Theoretical Cosmology

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PhD Lecture; ICG Portsmouth; December 2018

Outline

We will in these lectures go through the basics of **theoretical cosmology** focusing on questions like

- How do we **mathematically model** our Universe and the matter in in.
- What are the **assumptions** going into it (cosmological principle; matter as perfect fluid; gravity described by general relativity; …) and the **observational evidence** for this.
- How do we derive the **equations** governing the evolution of the universe. We will do a full derivation of the Friedmann equations.
- How do matter and radiation **propagate** in such a universe and what are the different ways of measuring **distances**.
- What are the main **observables** for background cosmology and how can a cosmological model can be confronted with such observations in order to determine the free parameters of the model.
- What is our **fiducial model** of cosmology today

Introduction

Cosmology: the study of the large scale properties of the universe as a whole

Ancient as a philosophical discipline, but fairly new as a quantitative science

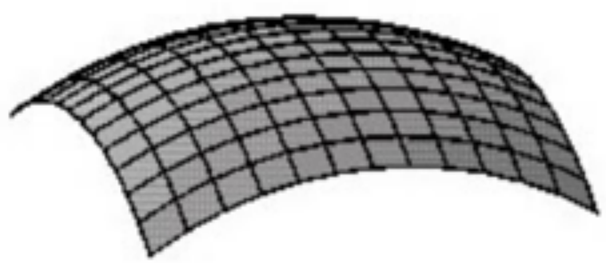
Key insight by Einstein (1916):

Gravity = Curvature of spacetime

Lead to concrete mathematical models for the evolution of the universe as a whole

Mathematical model for the Universe

Spacetime is a four dimensional **manifold**



Fundamental quantity: **metric**
Tells us how to measure distances
Metric often presented by writing down **line-element**

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Specify a **theory of gravity** to get evolution equations (PDEs) for the metric in terms of matter / energy content.

(NB: in these notes we will use units where c = 1)

Theory of gravity

We can specify the theory by either giving the **field-equations** (EOM), e.g. for General Relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

…or specifying a **Lagrangian** for which EOM follows from variation of action

$$S = \underbrace{\int \sqrt{-g}\, \mathrm{d}^4 x \frac{R}{16\pi G}}_{\text{gravity}} + \underbrace{\int \sqrt{-g}\, \mathrm{d}^4 x \left[\mathcal{L}^{\text{electron}} + \mathcal{L}^{\text{photon}} + \ldots\right]}_{\text{matter}}$$

For practical cosmology one only needs field-equations. For theoretical purposes (modifying gravity, adding new matter fields etc.) the latter is very useful to know. We have:

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}$$

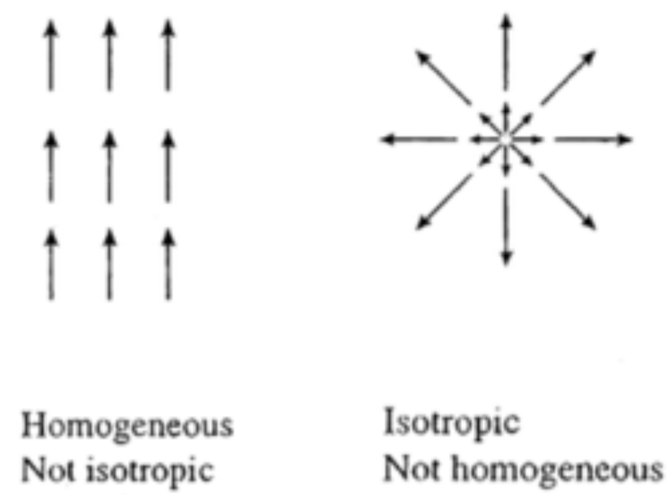
Cosmological principle

Einsteins equations are complicated! Analytical solutions only known for problems with high amount of **symmetry**. Even worse in alternative models

The main assumption in formulating a model for the universe is the **cosmological principle**

Homogenity: the universe is (statistically) the same no matter where you are in the universe

Isotropy: the universe (statistically) looks the same in any direction



Not really a principle that we have to assume, it’s more of a **convenient assumption**!

