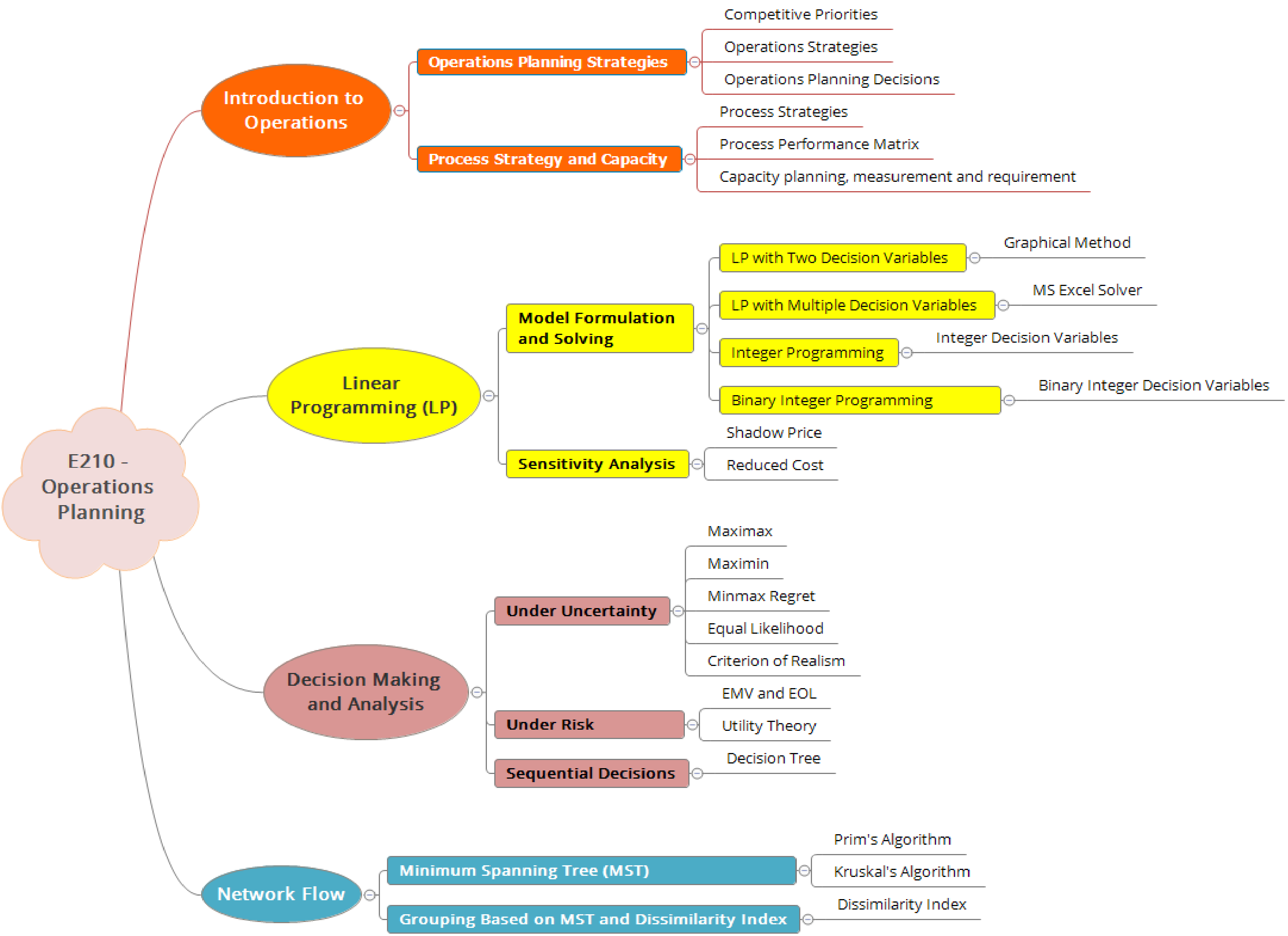
Lesson 03

The Graphical Method

E210 – Operations Planning

SCHOOL OF ENGINEERING

E210 Operations Planning Topic Tree



Scenario – Decorative Vase and Jar

- ArtDecor is a company specialized in producing porcelain-ware, such as dinner set, tea set and home decorative products. Based in Japan, ArtDecor has a number of manufacturing plants in Asia.
- Recently, ArtDecor opened a new manufacturing plant in Thailand. As a start, the plant focuses on producing two home decorative products: vase and Jar.
- As the Operations Manager, Andy is tasked to plan the right product mix (quantity of each product) to produce for the coming week to achieve maximum profit.
- The main resources consumed to produce the decorative vase and jar are clay, labour time and machine time. Andy's experiences told him that the machine time is more than enough so it is not a concern. Table below summarizes the amount of clay and labour hours needed to produce one vase and one jar, with expected profit earned from each of the products sold.

	Clay (Kilograms)	Labor (Hours)	Profit ($)
Vase	1.2	4	75
Jar	0.6	6	60

- Andy understands that the suppliers can provide up to 120 kg of clay each week. Currently, the plant has a maximum of 720 total labour hours available weekly.
- Andy recalls that the product mix can be determined using a mathematical model and a graphical approach.

Scenario – Tasks of the day

Assume that you are a consultant helping Andy plan the right product mix (quantity of each product) to produce for the coming week. Help him

- Determine the optimum quantity to produce for each product in order to maximize the weekly profit.
- Understand how the product mix can be determined using mathematical model and a graphical approach.
- Determine what is limiting the company from making more profit and how the profit will be affected if he could have more resources.
- Understand how sensitive the company's profit from the two products is to the unit profit of the vase.

Scenario Definition Template

- What do we know?

- What do we not know?

- What do we need to find out?

Mathematical Models

- Set of mathematical relationships, equations and logical assumptions as a representation of real world decision making problem.

- An example is mathematical programming, also referred to as optimization.

- Models have been applied successfully in the areas of military, industry, transportation, economics, health care systems, etc.

Mathematical Models

- 3 stages in the application of mathematical models to real-world problems:
  - Translating problem situation into a formal mathematical model,
  - Solving mathematical problem, and
  - Interpreting solution for the original situation

- Linear Programming (LP) is a type of mathematical technique used to allocate resources among competing demands in an optimal way.

Linear Programming (LP)

Identifying an LP problem:

- There are limited resources *(e.g. a finite number of hours available for each machine)*
- There is an explicit objective function (*e.g. worth of each variable and what is the goal in solving the problem?*)
- The equations are linear *(e.g. no cross-products)*
- The resources are homogenous *(e.g. everything is in one unit of measure, such as machine hours)*
- The decision variables are divisible and non-negative *(e.g. we can make fractional part of a product. If deemed undesirable, then integer programming would be used)*

Linear Programming (LP)

- Mathematical model designed to optimize the use of limited resources.
  - E.g. maximize profit or minimize cost

- Assumes linearity of all model equations.
  - Equations are of the form:

    $$ax + by + \ldots + cz = d$$

    where a, b, c and d are constants(coefficient)

- LP models can be efficiently solved in Excel spreadsheet or other analytical software.

LP Model Formulation Process

- Understand the problem.
- Define the **Decision Variables**.
- State the **Objective Function** in terms of the decision variables.
- Formulate each **Constraint** in terms of the decision variables.
- Identify any upper or lower bounds on the decision variables or if they should be integers.

LP Model Formulation

- Helpful questions to identify the major components of the Linear Programming model:

1. What am I trying to decide? (**Decision Variables**)
   - E.g. Number of Product A to make, Number of staff to hire

2. What is the aim? (**Objective Function**)
   - E.g. Maximize Profit, Minimize Cost

3. What limitations or requirements restrict the values of the decision variables? (**Constraints**)
   - E.g. Machine Time <= 50 hrs per week
   - Number of units to ship overseas >= 100

LP Formulation - Decision Variables & Objective Function

- The decision variables must represent a physical measurable quantity (number of units, weight, volume, etc.). The objective function is formulated using decision variables and coefficients. Example:
  - Decision variables are:

    Let X_1 = Number of product A to make

    Let X_2 = Number of product B to make
  - Objective function is:

    Maximize profit $Z = 100X_1 + 150X_2$

Objective function coefficients for X_1 and X_2 respectively, i.e., each unit of X_1 brings \$100 of profit and each unit of X_2 brings \$150 of profit. In the case where the objective function is to minimize cost, cost coefficients should be used.

LP Formulation - Constraints

- Each constraint in an LP represents a relation between two measurable quantities: Limited resource (≤), Minimum performance (≥), and Conservation/ balance (=).
- Example: A total of 150 labour hours are available for the production of X_1 and X_2

$$20X_1 + 10X_2 \leq 150$$

Production of each unit of X_1 requires 20 labour hours and each unit of X_2 requires 10 labour hours. They are constraint coefficients.

- **Special note**: Most frequent error in constraint formulation stage is "reversal of variables".

  How to represent “There is one teacher for every twelve students”?

Answer: S = 12T (S: Number of students)
(T: Number of teachers)

Check: When T=1, S = 12(1) = 12

Test Yourself – Mathematical Relationships

Translate the following statements into mathematical expressions.

a) 1 teacher will be in-charge of 20 students

S=20T

b) The number of students must be at least 15 times the number of teachers

S>=15T

c) The proportion of female students should not exceed 50% of all students (male and female)

F/(F+M)<=50%

Test Yourself – LP Formulation

Consider the scenario given, identify the major components in the LP formulation.

- Andy wants to know **how many vases** and **how many jars** to produce (**Decision**) per week so that,
- He could earn **as much profit as possible.**
  - Maximize profit (**Objective**). With each vase earning $75 and each jar earning $60.
- He has **limited** supply of clay (120 kg) and available labour hours (720 hours) (**Constraints**).
- He needs 1.2 kg of clay and 4 hours of labour to produce one vase and needs 0.6 kg of clay and 6 hours of labour to produce one Jar.

Test Yourself – LP Formulation

Consider the scenario given, formulate the LP model by stating the decision variables, constructing the objective function and formulating the constraints.

Decision Variables

Let X_1 be the number of decorative vases to produce per week
Let X_2 be the number of decorative jars to produce per week

Objective Function
Maximize profit $Z = 75X_1 + 60X_2$

Constraints

$1.2X_1 + 0.6X_2 \leq 120$ (Clay constraint)

$4X_1 + 6X_2 \leq 720$ (Labour constraint)

$X_1, X_2 \geq 0$ (Non–negativity constraints for the decision variables)

Solving a 2-decision variable LP Model using Graphical Method

1. Plot the ***constraints*** as equations in the graph.

2. Taking into account the **inequalities** of the constraints, indicate the feasible solution area.

Then either:

3. Plot the ***objective function*** and move this line in **parallel** away from the origin. Locate the ***extreme point*** of the feasible region.

4. Solve simultaneous equations at the extreme point to find the optimal solution values.

Or:

1. Solve simultaneous equations at each corner point of the feasible region to find the solution values at each point.

2. Substitute these values into the objective function to find the set of values that results in maximum/minimum Z value

Note: It may be possible to read the solution values directly from the graph instead of solving as simultaneous equations.

