

## MATH 494: HONORS ALGEBRA II

### Contents

1. JANUARY 4, 2017

### Rings

#### Definition 1.1.

a) A **ring** is a tuple  $(R, +, \cdot, 0)$  where:

- $R$  is a set
- $0 \in R$
- $+, \cdot : R \times R \rightarrow R, \quad (a, b) \mapsto a + b, a \cdot b$

subject to:

- $(R, +, 0)$  is an abelian group
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $(a + b) \cdot c = a \cdot c + b \cdot c$
- $a \cdot (b + c) = a \cdot b + a \cdot c$

b) A **ring with unity** is a tuple  $(R, +, \cdot, 0, 1)$ , where  $(R, +, \cdot, 0)$  is a ring, and  $1 \in R$  is subject to  $1 \cdot a = a \cdot 1 = a$  for all  $a \in R$ .

c) A ring  $(R, +, \cdot, 0)$  is called **commutative** if  $ab = ba$  for all  $a, b \in R$ .

d) A **field** is a commutative ring with unity  $(R, +, \cdot, 0, 1)$  such that  $(R \setminus \{0\}, \cdot, 1)$  is a group.

*Remark.*

- We don't really need to include 0,1 in notation: they are unique if they exist
- There is a notion of a **skew field**: ring with unity  $(R, +, \cdot, 0, 1)$  such that  $(R \setminus \{0\}, \cdot, 1)$  is a group. (This drops the commutative condition from the definition of a field).
- In French: *corps* is a skew field, and *corps commutatif* is a field.

**Fact 1.2.** Let  $R$  be a ring. For all  $a \in R$ ,  $0 \cdot a = 0$ .

*Proof.*  $(0 \cdot a) = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a \Rightarrow 0 = 0 \cdot a$  □

*Example.*

- $\mathbb{Z}$  is a ring, commutative, with unity
- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$  are fields
- $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$  where  $i^2 = j^2 = k^2 = ijk = -1$  are called the **Hamiltonian Quaternions** and are a skew-field
- $\mathcal{C}_c(\mathbb{R}) =$  functions on  $\mathbb{R}$  with compact support  
( $\text{supp}(f) = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$ ) is a commutative ring without unity
- $R = \{\star\}, 0 = 1 = \star$  is the **zero ring**.

**Fact 1.3.** If  $(R, +, \cdot, 0, 1)$  is a ring with unity and  $0 = 1$ , then  $R$  is the zero ring.

*Proof.* Take  $a \in R$ . Then  $a = a \cdot 1 = a \cdot 0 = 0$  by Fact ?? □

Convention: Unless otherwise noted, ring will refer to a commutative ring with 1.

**Definition 1.4.** Let  $R$  be a ring. Its **group of units** is

$$R^\times = \{a \in R \mid \exists b \in R : ab = 1\}$$

**Fact 1.5.**

- For  $a \in R^\times$ , there is a unique  $b \in R$  such that  $ab = 1$ . Write  $b = a^{-1}$ .
- For  $a, b \in R^\times$ ,  $a \cdot b \in R^\times$ .

*Proof.*

- Given  $b, b'$ , we have  $b = b \cdot 1 = b(ab') = (ba)b' = 1 \cdot b' = b'$ .
- $(a \cdot b) \cdot (b^{-1} \cdot a^{-1}) = 1$

□

*Example.*  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ ,  $\mathbb{Z}^\times = \{1, -1\}$

**Definition 1.6.** Let  $R, S$  be rings. A **morphism**  $\phi : R \rightarrow S$  is a map of sets  $\phi : R \rightarrow S$  satisfying

- $\phi(a + b) = \phi(a) + \phi(b)$
- $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$
- $\phi(1) = 1$

*Example.*  $\phi : \mathbb{Z} \rightarrow \mathbb{Z} \ u \mapsto 0$  is not a morphism of rings with 1. (it is a morphism of general rings).

**Fact 1.7.** For any ring  $R$  there is a unique morphism  $\phi : \mathbb{Z} \rightarrow R$ . Given  $z \in \mathbb{Z}$ , we write  $z_R$ , or simply  $z$  for its image under  $\phi$ .

*Example.*  $5 \in \mathbb{Z}$ ,  $5_{\mathbb{Q}} \in \mathbb{Q}$  usual number 5.  $5_{\mathbb{Z}/2\mathbb{Z}} = 1_{\mathbb{Z}/2\mathbb{Z}}$

**Definition 1.8.** Let  $R$  be a ring. A subset  $I \subset R$  is called an **ideal** if

- $I$  is a subgroup of  $(R, +, 0)$
- $a \cdot f \in I$  for all  $a \in R, f \in I$ .

**Definition 1.9.** Let  $R$  be a ring. A subset  $S \subset R$  is called a **subring** if

- $S$  is a subgroup of  $(R, +, 0)$
- $a \cdot b \in S$  for all  $a, b \in S$ .
- $1 \in S$ .

*Remark.*

- The only subset that is both a subring and an ideal is  $R$  itself. (reason: if  $1 \in I$ , then  $a \cdot 1 \in I$  for all  $a \in R$ , meaning  $I = R$ )
- $I = \{0\}, I = R$  are always ideals.
- In rings without unity, the 2 notions align closer: ideal becomes a special case of subring as  $1 \in S$  condition is dropped.

*Example.*

- Every subgroup of  $(\mathbb{Z}, +, 0)$  is an ideal of  $\mathbb{Z}$ .
- If  $F$  is a field, then  $\{0\}, R$  are the only ideals
- Let  $R = \mathcal{C}_C(\mathbb{R})$ ,  $S \in R$  subset.

$$I = \{f \in \mathcal{C}_C(\mathbb{R}) \mid f|_S = 0\}$$

is an ideal

**Definition 1.10.** An ideal  $I \in R$  is called **principal** if  $I = \{a \cdot r \mid r \in R\}$  for some  $a \in R$ . Then  $a$  is called a **generator**.

**Definition 1.11.** Let  $a_1, a_2, \dots, a_n \in R$ . An **ideal generated by**  $a_1, \dots, a_n$  is

$$(a_1, \dots, a_n) = \{a_1 r_1 + \dots + a_n r_n \mid r_i \in R\}$$

**Fact 1.12.** Given ideals  $I, J \subset R$  we have

- $I \cap J$  is an ideal
- $I + J = \{a + b \mid a \in I, b \in J\}$  is an ideal
- $I \cdot J = \left\{ \sum_{i=1}^n a_i b_i \mid a_i \in I, b_i \in J \right\}$  is an ideal

2. JANUARY 6, 2017

**Fact 2.1.** Let  $\varphi : R \rightarrow S$  be a morphism. Then

$$\ker(\varphi) = \{x \in R \mid \varphi(x) = 0\}$$

is an ideal.

*Proof.* (A Pranav Exclusive) We first show that the kernel is a subgroup of  $(R, +, 0)$ . Well, we first show that  $0 \in \ker(\varphi)$ . Well,

$$\varphi(0) = \varphi(0 + 0) = \varphi(0) + \varphi(0)$$

so, we have that  $\varphi(0) = 0$  and thus  $0 \in \ker(\varphi)$ . Next, we show that inverses are in the kernel as well.

If we have that  $\varphi(a) = 0$ , then we have

$$0 = \varphi(0) = \varphi(a + (-a)) = \varphi(a) + \varphi(-a) = \varphi(-a)$$

Now, we complete this step by proving closure. Assume  $a, b \in \ker(\varphi)$ . Then,

$$\phi(a + b) = \phi(a) + \phi(b) = 0 + 0 = 0$$

Thus, we have that the kernel is a subgroup. Now, we verify the second condition. Fix  $a \in R$  and  $f \in \ker(\varphi)$ . We have that

$$\phi(a \cdot f) = \phi(a) \cdot \phi(f) = \phi(a) \cdot 0 = 0$$

Thus, we have that  $a \cdot f \in \ker(\varphi)$ , meaning that  $\ker(\varphi)$  is an ideal. □

Question: Is every ideal the kernel of morphism?

