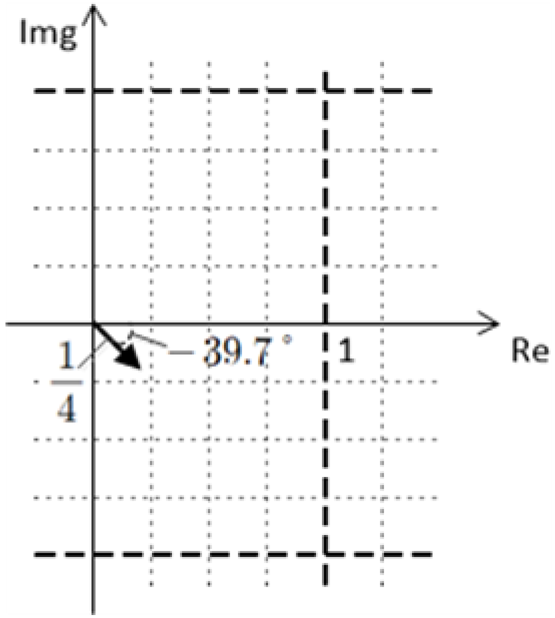
$$\frac{1}{4}e^{(-\ln 2)i}$$



$$\pi\, rad = 180^{\circ}$$

$$1\, rad = \left(\frac{180}{\pi}\right)^{\circ}$$

$$-\ln(2)\, rad = -\ln(2)\left(\frac{180}{\pi}\right) \approx -39.7^{\circ} \quad \text{(식5-99i)}$$



THANKS!

Any questions?

MY **BRIGHT** FUTURE

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