(Watch the video https://docs.google.com/file/d/0Bz-Yttbj61bxeW16a3R1NEQ5dVE/edit?usp=sharing)

Hands-on Session on Graphical Method (1)

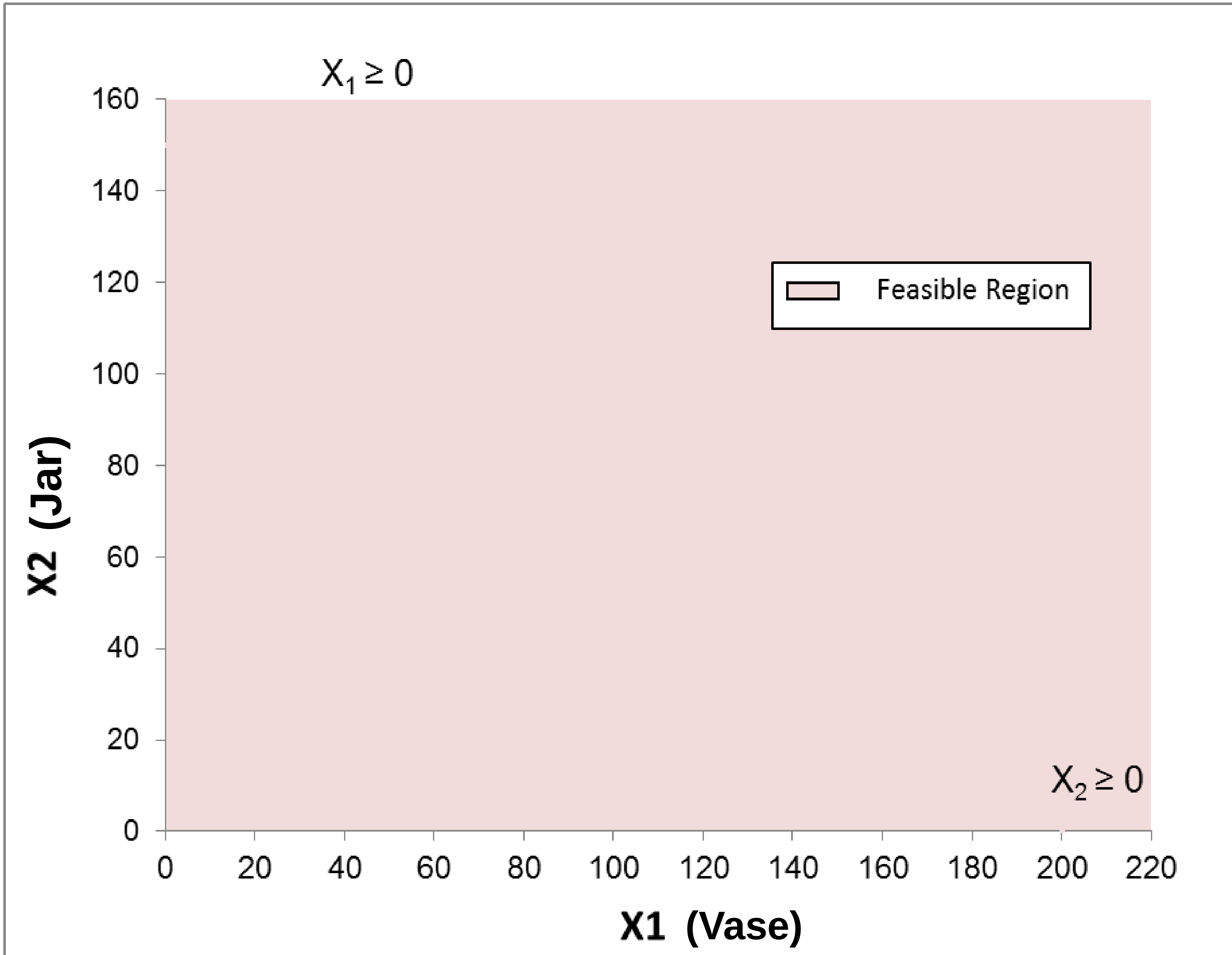
a) Refer to the graph on the graph paper provided to you, match the equation lines with the constraints that you have formulated.

b) What role does the inequality sign play in the graph?

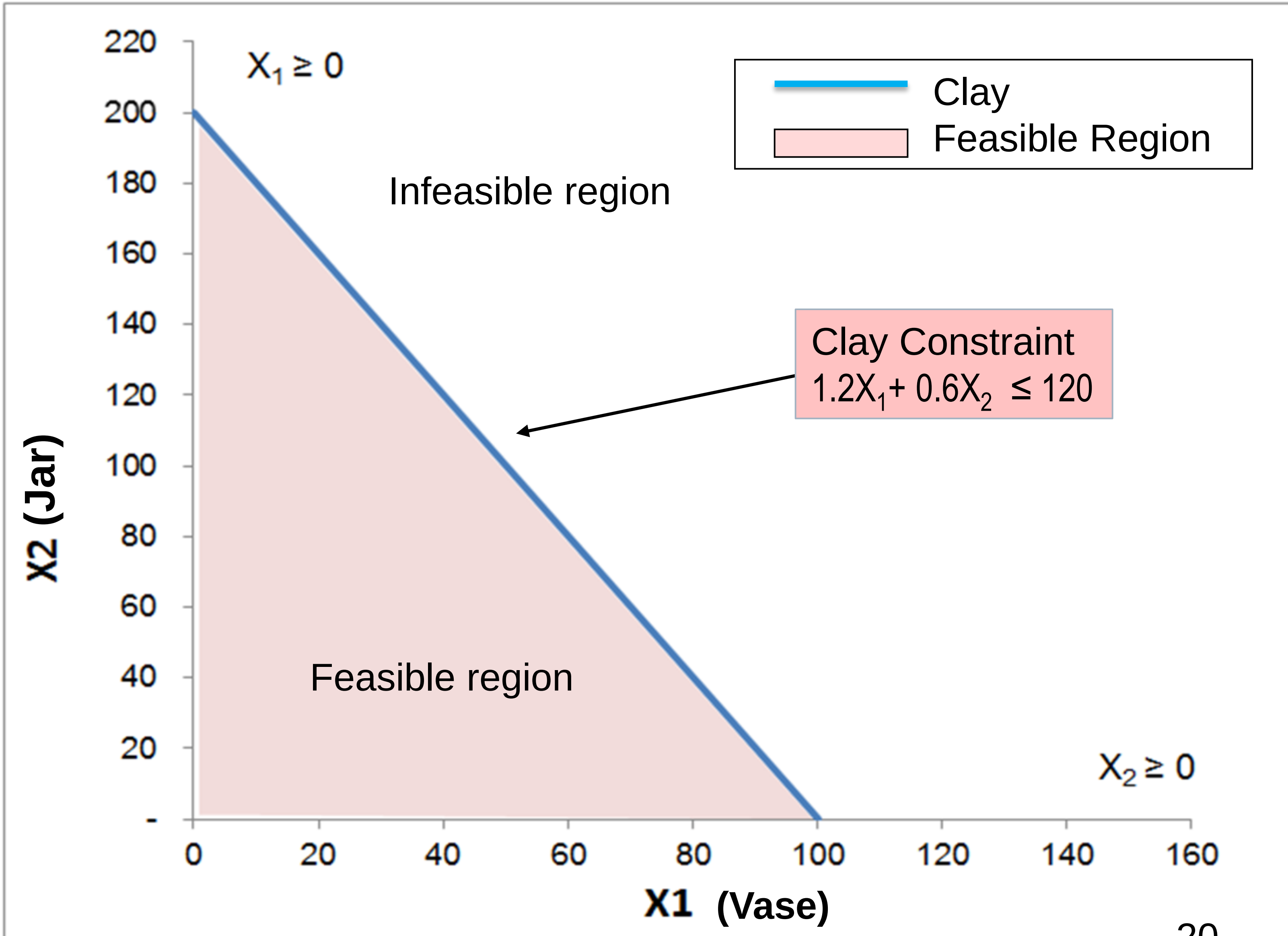
c) Shade the feasible solution region clearly in your graph.

- Students need to match the equation lines on the graph paper with the constraints formulated.
- To understand the inequality, substitute values such that the LHS is not equal to the RHS and then see where the point falls in the graph.
    - <= : below the line; towards the left side of the line
    - >= : above the line; towards the right side of the line

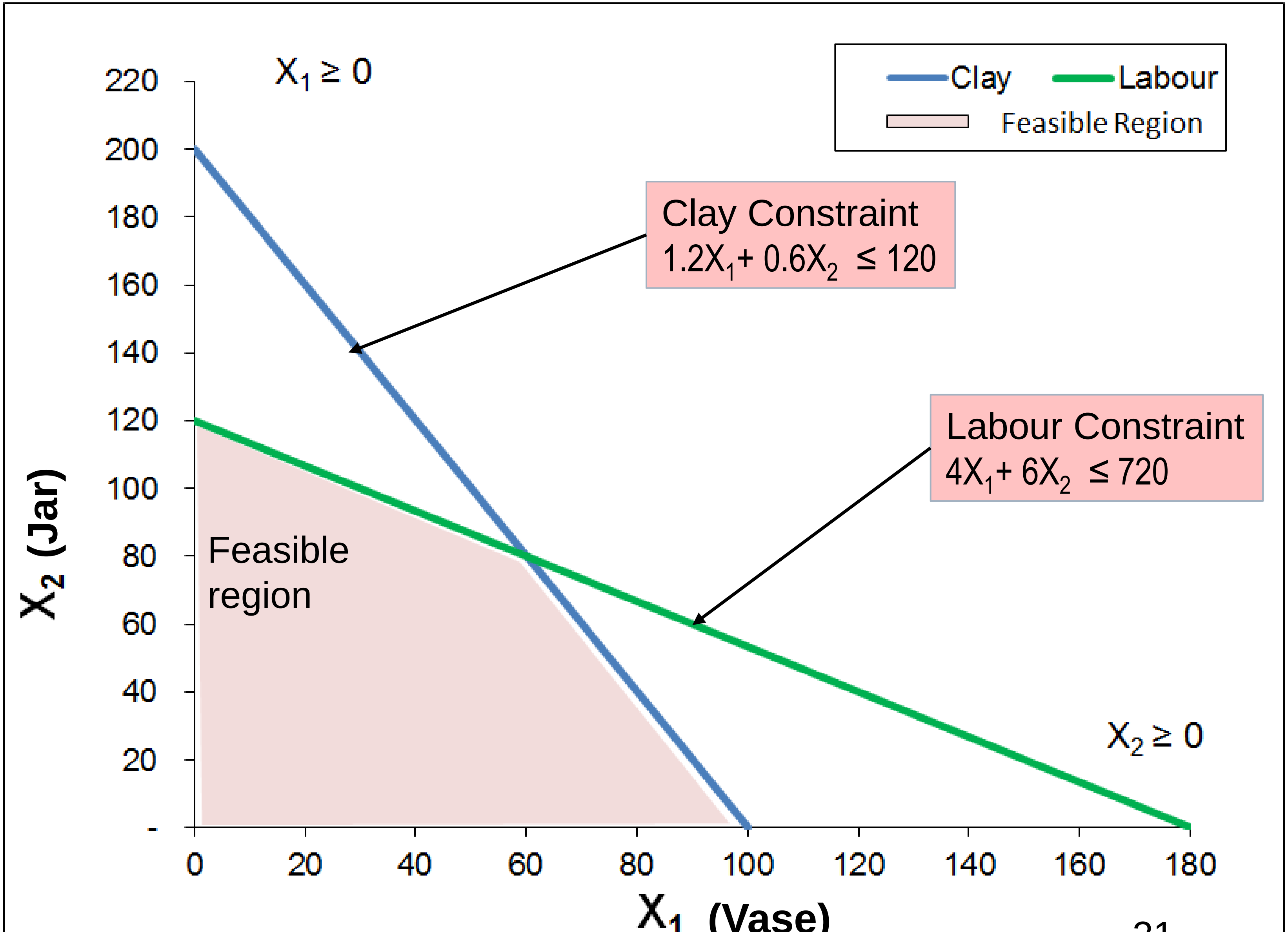
Graphical Solution: Non-negativity constraints for the number of Vases and Jars ($X_1 \geq 0$, $X_2 \geq 0$)



Graphical Solution: Clay Constraint ($1.2X_1 + 0.6X_2 \leq 120$)



Graphical Solution: Labour Constraint ($4X_1 + 6X_2 \le 720$)



Hands-on Session on Graphical Method (2)

Substitute an arbitrary number for Z (one suggestion: $2,400) in the objective function. Then plot the objective function line in the graph.

a) What happens to Z when you move this line upwards in parallel away from the origin?

b) Where is the furthest you can move without leaving the feasible solution region?

