Introduction to Theoretical Cosmology

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Overview

In these lectures we will cover the early universe. It's **way too much material** to be covered in only two hours so we will only be able to scratch the surface, but the aim is that after we are done you will have an overview of what happened in the early universe and understand the key physical concepts:

- Know the **basics of statistical mechanics** in an expanding universe
- Understand concepts like **decoupling** and **freeze-out**
- From this (with some particle physics input) be able to understand the **thermal history of the universe**
- Know about the key predictions of the hot big bang model: **nucleosynthesis** abundances and the **cosmic microwave background**

Units

In these lectures we will use units where the speed of light is unity, so e.g. Einsteins mass-energy relation is just

$$E = m$$

We will also set the units of temperature such that Boltzmann's constant k = 1. This means temperature and mass is given in terms of energy.

On the right gives formulas for how to convert between these units and SI.

- $\hbar = c = k_B = 1$

$$[Energy] = [Mass] = [Temperature] = [Length]^{-1} = [Time]^{-1}$$

$$1\ GeV = 10^3 MeV = 10^9 eV$$

$Energy$: $1\ GeV = 1.6022 \times 10^{-3}\ erg$
Mass: $1 GeV = 1.7827 \times 10^{-24} g$
$Temperature$: $1 GeV = 1.1605 \times 10^{13} K$
$Length$: $1 GeV^{-1} = 1.9733 \times 10^{-14} cm$
$Time$: $1 GeV^{-1} = 6.5822 \times 10^{-25}\ sec$

The Hot Big Bang Model

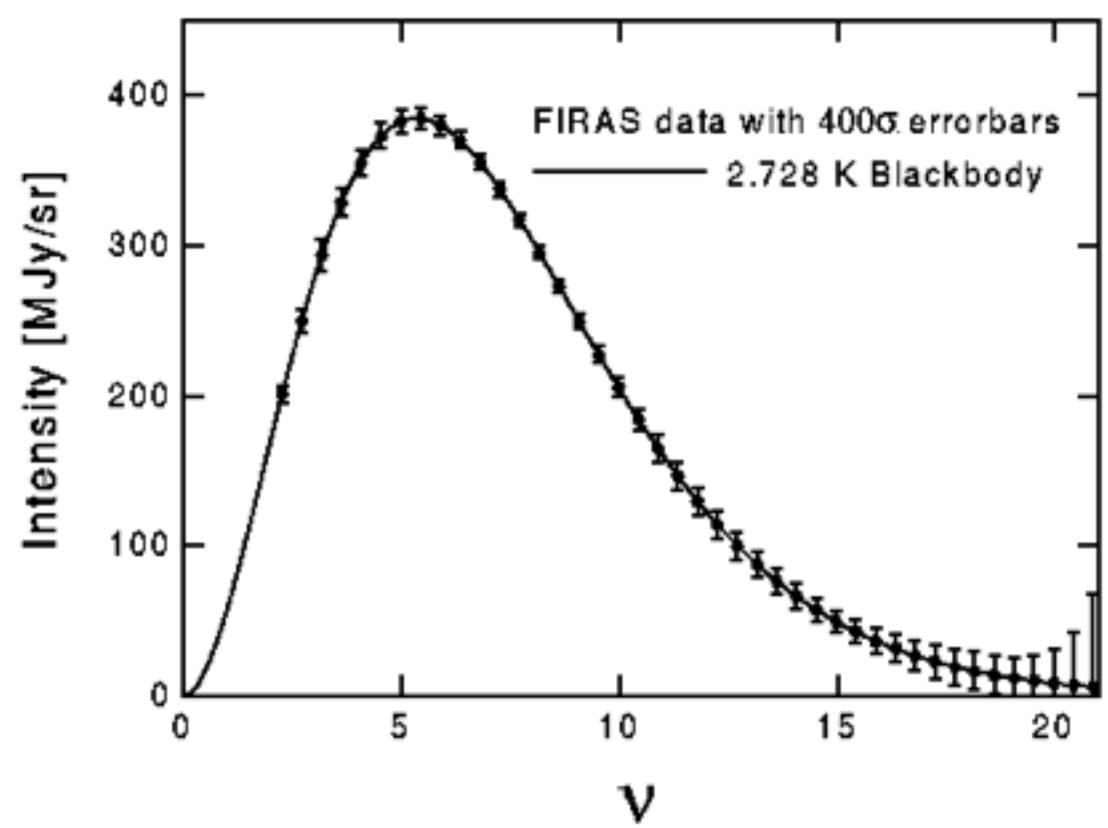
- We observe that our universe is filled with radiation (the Cosmic Microwave Background) which has an almost perfect black-body spectrum

$$\mathrm{d}n \propto \frac{E^2 \mathrm{d}E}{e^{\frac{E}{T}} - 1}$$

- Energy scales the same as temperature (1/a) and number density falls off a volume expansion.

- **Consequence:** a black-body spectrum (in absence of interactions) stays a black-body spectrum in an expanding universe with the peak of curve (which is closely related to temperate; Wien's law) redshifting away

- Temperature today is ~2.7 K. Observing the CMB gives us a lot of information about what happened in the early universe.



The Hot Big Bang Model

- Observations shows that our universe is expanding. It's found to contain matter, vacuum energy and a tiny bit of radiation energy (what we see as the CMB)

- Recall from last time that radiation and matter density scales as

$$\rho_{\rm radiation} \propto a^{-4} \qquad \rho_{\rm matter} \propto a^{-3}$$

- **<u>Consequence</u>**: the universe was radiation dominated in the early phases ($a << 1$)

- For a photon gas we have (**Stefan-Boltzman's law**)

$$\rho_{\rm radiation} \propto T^4 \implies T \propto a^{-1}$$

- **<u>Consequence:</u>** the universe gets both hotter and denser the further back in time we go

- To understand the universe at the early stages we need **statistical mechanics** and **quantum field theory** (interaction rates) in an expanding universe!

Statistical Mechanics

Not feasible to keep track of every particle in a large system. We aim for a statistical description.

Key quantity: the **distribution function** $f(\mathbf{p},\mathbf{x})$

which describes the phase-space density of particles. E.g. the number of particles with positions and momenta in a small cube around **(p,x)** is $\propto f(\mathbf{p},\mathbf{x})\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{x}$

For this lecture we are only interested in the background where we have no position dependence from now on.

Let's recall some key formulas. In **thermal equilibrium** we have:

$$n = \frac{g}{(2\pi)^3}\int f(\mathbf{p})\mathrm{d}^3\mathbf{p} \qquad \rho = \frac{g}{(2\pi)^3}\int f(\mathbf{p})E(\mathbf{p})\mathrm{d}^3\mathbf{p}$$

$$f^{(+\ \text{fermion})}_{(-\ \text{boson})} = \frac{1}{e^{\frac{E-\mu}{T}} \pm 1}, \qquad E(\mathbf{p}) = \sqrt{m^2+\mathbf{p}^2}$$

where **mu** is the chemical potential (energy that can be absorbed/released due to a change of the particle number in a chemical reaction), **g** is the number of internal degrees of freedom and **f** is the (**Fermi[+]Dirac** and **Bose[-]Einstein**) distribution function.

