Provably Good Sampling and Meshing of Lipschitz Surfaces

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ABSTRACT

In the last decade, a great deal of work has been devoted to the elaboration of a sampling theory for smooth surfaces. The goal was to ensure a good reconstruction of a given surface S from a finite subset E of S. The sampling conditions proposed so far offer guarantees provided that E is sufficiently dense with respect to the local feature size of S, which can be true only if S is smooth since the local feature size vanishes at singular points.

In this paper, we introduce a new measurable quantity, called the Lipschitz radius, which plays a role similar to that of the local feature size in the smooth setting, but which turns out to be well-defined and positive on a much larger class of shapes. Specifically, it characterizes the class of Lipschitz surfaces, which includes in particular all piecewise smooth surfaces such that the normal variation around singular points is not too large.

Our main result is that, if S is a Lipschitz surface and E is a sample of S such that any point p of S is at distance less than a fraction of the Lipschitz radius of S, then we obtain similar guarantees as in the smooth setting. More precisely, we show that the Delaunay triangulation of E restricted to S is a 2-manifold isotopic to S lying at bounded Hausdorff distance from S, provided that its facets are not too skinny.

We further extend this result to the case of loose samples. As a straightforward application, the Delaunay refinement algorithm we proved correct for smooth surfaces works fine and offers the same topological and geometric guarantees for Lipschitz surfaces.

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1. INTRODUCTION

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In the last decade, a great deal of work has been devoted to the elaboration of a sampling theory for surfaces. This research is motivated by the surface reconstruction problem which consists in constructing an approximation $\hat{S}$ of a surface S given a finite sample of points $E \subset S$. A prerequisite to the design of provably correct algorithms is the definition of sampling conditions E must satisfy. In their seminal work, Amenta and Bern [1] introduced the concept of ε-sample and gave the first provably correct algorithm for reconstructing surfaces in $\mathbb{R}^3$. A finite point set $E \subset S$ is an ε-sample of S if any point $p \in S$ is at distance at most $\varepsilon\ \mathrm{lfs}(p)$ from E, where $\mathrm{lfs}(p)$ denotes the distance from p to the medial axis of S. Since their paper, other algorithms have been proposed which are valid for ε-samples and sufficiently small ε. More recently, algorithms have been proposed to actually produce such ε-samples and, at the same time, good approximations of the underlying surface [5].

The main drawback of ε-samples is that they are only defined for smooth surfaces since lfs vanishes at singular points. It has been a major issue to sample, mesh and reconstruct non smooth surfaces. A step forward in this direction is the work by Chazal and Lieutier [9] and the related work by Cohen-Steiner, Edelsbrunner and Harer [12]. Given a sample E at Hausdorff distance ε from a non necessarily smooth surface S, they are able to compute from E some topological invariants of S provided that ε is smaller than a quantity called the weak feature size (wfs) of S. This nice result however does not lead to a good PL approximation of S.

In this paper, we introduce a new measurable quantity, called the Lipschitz radius, which plays a role similar to that of lfs in the smooth setting, but turns out to be well-defined and positive on a much larger class of shapes. Given a surface S, the k-Lipschitz radius at a point p, or $\mathrm{lr}_k(p)$, is the radius of the largest ball B centered at p such that $S \cap B$ is the graph of a k-Lipschitz bivariate function. The class of surfaces with positive k-Lipschitz radius coincides with the class of k-Lipschitz surfaces which has been extensively studied in various contexts and includes, in particular, all piecewise smooth surfaces with bounded normal deviation around singular points (the bound depending on k).

Our main result is that, if S has a positive k-Lipschitz radius (for some small enough k) and E is a sample of S such that any point $p \in S$ is at distance less than a fraction of $\mathrm{lr}_k(p)$, then we obtain the same guarantees as in the smooth setting. More precisely, we show that the Delaunay triangulation of E restricted to S is a 2-manifold isotopic to S lying at Hausdorff distance $O(\varepsilon)$ from S, as soon as its facets are not too skinny. We are also able to give tight bounds on the size of such ε-samples.

We further extend our results to loose ε-samples. Loose ε-sample have been introduced to analyze smooth surface meshing algorithms [5]. While this notion is weaker than the notion of ε-sample, we show that the previous results still hold for loose ε-samples. As a straightforward application, the Delaunay refinement algorithm we proved correct for smooth surfaces [5] works fine and offers the same topological and geometric guarantees for Lipschitz surfaces. Specifically, the output of the algorithm is a PL surface with an optimal number of vertices that is isotopic to S, and lies at Hausdorff distance $O(\varepsilon)$ from S.

To the best of our knowledge, this is the first provably correct algorithm for meshing nonsmooth surfaces. We are only aware of two related results. Dey, Li and Ray [13] have considered the problem of remeshing a polygonal surface S that approximates a smooth surface $\tilde{S}$. Although $\tilde{S}$ plays no role in their algorithm, it is heavily used in the analysis. However, our Lipschitz condition turns out to have some similarity with their (δ, μ)-flatness condition, and it is likely that their analysis can be done using our framework, without requiring the use of a smooth surface $\tilde{S}$. Another result related to ours is due to Chazal, Cohen-Steiner and Lieutier [6]. They consider the problem of constructing an approximation of a shape S from a given sample E lying at Hausdorff distance at most ε from S, for some sufficiently small ε. Specifically, they exhibit an offset of E that is isotopic to S. Differently from this paper, they do not consider the problem of actually constructing E and assume that E satisfies a uniform sampling condition.

After the recall of several well-known concepts in Section 2, we introduce the Lipschitz radius in Section 3. In Section 4, we review the local properties of Lipschitz surfaces. Our main approximation results are presented in Section 5, where we show that the restricted Delaunay triangulation of an ε-sample of a Lipschitz surface S is a good topological and geometric approximation of S, under some mild assumptions. We address the case of loose ε-samples in Section 6. Finally, we introduce our surface mesher and its variants in Section 7.

2. BACKGROUND

2.1 Surfaces and differentiability

We call S a *surface* if it is a compact C^0-continuous 2-dimensional submanifold (without boundary) of $\mathbb{R}^3$. This means that, for any point $p \in S$, there exists an open neighborhood $\mathcal{N}$ of p in $\mathbb{R}^3$ that can be mapped to the unit open ball B by some homeomorphism h, such that $h(p)$ is the origin o and $h(\mathcal{N} \cap S) = B \cap \mathbb{R}^2$ – see [3, §2.1.1]. The fact that S is a surface implies that $\mathbb{R}^3 \setminus S$ is composed of two disjoint open sets whose boundaries coincide with S: one of these open sets, called $\mathcal{O}^-$, is bounded. The other, $\mathcal{O}^+$, is unbounded. Note that $\mathcal{O}^+$ and $\mathcal{O}^-$ may be non connected. Let $\mathcal{O} = \mathcal{O}^- \cup \mathcal{O}^+ = \mathbb{R}^3 \setminus S$.

We say that S is *smooth* if it is $C^{1,1}$, *i.e.* if it is C^1-continuous and its normal satisfies a Lipschitz condition.

2.2 Restricted Delaunay triangulation

Given a finite point set E, we call $\mathrm{Vor}(E)$ and $\mathrm{Del}(E)$ respectively the Voronoi diagram and the Delaunay triangulation of E. For any face f of $\mathrm{Del}(E)$, $\mathrm{V}(f)$ denotes the Voronoi face dual to f.

DEFINITION 2.1. *The Delaunay triangulation of E* restricted *to S, or $\mathrm{Del}_{|S}(E)$ for short, is the subcomplex of $\mathrm{Del}(E)$ made of the facets of $\mathrm{Del}(E)$ whose dual Voronoi edges intersect S.*

This definition follows [5] and departs from the usual notion of restricted Delaunay triangulation, which includes all the Delaunay faces whose dual Voronoi faces intersect the surface.

A facet (resp. edge, vertex) of $\mathrm{Del}_{|S}(E)$ is called a *restricted Delaunay facet* (resp. *restricted Delaunay edge*, *restricted Delaunay vertex*). For a restricted Delaunay facet f, we call *surface Delaunay ball* of f any ball circumscribing f centered at some point of $S \cap \mathrm{V}(f)$. Note that the centers of the surface Delaunay balls are precisely the intersection points of S with the 1-skeleton graph $\mathrm{VG}(E)$ of $\mathrm{Vor}(E)$.

Given a vertex v of $\mathrm{Del}_{|S}(E)$, we call *star of v* and write $\mathrm{star}(v)$ the union of all the facets of $\mathrm{Del}_{|S}(E)$ incident to v. Given a facet f of $\mathrm{Del}_{|S}(E)$, we call *star of f*, or $\mathrm{star}(f)$ for short, the union of the stars of the vertices of f, i.e. the union of all the facets of $\mathrm{Del}_{|S}(E)$ that share a vertex or an edge with f (including f itself).

In the rest of the paper, $\mathbf{n}(f)$ denotes the direction of the normal to f.

3. SURFACES WITH POSITIVE LIPSCHITZ RADIUS

We will now define the class of surfaces that are dealt with in the rest of the paper. In Section 3.1, we introduce a new quantity, called *Lipschitz radius*, which serves as a local feature size for nonsmooth surfaces. The class of surfaces with positive Lipschitz radius coincides with the one of Lipschitz surfaces, defined in Section 3.2. This class is a subset of the objects with positive *weak feature size* (see Section 3.3), which will allow us to use some of the properties of these objects in our proofs.

3.1 Lipschitz radius

DEFINITION 3.1. *Given a surface S and a point $p \in S$, the k-Lipschitz radius of S at p, or $\mathrm{lr}_k(p)$ for short, is the maximum radius r such that $\mathcal{O}^- \cap B(p,r)$ is the intersection of $B(p,r)$ with the hypograph of some k-Lipschitz bivariate function f.*

An illustration of this definition is given in Figure 1. Recall that the hypograph of a real-valued bivariate function f is the set of points $(x,y,z) \in \mathbb{R}^3$ such that $z < f(x,y)$. The function f is k-Lipschitz if $\forall p,q \in \mathbb{R}^2$, $\frac{|f(p)-f(q)|}{\|p-q\|} \leq k$. Observe that, since S is compact without boundary, S is not the graph of any bivariate function. Therefore, $\mathrm{lr}_k(p)$ is finite, for any $p \in S$.