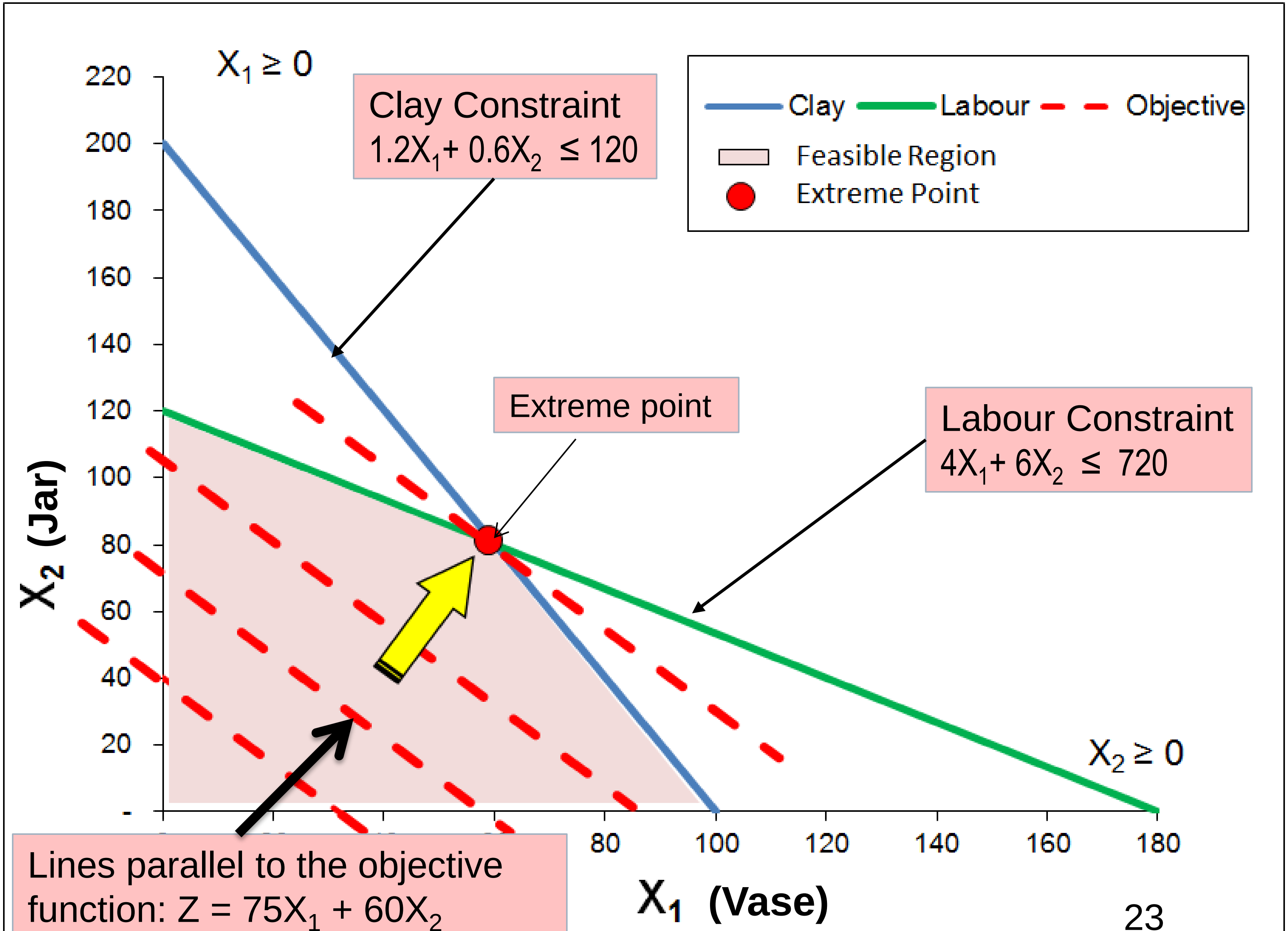
c) What is the significance of this extreme (furthest) point in the feasible region?

a) Z (the profit) changes when the objective function line moves. And since the objective is to maximize profit, the line should be moved as far right or upwards as possible without leaving the feasible region.

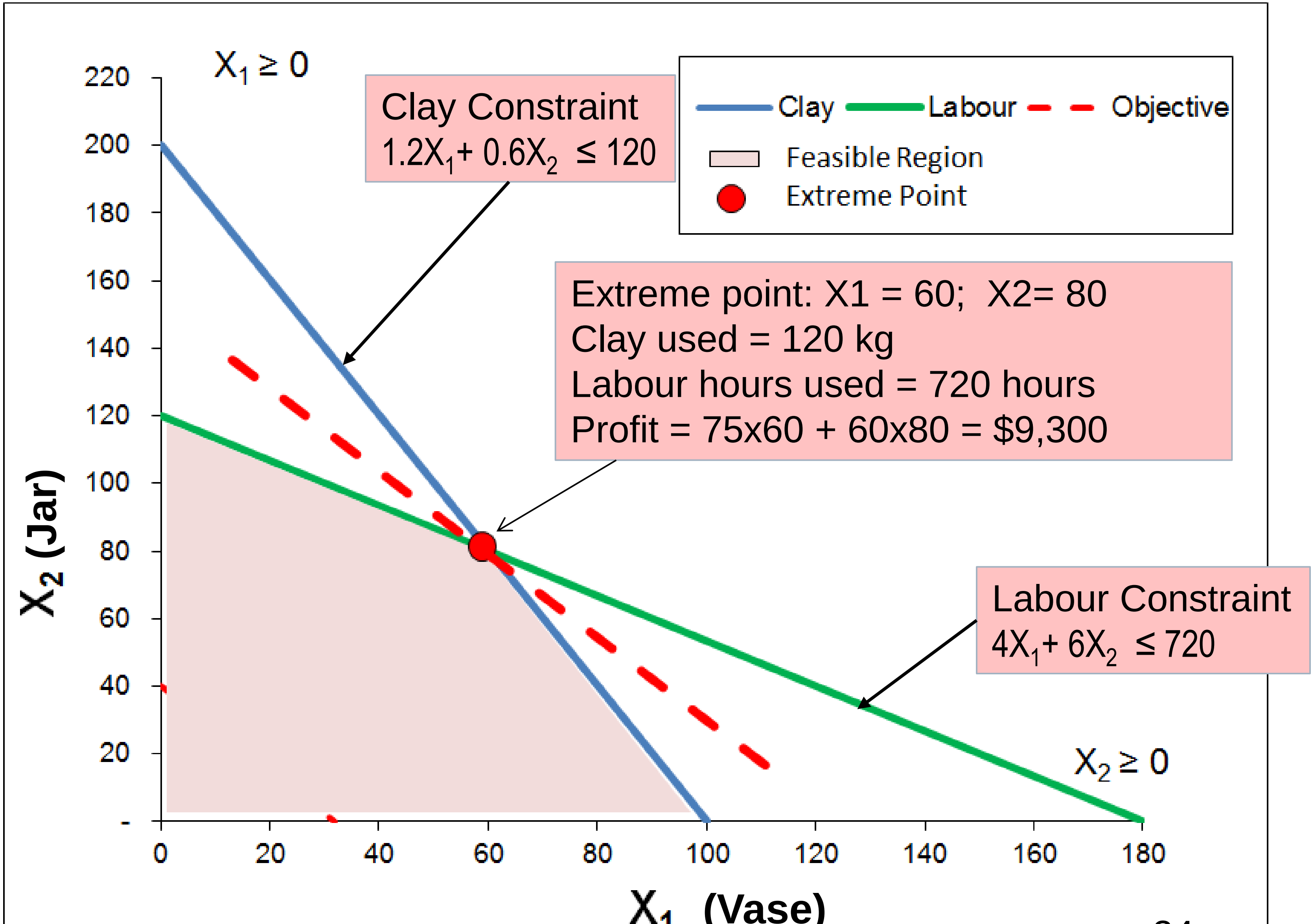
b) As the objective function line moves further, its value (right-hand side of equation) increases. Solution is found by moving the objective function right and upwards (keeping to its gradient) until the extreme point is reached in the feasible solution region.

c) This is the optimal solution that will give the maximum objective value.

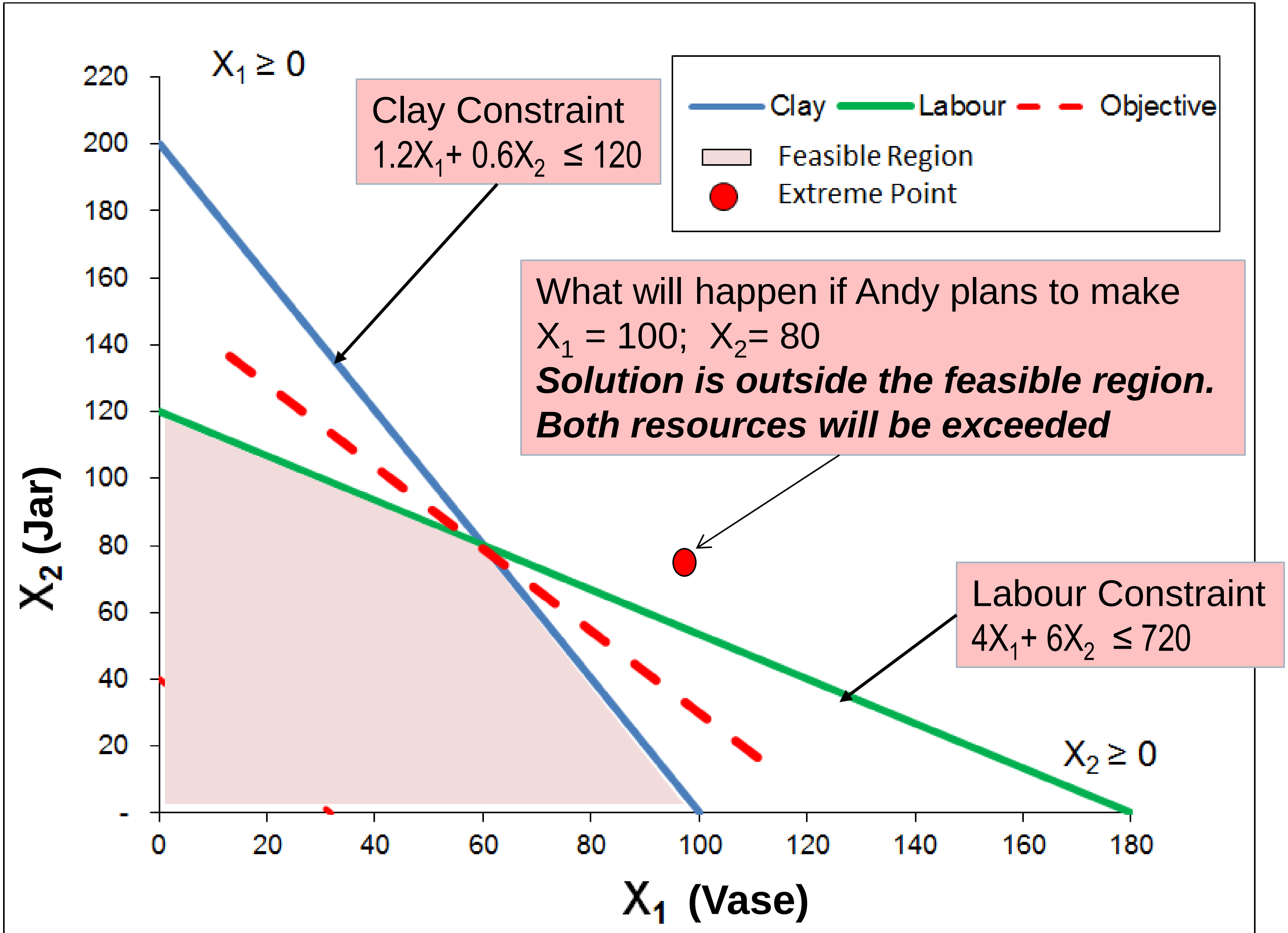
Graphical Solution: Possible Solutions for the Objective Function ($Z=75X_1+60X_2$)



Graphical Solution: Optimized Objective Function



Graphical Solution: Outside Feasible Region



Solutions

- The optimal solution at which profit is maximized is $X_1 = 60$, $X_2 = 80$.
- In other words, we recommend to make 60 vases and 80 jars per week which will yield a maximum profit of $9,300.

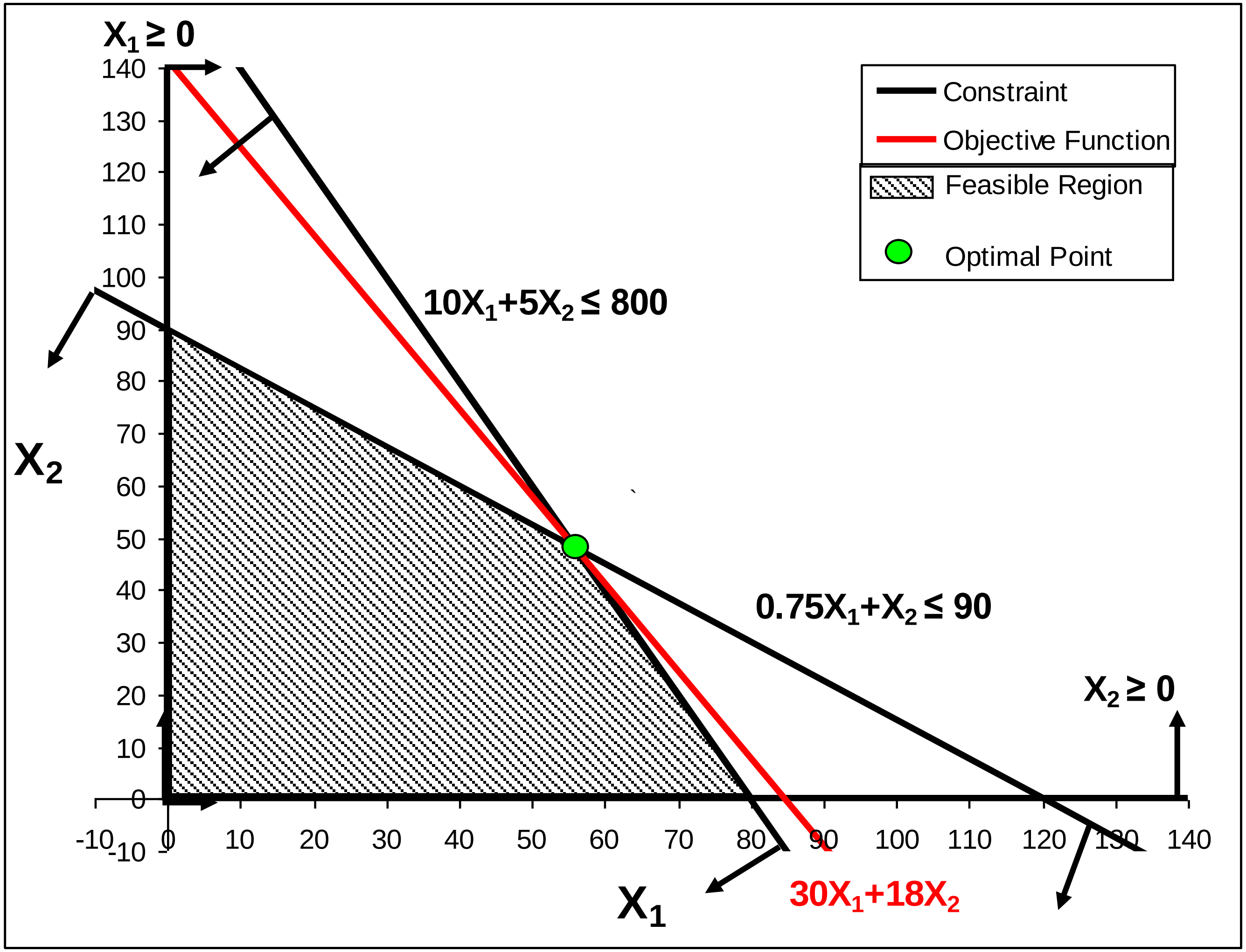
Profit = 75x60 + 60x80 = $9,300

- The assumptions are:
  - The coefficient and right-hand side values in the constraints are constant.
  - The linearity assumption holds for all values of the decision variables.