**Proposition 2.2.** Let  $R$  be a ring,  $I \subset R$  an ideal. Let  $R/I$  be the quotient of abelian groups and  $p : R \rightarrow R/I$  the canonical projection. Then there is a unique product map

$$\cdot : R/I \times R/I \rightarrow R/I$$

making  $R/I$  into a ring such that  $p$  is a morphism.

*Proof.* For  $p$  to be a morphism of rings, we need

- $p(1_R) = 1_{R/I}$
- The following diagram to commute

$$\begin{array}{ccc} R \times R & \xrightarrow{\cdot_R} & R \\ \downarrow p \times p & & \downarrow p \\ R/I \times R/I & \xrightarrow{\cdot_{R/I}} & R/I \end{array}$$

Uniqueness of  $\cdot_{R/I}$  follows from surjectivity of  $p \times p$  (each element in  $R/I \times R/I$  must go precisely to the result of the composition of  $p$  and  $\cdot_R$ )

For existence, define  $1_{R/I} = p(1_R)$  and  $(a + I) \cdot (b + I) \stackrel{\text{def}}{=} (a \cdot b) + I$ . We have to show this is well-defined (i.e it is independent of choice of  $a, b$ ).

Well, choose  $a', b'$  such that  $a' + I = a + I, b' + I = b + I$ . Thus,  $a' = a + i, b' = b + j$  for some  $i, j \in I$ . Then

$$(a' + I)(b' + I) = (a' \cdot b') + I = ((a + i) \cdot (b + j)) + I = (a \cdot b + a \cdot j + b \cdot i + i \cdot j) + I = a \cdot b + I$$

as we note that  $a \cdot j, b \cdot i$ , and  $i \cdot j$  are all in  $I$  as  $I$  is an ideal.

We have that all of the ring axioms for  $R/I$  are inherited from the ring structure on  $R$ .  $\square$

*Remark.*  $\ker(p) = I$

**Theorem 2.3. (Homomorphism Theorem):** Let  $\phi : R \rightarrow S$  be a morphism of rings,  $I \subset \ker(\phi)$  be an ideal of  $R$ . There is a unique morphism  $\bar{\phi} : R/I \rightarrow S$  such that  $\bar{\phi} \circ p = \phi$  i.e.

$$\begin{array}{ccc} & R/I & \\ p \nearrow & & \nwarrow \bar{\phi} \\ R & \xrightarrow{\phi} & S \end{array}$$

commutes. Moreover,  $\bar{\phi}$  is injective  $\iff \ker(\phi) = I$

*Proof.* All statements follow from looking at the abelian group  $(R, +, 0)$  and its subgroup  $I$ , except multiplicativity of  $\bar{\phi}$ .

(A Pranav Exclusive) Some justification: the uniqueness of this morphism follows because the projection map is surjective, meaning that in order for the composition to be commutative, we must have that each element in  $R/I$  maps exactly to where its associated element maps under  $\phi$ . Now, the existence. We simply need to check that the map  $\bar{\phi}$  that sends  $a + I$  to  $\phi(a)$  is well defined and is a morphism. We note that the additive morphism properties are inherited from the fact that  $\phi$  is a morphism itself. So, we check the well-definedness of  $\bar{\phi}$ . Pick 2 representatives of  $a + I$ , call them  $a + I$  and  $a' + I$ . We have that  $a' = a + i$  for  $i \in I$ . Then, we have that

$$\bar{\phi}(a' + I) = \bar{\phi}(a + i + I) = \bar{\phi}(a + I) + \bar{\phi}(i + I) = \bar{\phi}(a + I) + \bar{\phi}(I) = \bar{\phi}(a + I) + 0$$

as we have that  $\phi(i) = 0$  for all  $i \in I$  (since  $I \subset \ker(\phi)$ ). We finally verify the injective biconditional. Assume  $\bar{\phi}$  is injective. We already have that  $I \subset \ker(\phi)$ . Now, since  $\bar{\phi}$  is injective, its kernel is trivial, and is thus the identity of  $R/I$ , namely  $I$  itself. For any  $g \in \ker(\phi)$  we note that  $g + I$  must belong to the kernel of  $\bar{\phi}$ , meaning that  $g + I = I$  and thus  $g \in I$ . This gives us double containment and thus equality.

Now, assume that  $\ker(\phi) = I$ . We consider  $\ker(\bar{\phi})$ . This is exactly the collection  $\{a + I \mid a \in \ker(\phi)\}$ . Thus, this is  $\{a + I \mid a \in I\}$  and thus we have that  $\ker(\bar{\phi}) = I$ . Since the kernel of  $\bar{\phi}$  is trivial, we have that  $\bar{\phi}$  is injective.

Checking Multiplicativity: Let  $A, B \in R/I$ . Choose  $a, b \in R$  such that  $p(a) = A, p(b) = B$ . Then

$$\bar{\phi}(A \cdot B) = \bar{\phi}(p(a) \cdot p(b)) = \bar{\phi}(p(ab)) = \phi(ab) = \phi(a)\phi(b) = \bar{\phi}(p(a))\bar{\phi}(p(b)) = \bar{\phi}(A)\bar{\phi}(B)$$

$\square$

**Definition 2.4.** Let  $R$  be a ring.

- Let  $a, b \in R$ . We say that  $a$  **divides**  $b$  (denoted  $a \mid b$ ) if there is  $c \in R$  such that  $ac = b$ .
- We say  $0 \neq a \in R$  is a **zero divisor** if there is  $0 \neq b \in R$  such that  $ab = 0$ .
- We call  $R$  a **domain** (or **integral domain**) if it has no zero divisors.

**Fact 2.5.**  $a \mid b \iff (b) \subset (a) \iff b \in (a)$

*Proof.* (A Pranav Exclusive) We first show the first forward implication. Assume that  $a \mid b$ . Then, there is  $c \in R$  such that  $ac = b$ . Now, fix  $g \in (b)$ . It is of the form  $br$  for some  $r \in R$ . Thus, we have that  $g = (ac)r = a(cr)$ . Since  $cr \in R$ , we have that  $g \in (a)$ .

Next, we show the second forward implication. Assume that  $(b) \subset (a)$ . Well,  $b \in (b) \subset (a)$ .

Finally, we show that  $b \in (a)$  implies the original condition. Well, if  $b \in (a)$ , then  $b = ar$  for  $r \in R$ . This is exactly what it means for  $a \mid b$ ! Thus, we have shown equality of the above statements.  $\square$

**Fact 2.6.** (Cancellation Law) If  $a \neq 0 \in R$  is not a zero divisor, then for  $x, y \in R$

$$ax = ay \Rightarrow x = y$$

*Proof.*  $ax = ay \iff a(x - y) = 0$ .  $a \neq 0$  implies that  $x - y = 0$  as  $a$  is not a zero divisor.  $\square$

**Definition 2.7.** An ideal  $I \subsetneq R$  is called

- **prime** if  $a \cdot b \in I$  implies  $a \in I$  or  $b \in I$  for all  $a, b \in R$ .
- **maximal** if  $I$  and  $R$  are the only ideals containing  $I$ .

*Example.* In  $R = \mathbb{Z}$ , the ideals are of the form  $n\mathbb{Z}$ .  $n\mathbb{Z}$  is prime  $\iff n$  is prime or  $n = 0$ .