LEMMA 3.2. *lr_k is 1-Lipschitz.*

PROOF. Let p, q be two points of S. By definition of $\mathrm{lr}_k(p)$, for any $\eta > 0$, $\mathcal{O}^- \cap B(p, \mathrm{lr}_k(p) + \eta)$ is not the intersection of $B(p, \mathrm{lr}_k(p) + \eta)$ with the hypograph of any

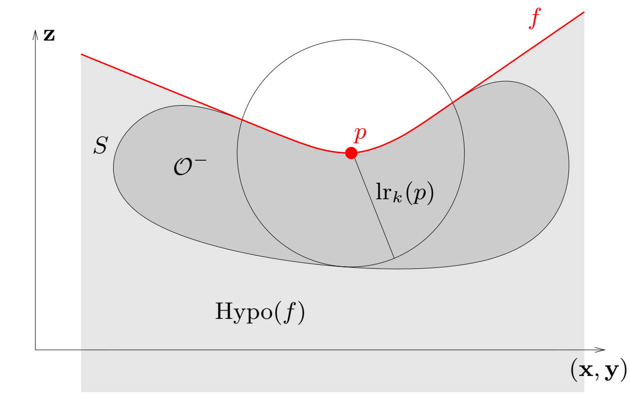

Figure 1: The k-Lipschitz bivariate function f and its associated oriented frame.

k-Lipschitz bivariate function. Now, $B(p, \mathrm{lr}_k(p) + \eta)$ is contained in the ball $B(q, \mathrm{d}(p, q) + \mathrm{lr}_k(p) + \eta)$. Thus, $\mathrm{lr}_k(q) \leq \mathrm{d}(p, q) + \mathrm{lr}_k(p) + \eta$. Since this is true for any $\eta > 0$, $\mathrm{lr}_k(q)$ is at most $\mathrm{d}(p, q) + \mathrm{lr}_k(p)$. $\square$

It follows from Lemma 3.2 that lr_k is continuous over S. Since S is compact, lr_k reaches its minimum at some point $p \in S$. We call this minimum the *k-Lipschitz radius of S*, or simply $\mathrm{lr}_k(S)$. Note that $\mathrm{lr}_k(S) \geq 0$. The set of surfaces with positive k-Lipschitz radius will be identified in the next section. Let us first give some examples (the proofs are deferred to the full version of the paper). We write $(\mathbf{n}, \mathbf{n}')$ for the modulus of the angle (measured in $[-\pi, \pi]$) between two vectors $\mathbf{n}$ and $\mathbf{n}'$.

THEOREM 3.3.

(i) If S is a C^1-continuous surface, then S has a positive k-Lipschitz radius, for any $k > 0$. If furthermore S is $C^{1,1}$, then we have: $\forall k > 0$, $\mathrm{lr}_k(S) \geq \frac{\arctan k}{1+\arctan k}\, \mathrm{rch}(S)$, where $\mathrm{rch}(S) > 0$ is the so-called reach of S, defined as the infimum of lfs *over S.*

(ii) Let $\theta < \frac{\pi}{2}$. If S is an oriented polyhedron without boundary, such that the normals $\boldsymbol{n}(f), \boldsymbol{n}(f')$ of any two non-disjoint facets f, f' of S satisfy $(\boldsymbol{n}(f), \boldsymbol{n}(f')) \leq \theta$, then S has a positive k-Lipschitz radius, for any $k \geq \frac{2 \sin \theta/2}{\sqrt{3 - 4 \sin^2 \theta/2}}$. Moreover, given $r > 0$, if for all $p \in S$ there is some direction $\boldsymbol{n}_p$ such that any facet f of S intersecting $B(p, r)$ satisfies $(\boldsymbol{n}(f), \boldsymbol{n}_p) \leq \arctan k$, then $\mathrm{lr}_k(S) \geq r$.

The proof of (ii) extends easily to the case where S is a piecewise smooth surface with bounded normal deviation around singular points.

Observe that the bound in (i) vanishes as θ tends to zero. This is coherent since S cannot be both 0-Lipschitz and compact without boundary. Once $k > 0$ is fixed, (i) states that $\mathrm{lr}_k(S)$ cannot be too small compared to $\mathrm{rch}(S)$. Inversely, $\mathrm{lr}_k(S)$ can be arbitrarily large compared to $\mathrm{rch}(S)$, even when S is $C^{1,1}$. Take the example of a planar curve made of two copies of the graph of $x \mapsto \sin x$, joined by two semi-circles – see Figure 2.

3.2 Lipschitz surfaces

It turns out that the set of surfaces with positive k-Lipschitz radius coincides with the set of k-Lipschitz surfaces, which

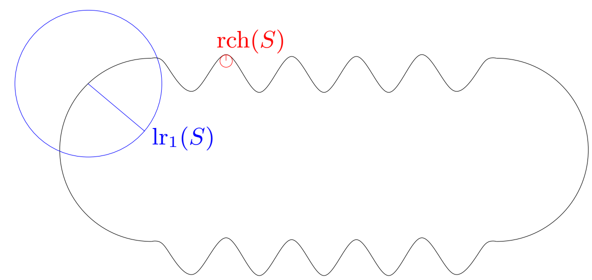

Figure 2: Comparing $\mathrm{lr}_1(S)$ with $\mathrm{rch}(S)$.

has been extensively studied in other contexts such as nonsmooth analysis [11, §7.3], elliptic PDE theory [17], or geometric measure theory [14, Ch. III]. Therefore, in Sections 3.3 and after, both terms will be used indifferently.

DEFINITION 3.4. *[11, §7.3]*
Let S be a surface, and let $\mathcal{O}^-$ be defined as in Section 2.1. S is a k-Lipschitz surface if $\mathcal{O}^-$ is locally the hypograph of some k-Lipschitz bivariate function, i.e. *for all $p \in \mathcal{O}^-$, there exists an open neighborhood $\mathcal{N}(p)$ of p in $\mathbb{R}^3$, an orthonormal frame (x, y, z) and a k-Lipschitz bivariate function $f : (x, y) \to \mathbb{R}$, such that $\mathcal{O}^- \cap \mathcal{N}(p)$ is the intersection of the hypograph of f with $\mathcal{N}(p)$.*

THEOREM 3.5. *A surface S is k-Lipschitz if and only if its k-Lipschitz radius is positive.*

PROOF. It is clear that, if a surface S has a positive k-Lipschitz radius, then for any $p \in S$ we have $\mathrm{lr}_k(p) > 0$. It follows that S satisfies the conditions of Definition 3.4 at any $p \in S$, by definition of $\mathrm{lr}_k(p)$. Hence, S is k-Lipschitz.
Conversely, if S is a k-Lipschitz surface, then by Definition 3.4, $\mathrm{lr}_k(p) > 0$ for any $p \in S$. Since lr_k is continuous (Lemma 3.2) and S is compact, there exists a point $p \in S$ such that $\mathrm{lr}_k(S) = \mathrm{lr}_k(p)$, which is positive. Hence, S has a positive k-Lipschitz radius. $\square$

A noticeable property of k-Lipschitz surfaces is that they are differentiable everywhere except on a set of measure zero. This fact follows easily from Rademacher's theorem [14, §3.1.6], and it implies in particular the following corollary, where $\tilde{S}$ denotes the set of points where the surface S is differentiable:

COROLLARY 3.6. *If S is a k-Lipschitz surface, then $\tilde{S}$ is* dense *in S, that is: $\forall p \in S$, $\forall \eta > 0$, $\tilde{S} \cap B(p, \eta) \neq \emptyset$.*

Given $p \in \tilde{S}$, we call $T(p)$ the tangent plane of S at p, and $\mathbf{n}(p)$ the unit vector orthogonal to $T(p)$ that points towards $\mathcal{O}^+$ (defined as in Section 2.1). This vector is called the normal of S at p.

Even though the normal of S is defined almost everywhere, making use of it in the proofs usually increases drastically the technicality of the arguments. Indeed, given $p \in S$, one cannot consider the normal of S at p, but at some point $q \in \tilde{S}$ arbitrarily close to p, by virtue of Corollary 3.6.

Instead, we define a pseudo normal $\mathbf{n}_k(p)$ at any point p of S. This pseudo normal depends on the constant k, and in the sequel it will play a role similar to that of the normal in the smooth setting.

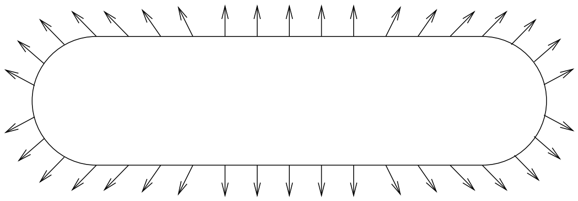

Figure 3: A planar curve (two semi-circles joined by straight-line segments) and its vector field $\mathbf{n}_1$.