Feasibility & Optimality

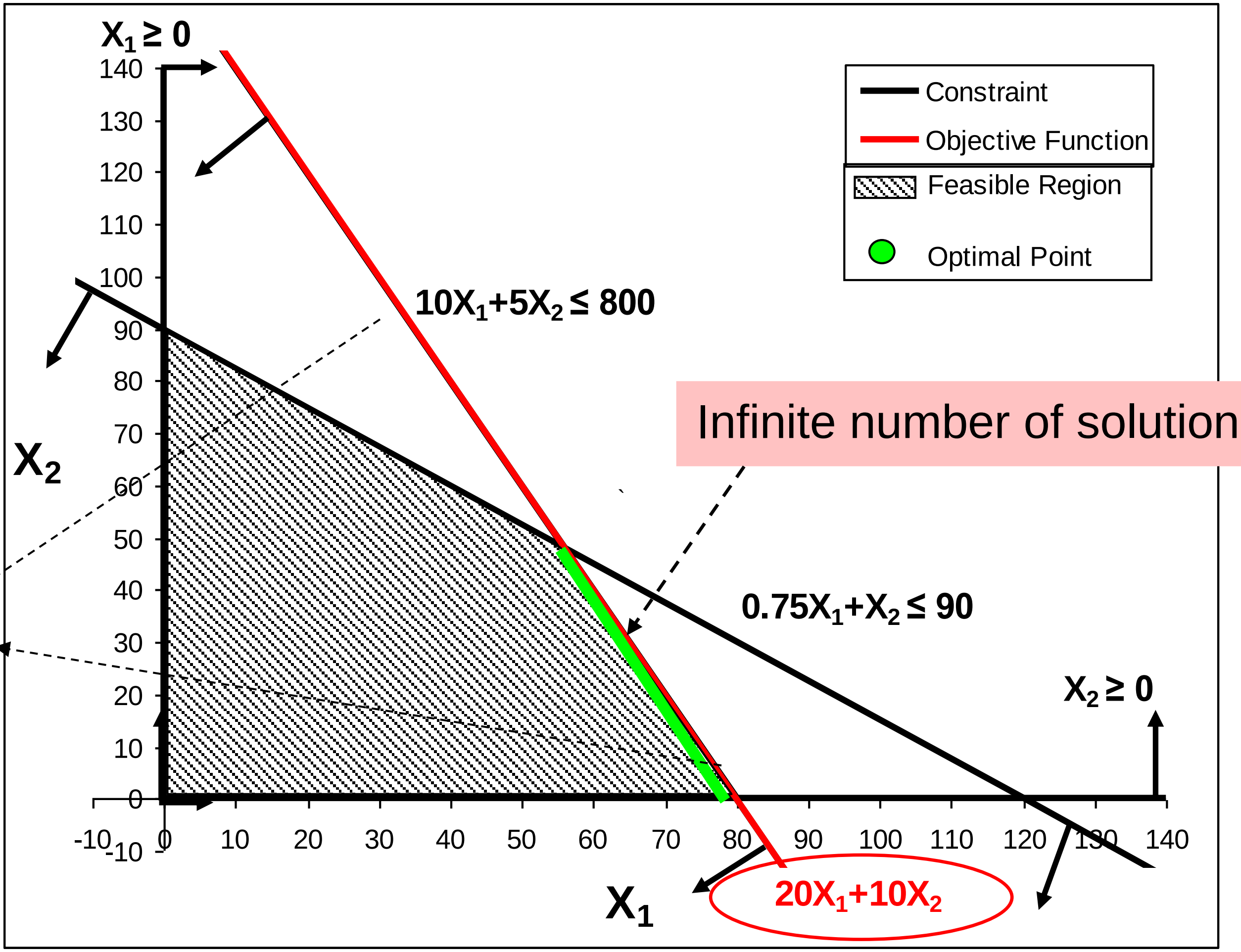
- Feasibility – Solutions that satisfy all constraints in the model.
- Optimality – Solutions that satisfy all constraints in the model and results in maximum (or minimum) objective function.
- An LP problem may have
  - One optimal solution
  - Multiple optimal solutions or
  - No optimal solution (e.g. unbounded solution)
  - No solution (not feasible)

One Optimal Solution



The optimal solution is an extreme point with the maximum objective value

Multiple Optimal Solutions

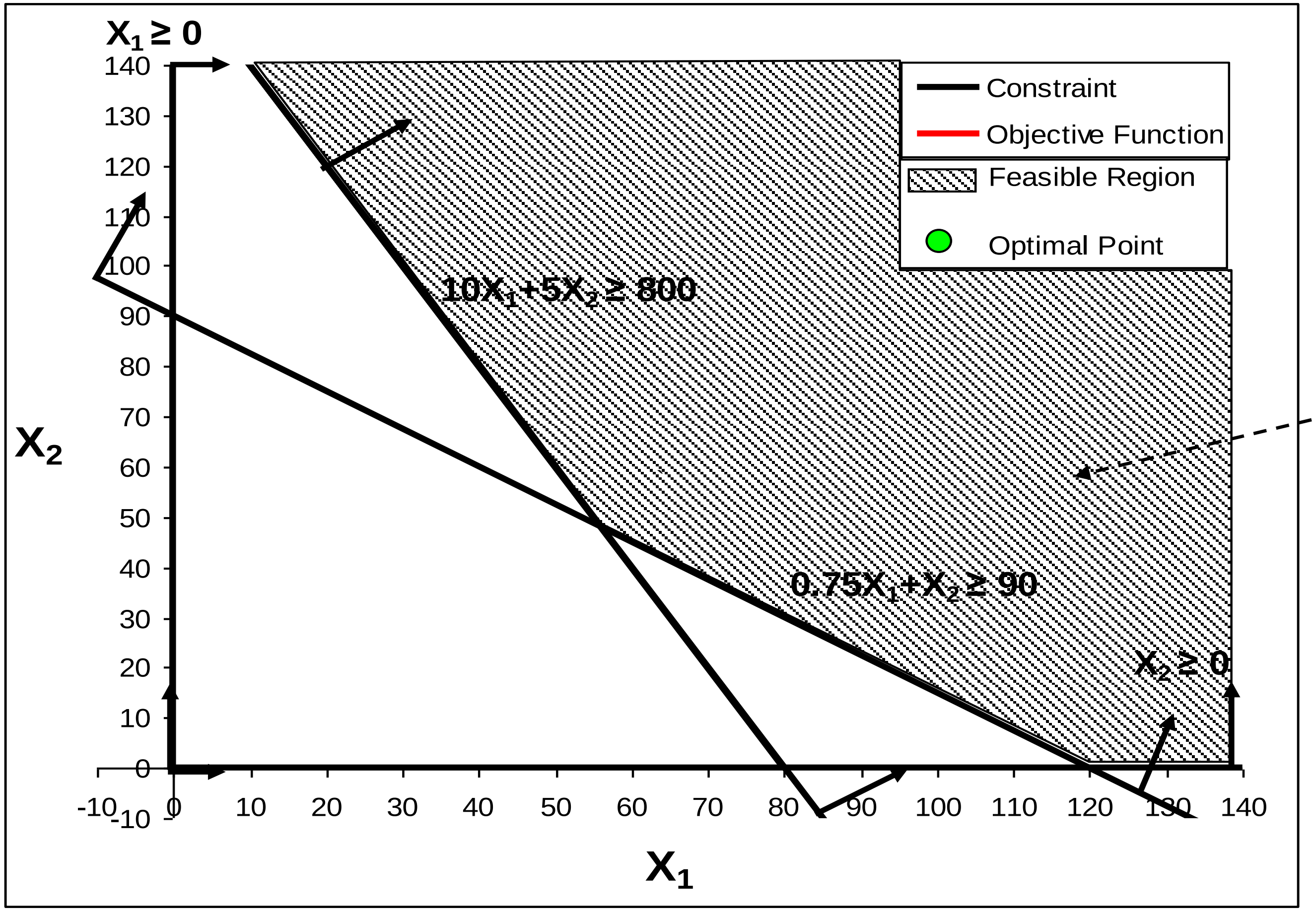


Slope of objective function is the same as the slope of a constraint

Infinite number of solutions

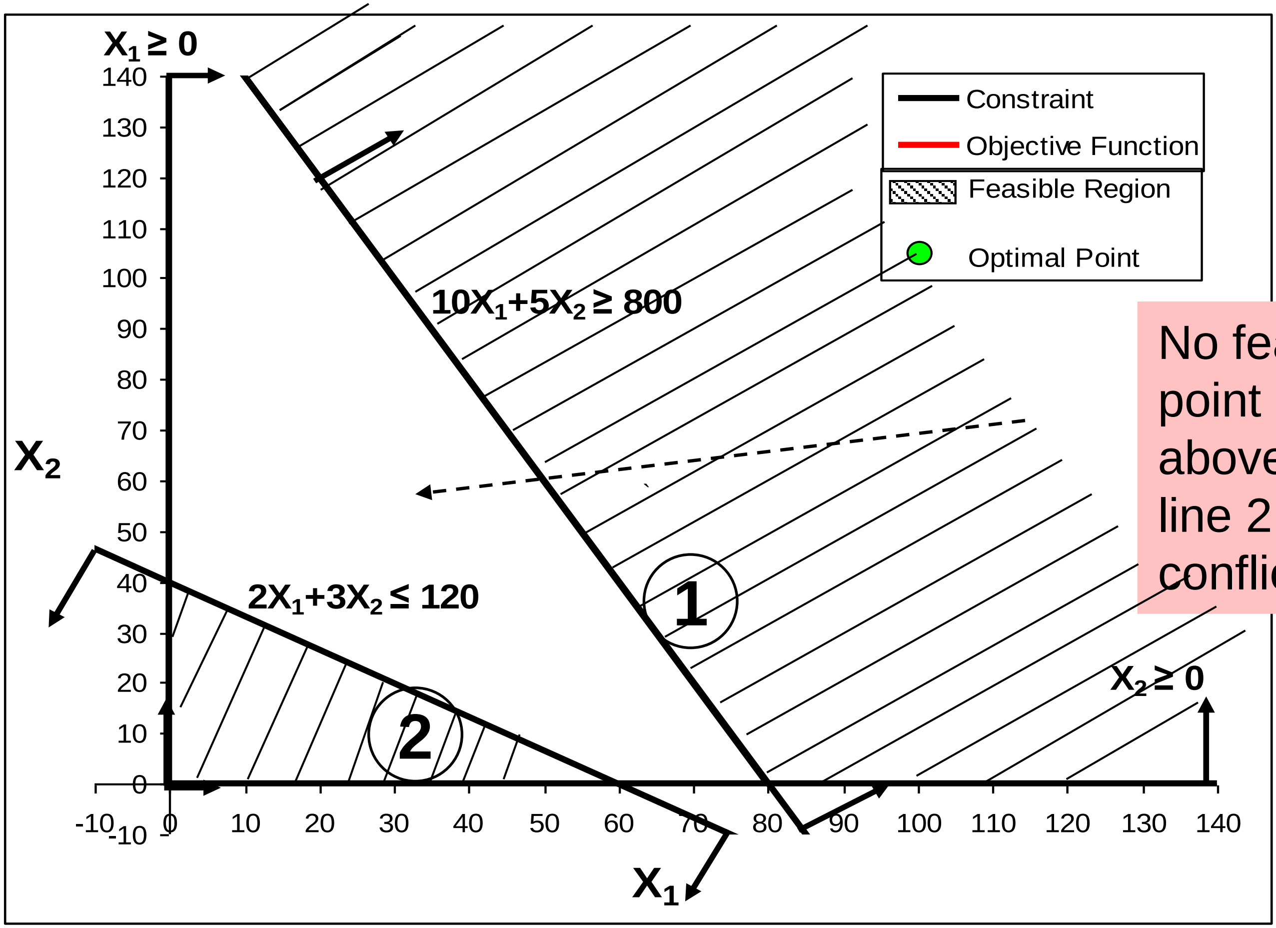
Note: In the next lesson, when you use Excel Solver, you will see that Excel Solver does not alert the user about the existence of alternate optimal solutions

Unbounded Solution



No optimal solution for maximization problem as the feasible region is unbounded and extends towards infinity. Optimal solution is not present.