*Proof.* (A Pranav Exclusive). We start with the forward direction. We proceed by contrapositive. Assume that  $n \neq 0$  and that  $n$  is not prime. Then,  $n$  is composite (we exclude  $n = 1$  as we must have a properly contained ideal by definition). Thus, we have that  $n = ab$  for some  $1 < a, b < n$ . Note that we have  $ab = n \in n\mathbb{Z}$ , but we have that both  $a$  and  $b$  are less than  $n$ , and thus there is no  $z \in \mathbb{Z}$  such that  $nz = a$  or  $nz = b$ . This means that  $n\mathbb{Z}$  is not prime, as we have found  $a, b$  such that  $ab \in n\mathbb{Z}$  but neither  $a$  nor  $b$  are in  $n\mathbb{Z}$ .

Now, the reverse direction. First, we show the condition for  $n$  prime. Assume that we have  $a, b \in \mathbb{Z}$  such that  $ab \in n\mathbb{Z}$ . This means that we have  $ab = nq$  for some  $q \in \mathbb{Z}$ . In particular, this means that  $n$  divides the product  $ab$ . However, we note that as  $n$  is prime, we have that  $n$  must divide  $a$  or  $b$  by Euclid's lemma. Thus, we have that either  $a = nr$  or  $b = nr$  (or both), which implies that  $a \in n\mathbb{Z}$  or  $b \in n\mathbb{Z}$ . Next, for  $n = 0$ . Well, if  $ab \in 0\mathbb{Z}$ , then  $ab = 0$ . This in  $\mathbb{Z}$  implies that either  $a$  or  $b$  is 0 and is also in  $n\mathbb{Z}$ . This completes the reverse direction.  $\square$

**Theorem 2.8.** Let  $R$  be a ring.

- i)  $R$  is a domain  $\iff \{0\}$  is prime.
- ii)  $R/I$  is a domain  $\iff I \subset R$  is a prime ideal.
- iii) Let  $\varphi : R \rightarrow S$  be a morphism,  $S$  a domain. Then  $\ker(\varphi)$  is prime. The converse is true if  $\varphi$  is surjective.
- iv)  $R$  is a field  $\iff \{0\}$  is maximal.
- v)  $R/I$  is a field  $\iff I \subset R$  is a maximal ideal.
- vi) Every field is a domain.
- vii) Every maximal ideal is prime.

*Proof.* We first claim that iii) implies ii) which in turn implies i). First, for iii) implies ii), we note that letting  $S$  be  $R/I$  (which means  $\varphi$  is the projection map  $p$  (which is definitely surjective)) gives us ii). (We have that  $\ker(p) = I$ ).

ii) implies i) simply by letting  $I$  be the zero ideal.

Now, we prove statement iii).

Let  $a, b \in R$  such that  $a \cdot b \in \ker(\varphi)$ . Then  $0 = \varphi(a \cdot b) = \varphi(a)\varphi(b)$ . Since we have that  $S$  is a domain, then we have no zero divisors, meaning that either  $\varphi(a) = 0$  or  $\varphi(b) = 0$ . This in turn implies that either  $a \in \ker(\varphi)$  or  $b \in \ker(\varphi)$ , so we have shown that  $\ker(\varphi)$  is a prime ideal. Now, the converse assuming surjectivity. We want to show that  $S$  has no zero divisors. Well, fix  $A, B \in S$  such that  $A \cdot B = 0$ . Since  $\varphi$  is surjective, we have  $a, b \in R$  such that  $\varphi(a) = A$  and  $\varphi(b) = B$ . Then, we have  $0 = \varphi(a)\varphi(b) = \varphi(ab)$ , meaning that  $ab$  is in

$\ker(\varphi)$ . Because we assume that  $\ker(\varphi)$  is prime, this in turn implies that either  $a$  or  $b$  is in  $\ker(\varphi)$  meaning that either  $\varphi(a) = 0$  or  $\varphi(b) = 0$ . This means that either  $A$  or  $B$  is 0, and thus  $S$  is a domain, as desired.

Next, note that v) implies iv). This comes from letting  $I$  be the zero ideal.

The proof of v) comes from the bijection

$$\{\text{ideals in } R \text{ containing } I\} \leftrightarrow \{\text{ideals in } R/I\}$$

This is a homework problem.

Now, we show vi). Assume that  $F$  is a field. Pick  $a, b \in F$  such that  $a \cdot b = 0$  with  $a \neq 0$ . We will show that  $b$  must be 0, thereby showing that  $F$  is a domain. Well, since  $a \neq 0$ , and  $F \setminus \{0\}$  is a group, we have that  $a^{-1}$  exists. Thus, we have that  $ab = 0$  implies that  $a^{-1}ab = 0$  and thus  $b = 0$ , as desired.

vii) follows from the facts vi), v) and ii). We have that

$$I \text{ is a maximal ideal} \xLeftrightarrow{\text{v}} R/I \text{ is a field} \xRightarrow{\text{vi}} R/I \text{ is a domain} \xLeftrightarrow{\text{ii}} I \text{ is prime.}$$

□

### 3. JANUARY 9, 2017

**Definition 3.1.** Let  $R$  be a domain. The canonical morphism  $\mathbb{Z} \rightarrow R$  of Fact ?? has a prime ideal as its kernel. By Thm ??, this is of the form  $p\mathbb{Z}$  with  $p$  prime or  $p = 0$ . We call  $p$  the **characteristic** of  $R$ .

*Example.*

$$\begin{aligned} \text{char}(\mathbb{Z}) &= 0 & \text{char}(\mathbb{Z}/3\mathbb{Z}) &= 3 \\ \text{char}(\mathbb{Q}) &= 0 & \text{char}(\mathbb{Z}/6\mathbb{Z}) &\text{ doesn't exist! } \mathbb{Z}/6\mathbb{Z} \text{ is not a domain.} \end{aligned}$$

**Lemma. (Zorn's Lemma)** (from Artin). An **inductive** (every totally ordered subset has an upper bound) partially ordered set  $S$  has at least one maximal element.

**Theorem 3.2.** Let  $R$  be a ring. Every proper ideal is contained in a max ideal.

*Proof.* Let  $I \subset R$  be a proper ideal. Let  $\mathcal{M}$  be the set of all proper ideals of  $R$  that contain  $I$ , with partial order given by inclusion.

Let  $\mathcal{C} \subset \mathcal{M}$  be a totally ordered subset.

**Claim.**  $J_0 = \left( \bigcup_{J \in \mathcal{C}} J \right) \in \mathcal{M}$

*Proof.* (of claim). We want to show that  $J_0$  is a proper ideal containing  $I$ . First, we show it is an ideal by showing closure of the subgroup and the ideal multiplicative closure. Let  $f_1, f_2 \in J_0$  and  $a \in R$ . Now, this means there is  $J_1, J_2 \in \mathcal{C}$  such that  $f_1 \in J_1$  and  $f_2 \in J_2$ . However, since  $\mathcal{C}$  is totally ordered, we have that the larger of  $J_1$  and  $J_2$  contains both  $f_1$  and  $f_2$ , meaning that we have the existence of  $J \in \mathcal{C}$  such that  $f_1, f_2 \in J$ . Since  $J$  is an ideal, we have that  $f_1 + f_2 \in J$  and that  $a \cdot f_1 \in J$ . This thus implies that since  $J \in \mathcal{C}$ , we have that  $a \cdot f_1$  and  $f_1 + f_2$  are both in  $J_0$ . Thus  $J_0$  is an ideal. Since  $I \in \mathcal{C}$ , we also have that  $I \subset J_0$ . Finally,  $J_0$  is not  $R$ , because otherwise  $1 \in J_0$ , which would mean that  $1 \in J$  for some  $J \in \mathcal{C}$ . This would then imply that  $J = R$ , which is not possible as  $J$  itself is a proper ideal. Thus, we have that  $J_0 \in \mathcal{M}$ . □

Thus, for every totally ordered subset of  $\mathcal{M}$ , we have the existence of an upper bound (namely  $J_0$ ). This gives us, by Zorn's Lemma, that  $\mathcal{M}$  has a maximal element. This maximal element is exactly what we wished to show existed. □

**Definition 3.3.** Let  $R, S$  be rings. Their product is the set  $R \times S$  with component-wise operations

- $(r, s) + (r', s') = (r + r', s + s')$
- $(r, s) \cdot (r', s') = (r \cdot r', s \cdot s')$
- $1_{R \times S} = (1_R, 1_S), 0_{R \times S} = (0_R, 0_S)$

*Remark.* Given morphisms  $\varphi_1 : R \rightarrow S_1, \varphi_2 : R \rightarrow S_2$ , we get a unique morphism  $\varphi_1 \times \varphi_2 : R \rightarrow S_1 \times S_2$ .