Evidence: distributions of galaxies (homogenity above ~ 100-200 Mpc/h) and the cosmic microwave background (isotropy).

Beyond homogeneity

Non-homogenous models have been considered.

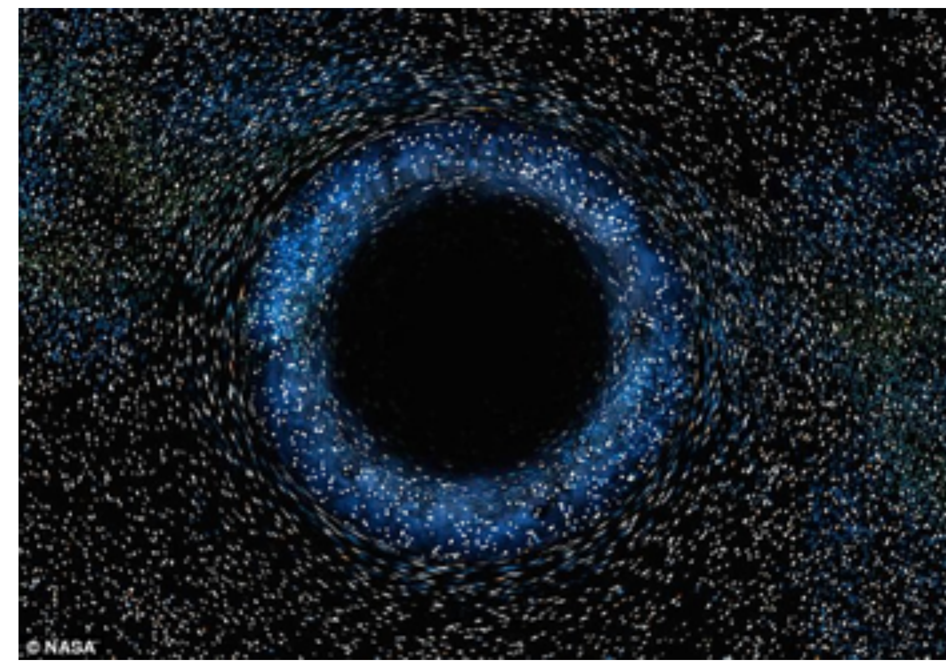
Example: **LTB models** was popular a few years ago

We live close to the centre of a big void

Can fit supernova data exactly by fitting density profile

Combining with CMB data we need very low H0 which is in tension with observations!

Final blow (of this particular idea, not inhomogeneity in general) by kinetic Sunyaev–Zel'dovich effect (scattering of CMB photons by electrons in galaxy clusters that have velocity wrt the CMB). See e.g. Bull et al.



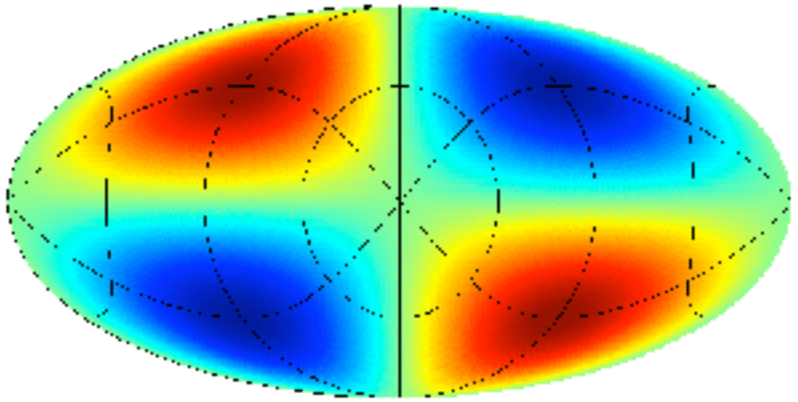
Beyond isotropy

Bianchi models are examples of homogenous, but anisotropic models

Simplest example has different expansion rates in the three spatial directions

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2 + b^2(t)\mathrm{d}y^2 + c^2(t)\mathrm{d}z^2$$

Highly constrained by the CMB: e.g. different expansion rates creates large quadrupole while the one we observe is less than one part in hundred thousand.