Definition 3.7. *Given $p \in S$, the pseudo normal of S at p, noted $\boldsymbol{n}_k(p)$, is the $\boldsymbol{z}$ vector of an oriented orthonormal frame $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in which $\mathcal{O}^- \cap B(p, \mathrm{lr}_k(p))$ is the intersection of $B(p, \mathrm{lr}_k(p))$ with the hypograph of a k-Lipschitz function $f(x, y)$. The pseudo tangent plane at p, noted $T_k(p)$, is the plane orthogonal to $\boldsymbol{n}_k(p)$ that passes through p.*

Note first that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ may not be the only frame in which $\mathcal{O}^- \cap B(p, \mathrm{lr}_k(p))$ is the hypograph of some k-Lipschitz function, thus $\mathbf{n}_k(p)$ is not uniquely defined. In particular, if $p \in \tilde{S}$, then $\mathbf{n}_k(p)$ may not coincide with $\mathbf{n}(p)$, and $\mathbf{n}(p)$ may not even be a valid choice for $\mathbf{n}_k(p)$, as illustrated in Figure 3 for $k = 1$. However, the same local and global properties hold whatever choice we make for $\mathbf{n}_k(p)$. Therefore, from now on we choose arbitrarily one possible $\mathbf{n}_k(p)$ and we will stick to this choice in the rest of the paper.

Note also that, in frame $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, any vertical line l intersects $S \cap B(p, \mathrm{lr}_k(p))$ at most once, and every point of $l \cap B(p, \mathrm{lr}_k(p))$ lying above $S \cap B(p, \mathrm{lr}_k(p))$ is located in $\mathcal{O}^+$, while every point of $l \cap B(p, \mathrm{lr}_k(p))$ lying below is located in $\mathcal{O}^-$. Since the normals of S are oriented towards $\mathcal{O}^+$, this means that, at every point of $\tilde{S} \cap B(p, \mathrm{lr}_k(p))$, the normal of S has a non-negative inner product with $\mathbf{n}_k(p)$.

3.3 Weak feature size

Chazal and Lieutier [8] have introduced the notion of *weak feature size*. It turns out that, when the surface S is k-Lipschitz, its lr_k is k-Lipschitz radius is related to the weak feature size.

Let d_S denote the distance function to S. In [16, §3.1], the author derives from d_S a vector field $\boldsymbol{\nabla} : \mathcal{O} \to \mathbb{R}^3$ defined as follows:

$$\forall p \in \mathcal{O},\ \boldsymbol{\nabla}(p) \ = \ \frac{1}{\mathrm{d}_S(p)}\ (\mathbf{p} - \mathbf{c}(p)) \tag{1}$$

where $c(p)$ is the center of the smallest ball $B(c(p), r(p))$ that contains all the points of S that are closest to p. The following result, proved in [16, §5.1], will be useful:

$$\forall p \in \mathcal{O},\ \|\boldsymbol{\nabla}(p)\|^2 \ = \ 1 - \frac{r(p)^2}{\mathrm{d}_S(p)^2} \tag{2}$$

A point $p \in \mathcal{O}$ is a critical point of $\boldsymbol{\nabla}$ if $\|\boldsymbol{\nabla}(p)\| = 0$. As emphasized in [8], p is critical if and only if it belongs to the convex hull of its nearest neighbors on S. We call *weak feature size of* $\mathcal{O}$, or simply $\mathrm{wfs}(\mathcal{O})$, the infimum over S of the distance to the set Φ of critical points of $\boldsymbol{\nabla}$: $\mathrm{wfs}(\mathcal{O}) \ = \ \min\{\mathrm{d}(p, q) \mid p \in S,\ q \in \Phi\}$.

Theorem 3.8. *If S is a k-Lipschitz surface, then $\mathcal{O} = \mathbb{R}^3 \setminus S$ has a positive weak feature size. More precisely,*

$$\forall p \in S,\ \mathrm{lr}_k(p) \leq 2\ \mathrm{d}(p, \Phi) \tag{3}$$

which implies that $\mathrm{lr}_k(S) \leq 2\mathrm{wfs}(\mathcal{O})$.

The theorem is an easy corollary of the following technical result, which controls the vector field $\boldsymbol{\nabla}$ in the vicinity of the surface:

Lemma 3.9. *If S is a k-Lipschitz surface, then for any $p \in S$ and any $q \in \mathcal{O} \cap B(p, \frac{1}{2}\ \mathrm{lr}_k(p))$, we have $\|\boldsymbol{\nabla}(q)\| \geq \cos\theta > 0$, where $\theta = \arctan k \in [0, \pi/2[$. Furthermore,*
if $q \in \mathcal{O}^+$, then $\ (\boldsymbol{\nabla}(q), \boldsymbol{n}_k(p)) \leq \theta$
if $q \in \mathcal{O}^-$, then $\ (\boldsymbol{\nabla}(q), -\boldsymbol{n}_k(p)) \leq \theta$

The proof uses the so-called cocone lemma 4.1, whose statement and proof are deferred to Section 4.

Proof of Lemma 3.9. We assume without loss of generality that $q \in \mathcal{O}^+$, the case $q \in \mathcal{O}^-$ being symmetric. We call $q_1, \cdots, q_k$ the nearest neighbors of q on S.

For all i, we have $\mathrm{d}(p, q_i) \leq \mathrm{d}(p, q) + \mathrm{d}(q, q_i) \leq 2\mathrm{d}(p, q)$ since p lies on S. Hence, q_i belongs to $B(p, \mathrm{lr}_k(p))$. It follows that $S \cap B(p, \mathrm{lr}_k(p))$ lies outside the double cone $K(q_i)$ of apex q_i, of axis aligned with $\mathbf{n}_k(p)$ and of half-angle $\frac{\pi}{2} - \theta$, by the cocone lemma 4.1. Let $K^+(q_i)$ be the cone of $K(q_i)$ that lies on the same side of S as q. Let $K^-(q_i)$ be the other cone of $K(q_i)$, and q' be a point of $K^-(q_i)$ closest to q. We claim that $q' = q_i$. Indeed, by assumption, q lies in $\mathcal{O}^+$ whereas $K^-(q_i) \setminus \{q_i\}$ is included in $\mathcal{O}^-$. Thus, if $q' \neq q_i$, then the open segment $]q, q'[$ intersects S. For any $q'' \in S \cap]q, q'[$, we then have $\mathrm{d}(q, q'') < \mathrm{d}(q, q') \leq \mathrm{d}(q, q_i)$, which contradicts the fact that q_i is a nearest neighbor of q on S. Hence, $q' = q_i$, which means that q_i is the point of $K^-(q_i)$ closest to q. It follows that q belongs to the cone of apex q_i, of axis $\mathbf{n}_k(p)$ and of half-angle θ. Equivalently,

$$(\mathbf{q} - \mathbf{q}_i,\ \mathbf{n}_k(p)) \leq \theta \tag{4}$$

Now, since $c(q)$ is the center of the smallest ball containing the q_i, $c(q)$ lies in the convex hull of the q_i. Hence, $(\mathbf{q} - \mathbf{c}(q)) \cdot \mathbf{n}_k(p)$ is a convex combination of the $(\mathbf{q} - \mathbf{q}_i) \cdot \mathbf{n}_k(p)$, which are all at least $\mathrm{d}_S(q)\ \cos\theta$, by (4). Using (1), we get

$$\|\boldsymbol{\nabla}(q)\| = \frac{\|\mathbf{q} - \mathbf{c}(q)\|}{\mathrm{d}_S(q)} \geq \frac{(\mathbf{q} - \mathbf{c}(q)) \cdot \mathbf{n}_k(p)}{\mathrm{d}_S(q)} \geq \cos\theta$$

In addition, the q_i lie in the cone of apex q, of axis aligned with $-\mathbf{n}_k(p)$ and of half-angle θ. Since this cone is convex, it contains $c(q)$, which is a convex combination of the q_i. Hence, $(\boldsymbol{\nabla}(q), \mathbf{n}_k(p)) \leq \theta$, which ends the proof of the lemma. □

4. LOCAL PROPERTIES OF LIPSCHITZ SURFACES

Like the distance to the medial axis in the smooth case, lr_k allows to predict the local behaviour of a k-Lipschitz surface. From one fundamental lemma (namely, the cocone lemma 4.1), it is possible to work out several local properties of Lipschitz surfaces that are similar to those already known in the smooth setting. The arguments in the proofs of Sections 5 and 6 are built on top of these local properties.

Let S be a k-Lipschitz surface, for some fixed k. For convenience, we define $\theta = \arctan k \in [0, \pi/2[$. The following local properties hold in a neighborhood $D_p = S \cap B(p, \mathrm{lr}_k(p))$ of a point p of S.

Lemma 4.1 (Cocone).
With the above notations, for any $q \in D_p$, D_p lies outside the double cone of apex q, of axis aligned with $\boldsymbol{n}_k(p)$ and of half-angle $\frac{\pi}{2} - \theta$. Moreover, if $q \in \tilde{S}$, then the angle $(\boldsymbol{n}(q), \boldsymbol{n}_k(p))$ is at most θ.

Proof. Given $q, q' \in D_p$, we call $\bar{q}$ and $\bar{q'}$ their orthogonal projections onto $T_k(p)$. Since D_p is the graph of a k-Lipschitz bivariate function f defined over $T_k(p)$, the angle α between line (q, q') and plane $T_k(p)$ is given by:

$$\tan\alpha = \frac{|f(\bar{q}) - f(\bar{q}')|}{\mathrm{d}(\bar{q}, \bar{q}')} \leq k = \tan\theta \qquad (5)$$

Hence, we have $\alpha \leq \theta$, which means that q' lies outside the double cone of apex q, of axis aligned with $\mathbf{n}_k(p)$ and of half-angle $\frac{\pi}{2} - \theta$.

Let us now assume that $q \in \tilde{S}$. Eq. (5) holds for any $q' \in D_p \setminus \{q\}$. In particular, as q' approaches q, the angle between line (q, q') and $T_k(p)$ remains bounded by θ. As a consequence, the angle between the tangent plane $T(q)$ and $T_k(p)$ is at most θ. Since $\mathbf{n}_k(p)$ is oriented such that $\mathbf{n}_k(p) \cdot \mathbf{n}(q) \geq 0$ (see Section 3.2), we get: $(\mathbf{n}(q), \mathbf{n}_k(p)) \leq \theta$, which concludes the proof of the lemma. $\square$

The next result is an equivalent of Lemma 7 of [1] in the Lipschitz setting.