*Remark.* Given  $I, J \subset R$  ideals we have

$$I \cdot J \subset I \cap J \subset I, J \subset I + J$$

**Definition 3.4.** Two ideals  $I, J \subset R$  are **coprime** if  $I + J = R$ .

**Theorem 3.5. (Chinese Remainder Theorem)** Let  $R$  be a ring,  $I_1, \dots, I_n \subset R$  be pairwise coprime ideals. Then the natural morphism

$$p : R \rightarrow R/I_1 \times R/I_2 \times \dots \times R/I_n$$

factors through the quotient  $R/(I_1 \cap I_2 \cap \dots \cap I_n)$  and induces an isomorphism of rings

$$\bar{p} : R/(I_1 \cap I_2 \cap \dots \cap I_n) \rightarrow R/I_1 \times R/I_2 \times \dots \times R/I_n$$

Moreover,  $I_1 \cdot I_2 \cdot \dots \cdot I_n = I_1 \cap I_2 \cap \dots \cap I_n$

*Proof.* As  $p$  is the natural morphism to a product of rings, we let  $p = p_1 \times p_2 \times \dots \times p_n$ , where each  $p_i$  is the projection morphism from  $R$  to  $R/I_i$ . Now, we can say that  $\ker(p) = \{r \in R \mid 0 = p_1(r), 0 = p_2(r), \dots, 0 = p_n(r)\}$ . Well, since each  $p_i$  by definition has kernel exactly  $I_i$ , this is the same as saying that  $\ker(p) = \{r \in R \mid r \in I_1 \cap I_2 \cap \dots \cap I_n\}$ .

By the homomorphism theorem (??), we have that  $p$  factors through  $R/I_1 \cap I_2 \cap \dots \cap I_n$  and also induces an injective ring morphism  $\bar{p} : R/I_1 \cap \dots \cap I_n \rightarrow R/I_1 \times \dots \times R/I_n$ .

**Claim.**  $\bar{p}$  is also surjective, and hence a isomorphism.

*Proof.* (of claim) We note that since each of the ideals are coprime, we have that  $I_1 + I_k = R$ . Now, we also note that  $R \cdot R = R$ . Thus, we can express

$$R = (I_1 + I_2) \cdot (I_1 + I_3) \cdot \dots \cdot (I_1 + I_n)$$

expanding the product, we note that by the earlier remark that any term containing an  $I_1$  (which is almost all of them) will be contained in  $I_1$ . The only term that is outside arises from selecting the second term in every single term of the product, so we can write that the above expression is

$$\subset I_1 + (I_2 \cdot I_3 \cdot \dots \cdot I_n)$$

Now, since  $R \subset I_1 + (I_2 \cdot I_3 \cdot \dots \cdot I_n)$ , we can take  $v_1 \in I_1$  and  $u_1 \in I_2 \cdot \dots \cdot I_n$  such that  $u_1 + v_1 = 1$ . Now, since  $u_1 \in I_2 \cdot \dots \cdot I_n$ ,  $u_1 \in I_j$  for  $j \neq 1$ . Thus, we can say that  $u_1$  maps to  $0_{R/I_j}$  under the projection map, as it is in the kernel.

Similarly, since  $u_1 = 1 - v_1$ , with  $v_1 \in I_1$ , we have that  $u_1 \in 1 + I_1$ , meaning that  $u_1$  maps to  $1_{R/I_1}$  under the projection map.

So, we have (abusing notation) that  $u_1 = 1$  in  $R/I_1$  and  $u_1 = 0$  in  $R/I_j$  for  $j \neq 1$  (really, as we showed above, it belongs to the associated cosets).

Now, we can repeat this construction with any  $I_i$  instead of  $I_1$ . Thus, we get for each such construction a  $v_i \in I_i$  and  $u_i \in I_1 \cdot I_2 \cdot \dots \cdot \widehat{I_i} \cdot \dots \cdot I_n$ . With this construction, we now have the existence of the  $u_i$  that belong to the 1 coset in exactly  $R/I_i$  and the 0 coset in all remaining  $R/I_j$ . With this, we can prove surjectivity. Fix any  $(x_1, \dots, x_n) \in R/I_1 \times \dots \times R/I_n$ . We have that there exists an associated  $r_1, \dots, r_n \in R$  such that  $p_1(r_1) = x_1, \dots, p_n(r_n) = x_n$ . Now, if we consider the element  $r \in R$  that equals  $u_1 r_1 + u_2 r_2 + \dots + u_n r_n$ , note that  $p(r) = (p_1(r), p_2(r), \dots, p_n(r))$ . However, since the  $u_i$  map to 1 under  $p_i$  and to 0 otherwise, this maps

precisely to  $(x_1, \dots, x_n)$ . Thus, we have that  $p(r)$  maps to the desired element in the product, meaning that the associated coset will map to the desired element under  $\bar{p}$ . This proves surjectivity.  $\square$

Thus, we have that  $\bar{p}$  is an isomorphism. Now, we show the second part of the statement. Well, we know by definition that  $I_1 \cdot I_2 \cdots I_n \subset I_1 \cap \cdots \cap I_n$ . So, we simply need to show the other containment, which we do by induction on  $n$ .

$n = 1$ :  $I_1 \subset I_1$ .

$n = 2$ : Take  $u_1 \in I_1$  and  $u_2 \in I_2$  such that  $1 = u_1 + u_2$  (this exists as  $I_1 + I_2 = R$ .) Now, for any  $u \in I_1 \cap I_2$ , we have

$$u = u \cdot 1 = u \cdot (u_1 + u_2) = u \cdot u_1 + u \cdot u_2$$

Since  $u \in I_1$  and  $u \in I_2$ , we have  $u \cdot u_1 \in I_2 \cdot I_1$  and  $u \cdot u_2 \in I_1 \cdot I_2$ . Thus, we have the sum in  $I_1 \cdot I_2$ . This gives us  $I_1 \cap I_2 \subset I_1 \cdot I_2$ .

Now, for general  $n$ . By the inductive hypothesis, we have that  $I_1 \cap I_2 \cdots I_n \subset (I_1 \cdots I_{n-1}) \cap I_n$ . From the claim above, we know that  $R = (I_1 \cdots I_{n-1}) + I_n$ . This implies thus that the ideals  $(I_1 \cdots I_{n-1})$  and  $I_n$  are coprime. Thus, applying the  $n = 2$  case on these 2 ideals, we have that  $(I_1 \cdots I_{n-1}) \cap I_n \subset (I_1 \cdots I_{n-1}) \cdot I_n$ , thereby proving the desired result.  $\square$

#### 4. JANUARY 11, 2017

*Remark.* • Any field is a domain.

- Any subring of a domain is a domain.
- Any subring of a field is a domain.