No Feasible Solution



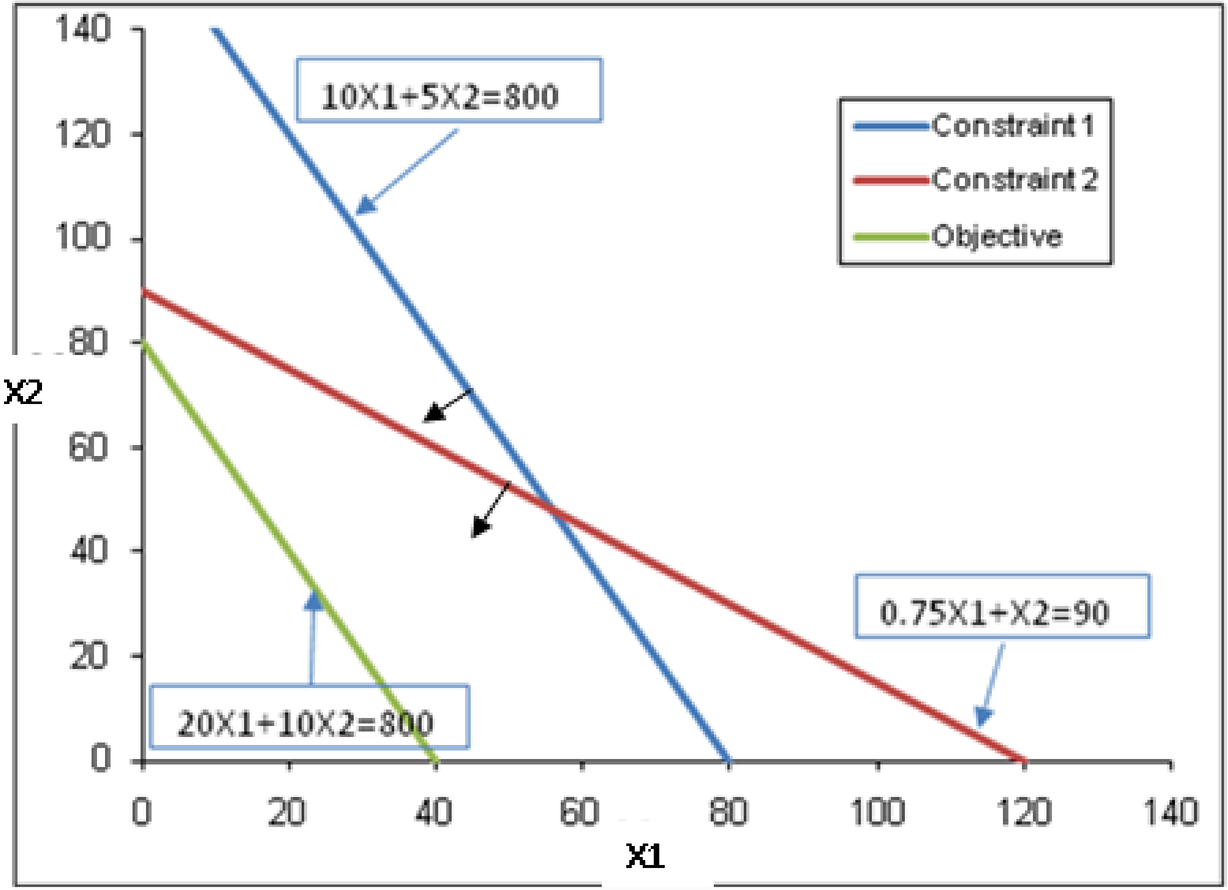
No feasible region: no point simultaneously lies above line 1 and below line 2. Constraints are in conflict.

Think-Pair-Share: Question 1

Analyze the following LP problem and comment on feasibility and optimality:

Max Z = 20X1+10X2

Subject to
10X1+5X2 <= 800
0.75X1+X2<= 90
X1, X2>=0



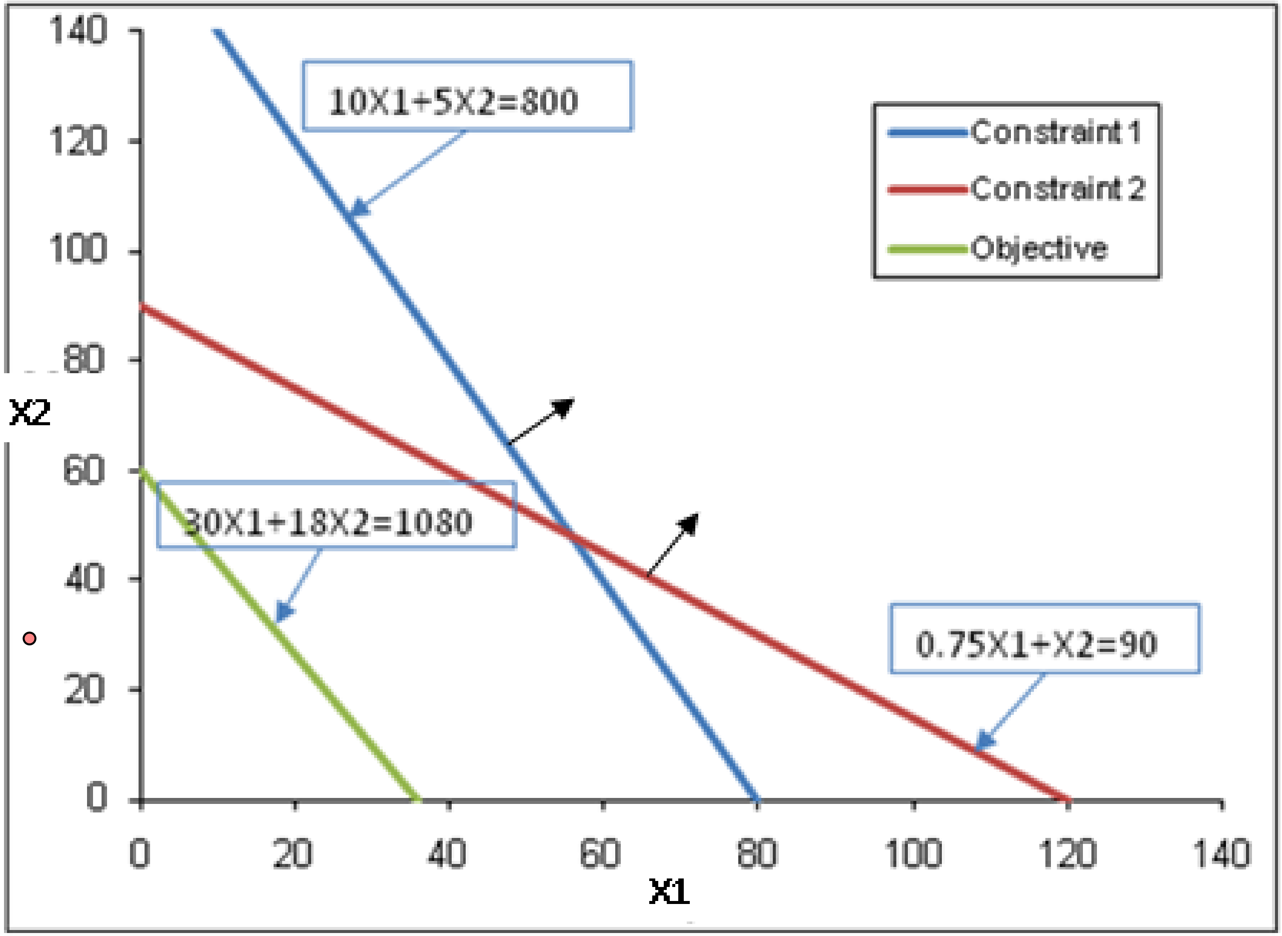
Multiple Optimal Solution

Think-Pair-Share: Question 2

Analyze the following LP problem and comment on feasibility and optimality:

Max Z = 30X1+18X2

Subject to
10X1+5X2 >= 800
0.75X1+X2>= 90
X1, X2>=0



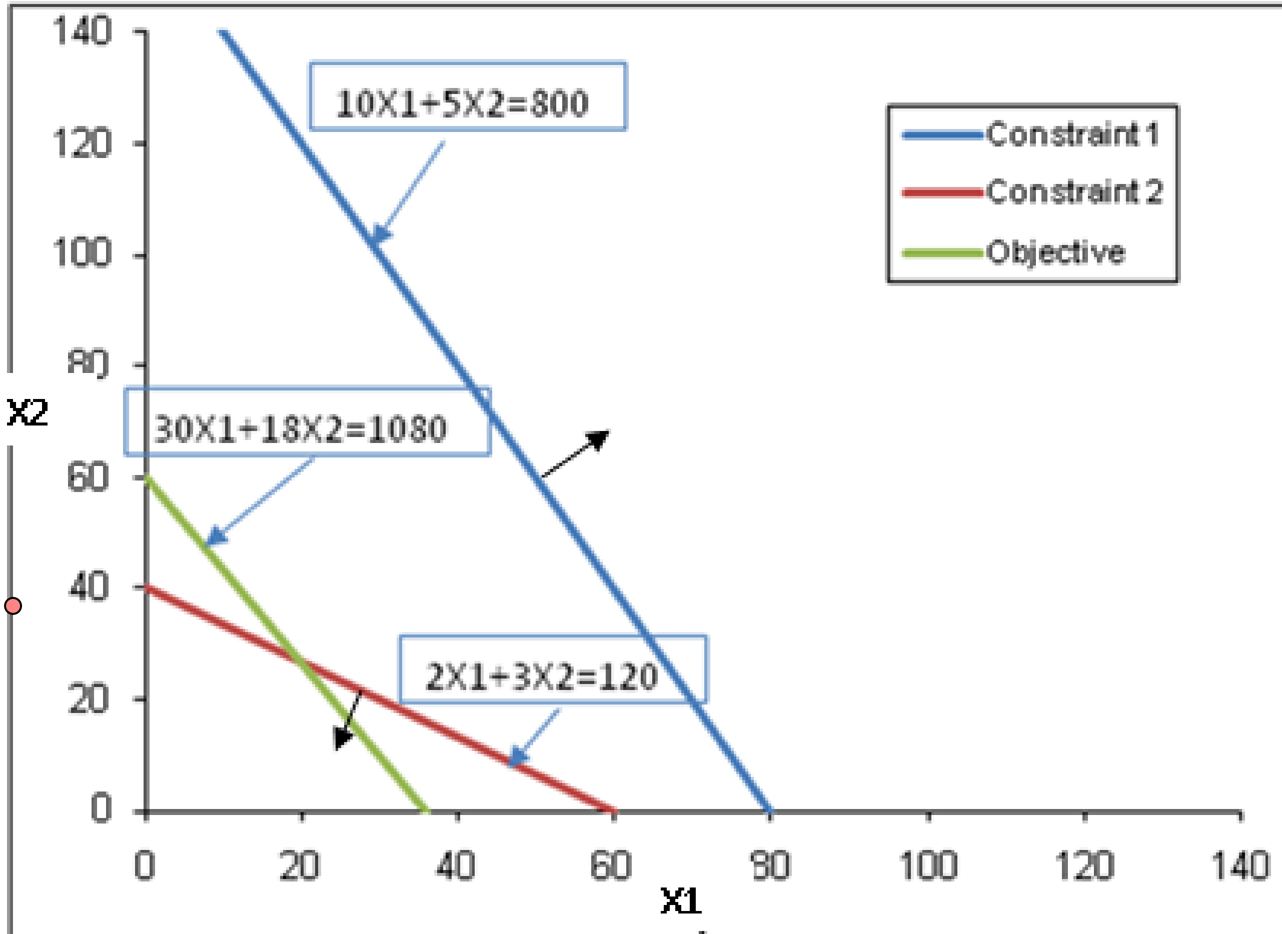
Unbounded Solution

Think-Pair-Share: Question 3

Analyze the following LP problem and comment on feasibility and optimality:

Max Z = 30X1+18X2

Subject to
10X1+5X2 >= 800
2X1+3X2<= 120
X1, X2>=0



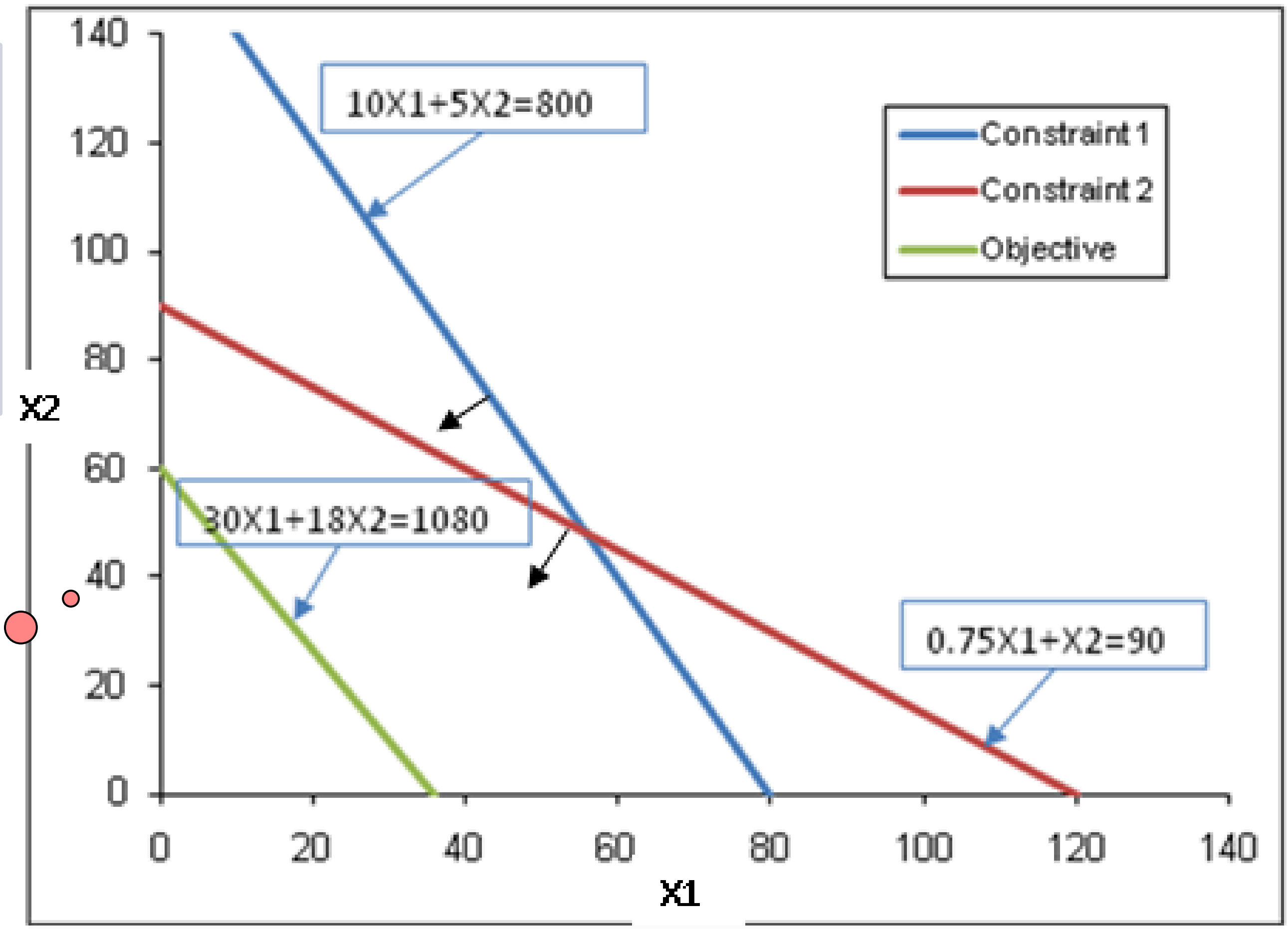
No Feasible Solution

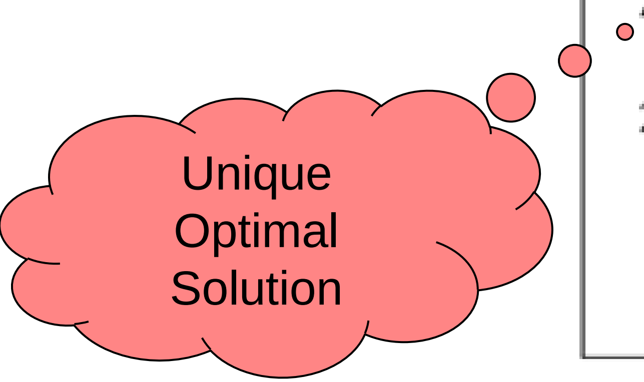
Think-Pair-Share: Question 4

Analyze the following LP problem and comment on feasibility and optimality:

Max Z = 30X1+18X2

Subject to
10X1+5X2 <= 800
0.75X1+X2<= 90
X1, X2>=0





Assumptions of Linear Programming (LP)

What will happen if this assumption does not hold?

- The ***Certainty*** assumption:
  - Each model parameter is known with certainty.
  - Model parameters here refer to objective function coefficients, right-hand side values of model constraints, and constraint coefficients.
- The ***Proportionality*** assumption:
  - The contribution to the objective function (left-hand side of each constraint) from each decision variable is proportional to the value of the decision variable.
- The ***Additivity*** assumption:
  - Total contributions to the objective function (left-hand side of each constraint) is the sum of individual contributions from each variable.
- The ***Continuity*** assumption:
  - Variables can take on any value within a given feasible range.

Role of Sensitivity Analysis of Optimal solution

- When formulating LP models, the model parameters are assumed to be **known with certainty**
- In reality, model parameters are simply estimates (best guesses) that are subject to changes.
- Sensitivity analysis is the analysis of parameter changes and their effects on the model solution.
- Sensitivity analysis provides a better picture of how the solution to a problem will change if different parameters in the model change.

Sensitivity Analysis

- Sensitivity analysis aids in answering questions such as:
  - **Range of optimality -** The range of values that the **objective function coefficients** can assume without changing the optimal solution.
  - **Range of feasibility -** The range of values for a right-hand side of a constraint, in which the shadow prices for the constraints remain unchanged - the valid range of **shadow prices.**
  - **Reduced Cost -** The impact of forcing a variable which is currently zero to be non-zero on the optimal objective function value.
  - **Shadow price -** The impact of increasing or decreasing the right-hand side value of various constraints on the optimal objective function value.
  - The impact of changing the **constraint coefficients** on the optimal solution to the problem.

Sensitivity Analysis

- The information and insights obtained through sensitivity analysis are valuable to management
  - They provide an indication of the degree of flexibility that is inherent in an operating environment.
- Almost all commercial software for linear programming provides
  - Right-hand-side ranging information
  - Objective coefficient ranging information.
  - Other sensitivity analysis, such as adding new variable and constraint, is dealt by solving the modified linear programming problem.

Hands-on Session on Sensitivity Analysis (1)

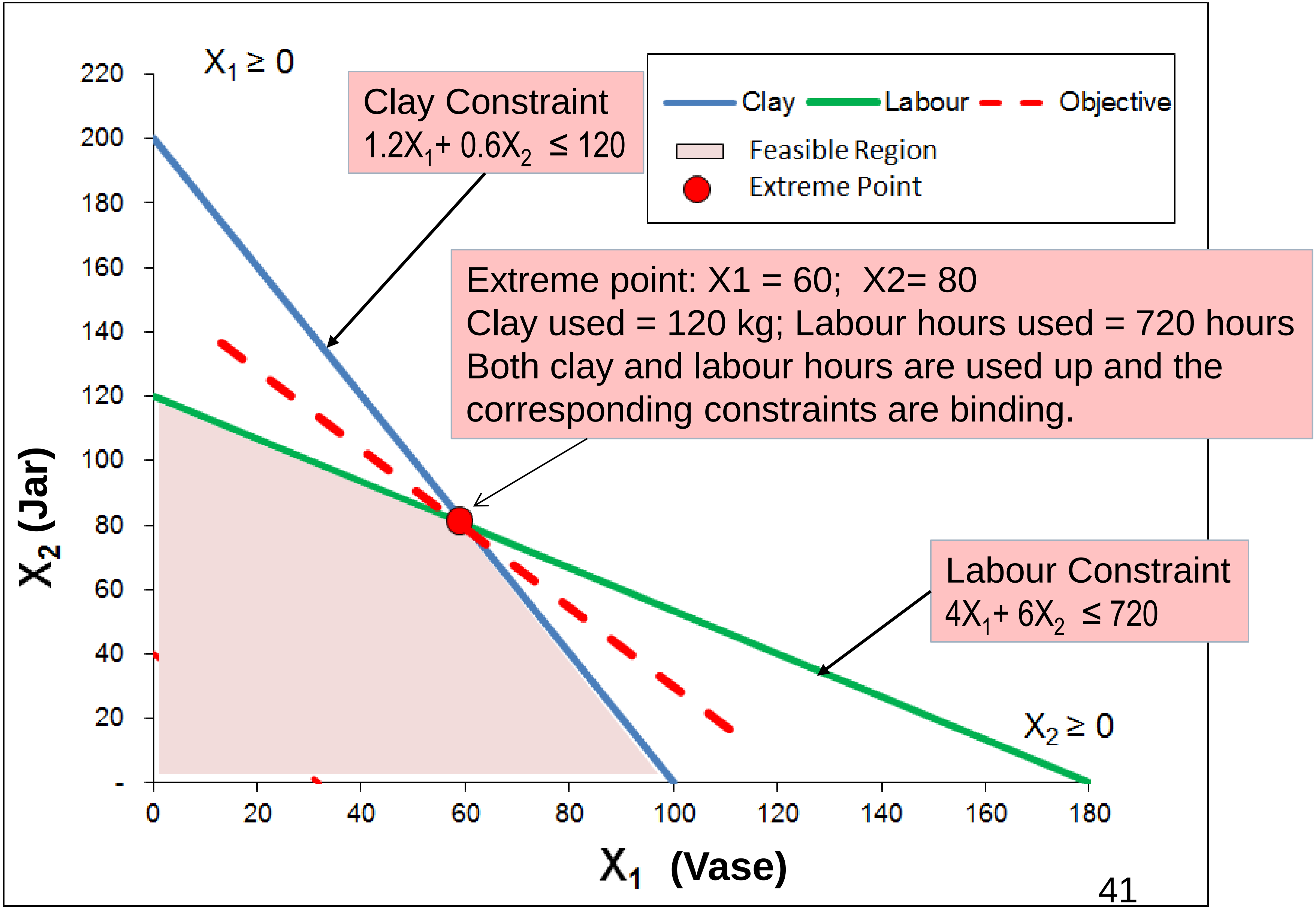
Consider the scenario given, how does Andy know if his resources have been used up? What do you call the constraints corresponding to the resources being used up?

A resource is used up if the optimal solution is on the equation line of the constraint. The corresponding constraint is binding.

A constraint is **binding** if the left-hand side and right-hand side of the constraint are **equal** when the optimal values of the decision variables are substituted into the constraint.

A constraint is **nonbinding** if the left-hand side and the right-hand side of the constraint are **unequal** when the optimal values of the decision variables are substituted into the constraint.

Graphical Solution: Binding Constraints



Hands-on Session on Sensitivity Analysis (2)

Consider the scenario given, assume that the weekly available amount of clay is increased from 120 to 144 kg.

a) Will the feasible region change?

b) Determine the new optimal solution and calculate the corresponding profit.