Lemma 4.2 (Triangle Normal).
With the above notations, for any triangle $f = (u, v, w)$ such that $u, v, w \in D_p$, the angle α between $\boldsymbol{n}_k(p)$ and the line orthogonal to the plane aff(u, v, w) satisfies $\sin\alpha \leq 2\varrho\sin\theta$, where ϱ is the radius-edge ratio[1] of f. If $\varrho \leq \frac{1}{2\sin\theta}$, then $\alpha \leq \arcsin(2\varrho\sin\theta)$.

Proof. Since the radius-edge ratio of f is ϱ, then it is well-known that the smaller inner angle of f has value $\beta = \arcsin\frac{1}{2\varrho}$.

Consider any vertex of f, say u. By the cocone lemma 4.1, v and w lie outside the double cone $K(u)$ of apex u, of axis aligned with $\mathbf{n}_k(p)$ and of half-angle $\frac{\pi}{2} - \theta$. Since aff(u, v, w) passes through the apex of $K(u)$, it intersects $K(u)$ either along a single point, or along a single line, or along a double wedge. If $K(u) \cap \mathrm{aff}(u, v, w)$ is a single point or a single line, then $\alpha \leq \theta$, which implies that $\sin\alpha < 2\varrho\sin\theta$, since the radius-edge ratio of a triangle is always at least $1/\sqrt{3}$, this lower bound being achieved when the triangle is equilateral.

If $K(u) \cap \mathrm{aff}(u, v, w)$ is a double wedge $K'(u)$, then the half-angle θ' of this double wedge depends on α and θ. We endow $\mathbb{R}^3$ with an oriented orthonormal frame $(u, \mathbf{x}, \mathbf{y}, \mathbf{z})$, such that the z-axis is aligned with $\mathbf{n}_k(p)$ and that the line of intersection between aff(u, v, w) and the xy-plane is aligned with the y-axis. In this frame, the equation of the boundary of $K(u)$ is $z^2 = \tan^2\theta\ (x^2 + y^2)$, and the equation of aff(u, v, w) is $z = x\ \tan\alpha$. Thus, inside aff(u, v, w) (which we endow with an oriented orthonormal frame $(u, \mathbf{X}, \mathbf{y})$), the equation of the boundary of $K'(u)$ is $y = \pm\frac{1}{\sin\theta}\sqrt{\sin^2\alpha - \sin^2\theta}\ X$. Hence, the half-angle of $K'(u)$ is

$$\theta' = \arctan\left(\frac{1}{\sin\theta}\sqrt{\sin^2\alpha - \sin^2\theta}\right). \qquad (6)$$

Since v and w lie outside $K(u)$, inside aff(u, v, w) they do not belong to $K'(u)$. Moreover, since we took u as being any vertex of f, we can now assume without loss of generality that the X-coordinate of u lies between those of v and of w, which implies that v and w do not belong to the same wedge of aff$(u, v, w) \setminus K'(u)$. Then, the inner angle $\hat{u}$ of f is at least $2\theta'$. Since all the angles of the triangle are at least β, we have $\hat{u} \leq \pi - 2\beta$, which implies that $\theta' \leq \frac{\pi}{2} - \beta$. This yields $\sin\alpha \leq \frac{\sin\theta}{\sin\beta}$, by (6). The lemma follows, since $\beta = \arcsin\frac{1}{2\varrho}$. $\square$

[1] which is the ratio between the circumradius of f and the length of its shortest edge.

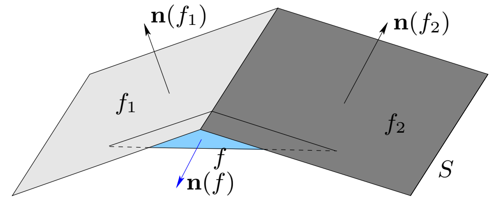

Figure 4: Controlling the normal of a triangle.

Observe that it is necessary to bound ϱ in order to control the normal of f. Figure 4 shows a counter-example, where ϱ is too big for the normal of f to be controlled.

The last result of the section is an equivalent of Lemma 3 of [1] in the Lipschitz setting.

Lemma 4.3 (Normal Variation).
$\forall p, q \in S$ s.t. $\mathrm{d}(p, q) < \mathrm{lr}_k(p)$, $(\boldsymbol{n}_k(p), \boldsymbol{n}_k(q)) \leq 2\theta$.

Proof. Let $r = \mathrm{lr}_k(p) - \mathrm{d}(p, q) > 0$. By Corollary 3.6, there exists some point $q' \in \tilde{S}$ lying in $B(q, r)$. By the triangle inequality, q' is located in $B(p, \mathrm{lr}_k(p))$. Moreover, we have $r \leq \mathrm{lr}_k(q)$ since lr_k is 1-Lipschitz (Lemma 3.2), hence q' also lies in $B(q, \mathrm{lr}_k(q))$ It follows by Lemma 4.1 that $(\mathbf{n}_k(p), \mathbf{n}(q'))$ and $(\mathbf{n}(q'), \mathbf{n}_k(q))$ are both at most θ, which implies that $(\mathbf{n}_k(p), \mathbf{n}_k(q)) \leq 2\theta$. $\square$

5. ε-SAMPLES

Definition 5.1. *Let S be a surface and ε be a positive function defined over S. A finite point set $E \subset S$ is an* ε-sample *of S if: $\forall p \in S$, $E \cap B(p, \varepsilon(p)) \neq \emptyset$.*

This definition is the same as the one introduced by Amenta and Bern [1] and used in all papers on certified surface meshing or reconstruction. The only difference is that we provide guarantees for ε less than a fraction of lr_k, instead of a fraction of the so-called local feature size lfs (which measures the distance to me medial axis of S). This makes our results meaningful for all k-Lipschitz surfaces, and not only for $C^{1,1}$ surfaces.

In this abstract, we will take a uniform $\varepsilon < \mathrm{lr}_k(S)$, for simplicity. This is no real loss of generality since ε is only an upper bound on the local density of the point sample. Moreover, thanks to the fact that the Lipschitz radius is 1-Lipschitz (Lemma 3.2), our proofs can be easily extended to the non uniform case, at the cost of additional technical detail we defer to the full version of the paper. Note that the non-uniform version assumes that $\varepsilon < \min\{\mathrm{wfs}(S), \mathrm{lr}_k\}$, for the proof of isotopy (Theorem 5.5) to hold.

Throughout Section 5, S is a k-Lipschitz surface and E is an ε-sample of S. The case of a loose ε-sample will be addressed in Section 6. Since S is fixed, k and $\theta = \arctan k$

are fixed constants. We assume that E satisfies the following hypotheses:

H1 E is an ε-sample of S, with $\varepsilon < \frac{1}{7}\,\mathrm{lr}_k(S)$;

H2 The facets of $\mathrm{Del}_{|S}(E)$ have radius-edge ratios of at most ϱ, where $\varrho < \frac{\cos(2\theta)}{2\sin\theta}$.

H1 imposes that E be dense with respect to $\mathrm{lr}_k(S)$. H2 imposes that the restricted Delaunay facets of E be not too skinny. Once the surface S (and hence the angle θ) is fixed, H2 gives an upper bound on ϱ. This assumption is mandatory to control the normals of the facets of $\mathrm{Del}_{|S}(E)$, as illustrated in Figure 4. It could be replaced by the following sparseness condition: *the points of E are pairwise farther than $h\varepsilon$, where h is a constant that does not depend on ε.* This condition is more restrictive than H2 since it forces the points of E to be uniformly sampled along S.

If the surface S has a positive reach, then it is k-Lipschitz for any $k > 0$ by Theorem 3.3 (i). Therefore, after choosing some positive k arbitrarily, H1 gives a looser condition on the sampling density than the one usually imposed through the local feature size (recall that $\mathrm{lr}_k(S)$ can be arbitrarily large compared to $\mathrm{rch}(S)$), while H2 gives an additional structural condition for our guarantees to hold.

The goal of this section is to show that, under H1–H2, we obtain the same guarantees on $\mathrm{Del}_{|S}(E)$ as what was known for $C^{1,1}$ surfaces using Amenta and Bern's ε-samples. We refer the reader to [5] for a comprehensive survey of the properties of ε-samples of smooth surfaces.

For any facet f of $\mathrm{Del}_{|S}(E)$, we call B_f the surface Delaunay ball of smallest radius that circumscribes f. Let c_f and r_f denote respectively the center and radius of B_f. We set $D_f = S \cap B_f$.

ORIENTATION CONVENTION 5.2. *For any facet $f \in \mathrm{Del}_{|S}(E)$, we orient f such that $\boldsymbol{n}(f) \cdot \boldsymbol{n}_k(p) \geq 0$ for all point $p \in S \cap B(c_f, \mathrm{lr}_k(S))$.*

The existence of such an orientation follows from H1, H2 and Lemmas 4.2 and 4.3. We leave the technical details to the full version of the paper. Note that it is not necessary to orient all the facets of $\mathrm{Del}(E)$, because only the facets of $\mathrm{Del}_{|S}(E)$ will play a role in the sequel.

We will now prove our main result, namely that, under H1–H2, $\mathrm{Del}_{|S}(E)$ is a compact surface without boundary (Section 5.1), at Hausdorff distance ε from S (Section 5.2) and isotopic to S (Section 5.3). Our proofs hold only for small enough θ (and hence small enough k). Indeed, for H2 to be satisfiable by some ϱ, $\frac{\cos(2\theta)}{2\sin\theta}$ must be greater than $1/\sqrt{3}$ (smallest possible radius-edge ratio of a triangle), which implies that $\theta < \arcsin\frac{\sqrt{7}-1}{2\sqrt{3}} \approx 28.4\,\mathrm{deg}$. In the case where S is a piecewise smooth surface, Theorem 3.3 (ii) states that the normal deviation around the singluar points of S must be less than $48.6\,\mathrm{deg}$ for θ to be sufficiently small. This bound on the normal deviation is somewhat pessimistic, since our experimental results [18, §6.4] show that ε-samples of piecewise smooth surfaces yield good topological and geometric approximations for normal deviations up to $\frac{\pi}{2}$.