Statistical Mechanics: Relativistic Limit

$$T \gg m, \mu \quad \text{and} \quad E(\mathbf{p}) \sim p \quad \text{so} \quad f = \frac{1}{e^{p/T} \pm 1}$$

and which gives us (**Stefan-Boltzmann's law**)

$$\rho \approx \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^2 \cdot p}{e^{p/T} \pm 1} \mathrm{d}p \propto T^4$$

The plus/minus above leads to a **difference between fermions and bosons** in the absolute numbers which corresponds to a factor of

$$\frac{7}{8} (= 1 - \frac{1}{2^D})$$

This will be important later when we will define and count effective number of degrees of freedom in the primordial plasma.

Statistical Mechanics: Non-Relativistic Limit

$$m \gg T \quad \text{and} \quad E(p) \approx m \quad \text{so} \quad \rho = m \cdot n$$

which gives us

$$f(p) \approx e^{-\frac{m-\mu}{T} - \frac{p^2}{2mT}} \implies n \propto T^{3/2} e^{-\frac{m-\mu}{T}}$$

Consequence: Number density **exponentially suppressed** as we lower temperature (to keep thermal equilibrium we would need interactions like pair production. Not possible when temperature is much lower than mass)

Statistical Mechanics: Entropy and the effective number of relativistic degrees of freedom

The second law of thermodynamics (ignoring chemical potential) tell us that entropy

$$\mathrm{d}s = \frac{1}{T}(\mathrm{d}(\rho V) + P\mathrm{d}V)$$

Combined with energy conservation (continuity equation) implies that entropy is conserved. The entropy of a relativistic gas is given by

$$s \propto \frac{\rho a^3}{T} \propto T^3$$

Once a particle goes non-relativistic the entropy it carries is negligible. The total entropy of the universe can therefore be written

$$s \propto g_{\text{eff}} T^3, \qquad g_{\text{eff}} = \sum_{\text{Relativistic Boson DOF}} + \frac{7}{8} \sum_{\text{Relativistic Fermion DOF}}$$

Can be used to compute the temperature in the primordial plasma at different stages.

Statistical Mechanics in an Expanding Universe: Temperature as measure of time

In the early universe it's convenient to use the temperature of the primordial plasma as a measure of time. This is possible since time is related one-to-one to temperature (and also to the scale-factor) in an expanding universe.

Recall that in natural units (c = k = 1) a temperature can be expressed as an energy. Combining **Hubbles law** with the energy density and temperature of a photon gas

$$\rho \propto T^4 \quad \text{and} \quad H^2 \propto 1/a^4 \quad \text{and} \quad T \propto 1/a$$

(and keeping track of the prefactors) we get the useful relation

$$\frac{t}{1\,\mathrm{sec}} \sim g_{\mathrm{eff}}^{-1/2} \left(\frac{1\,\mathrm{MeV}}{T} \right)^2$$

$\mathrm{T} = 1\mathrm{MeV}$ **correspond roughly to** 1 **second**

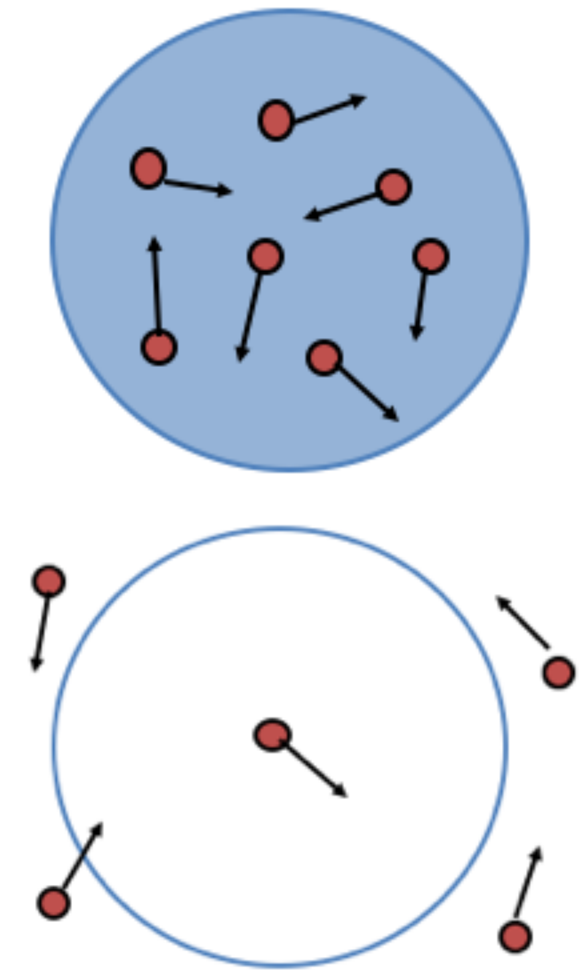
Key concept: Decoupling

In an expanding universe a particle can move at most the size of the horizon $(\sim 1/H)$

Let Γ be the interaction rate for some thermal process

If this interaction does not occur during a Hubble time $(1/\Gamma >> 1/H)$ then the gas can no longer be in thermal equilibrium