Friedmann–Lemaître–Robertson–Walker Metric

The cosmological principle implies that the metric takes the simple **FLRW** form (here in Cartesian coordinates)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}r^2 + K\frac{x_i^2\mathrm{d}x_i^2}{1-Kr^2}$$

a: **scale-factor**; the “radius” of the universe
K: **curvature**

Friedmann: Russian physicist. The expanding Universe
Lemaître: Belgian priest. The Big Bang theory
Robertson and **Walker**: American and British physicist / mathematicians. Classification of cosmological solutions

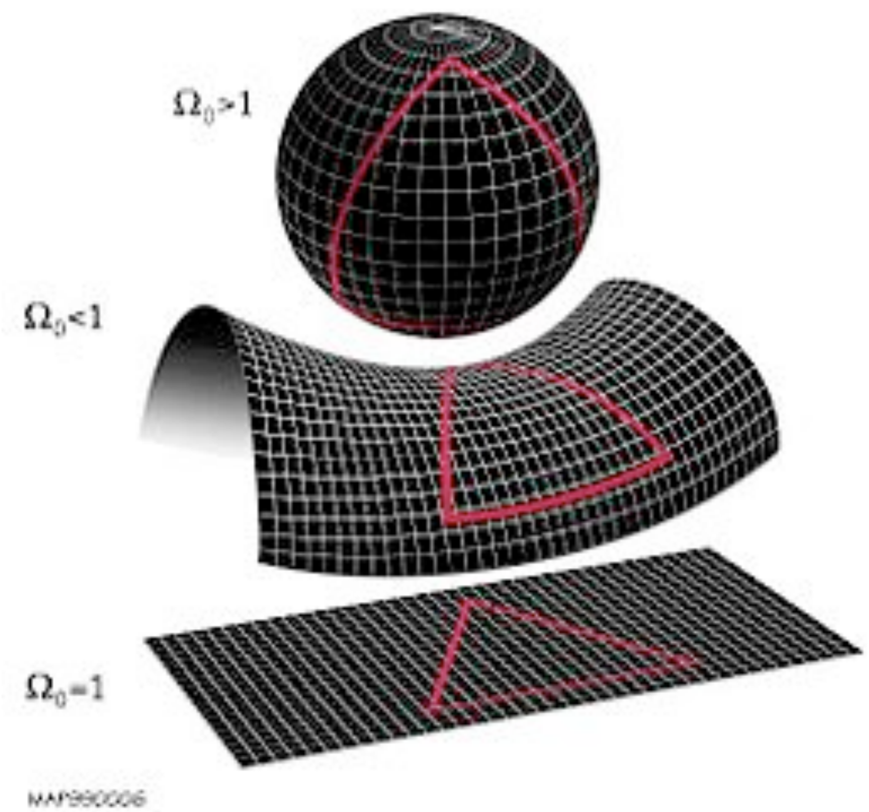
Three possible geometries

Magnitude of K determines how curved space is. Sign of K determines the type of geometry:

K>0: **Spherical** (finite)
K < 0: **Hyperbolic** (infinite)
K = 0: **Flat** (infinite)

Another way to view it: angles in triangle add up to > 180, < 180 or 180 degrees respectively

(Turns out curvature can be treated mathematically just as it was a matter fluid with a particular equation of state in a flat geometry)