5.1 Manifold

THEOREM 5.3. *Let S be a k-Lipschitz surface and $E \subset S$ be a finite point set. If E satisfies H1–H2, then $\mathrm{Del}_{|S}(E)$ is a compact surface without boundary, consistently oriented by the orientation convention 5.2.*

Our proof of the above result is the same as in the smooth setting. It uses the fact that two adjacent facets of $\mathrm{Del}_{|S}(E)$ cannot overlap when we project them onto the pseudo tangent plane of any of their common vertices. We first show that every edge of $\mathrm{Del}_{|S}(E)$ is incident to exactly two facets of $\mathrm{Del}_{|S}(E)$. Then, we show that the star of any vertex of $\mathrm{Del}_{|S}(E)$ is a simple polygon. These two properties imply that $\mathrm{Del}_{|S}(E)$ is a 2-manifold without boundary, because the relative interiors of the faces of $\mathrm{Del}_{|S}(E)$ are pairwise disjoint due to the fact that $\mathrm{Del}_{|S}(E)$ is an embedded simplicial complex. Finally, we prove that the orientation convention 5.2 induces a valid orientation of $\mathrm{Del}_{|S}(E)$.

The technical details of the proof can be found in [18, §1.2.1]. They differ slightly from the smooth setting, due essentially to the fact that the normal is replaced by the pseudo normal.

5.2 Hausdorff distance

THEOREM 5.4. *Let S be a k-Lipschitz surface and $E \subset S$ be a finite point set. If E is an ε-sample of S, then the Hausdorff distance between $\mathrm{Del}_{|S}(E)$ and S is at most ε.*

PROOF. The proof is standard. First, no point of S is farther than ε from E, since the latter is an ε-sample of S. Second, every facet of $\mathrm{Del}_{|S}(E)$ is circumscribed by some surface Delaunay ball, whose radius is at most ε because E is an ε-sample. Hence, no point of $\mathrm{Del}_{|S}(E)$ is farther than ε from S. $\square$

Differently from the smooth setting [5], the upper bound on the Hausdorff distance is in the order of ε and not ε^2. The reason is that, when S is smooth, it is locally squeezed between two tangent medial balls, yielding an order of ε^2 approximation. In the Lipschitz setting, these balls are replaced by tangent cones, which yield only an order of ε approximation.

5.3 Isotopy

THEOREM 5.5. *Let S be a k-Lipschitz surface and $E \subset S$ be a finite point set. If E satisfies H1–H2, then $\mathrm{Del}_{|S}(E)$ is isotopic to S.*

To prove Theorem 5.5, we use the following result[2], stated as Theorem 6.2 in [7]:

THEOREM 5.6. [7, Thm. 6.2]
Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ be two open subsets of $\mathbb{R}^3$ of positive weak feature size, whose boundaries $\partial\mathcal{O}$ and $\partial\hat{\mathcal{O}}$ are compact embedded surfaces. If the Hausdorff distance between $\partial\mathcal{O}$ and $\partial\hat{\mathcal{O}}$ is less than $\frac{1}{2}\,\min\{\mathrm{wfs}(\mathcal{O}),\ \mathrm{wfs}(\hat{\mathcal{O}})\}$, then $\partial\mathcal{O}$ and $\partial\hat{\mathcal{O}}$ are isotopic.

[2]Note that the original theorem [7, Thm. 6.2] requires that the open sets $\mathcal{O}$ and $\hat{\mathcal{O}}$ be bounded. However, since $\partial\mathcal{O}$ and $\partial\hat{\mathcal{O}}$ are compact, it is possible to bound $\mathcal{O}$ and $\hat{\mathcal{O}}$ with a sufficiently large sphere Σ while keeping $\mathrm{wfs}(\mathcal{O})$ and $\mathrm{wfs}(\hat{\mathcal{O}})$ unchanged. Then, by [7, Thm. 6.2], $\Sigma \cup \partial\mathcal{O}$ and $\Sigma \cup \partial\hat{\mathcal{O}}$ are isotopic, which means that $\partial\mathcal{O}$ and $\partial\hat{\mathcal{O}}$ are also isotopic.

In our context, we set $\mathcal{O} = \mathbb{R}^3 \setminus S$ and $\hat{\mathcal{O}} = \mathbb{R}^3 \setminus \mathrm{Del}_{|S}(E)$. All we have to do is to show that the Hausdorff distance between $\mathrm{Del}_{|S}(E)$ and S is less than $\frac{1}{2}\,\mathrm{wfs}(\mathcal{O})$ and less than $\frac{1}{2}\,\mathrm{wfs}(\hat{\mathcal{O}})$, and then to apply Theorem 5.6. Since, by Theorem 5.4, the Hausdorff distance between S and $\mathrm{Del}_{|S}(E)$ is at most ε, we simply need to prove the two following lemmas:

Lemma 5.7. *ε is less than half the weak feature size of $\mathcal{O}$.*

Proof. By H1, we have $\varepsilon < \frac{1}{7}\,\mathrm{lr}_k(S)$, which is at most $\frac{2}{7}\,\mathrm{wfs}(\mathcal{O}) < \frac{1}{2}\,\mathrm{wfs}(\mathcal{O})$ by Theorem 3.8. $\square$

Lemma 5.8. *ε is less than half the weak feature size of $\hat{\mathcal{O}}$.*

Proof. Let p be a point of $\mathrm{Del}_{|S}(E)$ and let f be a facet of $\mathrm{Del}_{|S}(E)$ that contains p. By H1, the surface Delaunay ball $B(c_f, r_f)$ of f has a radius $r_f \leq \varepsilon$. Let f' be a facet of $\mathrm{Del}_{|S}(E)$ that intersects $B(p, \mathrm{lr}_k(S) - 3\varepsilon)$. By H1, f' is circumscribed by a surface Delaunay ball of radius at most ε, included in $B(p, \mathrm{lr}_k(S) - \varepsilon)$ and hence in $B(c_f, \mathrm{lr}_k(S))$. Moreover, the radius-edge ratio of f is bounded by ϱ, by H2. Therefore, by the triangle normal lemma 4.2 and the orientation convention 5.2, we have $(\mathbf{n}(f'), \mathbf{n}_k(c_f)) < \arcsin(2\varrho \sin\theta)$.

Since this is true for all $p \in \mathrm{Del}_{|S}(E)$ and all facet f' of $\mathrm{Del}_{|S}(E)$ intersecting $B(p, \mathrm{lr}_k(S) - 3\varepsilon)$, and since $\mathrm{Del}_{|S}(E)$ is a polyhedron without boundary (Theorem 5.3), Theorem 3.3 (ii) states that $\mathrm{lr}_{k'}(\mathrm{Del}_{|S}(E)) \geq \mathrm{lr}_k(S) - 3\varepsilon$, where $k' = \tan(\arcsin(2\varrho\sin\theta))$. By Theorem 3.8, $\mathrm{wfs}(\hat{\mathcal{O}})$ is at least $\frac{1}{2}\,\mathrm{lr}_{k'}(\mathrm{Del}_{|S}(E)) \geq \frac{1}{2}\,(\mathrm{lr}_k(S) - 3\varepsilon)$, which is greater than 2ε since $\varepsilon < \frac{1}{7}\,\mathrm{lr}_k(S)$, by H1. $\square$

6. LOOSE ε-SAMPLES

The notion of loose ε-sample was first introduced in [5]. Let $\mathrm{VG}(E)$ denote the 1-skeleton graph of $\mathrm{Vor}(E)$.

Definition 6.1. *Given a surface S and a positive function ε defined over S, a finite point set $E \subset S$ is a* loose ε-sample *of S if:*

1. *$\forall p \in S \cap \mathrm{VG}(E)$, $E \cap B(p,\ \varepsilon(p)) \neq \emptyset$,*
2. *$\mathrm{Del}_{|S}(E)$ has vertices on all the connected components of S.*

Here again, we consider the specific case of a uniform bound ε, which is no real loss of generality. Since the centers of the surface Delaunay balls are precisely the intersection points of S with $\mathrm{VG}(E)$, Condition 1 of Definition 6.1 is satisfied if and only if every surface Delaunay ball $B(c, r)$ has a radius $r \leq \varepsilon$. Observe that Condition 1 alone is not sufficient to control the density of E since, without Condition 2, $\mathrm{VG}(E)$ may be empty – see [5, Fig. 1] for an example.

The proofs of Section 5 (except for the Hausdorff distance) do not make use of the full power of ε-samples and hold the same for loose ε-samples. To bound the Hausdorff distance, we need an additional condition on E:

H2bis The constants ϱ and θ in H2 also satisfy $\varrho < \frac{\sin\left(\frac{\pi}{3} - \theta\right)}{2\sin\theta}$.

Theorem 6.2. *If E is a loose ε-sample of a k-Lipschitz surface S, such that $\varepsilon < \frac{1}{7}\,\mathrm{lr}_k(S)$ and H2–H2bis are satisfied, then the Hausdorff distance between $\mathrm{Del}_{|S}(E)$ and S is at most $\frac{\varepsilon}{\cos^2\theta}$, where $\theta = \arctan k$.*

This theorem allows to relate the notions of ε-sample and loose ε-sample, like in the smooth setting [5]:

Corollary 6.3. *If E is a loose ε-sample of a k-Lipschitz surface S, such that $\varepsilon < \frac{1}{7}\,\mathrm{lr}_k(S)$ and H2–H2bis are satisfied, then E is an $\varepsilon\,\sqrt{1 + \frac{1}{\cos^4\theta}}$-sample of S, where $\theta = 2\arctan k$.*

As a consequence, if E is a loose ε-sample of S satisfying H2–H2bis, for some sufficiently small ε, then $\mathrm{Del}_{|S}(E)$ has all the nice properties stated in Section 5.