Is the opposite true?

**Theorem 4.1.** *Let  $R$  be a domain.*

1) *There exists a pair  $(i, K)$  with  $K$  a field,  $i : R \rightarrow K$  an injective morphism such that if  $(j, L)$  is another such pair, there exists a morphism  $l : K \rightarrow L$  such that  $j = l \circ i$ , which is to say that the following diagram commutes.*

$$\begin{array}{ccc} & R & \\ i \swarrow & & \searrow j \\ K & \xrightarrow{l} & L \end{array}$$

2) *If  $(i', K')$  is another pair as in 1) there exists a unique isomorphism  $\phi : K \rightarrow K'$  such that*

$$\begin{array}{ccc} & R & \\ i \swarrow & & \searrow i' \\ K & \xrightarrow{\phi} & K' \end{array} \quad \text{commutes.}$$

*Remark.* •  $(i, K)$  is an example of a “universal object”

- $(j, L)$  is called a test object
- $K$  is produced from  $R$ , just like the rationals are produced from the integers.

*Proof.* 2) Given two universal objects  $(i, K)$ ,  $(i', K')$ , apply 1) with  $(i, K)$  as the universal object, and  $(i', K')$  as a test object to get  $l : K \rightarrow K'$ . Do it the other way to get  $l' : K' \rightarrow K$ .

**Claim.**  $l \circ l' = id_{K'}$ ,  $l' \circ l = id_K$

*Proof.* Note that both  $l' \circ l$  and  $id_K$  make the diagrams



$$\begin{array}{ccc}
& R & \\
i \swarrow & & \searrow i \\
K & \xrightarrow{i l' \circ l} & L
\end{array}$$

$$\begin{array}{ccc}
& R & \\
i \swarrow & & \searrow i \\
K & \xrightarrow{i \text{id}_K} & L
\end{array}
\quad \text{commute.}$$

When  $(i, k)$  is both a universal object and a test object, we get  $l \circ l = \text{id}_K$ . □

1) Consider the set  $P = R \times R \setminus \{0\}$ . Introduce the relation  $(n, d) \sim (n', d') \iff nd' = n'd$ .

**Claim.**  $\sim$  is an equivalence relation.

*Proof.* Reflexive:  $(n, d) \sim (n, d) \iff nd = nd$ .

Symmetric  $(n, d) \sim (n', d') \iff nd' = n'd \iff n'd = nd' \iff (n', d') \sim (n, d)$ .

Transitive: Assume  $(n_1, d_1) \sim (n_2, d_2) \sim (n_3, d_3)$ . We want  $(n_1, d_1) \sim (n_3, d_3)$ . We have  $n_1 d_2 = n_2 d_1$ ,  $n_2 d_3 = n_3 d_2$  and want  $n_1 d_3 = n_3 d_1$ . We see that  $n_1 d_3 n_2 d_2 = n_1 d_3 n_2 d_3 = n_2 d_1 n_3 d_2 = n_3 d_1 n_2 d_2$ . Since  $R$  is a domain,  $n_2 d_2$  is not a zero-divisor. If  $n_2 d_2 \neq 0$ , then by Fact 6, Jan 6, we get  $n_1 d_3 = n_3 d_1$ . If  $n_2 d_2 = 0$ , then  $(d_2 \neq 0 \text{ and not a 0-divisor}) \implies n_2 = 0$ . For the same reason,  $n_1 = n_3 = 0$ . Again,  $n_1 d_3 = n_3 d_1$ . Either way, we are done. □

Put  $K = P / \sim$ . Write  $[n, d]$  for the image of  $(n, d) \in P$  in  $K$ . Define

$$[n, d] \cdot [n', d'] = [nn', dd']$$

$$[n, d] + [n', d'] = [nd' + n'd, dd']$$

$$0 = [0, 1], 1 = [1, 1]$$

$$i : R \rightarrow K, i(r) = [r, 1].$$

We leave as homework the verifications that  $+, \cdot$  are well defined, that  $K$  is a field, and that  $i$  is a morphism. Injectivity is obvious. Given  $(j, L)$ , define  $l : K \rightarrow L$  by

$$l([n, d]) = l(i(n)) \cdot l(i(d)^{-1}) = j(n)j(d)^{-1}.$$

Homework:  $l$  is well defined and a ring morphism. □

**Definition 4.2.** A pair  $(i, K)$  is called a (the) field of fractions (fraction field) of  $R$ .

**Definition 4.3.** 1) Let  $R$  be a ring. A polynomial in  $T$  over  $R$  is a formal expression  $a_n T^n + a_{n-1} T^{n-1} + \dots + a_0$ ,  $a_i \in R$ .

2) Given  $P(T) = a_n T^n + \dots + a_0$ ,  $Q(T) = b_m T^m + \dots + b_0$  define

$$(P + Q)(T) = (a_n + b_n)T^n + \dots + (a_0 + b_0)$$

$$(P \cdot Q)(T) = (c_m T^m + c_{m-1} T^{m-1} + \dots + c_0$$

where

$$c_k = \sum_{i+j=k} a_i \cdot b_j.$$

3) Given  $r \in R$  we have the constant polynomial  $r : (a_n T^n + \dots + a_0, a_0 = r, a_i = 0 \text{ for } i > 0)$ . In particular, we have  $0, 1$  as constant polynomials.

4) Let  $R[T]$  be the set of all polynomial in  $T$  over  $R$ .

**Fact 4.4.**  $(R[T], +, \cdot, 0, 1)$  is a ring. Moreover  $R \rightarrow R[T], r \rightarrow \text{constant polynomial } r$  is an injective morphism. The proof is left as an exercise to the reader.

**Definition 4.5.** Given  $0 \neq P \in R[T]$ , define  $\deg(P) = \min\{n | a_m = 0 \forall m > n\}$ ,  $\deg(0) = -\infty$ .

**Fact 4.6.** 1)  $\deg(P + Q) \leq \max(\deg(P), \deg(Q))$  with equality if  $\deg(P) \neq \deg(Q)$ . 2)  $\deg(P \cdot Q) \leq \deg(P) + \deg(Q)$  with equality if the leading coefficient of  $P$  (or  $Q$ ) is not a 0 divisor. 3) In particular, if  $R$  is a domain, so is  $R[T]$ . The proof is left as an exercise to the reader.

5. JANUARY 13, 2017

*Remark.* Any  $P \in R[T]$  gives a function  $R \rightarrow R$  by  $r \mapsto P(r) = a_n r^n + \dots + a_0$ . However,  $P$  is not necessarily determined by this function. For example, let  $R = \mathbb{Z}/p\mathbb{Z}$  where  $p$  is a prime and  $P(T) = T^p - T$ . Since  $x^p = x$  for all  $x \in R$ ,  $P$  and  $0$  give the same function. However,  $P \neq 0$ .

*Example.*

$P = T^2 + 3T - 2, Q = -T^2 + 3T - 7$  gives  $P + Q = 6T - 9$  ( $\deg(P + Q) < \max(\deg(P), \deg(Q))$ )

$R = \mathbb{Z}/4\mathbb{Z}, P = 2T^2 + 1, Q = 2T^3 + 3T$  gives  $PQ = 3T$  ( $\deg(PQ) < \deg(P) + \deg(Q)$ )

**Fact 5.1.** Let  $\phi : R \rightarrow S$  be a morphism and let  $s \in S$ . There exists a unique morphism  $\phi_s : R[T] \rightarrow S$  such that  $\phi_s(r) = \phi(r)$  for all  $r \in R$  and  $\phi_s(T) = s$ .