Modelling of matter

Matter is averaged over large regions
Assumed to be **perfect fluid**

Described by **energy density** and **pressure**

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}, \qquad (30)$$

Fundamental quantity: **equation of state** $\omega \equiv \frac{P}{\rho}$

Sidenote: the cosmological constant

For general relativity, taking the covariant derivative of the field equation gives us local energy conservation

$$\nabla^\mu G_{\mu\nu} \equiv 0 = 8\pi G \nabla^\mu T_{\mu\nu}$$

Since the covariant derivative of the metric is zero we can add a term (**the cosmological constant**)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

and still preserve this property. This also gives the same Newtonian limit. Main effect of this term is cosmologically, but it also modified say large distance limit of Schwarzschild (we can bound size of CC from planetary orbits!)

Vacuum energy gives the same type of term. Indistinguishable. Effective cosmological constant is sum of these two effects (**fine tuning problem**)

Modelling of matter

Conservation of energy in FLRW universe

$$\nabla^\mu T_{\mu\nu} = 0 \implies \dot{\rho} + 3H(1+\omega)\rho = 0 \implies \rho \propto \frac{1}{a^{3(1+\omega)}}$$

Normal matter : $\omega \approx 0 \implies \rho \propto \dfrac{1}{a^3}$ (volume dilution)

Radiation: $\omega = \dfrac{1}{3} \implies \rho \propto \dfrac{1}{a^4} = \dfrac{1}{a^3} \cdot \dfrac{1}{a}$ (dilution x redshift)

Vacuum energy / CC: $\omega = -1 \implies \rho \propto \mathrm{constant}$

Curvature: $\omega = -\dfrac{1}{3} \implies \rho \propto \dfrac{1}{a^2}$

(curvature acts just as if it were a matter fluid in the Friedmann equations for a flat geometry)

Averaging of the Einstein equations

When **assuming** matter can be described as a perfect fluid we have effectively averaged the RHS of Einstein equation:

$$\langle G_{\mu\nu}(g_{\mu\nu})\rangle = 8\pi G \langle T_{\mu\nu}\rangle$$

The FRLW metric is also the “averaged metric”, but

$$\langle G_{\mu\nu}(g_{\mu\nu})\rangle \neq G_{\mu\nu}(\langle g_{\mu\nu}\rangle)$$

since PDEs are non-linear and it’s the RHS we use when deriving equation.

This is the **averaging problem in cosmology** (which also includes the non-trivial problem of what it means to average a tensor in this setting)

The error we make here effectively introduces something that acts like a CC.
But (naive) estimates suggest it’s a small effect.
Has been proposed as potential solution to dark energy puzzle,
“Most likely” not a big problem for cosmology, but still not fully settled!

Friedmann equations

We have all the ingredients to compute evolution equations

Geometry (FLRW metric) + matter content (collection of perfect fluids) + a theory of gravity (GR)

Step 1: Compute Christoffel symbols $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\delta} (g_{\delta\beta,\alpha} + g_{\alpha\delta,\beta} - g_{\alpha\beta,\delta})$

Step 2: Compute Riemann tensor $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$

Step 3: Contract to get Ricci tensor and scalar $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}$

Step 4: Plug into Einstein equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

Friedmann equations

Assuming a flat (K=0) FRLW metric in cartesian coordinates then the non-zero **Christoffel symbols** are

$$\Gamma^0_{ij} = H a^2 \delta_{ij}, \quad \Gamma^i_{0j} = H \delta^i_j$$

Ricci tensor

$$R_{00} = -3\dot{H} - 3H^2, \quad R_{ij} = a^2(\dot{H} + 3H^2)\delta_{ij}$$

Ricci scalar

$$R = 6\dot{H} + 12H^2$$

Friedmann equations

Two independent equations

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00} \implies 3H^2 = \frac{8\pi G}{3} \sum_i \rho_i$$

and

$$R_{00} = 8\pi G (T_{00} - \frac{1}{2} g_{00} T) \implies \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i)$$

We also have energy conservation seen earlier, but it also follows from the two equation above.

