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Sampling and Reconstruction of Surfaces and Higher Dimensional Manifolds

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Abstract We present new sampling theorems for surfaces and higher dimensional manifolds. The core of the proofs resides in triangulation results for manifolds with boundary, not necessarily bounded. The method is based upon geometric considerations that are further augmented for 2-dimensional manifolds (i.e surfaces). In addition, we show how to apply the main results to obtain a new, geometric proof of the classical Shannon sampling theorem, and also to image analysis.

Keywords Image sampling · Image reconstruction · Geometric approach · Fat triangulation · Image manifolds

1 Introduction

Sampling is an essential preliminary step in processing of any continuous signal by a digital computer. Undersampling causes distortions due to aliasing of the post processed sampled data. Oversampling, on the other hand, results in time and memory consuming computational processes which, at the very least, slows down the analysis process. It is therefore important to have a measure which is instrumental in determining what is the optimal sampling rate. For one-dimensional signals such a measure exists, and, consequently, the optimal sampling rate is given by the fundamental sampling theorem of Shannon, that yielded the foundation of information theory and led technology into the digital era. Shannon's theorem asserts that a signal can be perfectly reconstructed from its samples, given that the signal is band limited within some bound on its highest frequency. Ever since the proof of Shannon's theorem was introduced in the late 1940s, deducing a similar sampling theorem for higher dimensional signals has become an essential problem related to various aspects of signal processing. This is further emphasized by the vast interest and numerous applications of image processing and by the growing need for fast yet accurate techniques for processing high dimensional data, such as medical and satellite images.

In this paper we present new sampling theorems for manifolds of dimensions ≥ 2. These theorems are derived form fundamental studies in three areas of mathematics: differential topology, differential geometry and geometric analysis. Both classical and recent results in these areas are combined to yield a rigorous and comprehensive sampling theory for such manifolds.

We first present sampling theorems for surfaces (dimension 2) and then for higher dimensional manifolds. In the case of surfaces, we account for surfaces that are at least $\mathcal{C}^2$, with bounded principal curvatures. This condition is, in a way, analogous to band limited signals in the case of one dimension (the classical Shannon sampling theorem). We then present a sampling theorem for surfaces that are not $\mathcal{C}^2$, and we proceed to present sampling theorems for manifolds of dimension ≥ 3. The main reasons for such a differentiated treatment of surfaces and of higher dimensional manifolds is that the geometry of surfaces is much more intuitive than that of manifolds of dimension ≥ 2. Therefore, the main ideas behind the given theorems, are more accessible in this

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case. Apart from this, there is also a deeper reason to distinguish between surfaces and higher dimensional manifolds: it is rooted in the geometrical richness of manifolds of dimensions ≥ 3, as compared with surfaces. This richness reflects on the present work through the variety of curvature measures applicable to manifolds of dimensions >2. In higher dimensions we can consider scalar, sectional and Ricci curvatures, each of which with its specific geometrical meaning and computational considerations. As a result, and due to the crucial role curvature plays in this whole work, when setting sampling theorems for high dimensional manifolds we first need to have a good understanding of which of the possible curvatures we would like to use.

The geometric sampling methods introduced herein are based on the existence of *fat* (see Sect. 2) triangulations of manifolds. Recently a surge in the study of fat triangulations and manifold sampling in computational geometry, computer graphics and their related fields has generated a considerable number of publications (e.g. [3, 7, 15, 16, 24, 26, 31], to name a few). For instance, in [3] Voronoi filtering is used for the construction of fat triangulations of compact, $\mathcal{C}^2$ surfaces embedded in $\mathbb{R}^3$. Note that Voronoi cell partitioning is also employed in "classical" sampling theory (see [40]). Further, [15] used these ideas for manifold reconstruction from point samples. In [26] a heuristic approach to the problem of the relation between curvature and sampling density is given. Again, in these studies the manifolds are assumed to be smooth, compact n-dimensional hyper-surfaces embedded in $\mathbb{R}^{n+1}$.

Our results extend the class of manifolds for which fat meshes and "good" samplings exist. Both classical and recent results in these areas are combined to yield a rigorous and comprehensive sampling theory for such manifolds. The sampling problem is fully integrated with fundamental mathematical concepts. The method proposed herein is developed with reference to fundamental results in differential topology, geometry and geometric analysis, and hence inherits mathematical rigour. This yields a rigorous and comprehensive sampling theory for manifolds. Such a study of the sampling problem, fully integrated with a fundamental mathematical approach is given here for the first time.