Proof of the corollary. Let p be a point of S and p' a nearest neighbor of p on $\mathrm{Del}_{|S}(E)$. By Theorem 6.2, $\mathrm{d}(p, p')$ is at most $\frac{\varepsilon}{\cos^2\theta}$. If p' is a vertex of $\mathrm{Del}_{|S}(E)$, then $p' \in E$ and the result is proved. Else, let f be a facet of $\mathrm{Del}_{|S}(E)$ that contains p', and let $v \neq p'$ be a vertex of f closest to p'. Since E is a loose ε-sample, the circumradius of f is at most ε. Thus, $\mathrm{d}(p', v) \leq \varepsilon$, because $p' \in f$. If p' belongs to an edge of f, then the lines (p, p') and (p', v) are perpendicular. Otherwise, p' belongs to the relative interior of f, thus the line (p, p') is perpendicular to the plane $\mathrm{aff}(f)$ and hence also to the line (p', v). In both cases, we have:

$$\mathrm{d}(p, E) \leq \mathrm{d}(p, v) = \sqrt{\mathrm{d}^2(p, p') + \mathrm{d}^2(p', v)} \leq \sqrt{\frac{\varepsilon^2}{\cos^4\theta} + \varepsilon^2}$$

$\square$

The rest of Section 6 is devoted to the proof of Theorem 6.2, which holds in a slightly more general setting:

Proposition 6.4. *Let S be a k-Lipschitz surface and $\hat{S}$ be an oriented simplicial surface without boundary, such that:*

(a) the vertices of $\hat{S}$ belong to S,
(b) $\hat{S}$ intersects every connected component of S,
(c) every facet of $\hat{S}$ has a circumradius of at most ε, where $\varepsilon < \frac{1}{7}\,\mathrm{lr}_k(S)$,
(d) for all facet f of $\hat{S}$ and all vertex v of $\mathrm{star}(f)$, one has $(\boldsymbol{n}(f), \boldsymbol{n}_k(v)) < \frac{\pi}{3} - \theta$, where $\theta = \arctan k$.

Then, the Hausdorff distance between S and $\hat{S}$ is at most $\frac{\varepsilon}{\cos^2\theta}$, where $\theta = \arctan k$.

This result, combined with Corollary 6.3 and Theorem 5.5, yields an equivalent of [2, Thm 19] in the Lipschitz setting.

It can be easily seen that, if S and E satisfy the hypotheses of Theorem 6.2, then S and $\mathrm{Del}_{|S}(E)$ satisfy those of Proposition 6.4, which reduces the proof of the theorem to that of the proposition.

Let $\mathcal{T}_\varepsilon = \left\{q \in \mathbb{R}^3 \mid \mathrm{d}(q, S) < \varepsilon\right\}$ be the tubular neighborhood of width ε around S. From hypotheses (a) and (c) we deduce that $\hat{S}$ is included in $\mathcal{T}_\varepsilon$, making the semi Hausdorff distance from $\hat{S}$ to S at most ε. Our strategy for bounding the semi Hausdorff distance from S to $\hat{S}$ has some similarity with the one adopted in the smooth setting [15, Section 5]. It consists in *pushing* the points of $\hat{S}$ along some continuous flow towards S, and showing that every point of S is eventually reached by some point of $\hat{S}$. The drawback of the flow along the normals of S, defined in [15] and used in the smooth setting, is that it is not defined on the medial axis of S, which, in the present case, may intersect $\mathcal{T}_\varepsilon$ for any positive value of ε, since S is not assumed to be smooth. Therefore, this flow is not well defined over $\mathcal{T}_\varepsilon$ and cannot be used in our context. This is why we define a new flow ϕ that was first introduced by Lieutier [16]. This flow has the advantage of being well-defined and continuous over $\mathcal{T}_\varepsilon \setminus S$.

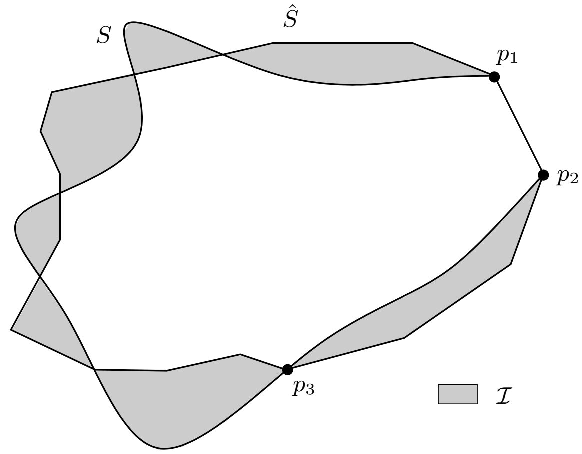

Figure 5: The set $\mathcal{I}$.

However, ϕ is not defined over S. Therefore, our proof of Proposition 6.4 proceeds in three steps:

- in Section 6.2, we consider the set $\mathcal{I}$ of points of $\mathcal{O} = \mathbb{R}^3 \setminus S$ whose flow lines intersect $\hat{S}$. We prove that $\mathcal{I}$ is a union of connected components of $\mathbb{R}^3 \setminus (S \cup \hat{S})$, as illustrated in Figure 5.
- in Section 6.3, we consider the sets $S \cap \partial\mathcal{I}$ and $S \cap \hat{S}$, and we prove that their union covers S. As a consequence, every point $p \in S$ belongs either to $\hat{S}$ or to $\partial\mathcal{I}$. In the latter case, one can find a point of $\mathcal{O}$ arbitrarily close to p whose flow line intersects $\hat{S}$.
- using this last observation, we can conclude the proof of the proposition in Section 6.4, by bounding the distance travelled along a flow line before reaching $\hat{S}$.

6.1 The flow

For any $d > 0$, we call $\mathcal{O}_d$ the subset of $\mathcal{O}$ made of the points that are farther than d from S. We have: $\mathcal{O}_d = \mathcal{O} \setminus \bar{\mathcal{T}}_d$, where $\mathcal{T}_d$ is the tubular neighborhood of S of width d.

It is proved in [16] that Euler schemes, using the vector field $\boldsymbol{\nabla}$ defined in Section 3.3, converges uniformly towards a continuous flow $\phi : \mathbb{R}^+ \times \mathcal{O} \to \mathcal{O}$ that verifies:

$$\forall t \in \mathbb{R}^+,\ \forall p \in \mathcal{O},\ \phi(t,p) = p + \int_{t'=0}^{t} \boldsymbol{\nabla}(\phi(t',p))\, dt' \quad (7)$$

Intuitively, the real variable t stands for the time, while the other variable is the position in space. It follows from Eq. (7) that the *stationary points* of ϕ (*i.e.* the points $p \in \mathcal{O}$ such that $\phi(t,p) = p\ \forall t \in \mathbb{R}^+$) are the critical points of $\boldsymbol{\nabla}$, *i.e.* the points of Φ.

For any $p \in \mathcal{O}$, we call *flow line of* p and note $\Lambda(p)$, the trajectory of p along ϕ:

$$\Lambda(p) = \phi(\mathbb{R}^+, p) = \{\phi(t,p) \mid t \geq 0\}$$

The flow ϕ enjoys several properties, including:

F1. [16, Lemma 4.12]
For any $p \in \mathcal{O} \setminus \Phi$, the distance to S increases strictly along $\Lambda(p)$, that is, the map $t \mapsto \mathrm{d}_S(\phi(t,p))$ is strictly increasing. Moreover,

$$\forall p \in \mathcal{O},\ \forall t \in \mathbb{R}^+,\ \mathrm{d}_S(\phi(t,p)) = \mathrm{d}_S(p) + \int_{t'=0}^{t} \|\boldsymbol{\nabla}(\phi(t',p))\|^2\, dt' \quad (8)$$

F2. [16, Lemma 4.13]
For any $p \in \mathcal{O}$, the map $t \mapsto \phi(t,p)$ is 1-Lipschitz. Moreover,

$$\forall d > 0,\ \forall t \geq 0,\ \forall p,q \in \mathcal{O}_d,\ \mathrm{d}(\phi(t,p), \phi(t,q)) \leq e^{t/d}\, \mathrm{d}(p,q) \quad (9)$$

The fact that ϕ is continuous implies that $\Lambda(p)$ is a connected arc, for any $p \in \mathcal{O}$. If $p \in \Phi$, then $\Lambda(p)$ is reduced to a point. Otherwise, by F1, the distance to S increases strictly along $\Lambda(p)$, thus $\Lambda(p)$ does not self-intersect. It follows also from F1 that, if $p \in \mathcal{T}_\varepsilon \setminus S$, then $\Lambda(p)$ cannot leave and then re-enter $\mathcal{T}_\varepsilon$. Therefore, $\Lambda(p) \cap \mathcal{T}_\varepsilon$ is a simple arc. If $p \notin \mathcal{T}_\varepsilon$, then $\Lambda(p) \cap \mathcal{T}_\varepsilon$ is empty. The next result bounds the time spent before a point moving along a flow line leaves $\mathcal{T}_\varepsilon$:

Lemma 6.5.