We can see that acceleration of the universe requires:

$$\omega < -\frac{1}{3}$$

Friedmann equations

We often describe the energy densities in terms of **density parameters**

$$\Omega \equiv \frac{8\pi G\rho}{3H^2}$$

With curvature and CC described as part of matter content the density parameters satisfy $$\sum_i \Omega_i \equiv 1$$

The first Friedmann equation can be written

$$\frac{H^2(a)}{H_0^2} = \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda 0} + \dots$$

The free parameters (Hubble parameter and density parameters) are to be fitted by observations

Light/matter propagation in a FLRW universe

Having discussed how the universe itself evolves, we need to discuss how matter/photons propagate in the universe

Geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{d^\alpha}{d\tau}\frac{d^\beta}{d\tau} = 0$$

Photons travel on **0-geodesics**:

$$\mathrm{d}s^2 = 0$$

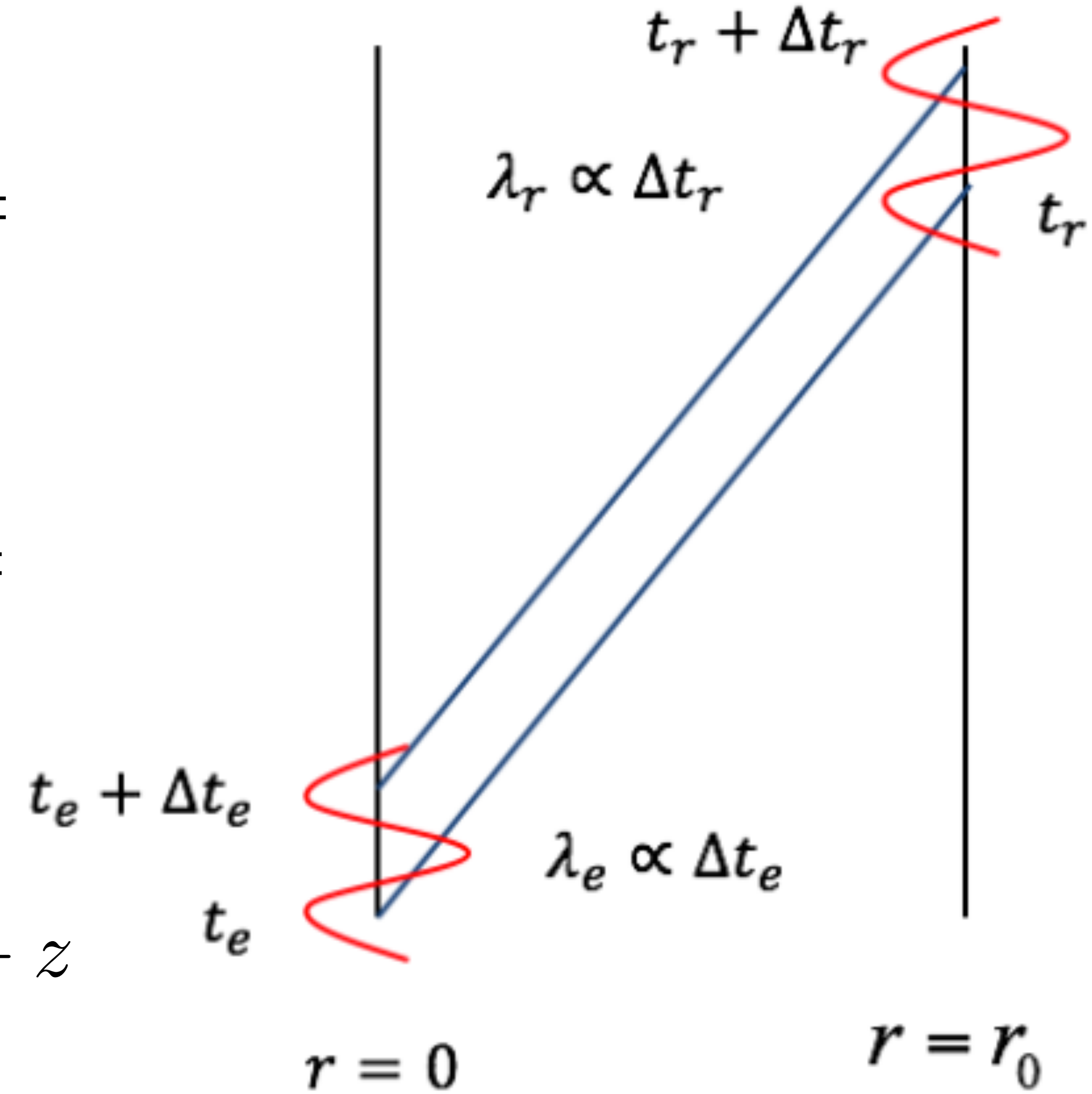
Light propagation in a FLRW universe

Consider the path of a photon

$$\int_{t_e}^{t_r} \frac{\mathrm{d}t}{a(t)} = \int_0^{r_0} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}$$

$$\int_{t_e+\Delta t_e}^{t_r+\Delta t_r} \frac{\mathrm{d}t}{a(t)} = \int_0^{r_0} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}$$