The paper is organized as follows: In Sect. 2 we review some preliminary results relevant to the theory. We first recall briefly some aspects of classical sampling theory. We then present the most relevant results from differential topology that play a central role in the theoretical background of our theory. More precisely, we focus on *PL*-approximation of smooth manifolds and on its counterpart of smoothing *PL*-manifolds. These results are directly adopted in order to show that our proposed reconstruction method is accurate and also to overcome the problem of non-smoothness. In Sect. 3 we provide some additional background results, combining both differential geometry and the theory of quasi-regular mappings. These results, both classical such as those of S.S. Cairns, starting from the early 1930s, and new, due to K. Peltonen from the 1990s and to E. Saucan from 2000s will be later adopted to give the existence of sampling for manifolds. The main results regarding sampling of manifolds are presented in Sect. 4. In Sect. 5 we show how to apply the surfaces/manifolds sampling results to obtain a new, geometric proof of the classical Shannon sampling theorem, and also in the analysis of images. In Sect. 6 we present some computational results regarding the implementation of our sampling and reconstruction theorems in the case of analytical surfaces. In the final section we examine some delicate aspects of our study, and discuss extensions of this work, relating both to geometric aspects of sampling, as well as to its relationship with classical sampling theory.

2 Preliminaries

2.1 Shannon's Theorem and Sampling Theory

We do not present here in detail the classical Whittaker-Kotelnikov-Nyquist-Shannon theorem (Shannon's Theorem, for short), but restrict ourselves to bringing the following version:

Theorem 2.1 *Let $f \in L^2(\mathbb{R})$, such that supp $(\hat{f}) \subseteq [-\pi, \pi]$, where $\hat{f}$ denotes the Fourier transform of f. Then*

$$f(x) = \sum_{t\in\mathbb{Z}} f(t)\,\mathrm{sinc}(x-t), \tag{2.1}$$

where $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$.

The classical Shannon theorem pertains to *band limited signals*. Various generalizations of it were proposed (see [1, 2, 6, 32, 40, 44, 45], amongst others).

We conclude this brief overview of Shannon's theorem with a few remarks relevant to the sequel:

1. Since (2.1) expresses f as an infinite series, it follows that obtaining a perfect reconstruction of f by applying Shannon's theorem requires an infinite length (duration) of a signal. In fact, to begin with, to be band limited, a signal has to be of an infinite duration.
2. Mathematically, Shannon's theorem belongs to the field of interpolation (see, e.g. [6, 40]). The main—and surprising—fact is that linear interpolation (the *secant approximation*, to be more precise—see Sects. 2.2, 2.3 below) basically suffices to faithfully reconstruct manifolds.
3. The quest for reproducing kernels is natural. However, not every family of functions admits such kernels

(see [5], pp. 380–381). Moreover, surfaces (and a fortiori higher dimensional manifolds) are geometric objects with far "wilder" smoothness properties than signals, as usually considered (see, e.g. [21]). Therefore, a general theory of reproducing kernels for manifolds seems difficult and remains, at this stage, yet to be developed.
4. Shannon's theorem is equivalent to a variety of seemingly unrelated results in classical Mathematical Analysis (see [21]). It is plausible, and indeed probable, that precisely these variations on the given theme can shed some more light on all the aspects of a sampling theory for surfaces.

2.2 Background on PL-Topology

We first recall a few classical definitions and notations:

Definition 2.2 Let $a_0,\dots,a_m \in \mathbb{R}^n$. $\{a_i\}_{i=1}^m$ are said to be *independent* iff the vectors $v_i = a_i - a_0, i = 1,\dots,m$; are linearly independent.

The set $\sigma = a_0a_1\dots a_m = \{x = \sum \alpha_i a_i | \alpha_i \geq 0, \sum \alpha_i = 1\}$ is called the *m-simplex spanned by* $a_0,\dots,a_m$. The points $a_0,\dots,a_m$ are called the *vertices* of σ.

The numbers α_i are called the *barycentric coordinates* of σ. The point $\tilde{\sigma} = \frac{1}{m+1}\sum \alpha_i$ is called the *barycenter* of σ.