We call this **decoupling**. Happens when $\Gamma < H$



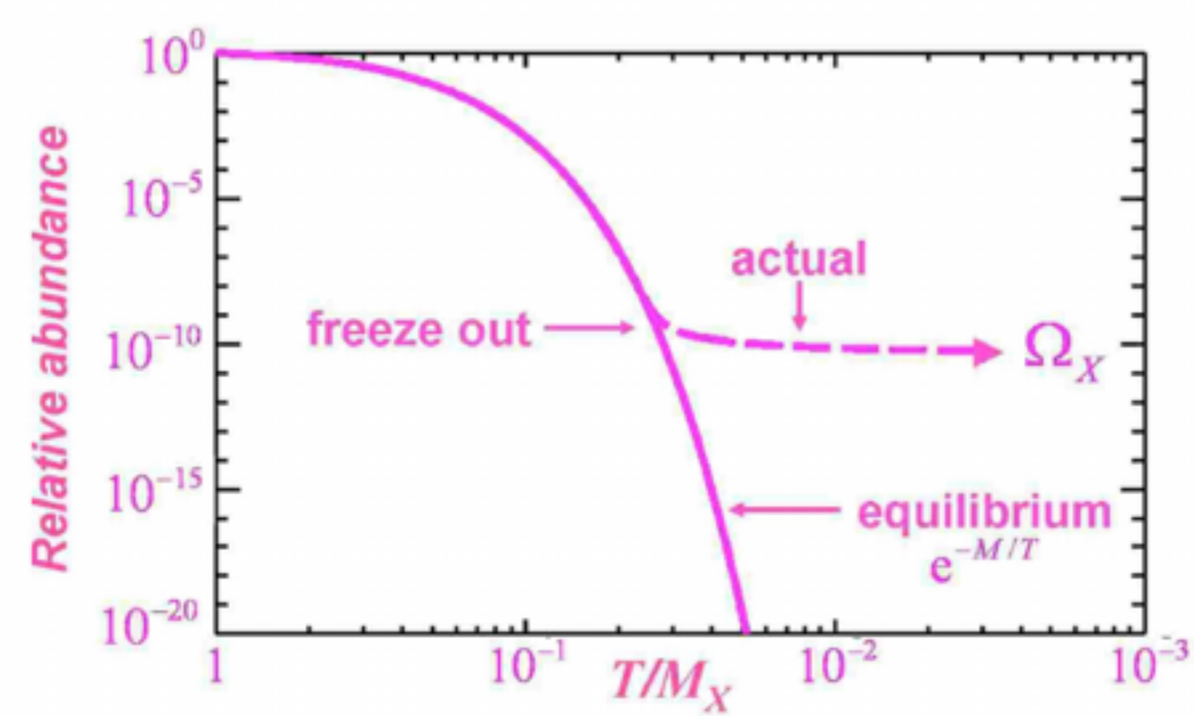
Key concept: Freeze-Out

Recall that in thermal equilibrium, the number of massive (non-relativistic) particles is **exponentially suppressed**

$$n \propto T^{3/2} e^{-\frac{m-\mu}{T}}$$

Once they decouple from the thermal bath (which will happen when the number density gets low enough), the number density of particles is just diluted by the expansion.

This means the number density of particles is no longer affected by equilibrium physics so it **"freezes out"** at decoupling.



Boltzmann Equation

The key equation for describing the evolution of the distribution function for the different particle species we have in the primordial plasma is the (relativistic) **Boltzmann equation**

$$\frac{df}{dt} = C[f] \quad \text{where the RHS describes interactions}$$

If no interactions then the distribution function is conserved. For cosmology (and a typical interaction 1+2 => 3+4) this equation takes the form

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)$$

Cosmology term:
Describes the expansion of the universe

Particle physics term:
interaction rate (Γ) for the process in question

For massive particle when the RHS is negligible (**decoupling**) the evolution of **n** is given by LHS which is just volume dilution (**freeze-out**).

Thermal history of the universe

The story of the universe at the earliest phases is unknown. We believe there was a period of inflation leaving the universe in a very homogenous (and flat) state.

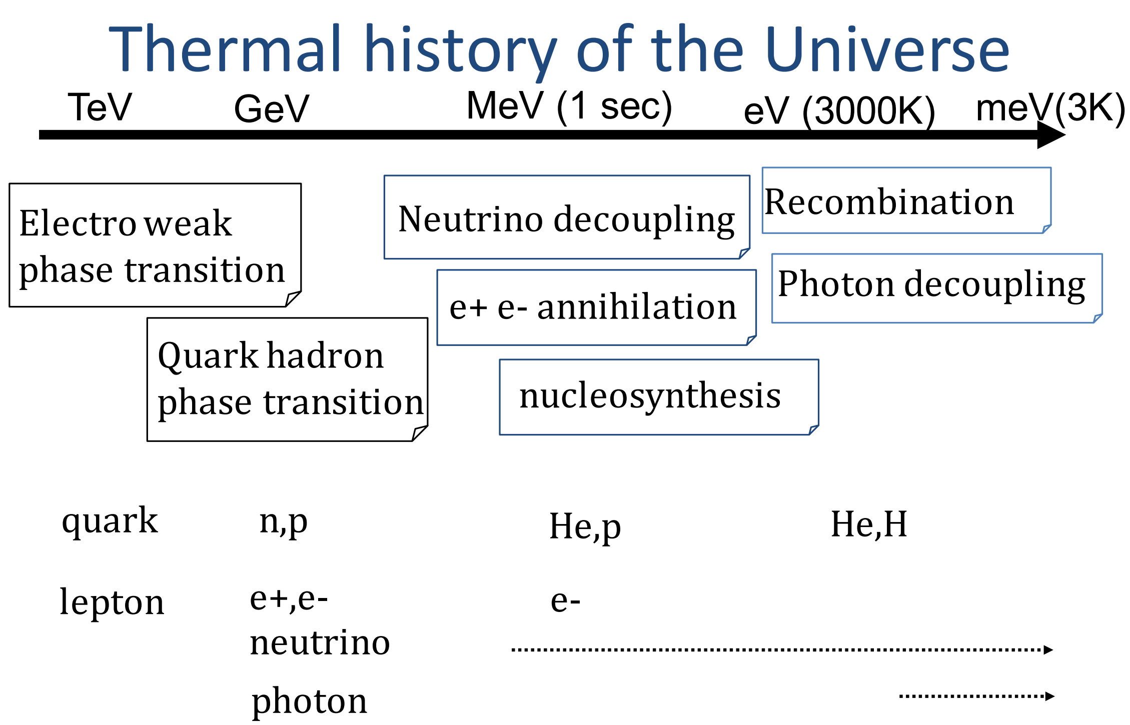
The universe is in the radiation dominated regime and so hot and dense that all particles are relativistic. The first important event that happen is the **electroweak phase transition**. After which the universe consists of free quarks and leptons + force carriers like photons, gluons, W, Z and the Higgs