$$\text{For small } \Delta t \implies \frac{a(t_r)}{a(t_e)} = \frac{\lambda_r}{\lambda_r} \equiv 1+z$$



Redshift is fractional change in wavelength due to expansion

Redshift and scale-factor as measure of time

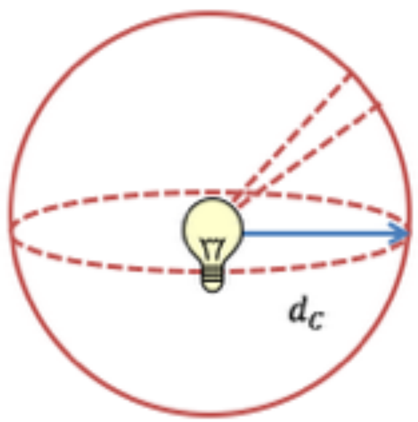
In expanding universe there is a 1-1 relation between **cosmic time**, **scale-factor** and **redshift**. We can use either to describe a time

$$t = \int_0^a \frac{\mathrm{d}a}{aH(a)} \qquad\qquad z = \frac{1}{a} - 1$$

Distance measures in a FLRW universe

Co-moving distance: distance between two points measured along a path defined at the present cosmological time

$$d_C(t) \equiv \int_t^{t_{\mathrm{today}}} \frac{\mathrm{d}t}{a(t)} = \int_a^1 \frac{\mathrm{d}a}{a^2 H(a)}$$

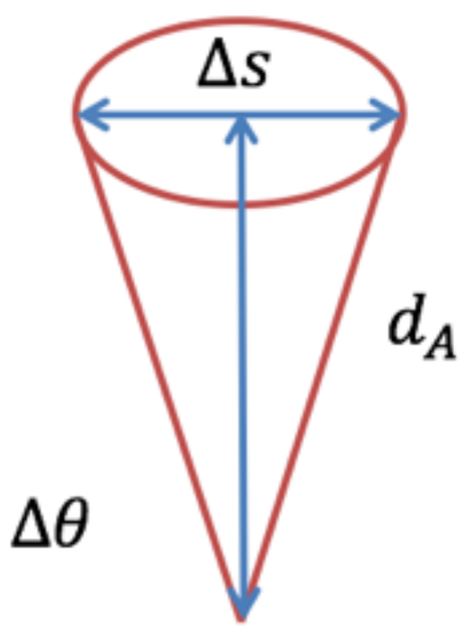


Luminosity distance (here assuming flatness):

$$F \equiv \frac{L}{4\pi d_L^2} = \frac{L}{4\pi d_C^2 (1+z)^2} \implies d_L = (1+z) d_C$$

Angular diameter distance (here assuming flatness):

$$d_A \equiv \frac{\Delta s}{\Delta\theta} = \frac{a d_C \Delta\theta}{\Delta\theta} = \frac{d_C}{1+z}$$



Distance measures in a FLRW universe

For small z all distances are the same and reduces to **Hubble's law**

$$d \sim \frac{cz}{H_0} \implies v = H_0 d \quad \text{with} \quad v = cz$$

Note: the velocity-redshift relation above is just the relation we have for the Doppler effect.