If $\{a_0,\dots,a_k\} \subseteq \{a_0,\dots,a_m\}$, then $\tau = a_0\dots a_k$ is called a *face* of σ, and we write $\tau < \sigma$.

Definition 2.3 Let $A, B \subset \mathbb{R}^n$. We define the join $A * B$ of A and B as $A * B = \{\alpha a + \beta b | a \in A, b \in B; \alpha,\beta \geq 0, \alpha + \beta = 1\}$. If $A = \{a\}$, then $A * B$ is called the *cone* with *vertex* a and *base* B.

Definition 2.4 A collection K of simplices is called a *simplicial complex* if

1. If $\tau < \sigma$, then $\tau \in K$.
2. Let $\sigma_1, \sigma_2 \in K$ and let $\tau = \sigma_1 \cap \sigma_2$. Then $\tau < \sigma_1, \tau < \sigma_2$.
3. K is locally finite.

$|K| = \bigcup_{\sigma \in K} \sigma$ is called the *underlying polyhedron* (or *polytope*) of K.

Definition 2.5 A complex K' is called a *subdivision* of K iff

1. $K' \subset K$;
2. if $\tau \in K'$, then there exists $\sigma \in K$ such that $\tau \subseteq \sigma$.

If K' is a subdivision of K we denote it by $K' \lhd K$.

Let K be a simplicial complex and let $L \subset K$. If L is a simplicial complex, then it is called a *subcomplex* of K.

Definition 2.6 Let $a \in |K|$. Then

$$St(a, K) = \bigcup_{\substack{a \in \sigma \\ \sigma \in K}} \sigma$$

is called the *star* of $a \in K$.

If $S \subset K$, then we define: $St(S, K) = \bigcup_{a \in S} St(a, K)$.

Definition 2.7 Let $\sigma = a_0a_1\dots a_m$ and let $f : \sigma \to \mathbb{R}^p$. The map f is called *linear* iff for any $x = \sum \alpha_i a_i \in \sigma$, it holds that $f(x) = \sum \alpha_i f(a_i)$.

Let K, L be complexes, and let $f : |K| \to |L|$. Then f is called *linear* (relative to K and L) iff for any $\sigma \in K$, $\tau = f(\sigma) \in L$.

The map $f : K \to L$ is called *piecewise linear* (PL) iff there exists a subdivision K' of K such that $f : K' \to L$ is linear.

If (i) $f : K \to L$ is a homeomorphism of $|K|$ onto $|L|$, (ii) $f|_\sigma$ is linear and (iii) $\tau = f|_\sigma \in L$, for any $\sigma \in K$, then f is called a *linear homeomorphism*.

Definition 2.8 A *cell* γ is a bounded subset of $\mathbb{R}^n$ defined by:

$$\gamma = \{x \in \mathbb{R}^n | \sum_j \alpha_{ij} x_j \geq \beta_i; i = 1,\dots,p\},$$

for some constants $\alpha_{i,j}$ and β_i.

The *dimension* m of γ is defined as $\min\{dim \Pi | \gamma \subset \Pi$, Π *being a hyperplane in*$\mathbb{R}^n\}$.

Let γ be an m-dimensional cell. The $(m-1)$-cells β_j of $\partial\gamma$ are called its $(m-1)$-*faces*, the $(m-2)$-faces of each β_j are called the $(m-2)$-*faces* of γ, etc. By convention $\emptyset$ and γ are also faces of γ.

A *cell complex* is defined in the same manner as a simplicial complex, more precisely, a cell complex K is a collection of cells that satisfy conditions 1–3 of Definition 2.4.

Subcomplexes are also defined in analogy to the simplicial case. In particular, the *q-skeleton* K^q of K, $K^q = \{\gamma | \gamma \in K, \dim \gamma \leq q\}$ is a subcomplex of K.

Lemma 2.9 *Let K be cell complex. Then, K has a simplicial subdivision.*

Proof See [29], Lemma 7.8. □

We next define the concept of *embedding* for complexes, but first we need some basic definitions:

Definition 2.10 Let K be a simplicial complex.

1. $f : |K| \to M^n$ is $\mathcal{C}^r$ differentiable (relative to $|K|$) iff $f|_\sigma \in \mathcal{C}^r(\sigma)$, for any simplex $\sigma \in K$.
2. $f : |K| \to M^n$ is non-degenerate iff $rank(f|_\sigma) = \dim(\sigma)$, for any simplex $\sigma \in K$.