*Proof.* If  $\phi_s$  is any such morphism then  $\phi_s(a_n T^n + \dots + a_0)$  must equal  $\phi(a_n)s^n + \dots + \phi(a_0)$ . This proves uniqueness and existence (upon checking that this is a morphism).  $\square$

*Example.*

- If  $\phi = \text{id} : R \rightarrow R$  then we get evaluation morphism  $R[T] \rightarrow R$  given by  $P \mapsto P(s)$ .
- Let  $I \subseteq R$  be an ideal and let  $\phi : R \rightarrow R/I \hookrightarrow R/I[T]$  and let  $s = T$ . We get “reduction mod  $I$ ” morphism  $R[T] \rightarrow R/I[T]$ .

*Remark.* (def. 1.5)  $a \in R$  is **nilpotent** if  $a^n = 0$  for some  $n \in \mathbb{N}$

**Proposition 5.2.** Let  $P = a_n T^n + \dots + a_0 \in R[T]$ . We have  $P \in R[T]^\times$  iff  $a_0 \in R^\times$  and  $a_1, \dots, a_n$  are nilpotent.

*Proof.*

Assume that  $R$  is a domain. We have that  $P$  is a unit iff there exists  $Q \in R[T]$  such that  $PQ = 1$ . By 1-11 Fact 6,  $0 = \deg(1) = \deg(PQ) = \deg(P) + \deg(Q)$  ( $R$  is a domain so the leading coefficient of  $P$  (alternatively  $Q$ ) is not a zero divisor). Thus  $\deg(P), \deg(Q) = 0$ . Thus  $a_1, \dots, a_n = 0$  are nilpotent and  $a_0 \in R^\times$ .

Let  $R$  be a general ring. Let  $\mathcal{P} \subseteq R$  be a prime ideal. Since  $P$  is a unit in  $R[T]$ , the image of  $P$  in  $R/\mathcal{P}[T]$  is a unit. Since  $R/\mathcal{P}$  is a domain by 1-6 thm. 8, by the above argument  $a_1, \dots, a_n = 0_{R/\mathcal{P}}$  and thus  $a_1, \dots, a_n \in \mathcal{P}$ . Since this holds for all  $\mathcal{P}$ , by HW we have that  $a_1, \dots, a_n$  are nilpotent.  $\square$

**Lemma 5.3.** Let  $P \in R[T]$  and  $r \in R$ . We have  $P(r) = 0$  iff  $(T - r) \mid P$ .

*Proof.* The backward direction is clear. Apply fact 1 with  $S = R[T]$ ,  $\phi : R \hookrightarrow R[T]$ ,  $s = T + r$  to get a morphism  $R[T] \rightarrow R[T]$ . This is an isomorphism with inverse given by the same construction with  $s = T - r$ . Under this isomorphism,  $P \mapsto Q$  with  $Q(0) = 0$ . Thus  $Q(T) = b_n T^n + \dots + b_1 T$  so  $T \mid Q$ . Taking the preimage under the above isomorphism, we have  $(T - r) \mid P$ .

Gitlin's thoughts: The constructed isomorphism can be thought of as the map  $R[T] \rightarrow R[T]$  which "replaces every  $T$  with  $T + r$ ." Thus  $Q(x) = P(x + r)$  for all  $x \in R$ . In particular,  $Q(0) = P(r)$  which is 0. The inverse map is the map  $R[T] \rightarrow R[T]$  which "replaces every  $T$  with  $T - r$ ." In particular, the preimage of  $Q(T) = b_n T^n + \dots + b_1 T$  is  $b_n(T - r)^n + \dots + b_1(T - r)$ . □

**Proposition 5.4.** *Let  $P, D \in R[T]$ . Assume that  $D \neq 0$  and that the leading coefficient of  $D$  is a unit. There exist unique  $Q, Z \in R[T]$  with  $\deg(Z) < \deg(D)$  such that  $P = QD + Z$ .*

*Proof.*

Choose  $Q$  so that  $\deg(Z)$  is minimal where  $Z = P - QD$ . We claim  $\deg(Z) < \deg(D)$ . Suppose not. Let  $D = d_n T^n + \dots + d_0$  and  $Z = z_m T^m + \dots + z_0$  with  $m \geq n$ . Note that  $P - (Q + z_m d_n^{-1} T^{m-n})D = Z - (z_m d_n^{-1} T^{m-n})D$  has degree less than  $\deg(Z)$ , contradicting the minimality of  $Z$ . This shows existence.

Gitlin's thoughts: The set of "candidates" is the set of elements of  $R[T]$  that have the form  $P - *D$  where  $*$  varies over  $R[T]$ . Clearly  $P - (Q + z_m d_n^{-1} T^{m-n})D$  is a candidate. Furthermore, the leading term  $z_m T^m$  of  $Z$  cancels with the leading term  $z_m d_n^{-1} T^{m-n} \cdot d_n T^n = z_m T^m$  of  $(z_m d_n^{-1} T^{m-n})D$  in the subtraction  $Z - (z_m d_n^{-1} T^{m-n})D$  so the degree of  $Z$  is at least one more than the degree of  $Z - (z_m d_n^{-1} T^{m-n})D$ .

Let  $Q', Z'$  be another such pair. We have  $QD + Z = P = Q'D + Z'$  so  $(Q - Q')D = Z' - Z$ . Thus (1-11 fact 6)  $\deg(D) > \max(\deg(Z'), \deg(Z)) \geq \deg(Z' - Z) = \deg((Q - Q')D) = \deg(Q - Q') + \deg(D)$  (the leading coefficient of  $D$  is a unit and thus not a divisor of zero). This means  $\deg(Q - Q') = -\infty$  so  $Q - Q' = 0$  so  $Q = Q'$ . Thus  $Z = P - QD = P - Q'D = Z'$ . This shows uniqueness.

Gitlin's thoughts: My uniqueness proof likely differs from the one given in class. Sorry Tasho. I couldn't follow your inequalities. □

6. JAN. 18, 2017

**Lemma 6.1.** *(This is Lemma 3 from Jan. 13) Given  $P \in R[T], r \in R$ , then  $P(r) = 0 \Leftrightarrow (T - r) \mid P$ .*

**Proposition 6.2.** *(This is Proposition 4 from Jan. 13) Given  $P, D \in R[T]$  such that  $D$  has a unit as a leading coefficient, there exists unique  $Q, Z \in R[T]$  with  $\deg(Z) < \deg(D)$  such that  $P = DQ + Z$ .*

*Proof.* This proof rehashes the one given in class by Tasho to rectify any mistakes or lack of clarity in the one given on Jan. 13 (no offense Gitlin). Following the proof given on Jan. 13 (before the proof of uniqueness), we have the following string of inequalities:

$$\deg(D) > \deg(Z) \geq \deg(Z - Z') = \deg(D) + \deg(Q - Q').$$

This implies that  $\deg(Q - Q') = -\infty = \deg(Z - Z')$ , so  $Q - Q' = Z - Z' = 0$ , i.e.  $Q = Q'$  and  $Z = Z'$ , completing the proof. □

The following is a proof Tasho gave of Lemma 3 from Jan. 13 which utilizes Proposition 4 from Jan. 13.

*Proof.* As per Proposition 4, write  $P(T) = (T - r) \cdot Q(T) + Z(T)$  with  $\deg(Z) < \deg(T - r) = 1 \Rightarrow Z \in R$ . We have then  $P(r) = (r - r) \cdot Q(r) + Z(r) = Z$ , but since  $P(r) = 0$  we have now  $Z = 0 \Leftrightarrow (T - r) \mid P$ . □

**Definition 6.3.** Let  $P \in R[T], r \in R$ . Define **the multiplicity of  $r$  in  $P$**  by  $\max\{n \in \mathbb{N} : (T - r)^n \mid P\}$ .

