第 4 章　上下文无关语言

(Part 2 of 2)

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Outline

1. Context-Free Grammars
2. Pushdown Automata
3. Non-Context-Free Languages

Context-Free Languages

regular languages 正则语言

- finite automata: DFA / NFA
- regular expressions

some simple languages, such as $\{0^n1^n \mid n \geq 0\}$, are ***not*** regular languages.

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context-free languages 上下文无关语言

- pushdown automata 下推自动机

- first used in the study of human languages
- in the specification and compilation of programming languages
  - ***parser***
  - the construction of a parser from a context-free grammar

Outline

1. Context-Free Grammars
   - Formal Definition of a Context-Free Grammar
   - Examples of a Context-Free Grammar
   - Designing Context-Free Grammars
   - Ambiguity
   - Chomsky Normal Form
2. Pushdown Automata
3. Non-Context-Free Languages

Outline

1. Context-Free Grammars

2. Pushdown Automata
   - Formal Definition of a Pushdown Automaton
   - Examples of Pushdown Automata
   - Equivalence With Context-Free Grammars

3. Non-Context-Free Languages

Pushdown Automata 下推自动机

Pushdown Automata (PDA): we introduce a new type of computational model.

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Pushdown Automata (PDA): we introduce a new type of computational model.

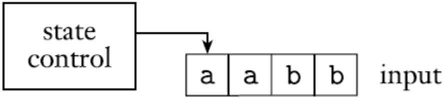
- like NFA but have an extra component called a ***stack***.
- the stack provides additional memory beyond the finite amount available in the control.
- the stack allows PDA to recognize some nonregular languages.

PDA are equivalent in power to CFG

- two options for proving that a language is context free
  - give either a CFG generating it (generator)
  - or a PDA recognizing it (recognizer)

Pushdown Automata 下推自动机

DFA/NFA vs. PDA



Schematic of DFA/NFA

Pushdown Automata 下推自动机

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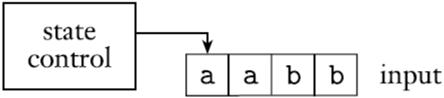

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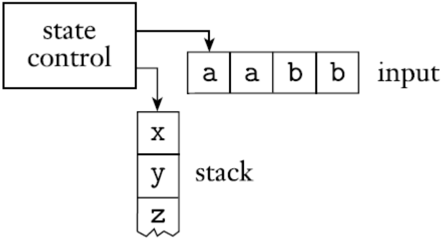

Schematic of PDA

Pushdown Automata 下推自动机

PDA can write symbols on the stack and read them back later.

Writing a symbol ***"pushes down"*** all the other symbols on the stack.

- all access to the stack may be done only at the top: "last in, first out"
- ***pushing***: writing a symbol on the top of the stack
- ***popping***: removing a symbol on the top of the stack

A stack can hold an unlimited amount of information.

the language $\{0^n1^n \mid n \geq 0\}$

- a DFA/NFA is unable to recognize it.
- A PDA is able to recognize it.

Pushdown Automata 下推自动机

Deterministic and nondeterministic PDA are ***not*** equivalent in power.

- Nondeterministic PDA recognize certain languages that ***no*** deterministic PDA can recognize.
- Recall that DFA and NFA do recognize the same class of languages.
- So the pushdown automata situation is different.
- We focus on nondeterministic PDA because these automata are equivalent in power to CFG.

Formal Definition of a Pushdown Automaton

定义 (PDA (下推自动机))

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6. $F \subseteq Q$ is the set of ***accept states***.

Formal Definition of Computation for a PDA

A PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ computes as follows.

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  3. $r_m \in F$

Examples of Pushdown Automata

例 (PDA M_1 recognizes the language $\{0^n 1^n \mid n \geq 0\}$)

Let M_1 be $(Q, \Sigma, \Gamma, \delta, q_1, F)$, where

$Q = \{q_1, q_2, q_3, q_4\}$

$\Sigma = \{0, 1\}$

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Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									

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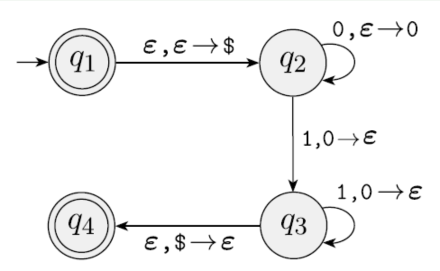

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A pushdown automaton that recognizes the language

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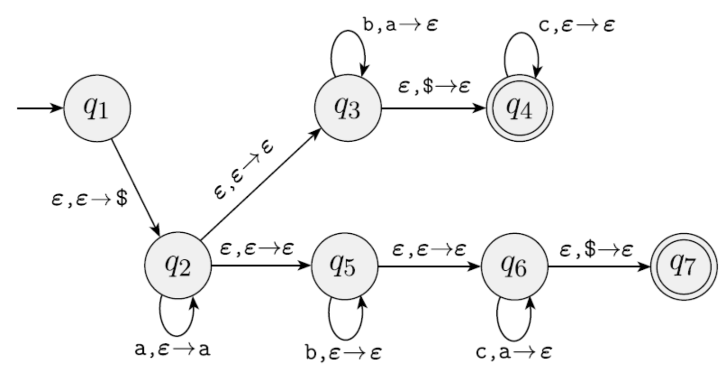

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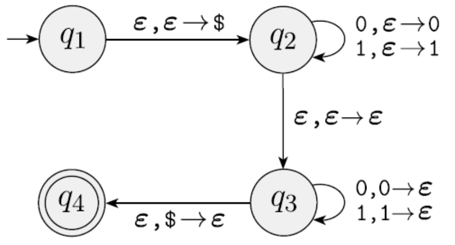

Equivalence With Context-Free Grammars

定理

A language is context free if and only if some pushdown automaton recognizes it.

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If a language is context free, then some pushdown automaton recognizes it.

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If a pushdown automaton recognizes some language, then it is context free.

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推论

Every regular language is context free.

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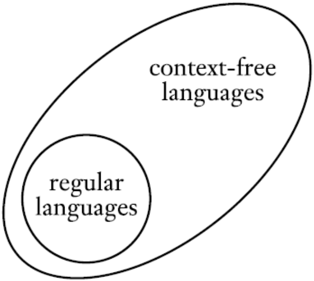

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2. *$|vy| > 0$, and*
3. *$|vxy| \leq p$.*