History of cosmological models

We have seen that in our framework a cosmological model is fully specified by specifying the energy content

Historically one of the first models considered was
Einsteins static universe

With matter + curvature + cosmological constant we can make a static universe

$$\sum \rho_i = 0 \quad \sum_i \rho_i + 3p_i = 0 \implies H^2 = 0 \quad \text{and} \quad \ddot{a} = 0$$

…but it's fundamentally unstable to perturbations!
Einstein called this his greatest blunder.

History of cosmological models

The static universe was considered appealing as it has no beginning that needs to be explained

One cannot have a stable static universe, however if matter was spontaneously created as to keep the energy density constant then we would have an expanding universe that looks static: a **steady state universe** (Hoyle 1933)

Creation rate needed is **really tiny** so no problem with experiments.

Big problem of explaining observations of redshift dependence of radio sources

<u>Final blow</u>: the discovery of the CMB showing the Universe had a beginning

History of cosmological models

The standard model was long a purely matter dominated model (**Einstein de-Sitter model**)

Was realised (e.g. missing matter in galaxies Oort 1932 and galaxy clusters Zwicky 1933; rotation curves Rubin 1970s; …) most of the matter needed to be in form of non-baryonic **cold dark matter**

Early to mid 1990s it was realised that one likely needs to add something like a **cosmological constant** (CC)

Accelerated expansion of the universe discovered in 1997 with supernova observations.

Biggest unsolved problem in cosmology today: is this acceleration due to a pure CC of some new form of matter (**dark energy**) or modifications of gravity or something else.

Determining the cosmological model

Observational data constrains the parameters of a model.
Some fundamental observables and some key features:

Supernova Type 1 (SN1A): we know (can standarize) the luminosity of type 1 supernova. We can measure the luminosity distance and then fit these to a given cosmological model.

Cosmic Microwave Background (CMB): typical size of fluctuations / location of first peak gives constraint on flatness and amount of vacuum energy +

Baryonic acoustic oscillations (BAO): gives us how much matter we have in the universe +

The **best-fit values** gives us the parameters of the model we try to fit.

If fit is not consistent across different observables => not a viable model.

Comparing **goodness-of-fit** across different models weighted up against (often subjective) criteria such as how **"natural"** is one model over another + **how many free parameters** etc. gives a way of determining what the **"best" model** is.

"Seek simplicity and distrust it"; "Less is more"

The standard model of cosmology today

Observations have lead us to the so-called **LCDM (Lambda-Cold-Dark-Matter)** model

Gravity is described by **General Relativity**

The geometry is described by a **flat FLRW** metric (no curvature)

Matter accounts for ~0.3 of the energy budget today, but only ~0.05 is in term of baryonic matter (stars, gas). The rest is **cold-dark-matter** (and a tiny bit being **massive neutrinos**)

Dark energy in the form of a **cosmological constant** account for ~0.7 of the energy budget today and has dominated since z~1.

Radiation (photons and massless neutrinos) account for only ~1/100.000 today. Temperature T = 2.7K. Dominated the energy budget at z >> 1000.

Impressively consistent fit across many different observables. No clear sign of departure from LCDM, thought there are some tensions.

Predicts that the universe started off in a hot dense state, the so-called **hot big bang**, and the physics of the early universe is something we will cover next lecture.

Supernova Cosmology Project
Suzuki, et al., *Ap.J.* (2011)