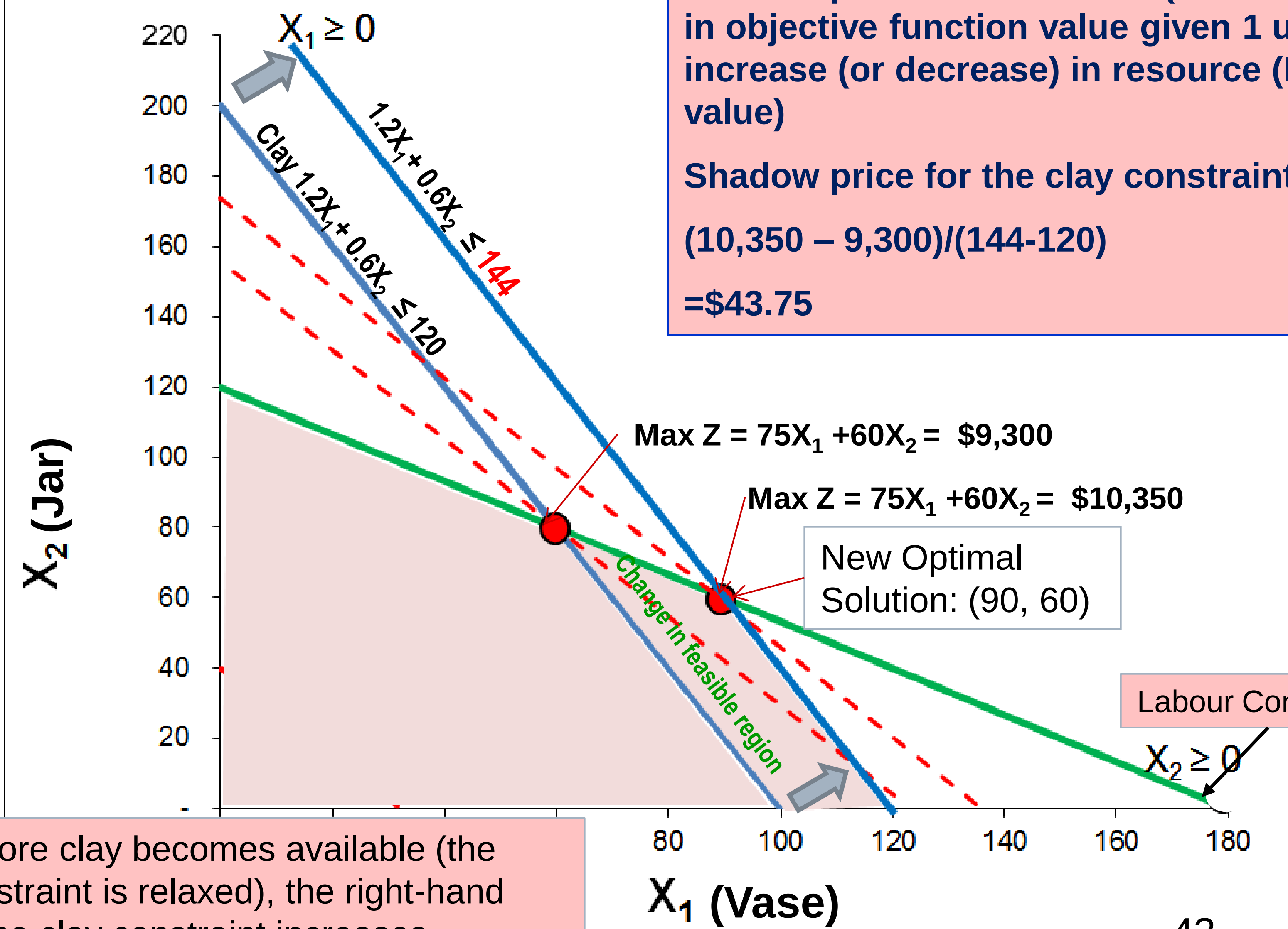
Sensitivity Analysis of Right-hand Side Value of Constraints

Shadow price: The increase (or decrease) in objective function value given 1 unit increase (or decrease) in resource (RHS value)

Shadow price for the clay constraint:

(10,350 – 9,300)/(144-120)

=$43.75



When more clay becomes available (the clay constraint is relaxed), the right-hand side of the clay constraint increases.

Hands-on Session on Sensitivity Analysis (3)

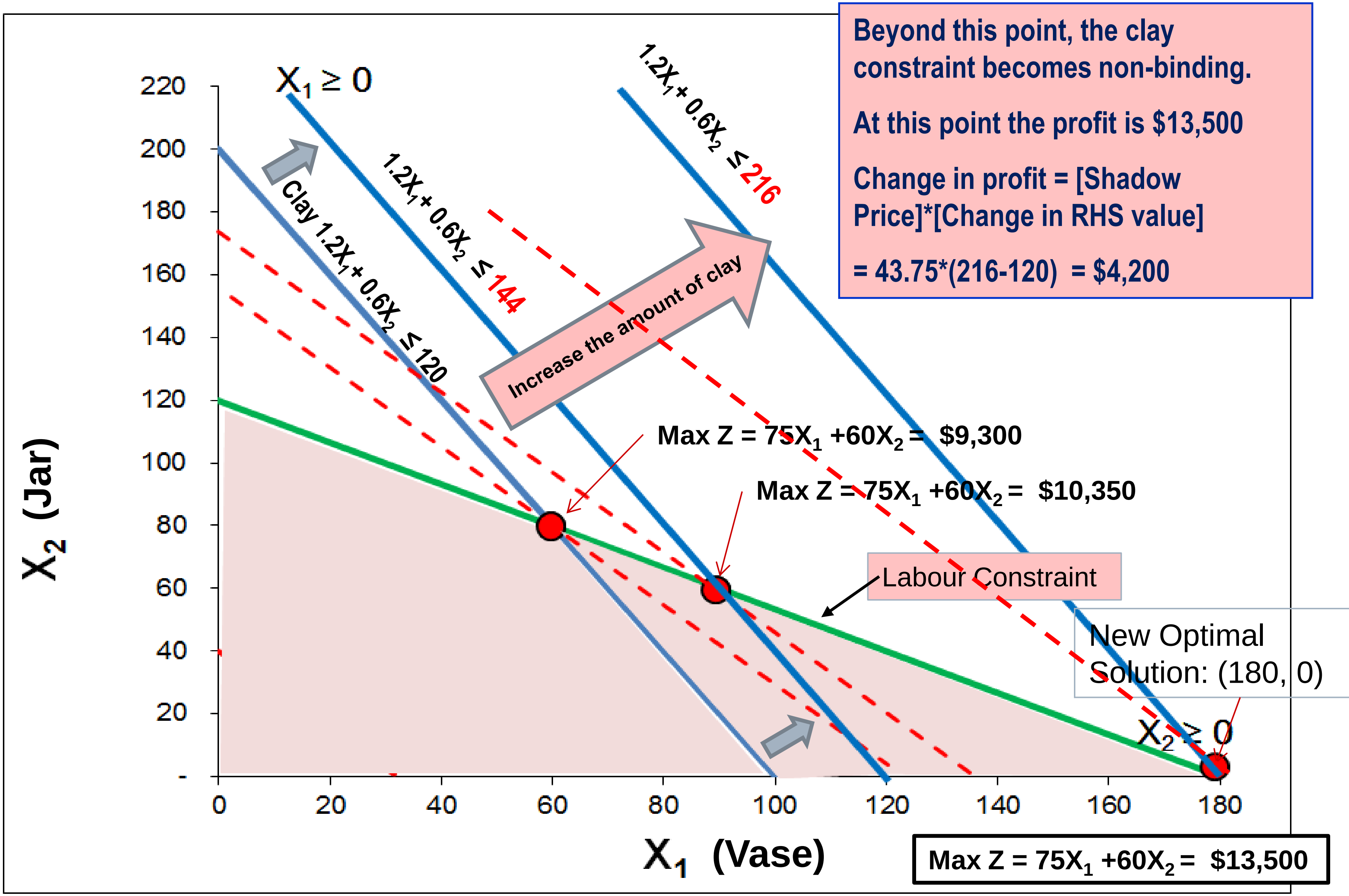
Consider the scenario given, assume that the weekly available clay is increased from 120 to 216 kg.

a) Determine the new optimal solution and calculate the corresponding profit.

b) How to calculate the change in profit based on the shadow price?

c) What will happen if the weekly available clay is increased beyond 216 kg?

Sensitivity Analysis of Right-hand Side Value of Constraints



Hands-on Session on Sensitivity Analysis (4)

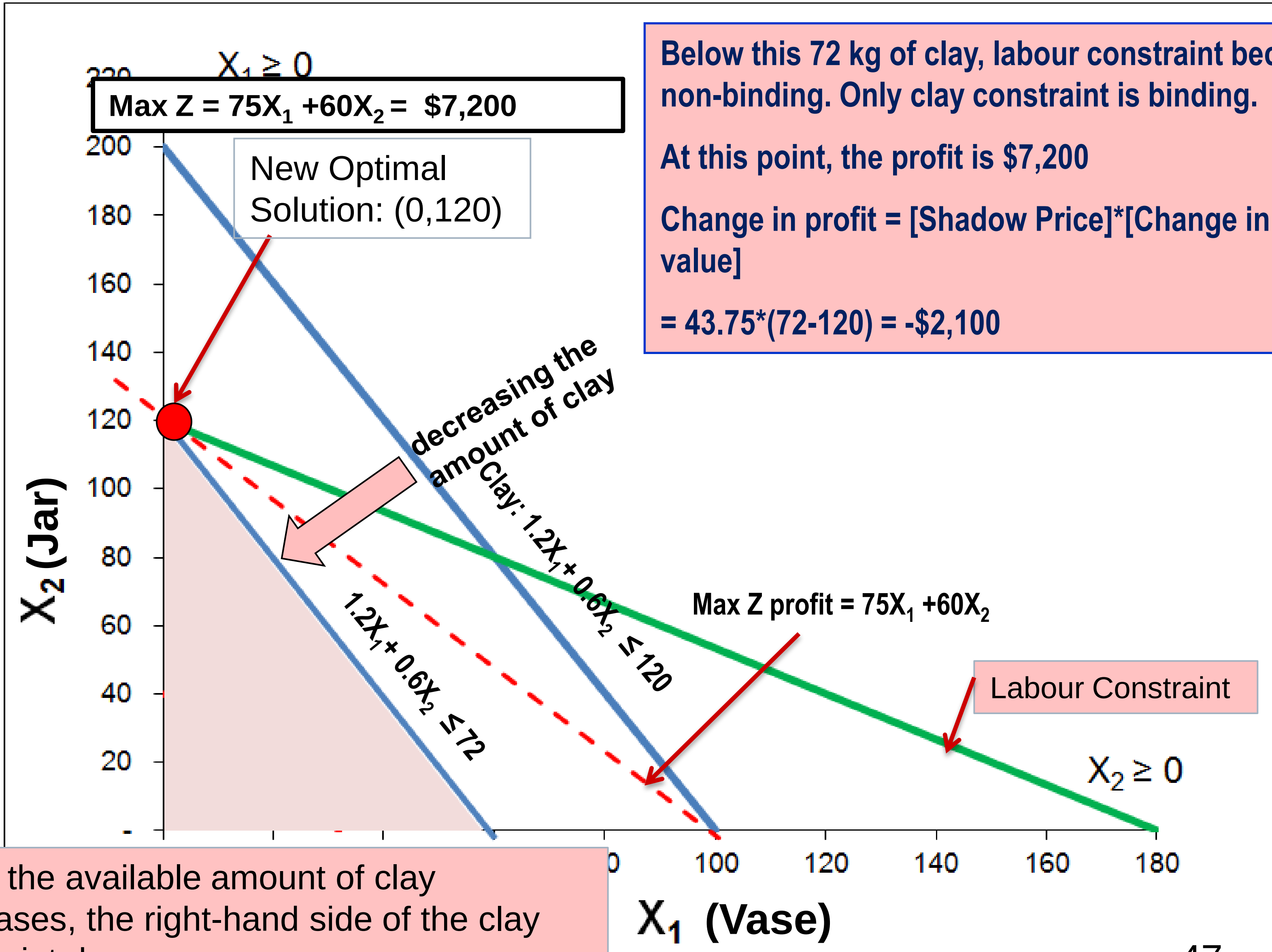
Consider the scenario given, assume that the weekly available clay is decreased from 120 to 72 kg.

a) Determine the new optimal solution and calculate the corresponding profit.

b) How to calculate the change in profit based on the shadow price?

c) What will happen if the weekly available clay is decreased to below 72 kg?

Sensitivity Analysis of Right-hand Side Value of Constraints



Below this 72 kg of clay, labour constraint becomes non-binding. Only clay constraint is binding.

At this point, the profit is $7,200

Change in profit = [Shadow Price]*[Change in RHS value]

= 43.75*(72-120) = -$2,100

When the available amount of clay decreases, the right-hand side of the clay constraint decreases.