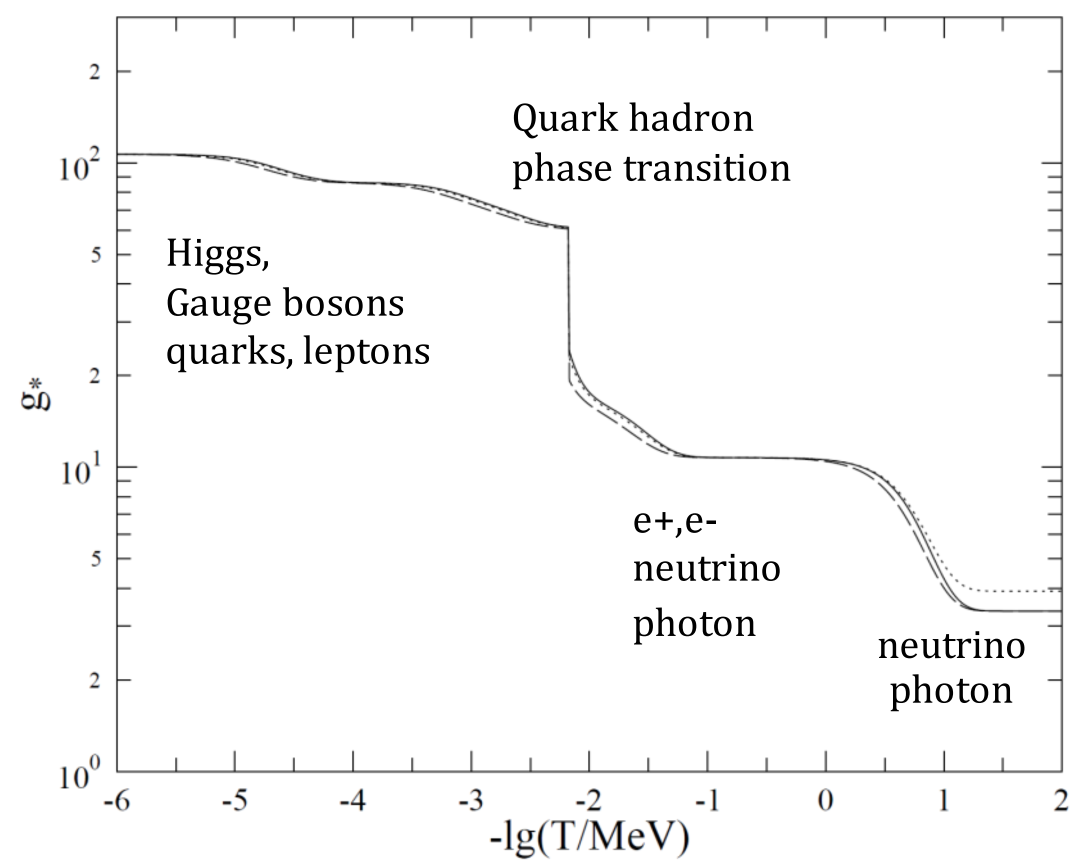
The temperature is initially so high that quarks cannot form baryons/hadrons and we have a quark-gluon plasma.

As the temperature drops we get the **QCD phase transition** where we form hadrons (mainly protons and neutrons).

[Sidenote: Some anti-symmetry between matter and anti-matter must be in place to ensure we have excess of normal matter. Active field of study: **baryogenesis**]

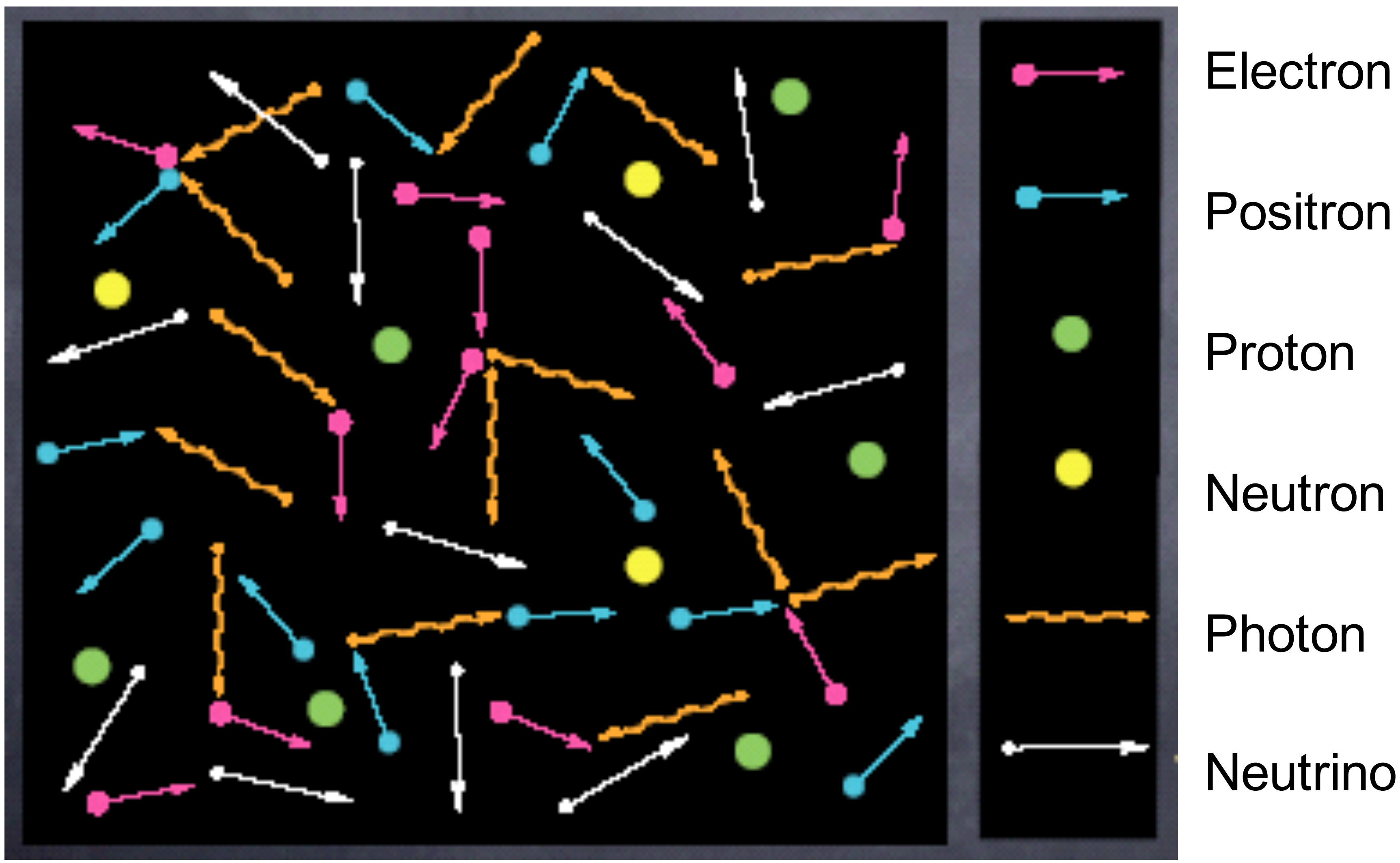
Thermal history of the universe





Timeline of the early universe (left) and the effective number of relativistic degrees of freedom as function of temperature/time (right)

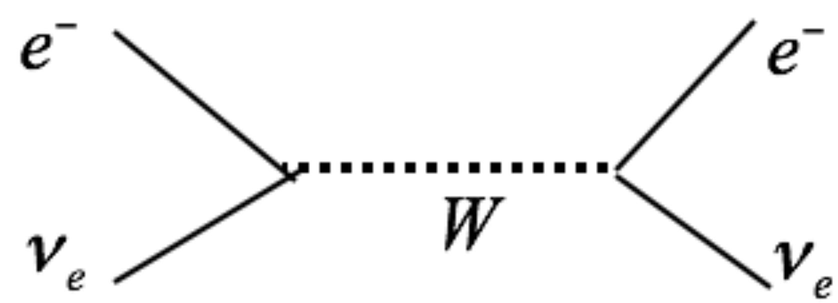
Thermal history of the universe



Our story begins after the QCD phase transition when the primordial plasma looks like this