**Corollary 1.** *Let  $R$  be a domain. Then  $0 \neq P \in R[T]$  has at most  $\deg(P)$ -many zeroes, counted with multiplicity.*

*Proof.* We induct on  $\deg(P)$ . As a base case take  $\deg(P) = 0$ :  $P \in R$  has no zeroes, as required. For an inductive step assume that the statement holds for all polynomials of degree less than  $\deg(P)$ . If  $P$  has no zeroes, we are done. Otherwise let  $r \in R$  be a zero of  $P$ . By Lemma 3 from Jan. 13,  $P = (T - r) \cdot Q(T)$  for some  $Q \in R[T]$  with  $\deg(Q) < \deg(P)$ . By inductive hypothesis  $Q$  has at most  $\deg(Q) = \deg(P) - 1$  zeroes. Since  $R$  is a domain, a zero of  $P$  is either  $r$  or a zero of  $Q$ , so  $P$  has at most  $\deg(Q) + 1 = \deg(P)$  zeroes, as required. This completes the inductive step, and thus the proof.  $\square$

The following example answers the question “does this fail when  $R$  is not a domain?”:

*Example.* If  $R = \mathbb{Z}/6\mathbb{Z}$  and we define  $P(T) = 3T$ , then  $\#\{\text{zeroes of } P\} = \#\{0, 2\} > 1 = \deg(P)$ .

**Definition 6.4.** A field  $k$  is called **algebraically closed** provided that every nonconstant  $P \in k[T]$  has a zero.

**Proposition 6.5.** *Let  $k$  be an algebraically closed field,  $P \in k[T]$ . Then*

$$P(T) = c \cdot (T - r_1)^{m_1} \cdots (T - r_n)^{m_n}$$

for  $c, r_1, \dots, r_n \in k, m_1, \dots, m_n \in \mathbb{N}$ .

*Proof.* This is proved on Homework 2. (Future readers: feel free to input the proof here when completed.)  $\square$

**Definition 6.6.** Define  $\mathbf{R}[T_1, \dots, T_n]$  recursively as  $(R[T_1, \dots, T_{n-1}])[T_n]$ . More concretely: a **monomial** in  $T_1, \dots, T_n$  is an expression  $T_1^{m_1} \cdots T_n^{m_n}$ , for  $m_i \in \mathbb{N}$ . A polynomial in  $T_1, \dots, T_n$  with coefficients in  $R$  is an  $R$ -linear combination of monomials; the set of all polynomials in  $T_1, \dots, T_n$  is  $R[T_1, \dots, T_n]$ .

Recall:  $P \in R[T] \leftrightarrow$  an eventually-zero sequence  $(a_0, a_1, \dots) \leftrightarrow$  a map  $\mathbb{N} \rightarrow R$  with finite support. So  $P \in R[T_1, \dots, T_n] \leftrightarrow$  a map  $\mathbb{N}^n \rightarrow R$  with finite support, so we may think of a polynomial  $P \in R[T_1, \dots, T_n]$  as

$$P(T) = \sum_{c_i \in \mathbb{N}} a_{c_1, \dots, c_n} \cdot T_1^{c_1} \cdots T_n^{c_n}.$$

**Fact 6.7.** *Given a ring morphism  $\phi: R \rightarrow S, s_1, \dots, s_n \in S$ , there exists a unique ring morphism*

$$\phi_{s_1, \dots, s_n}: R[T_1, \dots, T_n] \rightarrow S$$

such that  $\phi_{s_1, \dots, s_n}(r) = \phi(r)$  for  $r \in R$ , and  $\phi_{s_1, \dots, s_n}(T_i) = s_i$ .

*Proof.* We induct on  $n$ . As a base case let  $s \in S$ . Fact 5.1, Jan. 13 above guarantees the existence of a unique morphism  $\phi_s: R[T] \rightarrow S$  with the given properties. For an inductive case assume the statement holds for  $n - 1$ . By inductive hypothesis we have a unique morphism  $\phi_{s_1, \dots, s_{n-1}}: R[T] \rightarrow S$  satisfying the given properties. Applying the base case to  $\phi_{s_1, \dots, s_{n-1}}$  with  $s = s_n$  gives the desired (unique) morphism, completing the inductive step and thus the proof.  $\square$

**Definition 6.8.**

- The image of  $\phi_{s_1, \dots, s_n}$  is called the  **$R$ -subalgebra of  $S$  generated by  $s_1, \dots, s_n$**
- $s_1, \dots, s_n$  are called **algebraically independent** provided that  $\phi_{s_1, \dots, s_n}$  is injective.

To gain some intuition about algebraic independence, consider the algebraic relation

$$r_1 s_1^{m_{1,1}} \cdots s_n^{m_{1,n}} + \cdots + r_n s_1^{m_{n,1}} \cdots s_n^{m_{n,n}} = 0.$$

Taking preimages under  $\phi_{s_1, \dots, s_n}$  we have

$$\begin{aligned} \phi_{s_1, \dots, s_n}^{-1}(r_1 s_1^{m_{1,1}} \cdots + s_n^{m_{1,n}} + \cdots + r_n s_1^{m_{n,1}} \cdots s_n^{m_{n,n}}) &\in \phi_{s_1, \dots, s_n}^{-1}(0) = \ker(\phi_{s_1, \dots, s_n}) \\ \Rightarrow \phi^{-1}(r_1) T_1^{m_{1,1}} \cdots T_n^{m_{1,n}} + \cdots + \phi^{-1}(r_n) T_1^{m_{n,1}} \cdots T_n^{m_{n,n}} &\in \ker(\phi_{s_1, \dots, s_n}) \end{aligned}$$

thus if there exists such a relation that is nontrivial, there is a nonzero element in  $\ker(\phi_{s_1, \dots, s_n})$ , so it is not injective. (Note from Ben: this seems to assume that the coefficients  $r_i$  are in the image of  $\phi$ , which is not generally true. Will bring up with Tasho and edit as necessary.)

**Definition 6.9.** Let  $R$  be a ring,  $0 \neq a \in R$  not a unit.

- $a$  is called **irreducible** provided that  $a = bc \Rightarrow b \in R^\times$  or  $c \in R^\times$ .
- $a$  is called **prime** provided that  $a = bc \Rightarrow a \mid b$  or  $a \mid c$ .

**Fact 6.10.**  $R$  a domain  $\Rightarrow$  every prime element of  $R$  is irreducible.

*Proof.* Let  $p \in R$  be prime and write  $p = ab$ . Then without loss of generality we have  $p \mid a \Rightarrow a = pc$  for some  $c \in R$ , so we may rewrite  $p = pcb$ , so  $p(1 - bc) = 0$ . Since  $R$  is a domain and  $p \neq 0$ , we have  $1 = bc$ , so  $c = b^{-1} \Leftrightarrow b \in R^\times$ .  $\square$

*Remark.*

- If  $R$  is not a domain, prime elements need not be irreducible (see Homework 2).
- Even if  $R$  is a domain, irreducible elements need not be prime (see Homework 2).

**Definition 6.11.** Let  $R$  be a domain.

- (1)  $R$  has **factorization** provided that every  $0 \neq r \in R$  can be written  $\epsilon \cdot u_1 \cdots u_k$  with  $\epsilon \in R^\times$  and  $u_i \in R$  irreducible.
- (2) Write  $a \sim b \Leftrightarrow a = \epsilon \cdot b, \epsilon \in R^\times$ .
- (3)  $R$  has **unique factorization** provided that  $R$  has factorization and if  $\epsilon \cdot u_1 \cdots u_k = \mu \cdot v_1 \cdots v_m$  then  $k = m$  and there exists a permutation  $\sigma \in S_m$  such that  $u_i \sim v_{\sigma(i)}$ .
- (4) Such an  $R$  is called a **unique factorization domain**, or UFD.