Definition 2.11 Let σ be a simplex, and let $f : \sigma \to \mathbb{R}^n$, $f \in \mathcal{C}^r$. For $a \in \sigma$ we define $df_a : \sigma \to \mathbb{R}^n$ as follows: $df_a(x) = Df(a) \cdot (x - a)$, where $Df(a)$ denotes the *formal derivative* $Df(a) = (\partial f_i/\partial x^j)_{1\leq i,j\leq n}$, computed with respect to some orthogonal coordinate system contained in $\Pi(\sigma)$, where $\Pi(\sigma)$ is the hyperplane determined by σ. The map $df_a : \sigma \to \mathbb{R}^n$ does not depend upon the choice of this coordinate system.

Note that $df_a|_{\sigma\cap\tau}$ is well defined, for any $\sigma, \tau \in \overline{St}(a, K)$. Therefore, the map $df_a : \overline{St}(a, K) \to \mathbb{R}^n$ is well-defined and continuous. It is called the *differential* of f, in analogy to the case of differentiable manifolds.

Remark 2.12 In contrast to the differential case, the *tangent space* $T_{f(p)}(M^n)$ is a union of *polyhedral tangent cones*. It, therefore, does not possess a natural vector space structure (see [42], p. 196).

Definition 2.13 Let K be a simplicial complex, let M^n be a $\mathcal{C}^r$ submanifold of $\mathbb{R}^N$, and let $f : K \to M^n$ be a $\mathcal{C}^r$ map. Then, f is called

1. an *immersion*, iff $df_\sigma : \overline{St}(\sigma, K) \to \mathbb{R}^n$ is injective for each and every $\sigma \in K$;
2. an *embedding*, iff it is an immersion and a homeomorphism on the image $f(K)$;
3. a $\mathcal{C}^r$*-triangulation*, iff it is an embedding such that $f(K) = M^n$.

Remark 2.14 If the class of the map f is not relevant, f will be called simply a triangulation.

Definition 2.15 Let $f : K \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, and let $\delta : K \to \mathbb{R}^*_+$ be a continuous function. Then $g : |K| \to \mathbb{R}^n$ is called a δ*-approximation to* f iff:

(i) There exists a subdivision K' of K such that $g \in \mathcal{C}^r(K', \mathbb{R}^n)$;
(ii) $d_2(f(x), g(x)) < \delta(x)$, for any $x \in |K|$;
(iii) $d_2(df_a(x), dg_a(x)) \le \delta(a) \cdot d_2(x, a)$, for any $a \in |K|$ and for all $x \in \overline{St}(a, K')$.

(Here d_2 denotes the Euclidean distance on $\mathbb{R}^n$.)

Definition 2.16 Let K' be a subdivision of K, $U = \overset{\circ}{U}$, and let $f \in \mathcal{C}^r(K, \mathbb{R}^n), g \in \mathcal{C}^r(K', \mathbb{R}^n)$. g is called a δ*-approximation* of f (on U) iff conditions (ii) and (iii) of Definition 2.6 hold for any $a \in U$.

The most natural and intuitive δ-approximation to a given mapping f is the *secant map induced by* f:

Definition 2.17 Let $f \in \mathcal{C}^r(K)$ and let s be a simplex, $s < \sigma \in K$. Then, the linear map: $L_s : s \to \mathbb{R}^n$ defined by $L_s(v) = f(v)$, where v is a vertex of s, is called the *secant map induced by* f.

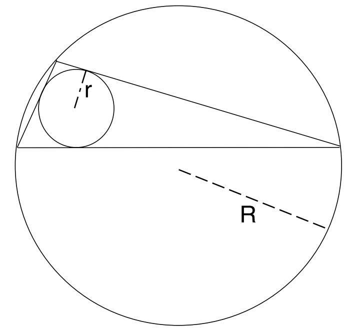

Fig. 1 Thin triangle—Peltonen's definition

2.3 PL-Approximation of Smooth Manifolds

We show in this section that the apparent "naive" secant approximation of surfaces (and higher dimensional manifolds) represents a good approximation, both in distances and in angles, provided the secant approximation induced by a triangulation satisfies a certain un-degeneracy condition called "fatness" (or "thickness").