Key Event 1: Neutrino decoupling

Neutrinos interact via the weak interactions



The interaction rate is given by

$$\Gamma \sim G_F^2 T^5 \quad \text{where} \quad G_F \approx m_W^{-2} \approx 1.17 \cdot 10^5 \ \text{GeV}^{-2}$$

Decoupling occurs when (after ~ one second):

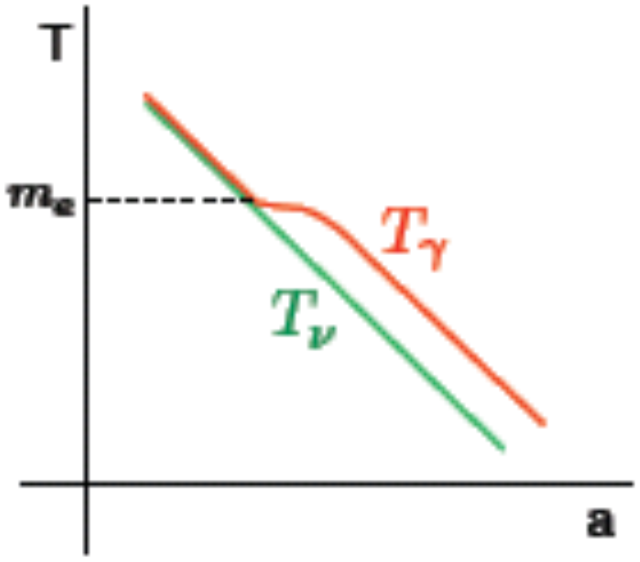
$$\frac{\Gamma}{H} \approx \left(\frac{T}{1.4 \, \text{MeV}} \right)^3 < 1$$

Key Event 2: Electron-positron annihilation

Electrons and positrons interact via the process $e^+ + e^- \to \gamma$

When the temperature has dropped below the mass of the electron ($\mathrm{T = 0.5\ MeV \sim 1\ sec}$) photons cannot decay to electron-positron pairs.

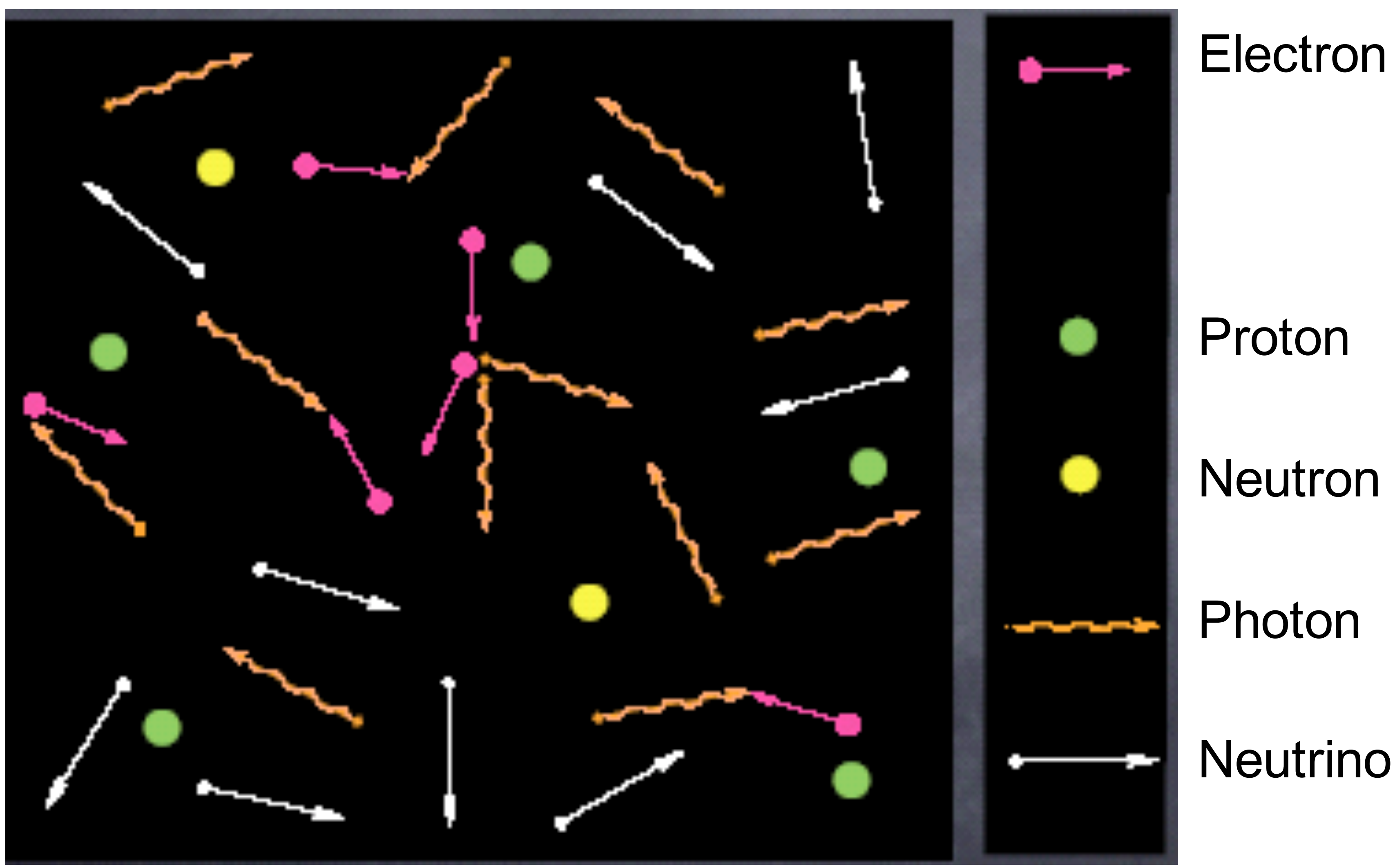
Tiny bits of electrons remains (**lepton asymmetry**)



The annihilation injects energy into the heat bath which slightly increases the temperature. Neutrinos have decoupled so does not get heated. Conservation of entropy gives us

$$T_\nu = T_\gamma \left(\frac{4}{11}\right)^{\frac{1}{3}}$$

Thermal history of the universe



After electron-position annihilation we are left with this
The neutrinos are just observers at this point

Key Event 3: Big Bang Nucleosynthesis

The next thing that happens is that light elements are formed. Before BBN neutrons and protons kept in equilibrium via weak interactions:

$$\text{scattering}\quad\left\{\begin{array}{l} n + e^{+} \rightarrow p + \bar{\nu}_e, \\ p + e^{-} \rightarrow n + \nu_e, \end{array}\right.$$