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**Proposition 7.1.** Let  $R$  be a domain with factorization. Then  $R$  has unique factorization if and only if every irreducible element is prime.

*Proof.* Say  $R$  has unique factorization. Let  $p$  be irreducible. We will show that  $p$  is prime. Let  $a, b \in R$  be such that  $p \mid ab$ . Using factorization, we write that  $a = \zeta u_1 u_2 \cdots u_k$ ,  $b = \mu v_1 \cdots v_l$ , and that  $\frac{ab}{p} = \alpha w_1 w_2 \cdots w_m$ . From the fact that  $\frac{ab}{p} \cdot p = ab$ , we have that  $\alpha p w_1 w_2 \cdots w_m = \zeta \mu u_1 \cdots u_k v_1 \cdots v_l$ . By uniqueness of factorization, we have that these expressions are permutations upto multiplication by units, so we have that  $p \sim u_i$  or  $p \sim v_j$ . In the first case, we have that  $p \mid a$  and in the second, we have that  $p \mid b$ . Assume now that every irreducible is prime. We will show uniqueness by induction on the length of the factorization.

Length = 0: Then,  $a$  is a unit, so  $a$  is not divisible by any irreducible.

Assume now that the statement holds for a length  $k$  factorization. Assume now that we have

$$a = \zeta u_1 \cdots u_{k+1} = \mu v_1 \cdots v_l$$

Now  $u_{k+1}$  divides  $v_1 \cdots v_l$ . But since  $u_{k+1}$  is prime,  $u_{k+1} \mid v_j$  for some  $1 \leq j \leq l$ . Reordering the factors, we have that  $u_{k+1} \mid v_l$ . Since  $v_l$  is irreducible, this thus implies that if  $u_{k+1}x = v_l$ ,  $x$  must be a unit, and thus  $v_l \sim u_{k+1}$ . Furthermore, we can write that  $\zeta u_1 \cdots u_k = \mu \left( \frac{v_l}{u_{k+1}} \right) v_1 \cdots v_{l-1}$ . Applying the inductive

hypothesis, we have  $l - 1 = k$  and the first  $k$  terms are simply a permutation of each other, while the last is simply a unit scaling away from the other. Thus, these are the same factorization.  $\square$

**Definition 7.2.** A ring is called **principal**, if every ideal is principal. If, in addition, the ring is a domain, it is a **Principal Ideal Domain**, or **PID**.

*Example.*  $\mathbb{Z}$  is a PID.  $\mathbb{Z}/n\mathbb{Z}$  are principal, any field is a PID. Quotients and products of principal rings are principals.

**Theorem 7.3.** *Every PID is a UFD*

*Proof.* Let  $R$  be a PID. We first show the existence of factorization. Say that  $a \in R$ ,  $a \neq 0$ ,  $a \in R^\times$ , and has no factorization. Then,  $a$  is not irreducible, (as otherwise, it itself is a factorization). So, we can right that  $a = a_0 = a_1 b_1$ . However, note that one of these must not have a factorization, as otherwise the product of their factorizations is a factorization of  $R$ . So, WoLOG assume that  $a_1$  does not have a factorization. Again,  $a_1$  is not irreducible, and we write  $a_1 = a_2 b_2$ . Recursively applying this argument, we have that  $a_0 = a_1 a_2 a_3 \dots$  an infinite sequence with  $a_i \neq 0$ ,  $a_i \notin R^\times$  and  $a_i \not\sim a_{i+1}$ , and  $a_{i+1} \mid a_i$ . In terms of the ideals, we have that

$$(a_0) \subsetneq (a_1) \subsetneq \dots$$

Now, we consider  $I = \bigcup_{i=0}^{\infty} (a_i)$ . Since we have a nested sequence of ideals, we have that  $I$  itself is an ideal.

We have that  $R$  is a principal ideal, which means that  $I = (c)$  for some  $c \in R$ . This means though that  $c \in I$ , which means that  $c \in (a_k)$  for some  $k \in \mathbb{N}$ . However, this means that for any  $n \geq k$ , we have that  $(c) \subset (a_k) \subset I = (c)$ , meaning that each  $(a_n) = (c)$ . However, we assumed proper inclusion above, and this is thus a contradiction. Thus, we have factorization.

Next, we verify uniqueness. We can show this by showing that every irreducible is prime. Let  $u$  be an irreducible. We want to show it is prime, so assume that  $u \mid ab$ . Assume that  $u \nmid a$ . We will show that  $u \mid b$ . Since we have that  $u \nmid a$ , we have that  $u \notin (a)$ , so we have that  $(a) \subsetneq (a, u)$ . Since  $R$  is principal, we have that  $(u, a) = (d)$ . Since  $u \in (d)$  we have that  $d \mid u$ . However, since  $u$  is irreducible,  $d$  must be a unit, or  $d \sim u$ . We know because  $(u) \subsetneq (u, a)$ , we have that  $d \sim u$  is impossible. Thus, assume that  $d$  is a unit. This means though that  $(u, a) = R$ . This means that we have  $1 \in R$ , so  $1 = \alpha u + \beta a$  (this is what it means to be in  $(u, a)$ ). Scaling this equality by  $b$ , we have that  $b = b\alpha u + \beta(ab)$ . Now, since  $u \mid ab$  and  $u \mid u$ ,  $u \mid b$ , as desired.  $\square$

**Definition 7.4.** A **Euclidean Domain** is a pair  $(R, H)$ , where  $R$  is a domain, and  $H : R \setminus \{0\} \rightarrow \mathbb{N}$  is a function such that

- (1)  $H(ab) \geq H(a)$
- (2) Given  $X, d \in R$  with  $d \neq 0$ , there are  $q, r \in R$  such that
  - (a)  $X = qd + r$
  - (b) either  $r = 0$  or  $H(r) < H(d)$

*Example.* If  $F$  is a field, then  $(F[T], \deg)$  is a Euclidean domain.

**Proposition 7.5.** *Every Euclidean domain is a PID.*

*Proof.* Let  $I \subset R$  be an ideal. WoLOG,  $I \neq \{0\}$ . Choose  $d \in I$  such that  $H(d)$  is minimal. To show that  $I = (d)$ , let  $a \in I$  and take  $q, r \in R$  such that  $a = qd + r$ . Then,  $r = a - qd \in I$ . If  $r \neq 0$ , we have that  $H(r) < H(d)$ , contradicting the minimality of  $d$ . Thus  $r = 0$ , and  $a = qd$ . Thus, every ideal is of the form  $(d)$ , showing it is principal.  $\square$

**Corollary 2.** *If  $F$  is a field,  $F[T]$  is a PID, and hence a UFD.*

**Definition 7.6.** Let  $R$  be a UFD. We call  $0 \neq P \in R[T]$  **primitive** if  $a \in R$  with  $a \mid P$  means that  $a \in R^\times$ .

*Remark.* This is equivalent to no irreducible  $p \in R$  divides all the coefficients of  $P$ . Intuitively, we can say the “gcd” of the coefficients of  $P$  is 1.

**Lemma 7.7. (Gauss)** *If  $P, Q$  are primitive, so is  $PQ$ .*

*Proof.* Let  $P \neq 0, Q \neq 0$  such that  $PQ$  is not primitive. Let  $p \in R$  prime such that  $p \mid PQ$ . Thus,  $PQ$  becomes 0 under the  $R[T] \rightarrow (R/(p))[T]$ .

But  $R/(p)$  is a domain by ??, and thus, by ??, we have that  $(R/(p))[T]$  is also a domain. Thus, we have that either  $P$  or  $Q$  is 0 in  $(R/(p))[T]$ , meaning it is also divisible  $p$ , as desired.  $\square$