2.3.1 Fat Triangulations

We first provide the following informal, intuitive definition:

Definition 2.18 A triangle in $\mathbb{R}^2$ is called *fat* (or φ*-fat*, to be more precise) iff all its angles are larger than a φ.

In other words, fat triangles are those that do not "deviate" too much from being equiangular (regular), hence fat triangles are not too "slim". This can be defined more formally by requiring that the ratio of the radii of the inscribed and circumscribed circles of the triangle is bounded from bellow by φ, i.e. $\frac{r}{R} \ge \varphi$, for some $\varphi > 0$, where r denotes the radius of the inscribed circle of τ (*inradius*) and R denotes the radius of the circumscribed circle of τ (*circumradius*) (Fig. 1).

One can easily check, by elementary methods, that the angle-condition and the radii condition are equivalent. Even if, perhaps, more intuitive, the angle condition is more difficult to properly formulate in higher dimension, therefore we opt for the following formal definition of fatness:

Definition 2.19 A k-simplex $\tau \subset \mathbb{R}^n$, $2 \le k \le n$, is φ*-fat* if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \ge \varphi$. A triangulation of a submanifold of $\mathbb{R}^n$, $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is φ*-fat* if all its simplices are φ*-fat*. A triangulation $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is *fat* if there exists $\varphi \ge 0$ such that all its simplices are φ*-fat*; for any $i \in \mathbf{I}$.

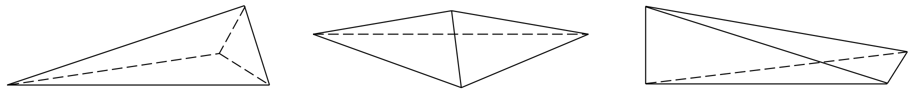

Fig. 2 "Slim" tetrahedra in $\mathbb{R}^3$

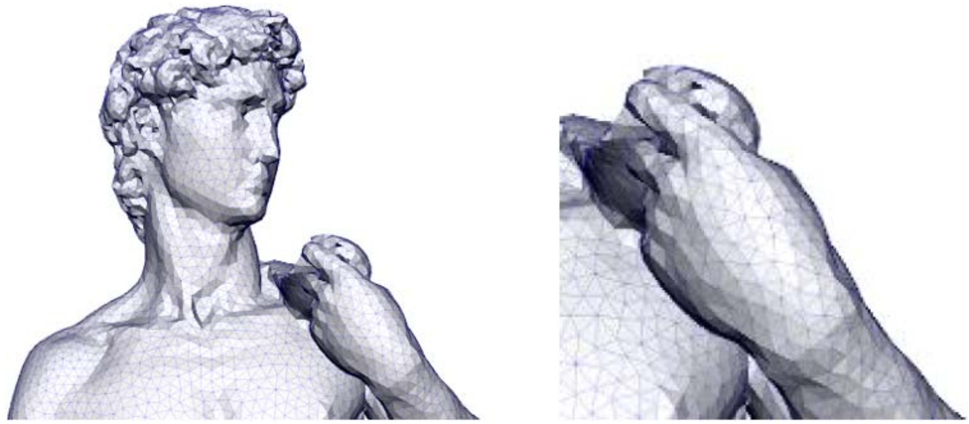

Fig. 3 Michelangelo's David model endowed with almost ideal triangulation: quasi-equilateral triangles of approximately equal size

Proposition 2.20 [14] *There exists a constant $c(k)$ that depends solely upon the dimension k of τ such that*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \le \min_{\sigma < \tau} \measuredangle(\tau, \sigma) \le c(k) \cdot \varphi(\tau), \tag{2.2}$$

and

$$\varphi(\tau) \le \frac{Vol_j(\sigma)}{diam^j\,\sigma} \le c(k) \cdot \varphi(\tau), \tag{2.3}$$

where φ denotes the fatness of the simplex τ, $\measuredangle(\tau,\sigma)$ denotes the (internal) dihedral angle *of the face $\sigma < \tau$ and $Vol_j(\sigma)$; $diam\,\sigma$ stand for the Euclidean j-volume and the diameter of σ respectively.* (*If* $\dim\sigma = 0$, *then* $Vol_j(\sigma) = 1$, *by convention*.)

Condition 2.2 is just the expression of fatness as a function of dihedral angles in all dimensions, while Condition 2.3 expresses fatness as given by "large area/diameter". Diameter is important since fatness is independent of scale.