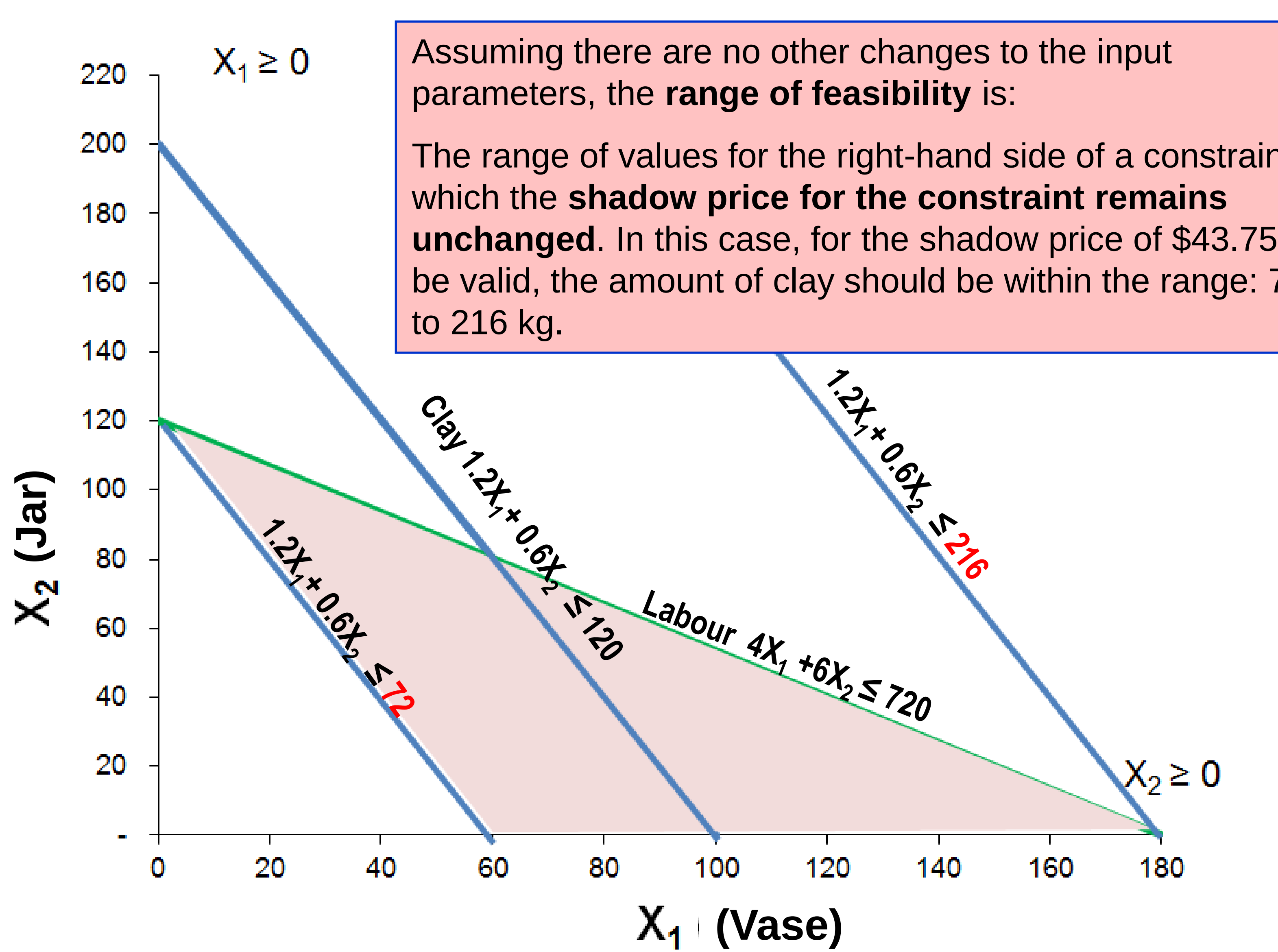
Hands-on Session on Sensitivity Analysis (5)

Consider the scenario given and the clay constraint,

a) What does the range of feasibility mean?

b) What is the range of feasibility for the right-hand side of the clay constraint?

Range of Feasibility – Clay



Assuming there are no other changes to the input parameters, the **range of feasibility** is:

The range of values for the right-hand side of a constraint, in which the **shadow price for the constraint remains unchanged**. In this case, for the shadow price of $43.75 to be valid, the amount of clay should be within the range: 72 to 216 kg.

Hands-on Session on Sensitivity Analysis (6)

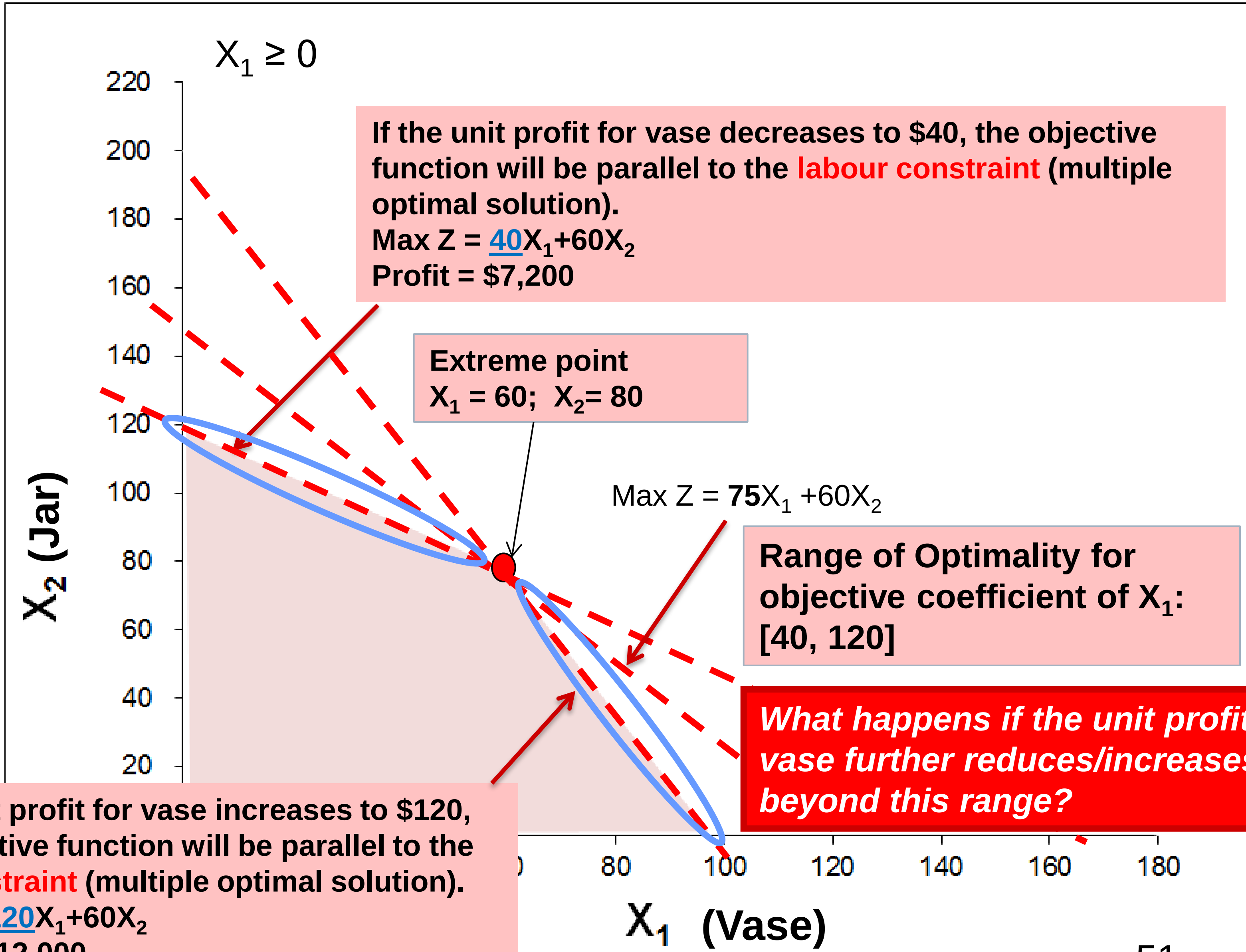
Consider the scenario given,

a) What would happen if the unit profit of the decorative vase is decreased from $75 to $40?

b) What would happen if the unit profit of the decorative vase is increased from $75 to $120?

c) What would happen if the unit profit of the decorative vase is further increased or decreased beyond this range?

Sensitivity Analysis of Objective Function Coefficients - Range of Optimality



If the unit profit for vase decreases to \$40, the objective function will be parallel to the labour constraint (multiple optimal solution).
Max Z = 40X_1+60X_2
Profit = \$7,200

Extreme point
X_1 = 60; X_2= 80

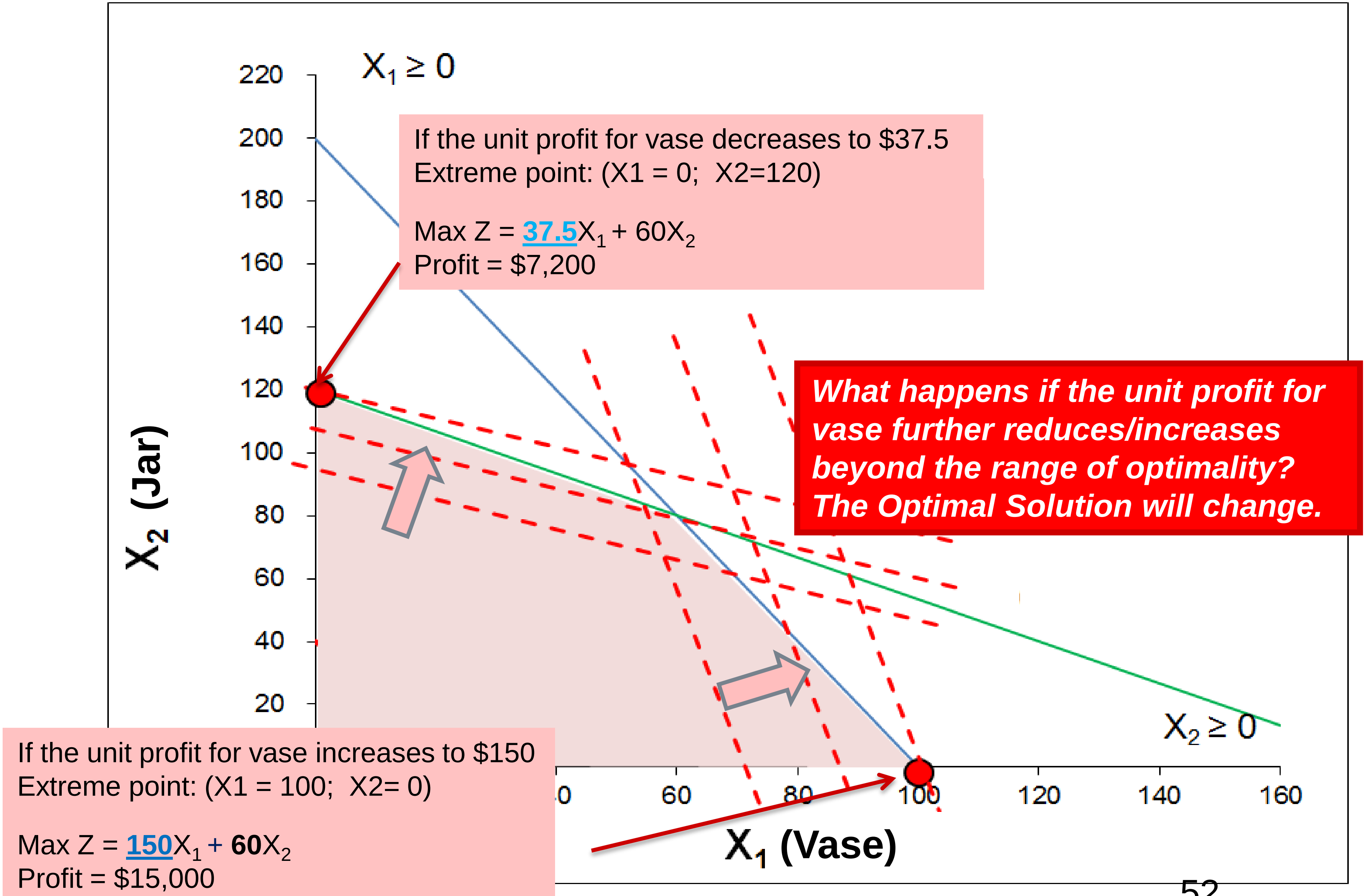
Range of Optimality for objective coefficient of X_1: [40, 120]

What happens if the unit profit for vase further reduces/increases beyond this range?

If the unit profit for vase increases to \$120, the objective function will be parallel to the clay constraint (multiple optimal solution).
Max Z = 120X_1+60X_2
Profit = \$12,000

Sensitivity Analysis of Objective Function Coefficients

- What happens if the unit profit further reduces/increases?



Validity of Model

- Verify the model – check the formula and assumptions
- Check that the model represents reality by comparing the result with the expected outcome
  - Use historical data to reconstruct the past and then determine how well the model and the resulting solution would have performed if they had been used.

Learning Objectives

At the end of the lesson, students should be able to:

- Identify problems that linear programming can handle.
- Identify the assumptions and elements of a linear programming problem. (Objective function, Decision variables and Constraints).
- Formulate and solve Linear Programming (LP) problems with two decision variables using graphical method.
- Perform sensitivity analysis on the objective function coefficients and right-hand side values of the constraints using graphical method.

Overview of E210 Operation Planning Module