(i) $\forall p \in \mathcal{T}_\varepsilon \setminus S,\ \forall t \geq \frac{\varepsilon - \mathrm{d}_S(p)}{\cos^2\theta},\ \phi(t,p) \notin \mathcal{T}_\varepsilon$

(ii) $\forall p \in \mathcal{O} \setminus \mathcal{T}_\varepsilon,\ \forall t \geq 0,\ \phi(t,p) \notin \mathcal{T}_\varepsilon$

Proof. Given $p \in \mathcal{T}_\varepsilon \setminus S$ and $t \in \mathbb{R}^+$ such that $\phi(t,p) \in \mathcal{T}_\varepsilon$, we know by F1 that $\phi(t',p)$ belongs to $\mathcal{T}_\varepsilon$ for any $t' \in [0,t]$. Since by hypothesis (a) we have $\varepsilon < \frac{1}{7}\ \mathrm{lr}_k(S) < \frac{1}{2}\ \mathrm{lr}_k(S)$, Lemma 3.9 and Eq. (8) imply that $\mathrm{d}_S(\phi(t,p)) \geq \mathrm{d}_S(p) + t\ \cos^2\theta$. Hence, the time t_ε at which $\mathrm{d}_S(\phi(t_\varepsilon,p)) = \varepsilon$ is at most $\frac{\varepsilon - \mathrm{d}_S(p)}{\cos^2\theta}$. This means that $\phi(t,p) \notin \mathcal{T}_\varepsilon$ for all $t \geq \frac{\varepsilon - \mathrm{d}_S(p)}{\cos^2\theta}$, hereby proving the lemma for $p \in \mathcal{T}_\varepsilon \setminus S$.

Given $p \in \mathcal{O} \setminus \mathcal{T}_\varepsilon$, F1 states that $\forall t \in \mathbb{R}^+$, $\mathrm{d}_S(\phi(t,p)) \geq \mathrm{d}_S(p) \geq \varepsilon$. Hence, $\phi(t,p) \notin \mathcal{T}_\varepsilon$, which proves the lemma for $p \in \mathcal{O} \setminus \mathcal{T}_\varepsilon$. $\square$

6.2 Flow lines intersecting $\hat{S}$

We define $\mathcal{I}$ as the set of points of $\mathcal{O}$ whose flow lines intersect $\hat{S}$. For convenience, we exclude the points of $\hat{S}$ from $\mathcal{I}$:

$$\mathcal{I} = \{p \in \mathcal{O} \setminus \hat{S} \mid \Lambda(p) \cap \hat{S} \neq \emptyset\}$$

Our aim is to prove that $\mathcal{I}$ is a union of connected components of $\mathcal{O} \setminus \hat{S}$, as illustrated in Figure 5. Since $\mathcal{I} \subseteq \mathcal{O} \setminus \hat{S}$, this comes down to proving that the boundary of $\mathcal{I}$ is included in $S \cup \hat{S}$.

Recall that $\hat{S}$ is included in $\mathcal{T}_\varepsilon$, by hypotheses (a) and (c). It follows that $\mathcal{I}$ is also included in $\mathcal{T}_\varepsilon$, by Lemma 6.5 (ii). We first show that the boundary of $\mathcal{I}$ lies in $S \cup \hat{S} \cup \partial\mathcal{T}_\varepsilon$:

Lemma 6.6. *For any $p \in \mathcal{I}$, there exists a positive value $r(p)$, vanishing only as p approaches S or $\hat{S}$ or $\partial\mathcal{T}_\varepsilon$, such that $B(p, r(p)) \subseteq \mathcal{I}$. As a consequence, $\partial\mathcal{I} \subseteq S \cup \hat{S} \cup \partial\mathcal{T}_\varepsilon$.*

Proof. Three major steps of the proof are stated as Claims 6.6.1, 6.6.2 and 6.6.3, whose proofs use H2–H2bis and are skipped in this abstract.

Let $p \in \mathcal{I}$. Since $p \notin S$, $\mathrm{d}_S(p)$ is positive. By F2, the restriction of ϕ to $\left[0, \frac{\varepsilon - \mathrm{d}_S(p)/2}{\cos^2\theta}\right] \times \mathcal{O}_{\mathrm{d}_S(p)/2}$ is 1-Lipschitz as a function of time, and κ-Lipschitz as a function of space, where $\kappa = \exp\left(\frac{2\varepsilon - \mathrm{d}_S(p)}{\mathrm{d}_S(p)\ \cos^2\theta}\right)$. Since $\mathcal{I} \subset \mathcal{T}_\varepsilon$, we have $\mathrm{d}_S(p) < \varepsilon$, which implies that $\kappa > 1/\cos^2\theta \geq 1$.

The function $q \mapsto \mathrm{d}(q, \partial\mathcal{T}_\varepsilon)$ is continuous over $\hat{S}$, thus it reaches its minimum δ since $\hat{S}$ is compact. This minimum is positive because $\hat{S} \subset \mathcal{T}_\varepsilon$. In addition, for any facet f of $\hat{S}$, the function $q \mapsto \mathrm{d}(q, \hat{S} \setminus \mathrm{star}(f))$ is positive and continuous over f, hence its minimum $m(f)$ over f is positive. Let $m = \min\{m(f),\ f \in \hat{S}\}$: m is positive since the (finitely many) $m(f)$ are. For any point $q \in \hat{S}$ and any facet f containing q, the distance of q to $\hat{S} \setminus \mathrm{star}(f)$ is at least m.

We set $r(p)$ as follows:

$$r(p) \;=\; \min\left\{\frac{\mathrm{d}_S(p)}{2\kappa},\ \varepsilon - \mathrm{d}_S(p),\ \frac{1}{3\kappa}\,\mathrm{d}(p,\hat{S}),\ \frac{\delta}{2\kappa},\ \frac{m}{2\kappa}\right\} \tag{10}$$

Note that $r(p)$ vanishes only if $\mathrm{d}_S(p) \to 0$ (p approaches S), or if $(\varepsilon - \mathrm{d}_S(p)) \to 0$ (p approaches $\partial\mathcal{T}_\varepsilon$), or if $\mathrm{d}(p,\hat{S}) \to 0$ (p approaches $\hat{S}$). We will prove that the open ball $B(p, r(p))$ is included in $\mathcal{I}$.

Let q lie in $B(p, r(p))$. Since $r(p) \le \varepsilon - \mathrm{d}_S(p)$, q belongs to $\mathcal{T}_\varepsilon$. Moreover, since $\kappa > 1$, we have $r(p) < \min\{\mathrm{d}_S(p),\ \mathrm{d}(p,\hat{S})\}$. Thus, $q \notin S \cup \hat{S}$. Let us prove that $\Lambda(q)$ intersects $\hat{S}$.

Since $p \in \mathcal{I}$, $\Lambda(p)$ intersects $\hat{S}$. Let $p' \in \Lambda(p) \cap \hat{S}$. We have $p' \neq p$ since $p \notin \hat{S}$. Let d' be defined by

$$d' = \min\left\{\frac{\mathrm{d}_S(p)}{2},\ \frac{\mathrm{d}(p,\hat{S})}{3},\ \frac{\delta}{2},\ \frac{m}{2}\right\}$$

We call $B^1_{p'}$ and $B^2_{p'}$ the open balls centered at p', of radii d' and $2d'$ respectively. Observe that $2d' \le \frac{2}{3}\ \mathrm{d}(p,\hat{S}) \le \frac{2}{3}\ \mathrm{d}(p,p')$. Moreover, since $\kappa > 1$, $r(p)$ is less than $\frac{1}{3}\ \mathrm{d}(p,\hat{S}) \le \frac{1}{3}\ \mathrm{d}(p,p')$. Hence,

$$B(p, r(p)) \cap B^2_{p'} = \emptyset \tag{11}$$

Claim 6.6.1. *$\Lambda(q)$ pierces $B^1_{p'}$,* i.e. *it enters and then leaves $B^1_{p'}$. Similarly, $\Lambda(q)$ pierces $B^2_{p'}$.*

Let f be a facet of $\hat{S}$ that contains p' and v be a vertex of f closest to p'. The distance from p' to v is at most the circumradius of f, which is bounded by ε, by hypothesis (c). Moreover, $2d'$ is at most $\mathrm{d}_S(p) < \varepsilon$. Therefore, $B^2_{p'}$ is included in $B(v, 2\varepsilon)$.

By Claim 6.6.1, $\Lambda(q) \cap B^1_{p'}$ is not empty. Let $q' \in \Lambda(q) \cap B^1_{p'}$. We call $K(q')$ the double cone of apex q', of axis aligned with $\mathbf{n}_k(v)$ and of half-angle θ. Since $q' \in B^1_{p'} \subset B^2_{p'}$, $K(q')$ intersects $\partial B^2_{p'}$ along two spherical patches $C_1(q')$ and $C_2(q')$, such that every connected curve included in $K(q')$ and joining $C_1(q')$ to $C_2(q')$ passes through q'. One arc of $\Lambda(q) \cap B^2_{p'}$ has this property, as stated in the next claim and illustrated in Figure 6:

Claim 6.6.2. *Let $\Lambda'(q)$ be the arc of $\Lambda(q) \cap B^2_{p'}$ that contains q'. $\Lambda'(q)$ lies in $K(q')$ and joins $C_1(q')$ to $C_2(q')$, with one endpoint in $C_1(q')$ and the other endpoint in $C_2(q')$.*

The next step is to show that such an arc intersects $\mathrm{star}(f)$:

Claim 6.6.3. *Inside $B^2_{p'}$, $C_1(q')$ and $C_2(q')$ are separated by* $\mathrm{star}(f)$, i.e. *every connected curve included in $B^2_{p'}$ and joining $C_1(q')$ to $C_2(q')$ intersects* $\mathrm{star}(f)$.

It follows from Claims 6.6.2 and 6.6.3 that $\Lambda(q)$ intersects $\mathrm{star}(f)$. Hence, $\Lambda(q) \cap \hat{S} \neq \emptyset$, which means that $q \in \mathcal{I}$. This ends the proof of Lemma 6.6. $\square$

By Lemma 6.6, the boundary of $\mathcal{I}$ is included in $S \cup \hat{S} \cup \partial\mathcal{T}_\varepsilon$. We now prove that, in fact, $\partial\mathcal{I}$ does not touch $\partial\mathcal{T}_\varepsilon$:

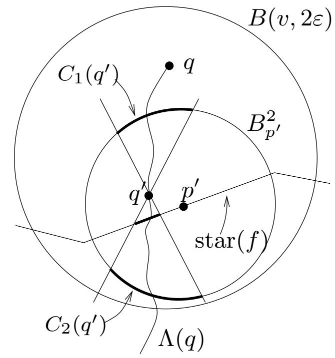

Figure 6: For the proof of Lemma 6.6.