One can gain some insight into the equivalence of all the definitions above, by analyzing the three-dimensional examples below (see Fig. 2). (See [16] for a complete classification of "slim" triangles in dimensions 2 and 3.)

Remark 2.21 The above definition is the one introduced in [33]. We employ it, as already noted, mainly for briefness. For other, equivalent definitions of fatness see [11, 12, 14], (based upon angles), [29] (the most similar to the one given above—see below) and [43] (based upon area/diameter).

Remark 2.22 In practice, the "fatness" φ of a triangulation is predetermined by some geometric condition, see Sect. 4 below.

Remark 2.23 As was already noted in the introduction, achieving a fat triangulation endowed, moreover, with simplices of almost equal diameter (see Fig. 3) is highly important in computer graphics and related fields. This is obtained via a process called "mesh improvement", akin to our "fattening" technique of a given triangulation. However, real (i.e. scanned images) produce non-fat (slim) triangulations with a high range of diameters—see Fig. 4 and, for an extreme case, Fig. 12, that illustrates the triangulation obtained from the CT scan of the human colon.

2.3.2 The Main Result

While, by Proposition 2.20, we could have employed any of the equivalent definitions of fatness, the computations in the proposition below are performed for

$$\varphi(\sigma) = \frac{r(\sigma)}{diam(\sigma)};$$

(where the notations are as above).

Proposition 2.24 ([29], Lemma 9.3) *Let $f : \sigma \to \mathbb{R}^n$ be of class $\mathcal{C}^k$. Then, for $\delta, \varphi_0 > 0$, there exists $\varepsilon > 0$, such that, for any $\tau < \sigma$, such that $diam(\tau) < \varepsilon$ and such that $\varphi(\tau) > \varphi_0$, the secant map L_τ is a δ-approximation to $f|\tau$.*

Proof We first show that (i) $F_b(x) = f(b) + Df(b)\cdot(x-b)$, where b denotes the barycenter of σ, is a $\delta/2$-approximation to f on a sufficient small neigbourhood of b. We then prove that (ii) if $\tau < \sigma$ satisfies the conditions from the statement of the theorem, then L_σ is a $\delta/2$-approximation to F_b. This two assertions suffice to prove the theorem.

Proof of (i) Follows immediately from the definition of Df. We impose the additional requirement $||f(x) -$