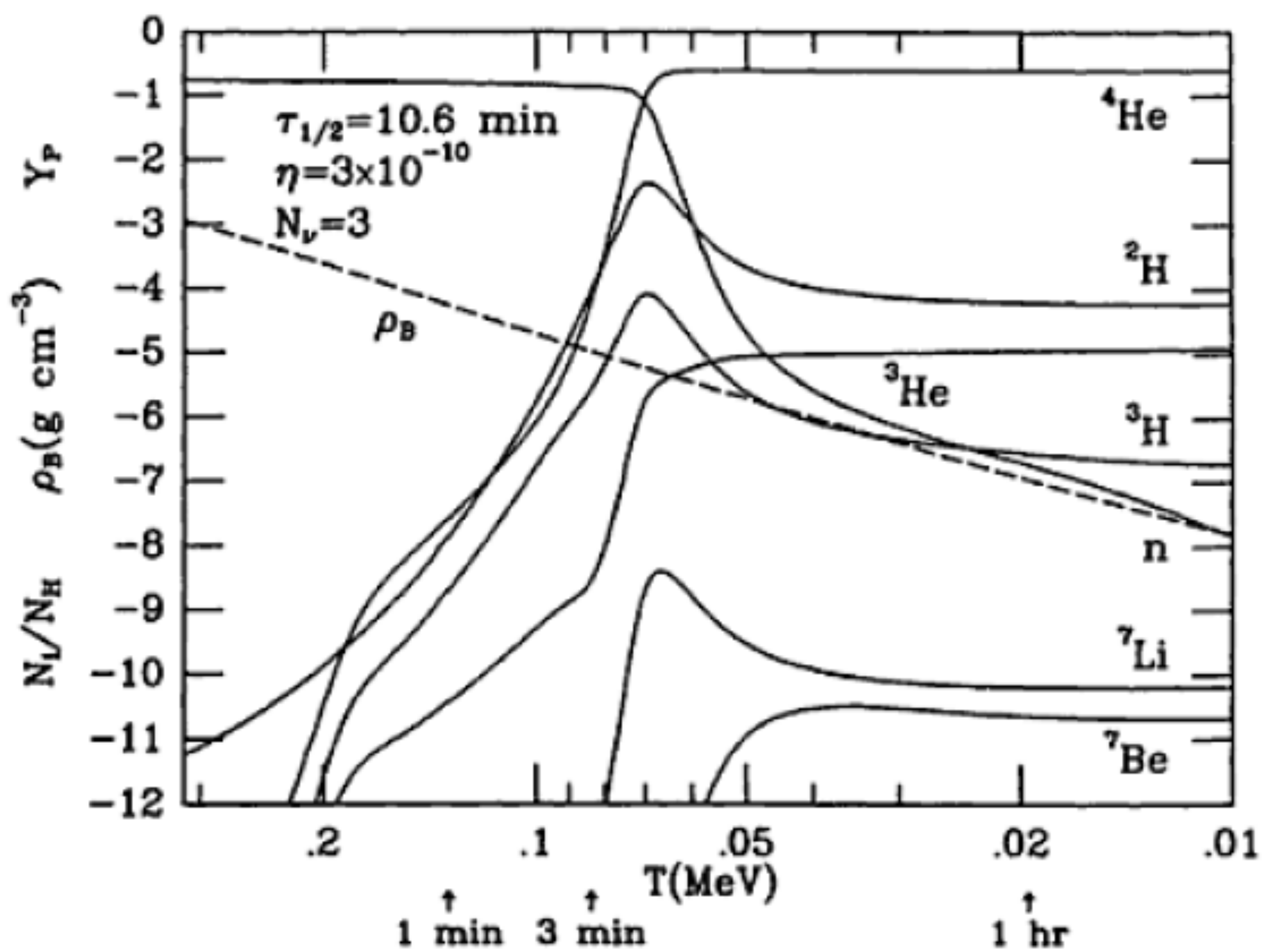
$$\text{decay}\qquad n \rightarrow p + e^{-} + \bar{\nu}_e$$

In BBN light elements are formed from combining free protons and neutrons in the processes:

Deuterium:	p + n → D+γ	$E = 2.2\ MeV$
Helium-3:	D + D → ^{3}He+n	$E = 7.7\ MeV$
Helium-4:	^{3}He + D → ^{4}He+p	$E = 28\ MeV$

Big Bang Nucleosynthesis

BBN happens in the first ~15min of the universe. Below you can see the evolution of the light element abundances



Big Bang Nucleosynthesis

We don't have time to go through this in detail, but let's mention some key points.

Recall number density of non-relativistic species is exponentially suppressed. Relative number of protons to neutrons therefore differ as there is a **slight mass difference** (neutron $1.3\ \mathrm{MeV}$ heavier than proton which has mass $938\ \mathrm{MeV}$).

Protons freeze out at $\mathrm{T} \sim 0.8\ \mathrm{MeV}$ and after that only neutron decay is important to change relative abundances. This leads **protons to outnumber neutrons** by a factor of 6-7 (naively 6, but closer to 7 due to very large photon-to-baryon ratio).

Almost all of the neutrons gets processed to Helium 4. With the neutron to proton ratio of 1/7 this means the abundance of helium to the total abundance is ~ 0.25

Hardly any elements heavier than Lithium gets produced. E.g. we need a tripple-alpha process to produce carbon 12 which requires a density of helium to be much higher than it is. This only happens in stars later on. BBN produces hydrogen+helium+lithium. Anything up to iron is produced in stars and the rest is produced in supernova explosions.