Lemma 6.7. $\partial\mathcal{I} \cap \partial\mathcal{T}_\varepsilon = \emptyset$.

Proof. Since $\hat{S}$ is compact and d_S is continuous, the restriction of d_S to $\hat{S}$ reaches its maximum. Let $p \in \hat{S}$ be such that $\forall p' \in \hat{S}$, $\mathrm{d}_S(p') \le \mathrm{d}_S(p)$. Since $\hat{S}$ is included in $\mathcal{T}_\varepsilon$, $\mathrm{d}_S(p)$ is less than ε. It follows that $\hat{S}$ is in fact included in $\mathcal{T}_{\delta'}$, for any δ' such that $\mathrm{d}_S(p) < \delta' < \varepsilon$. By Lemma 6.5 (ii), for any $q \notin \mathcal{T}_{\delta'}$, $\Lambda(q) \cap \mathcal{T}_{\delta'} = \emptyset$, hence $\mathcal{I}$ is included in $\mathcal{T}_{\delta'}$. As a consequence, $\partial\mathcal{I}$ is included in the topological closure of $\mathcal{T}_{\delta'}$, which does not intersect $\partial\mathcal{T}_\varepsilon$ since $\delta' < \varepsilon$. $\square$

Lemmas 6.6 and 6.7 imply that the boundary of $\mathcal{I}$ is included in $S \cup \hat{S}$, which concludes the proof of the main result of Section 6.2:

Lemma 6.8. *$\mathcal{I}$ is a union of connected components of $\mathcal{O} \setminus \hat{S}$.*

6.3 $\hat{S} \cup \partial\mathcal{I}$ covers S

Let $S_\mathcal{D} = S \cap \hat{S}$ and $S_\mathcal{I} = S \cap \partial\mathcal{I}$. Our aim is to prove the following lemma:

Lemma 6.9. *$S = S_\mathcal{D} \cup S_\mathcal{I}$.*

The sets $S_\mathcal{D}$ and $S_\mathcal{I}$ can be viewed in Figure 5: the arc $\overline{p_1p_2}$ of S belongs to $S_\mathcal{D}$, while the arc $\overline{p_2p_3}$ of S belongs to $S_\mathcal{I}$. We will show that $S_\mathcal{D} \cup S_\mathcal{I}$ has no boundary, for the topology of S induced by $\mathbb{R}^3$. This implies that $S_\mathcal{D} \cup S_\mathcal{I}$ is a union of connected components of S, and since $\hat{S}$ intersects every connected component of S (hypothesis (b)), we get $S_\mathcal{D} \cup S_\mathcal{I} = S$, which proves Lemma 6.9.

The boundary of $S_\mathcal{D} \cup S_\mathcal{I}$ is included in $\partial S_\mathcal{D} \cup \partial S_\mathcal{I}$. According to Lemma 6.8, $\mathcal{I}$ is a union of connected components of $\mathbb{R}^3 \setminus (S \cup \hat{S})$. Thus, $\partial S_\mathcal{I}$ is included in $S_\mathcal{D}$. Therefore, to prove that the boundary of $S_\mathcal{D} \cup S_\mathcal{I}$ is empty, we need only to show that $S_\mathcal{D}$ is included in the interior of $S_\mathcal{D} \cup S_\mathcal{I}$:

Lemma 6.10. *For any point $p \in S_\mathcal{D}$, there exists an open neighborhood $\mathcal{N}$ of p on S such that $\mathcal{N} \subseteq S_\mathcal{D} \cup S_\mathcal{I}$.*

The proof of this result is very similar in spirit to the proof of Lemma 6.6. Therefore, we skip it in this abstract.

6.4 End of the proof of Proposition 6.4

According to Lemma 6.9, for any point $p \in S$:

- either p belongs to $\hat{S}$, which means that $\mathrm{d}(p, \hat{S}) = 0$;

- or p belongs to $\partial\mathcal{I}$, which means that for any $\eta > 0$ there exists some point $p_\eta \in B(p,\eta) \setminus S$ such that $\Lambda(p_\eta) \cap \hat{S} \neq \emptyset$. Let $p'_\eta \in \Lambda(p_\eta) \cap \hat{S}$ and let $t_\eta \geq 0$ be the time at which $\Lambda(p_\eta)$ reaches p'_η. Since $\hat{S} \subset \mathcal{T}_\varepsilon$, we have $t_\eta \leq \frac{\varepsilon - \mathrm{d}_S(p_\eta)}{\cos^2\theta} < \frac{\varepsilon}{\cos^2\theta}$, by Lemma 6.5 (i). Since by F2 ϕ is 1-Lipschitz as a function of time, we deduce that $\mathrm{d}(p_\eta, p'_\eta) < \frac{\varepsilon}{\cos^2\theta}$, which implies that $\mathrm{d}(p,\hat{S}) < \eta + \frac{\varepsilon}{\cos^2\theta}$. Since this is true for any $\eta > 0$, $\mathrm{d}(p,\hat{S})$ is at most $\frac{\varepsilon}{\cos^2\theta}$.

Therefore, no point of S is farther than $\frac{\varepsilon}{\cos^2\theta}$ from $\hat{S}$, which concludes the proof of Proposition 6.4.

7. MESHING LIPSCHITZ SURFACES

In [5] it is proved that, for any input compact surface S and any input positive parameter ε, Chew's algorithm [10] outputs a sparse loose ε-sample E of S, such that the inner angles of the facets of $\mathrm{Del}_{|S}(E)$ are at least $\pi/6$ (or, equivalently, that the radius-edge ratios are at most 1). From the properties of loose ε-samples of smooth surfaces, one deduces that, if S is $C^{1,1}$ and ε is sufficiently small with respect to $\mathrm{rch}(S)$, then $\mathrm{Del}_{|S}(E)$ is a good topological and geometric PL approximation of S. By combining the structural theorems of Sections 5 and 6, one gets a result of same flavor for Lipschitz surfaces:

THEOREM 7.1. *If S is a $\tan\theta$-Lipschitz surface, for some $\theta < \arctan\frac{\sqrt{3}}{5} \approx 19.1\,\mathrm{deg}$, and if $\varepsilon < \frac{\cos^2\theta}{7\sqrt{1+\cos^4\theta}}\ \mathrm{lr}_k(S)$ (where $k = \tan\theta$), then Chew's algorithm outputs an $\varepsilon\sqrt{1+\frac{1}{\cos^4\theta}}$-sample E of S (Corollary 6.3), with $\varepsilon\sqrt{1+\frac{1}{\cos^4\theta}} < \frac{1}{7}\ \mathrm{lr}_k(S)$. Therefore, by Theorems 5.3, 5.4 and 5.5, $\mathrm{Del}_{|S}(E_F)$ is a manifold isotopic to S that lies at Hausdorff distance at most $\varepsilon\sqrt{1+\frac{1}{\cos^4\theta}}$ from S.*

If S is a piecewise smooth surface, then, by Theorems 3.3 (ii) and 7.1, Chew's algorithm can generate good PL approximations of S if the normal deviation around the singular points of S is not more than 33 deg. Experimental results show however that the algorithm can handle normal deviations up to $\frac{\pi}{2}$ in practice [18, §6.4].

Observe that we use exactly the same algorithm as in the smooth setting, except that the estimation of the local feature size (lfs) is replaced by the evaluation of the Lipschitz radius (lr_k). In particular, no adaptation is needed in areas close to the singularities of S, and the singular points of S are not required (but allowed) to be inserted in the output point set E.

The estimation of the Lipschitz constant k of S as well as $\mathrm{lr}_k(S)$ deserves a detailed analysis to be provided in the full version of the paper. Note however that $\mathrm{lr}_k(S)$ can be estimated quite easily and efficiently when S is an oriented polyhedron without boundary, thanks to Theorem 3.3 (ii).

As for the output point set E, it has been proved to be sparse in the smooth setting [5], which implies that $|E| = \Theta\left(\frac{\mathrm{Area}(S)}{\varepsilon^2}\right)$, where the constant in the Θ is independent from S and ε. The same bound holds in the Lipschitz case, by very similar arguments.

Finally, let us emphasize that Chew's algorithm has also been adapted to solve several related problems, such as probing unknown smooth objects in the plane or in 3-space [4], or meshing volumes bounded by smooth surfaces [19]. It is clear that these variants of the algorithm enjoy the same theoretical guarantees in the Lipschitz setting, without any change in the code.

8. CONCLUSION AND FUTURE WORK

We have introduced the notion of Lipschitz radius, which is a natural extension of the local feature size to the class of Lipschitz surfaces. The Lipschitz radius is a 1-Lipschitz function, and it is bounded away from zero on Lipschitz surfaces. We have shown that (loose) ε-samples enjoy the same theoretical guarantees in the Lipschitz setting as they do in the smooth setting, provided that ε is small enough with respect to the Lipschitz radius and that the inner angles of the facets of the restricted Delaunay triangulation are not too small. As a straightforward application, we have shown that Chew's algorithm and its variants can produce good PL approximations of Lipschitz surfaces. In addition to providing new results and, in particular, the first provably correct algorithm for meshing nonsmooth surfaces, we believe our analysis sheds new light onto the structural properties of the restricted Delaunay triangulation and the Delaunay refinement paradigm.

As future work, we would like to use our sampling conditions in the context of surface reconstruction, where the main difficulty is to identify the facets that belong to the restricted Delaunay triangulation since the underlying surface is unknown. The question of the noise is also to be addressed. Finally, we would also like to extend our results to hypersurfaces in higher dimensional space. Observe that the Lipschitz framework is not designed for higher codimensions.

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