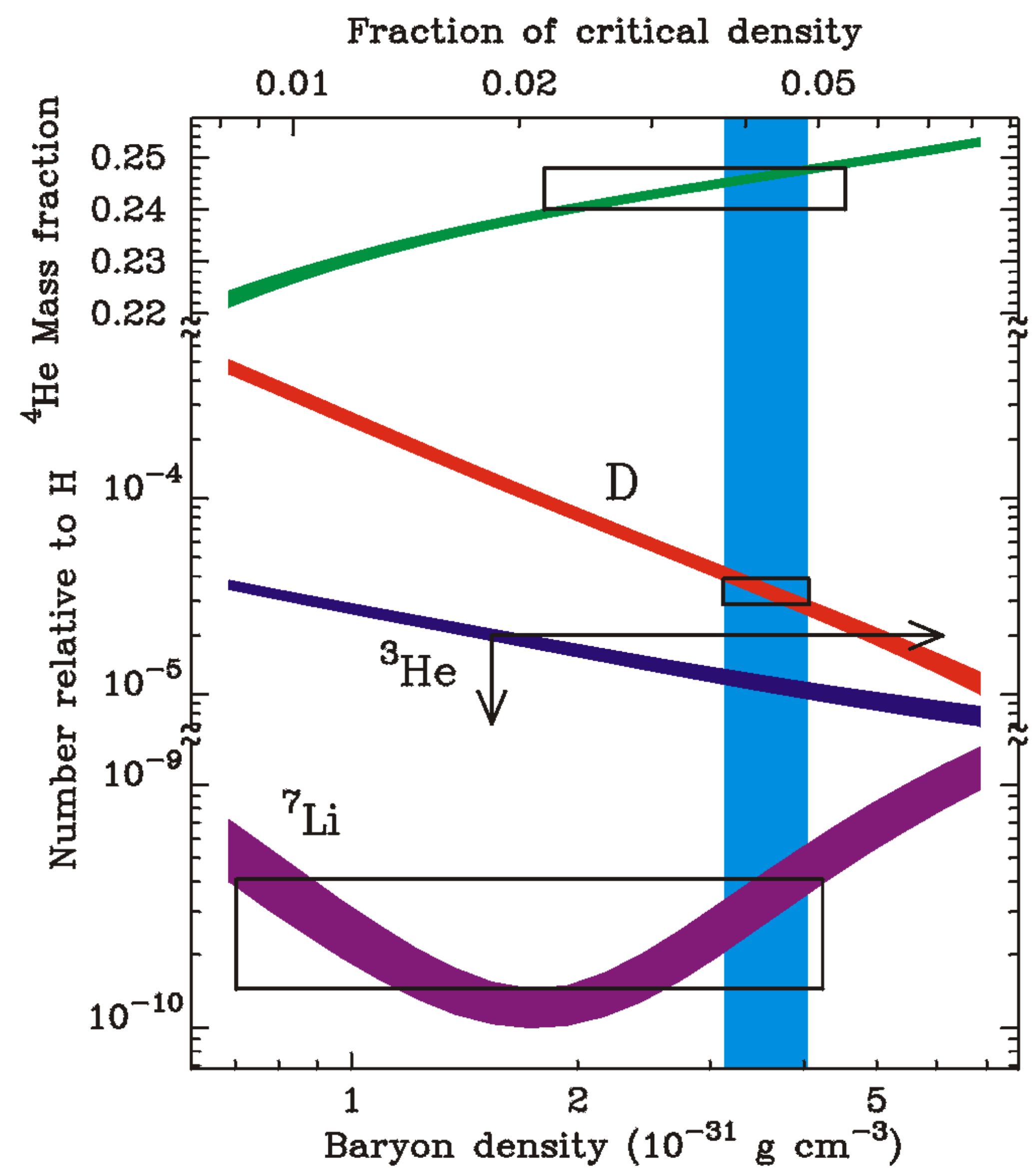
Big Bang Nucleosynthesis

We can compute what the relative abundances of light elements are and compare to observations (what is in stars and gas in the universe etc.). Matches very well with theoretical predictions!

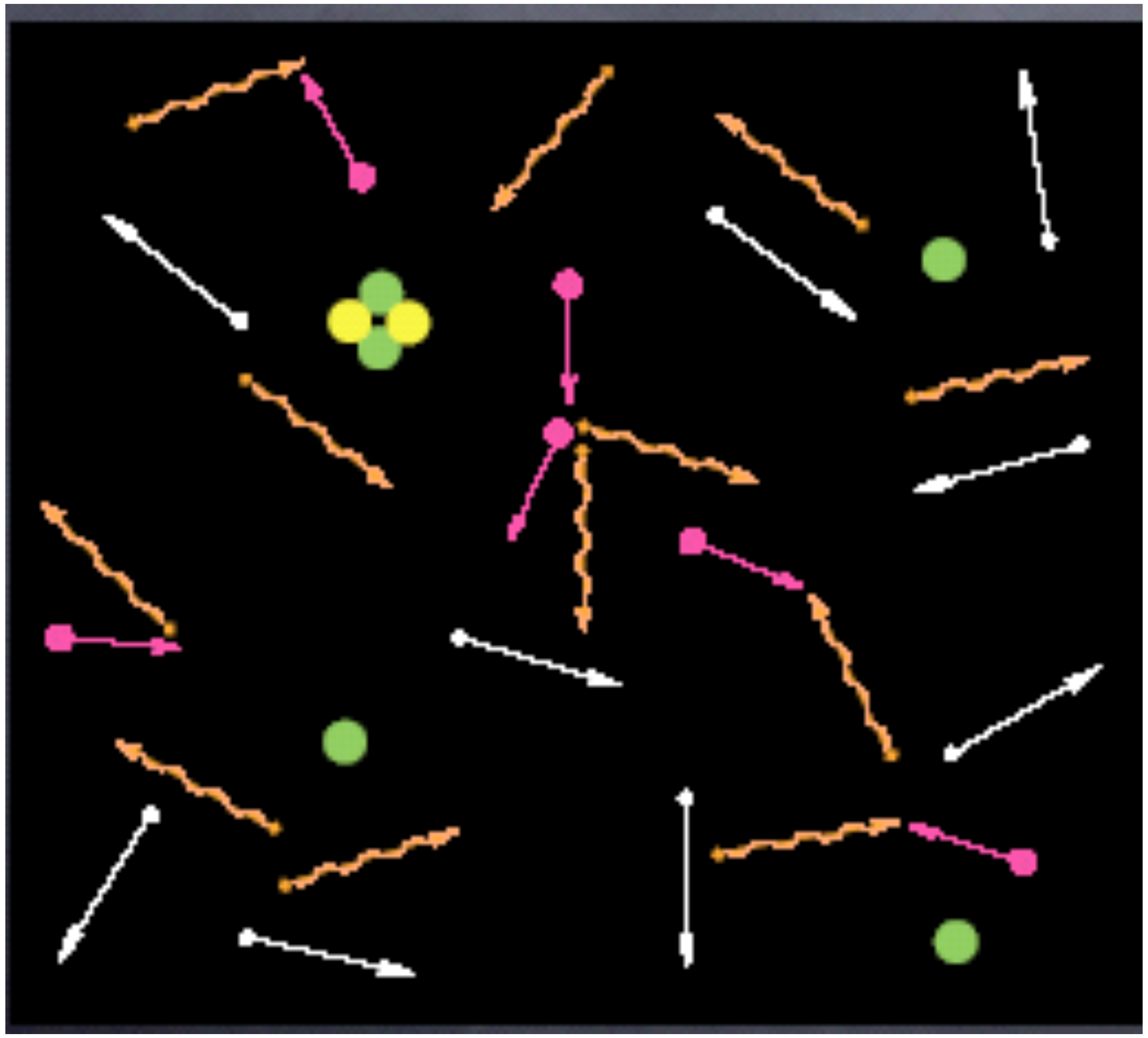
E.g. helium to hydrogen ratio is found to be ~ 0.25 compared to 0.22 - 0.25 observationally!

Strong evidence for the Big Bang model.

(There are some small tensions regarding the abundance of lithium: this is the co-called **lithium problem**)



Thermal history of the universe

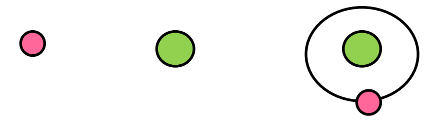


Now we have free-streaming neutrinos, interacting electrons and photons, and light nuclei

Key Event 4: Recombination and the CMB

The temperature is now finally low enough such that we can form atoms in the process

$$e^- + p \rightarrow H + \gamma$$

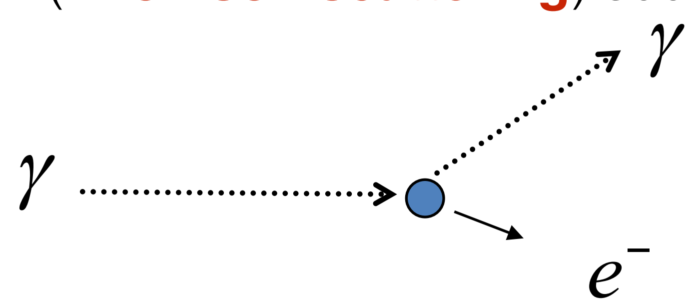


Naively we expect this to happen when the temperature gets below the ground state energy of hydrogen (13.6 eV), however due to the low baryon to photon ratio (one part in a billion) this is delayed to 0.1 eV . Precise treatment via **Boltzmann equation** (**Saha equation**).

This process is called **recombination** (or really just combination)

Recombination and the CMB

Most of the free electrons ends up in atoms. The main process keeping the electrons and photons in equilibrium (**Thomson scattering**) soon becomes inefficient

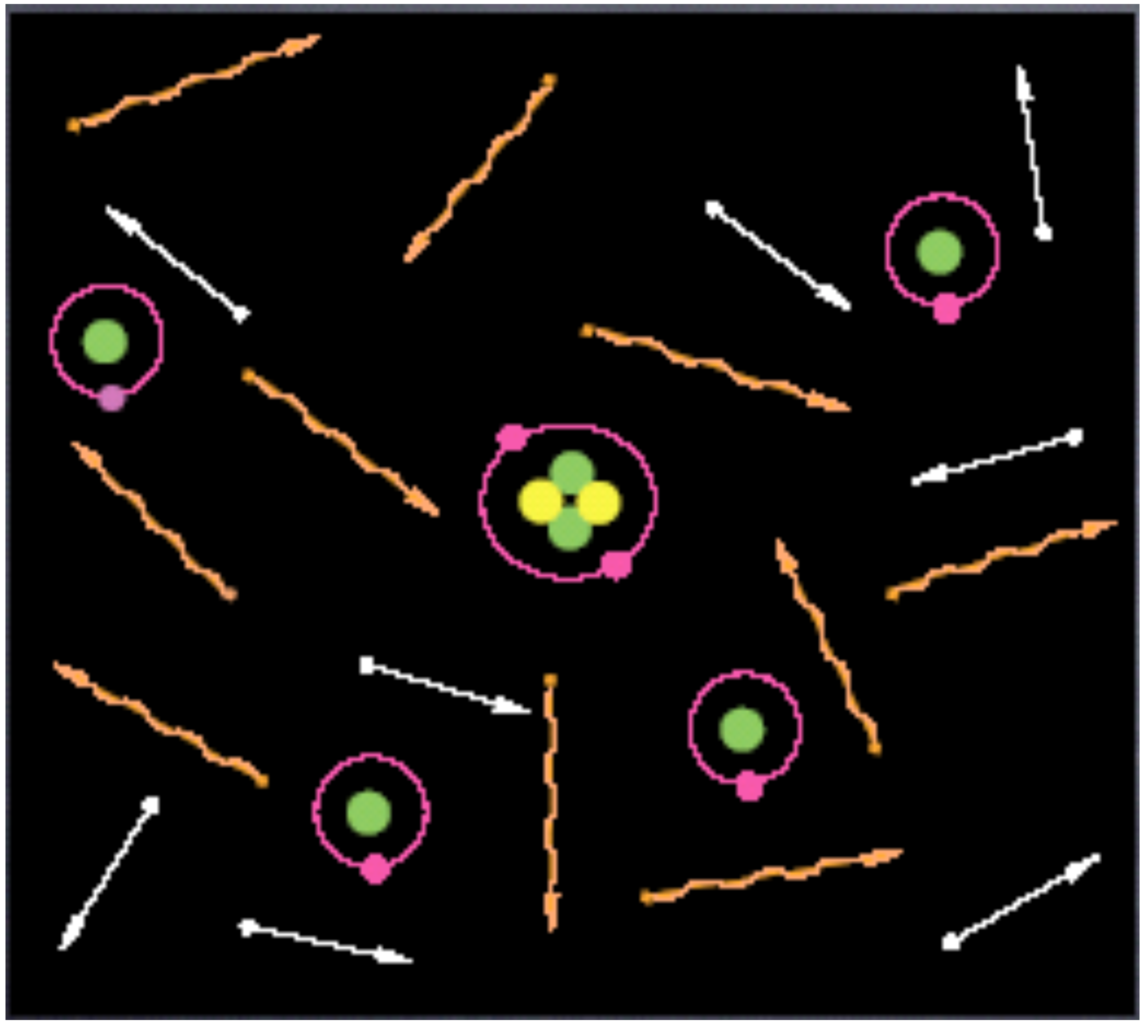


This leads the photons to **decouple** from the primordial plasma (happens roughly when abundance of free electrons is $\sim 1\%$ of total)

We call this the "**last scattering surface**" at redshift $z \sim 1100$ $(T \sim 3000K)$ and after this the photons free-stream towards us maintaining it's black-body spectrum while the temperature decreasing with the expansion of the universe.

Since photons are not in thermal equilibrium any interactions along the way will destroy the black-body shape. Happens for example in galaxy clusters (the **Sunyaev–Zeldovich effect**) and can be used to locate such clusters using observations of the CMB.

Thermal history of the universe



Now we have free-streaming neutrinos and photons, and atomic hydrogen and helium

The CMB is released!

Summary

The big-bang model is the realisation that our universe started up in very hot dense state. Combining **particle physics** with **general relativity** and **statistical mechanics** / thermodynamics we are able to describe the universe and it's contents from early on till today.

We have gone through (schematically) the evolution of the early universe from after the QCD phase transition till the formation of the CMB.

The Hot Big Bang model gives key predictions like the **abundance of light elements** and the **cosmic microwave background** that agree very well with observations.