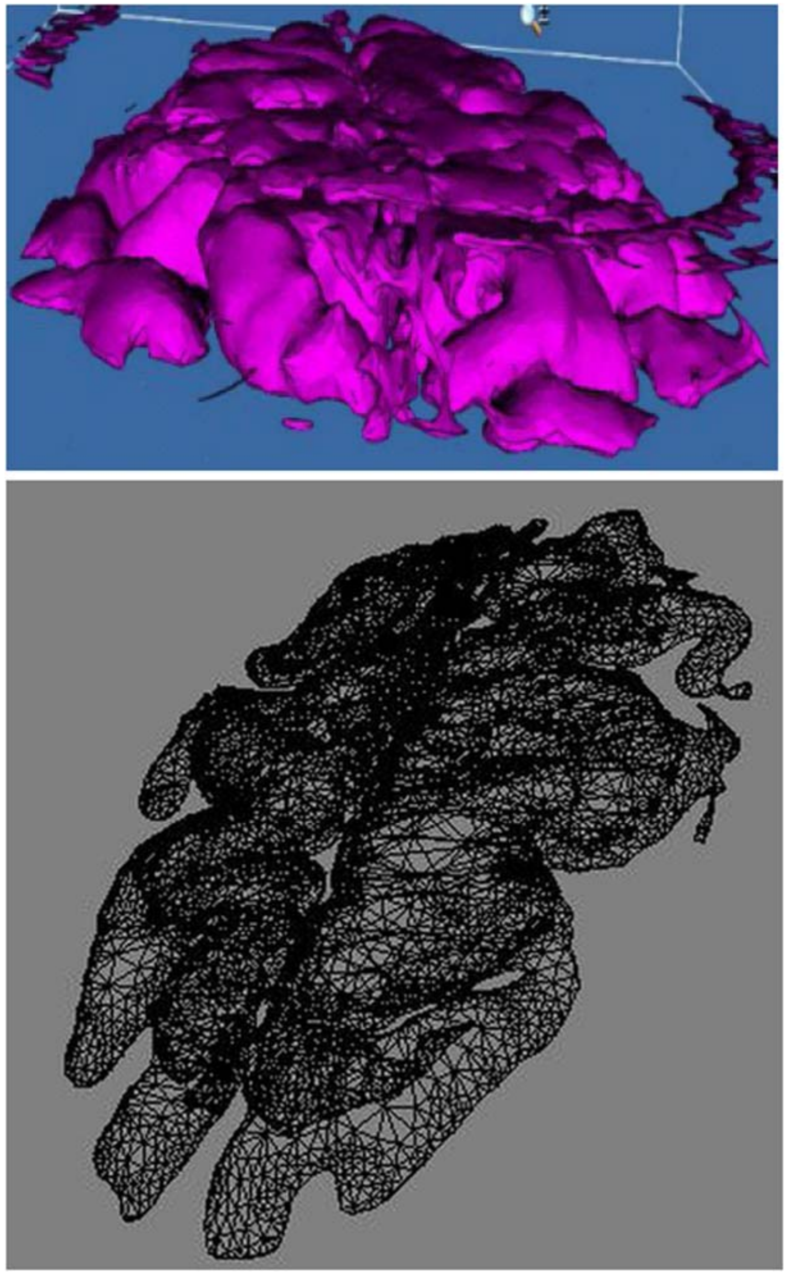

Fig. 4 Sampling (triangulation) of an MRI image of part of cerebral cortex surface. Note the uneven diameters and fatness of the simplices

$F_b(x)||/||x-b|| < \delta\varphi_0/4$, for $||x-b|| < \varepsilon$. (Here $||\cdot||$ denotes the Euclidean norm.)

Before we proceed further we need the following result: Let $L, F : \tau \to \mathbb{R}^n$ be linear maps, such that $||L(x) - F(x)|| < c$, for all $x \in \tau$. Then, it results immediately from (i) that $||DL(x)\cdot \mathbf{u} - DF(x)\cdot \mathbf{u}|| \le c/r(\tau)$, for all $\mathbf{u}$ in the plane of τ, $||\mathbf{u}|| = 1$.

Proof of (ii) Let $v_0, \dots, v_k$ be the vertices of τ, and let $x \in \tau, x = \sum \alpha_i v_i$. Then, by the linearity of L_s and F_s it follows that $L_s(x) = \sum \alpha_i L_s(v_i) = \sum \alpha_i f(v_i)$ and $F_b(x) = \sum \alpha_i F_b(v_i)$. Hence:

$$\|L_s(x) - F_b(x)\| = \left\| \sum \alpha_i \frac{\|f(x) - F_b(x)\|}{\|x-b\|} \right\|$$

$$\le \max \|f(x) - F_b(x)\|,$$

but $\|f(x) - F_b(x)\| < \delta/2$ and $\|L_s(x) - F_b(x)\| < \delta/2$, for all $\|x-b\| < \varepsilon$. Moreover, $\|f(x) - F_b(x)\|/\|x-b\| < \delta\varphi_0/4$, for all $\|x-b\| < \varepsilon$, and, since $\varphi_0 \le r(\tau)/diam(\tau)$, it follows that:

$$\begin{aligned}\|L_s(x) - F_b(x)\| &< \max \|v_i - b\|\delta\varphi_0/4 \\ &\le diam(\tau)\delta\varphi_0/4 \le \delta\, r(\tau)/4.\end{aligned}$$

This concludes the proof of (ii), and, hence, of the proposition. □

2.4 Smoothing of Manifolds

We proceed to address the problem of smoothing of manifolds, i.e. approximating a differentiable manifold of class $\mathcal{C}^r$, $r \geq 0$, by manifolds of class $\mathcal{C}^\infty$. Of special interest is the case where $r = 0$. This will be used in our development of our sampling theorem, and as a postprocessing step where, after reproducing a *PL* manifold out of the samples, to get a smooth reproduced manifold. Smoothing is also useful in preprocessing, when we wish to extend the sampling theorem to manifolds which are not necessarily smooth. Smoothing is, in this case, followed up by sampling of the smoothed manifold, yielding a set of samples representing the non-smooth manifold as well. (Our main reference here are [29], Chap. 4, and [22].)

Question 1 *What does smoothing of manifolds entail? This is much less obvious when an additional requirement of "geometric" approximation is imposed, e.g. when a proper curvature (Gauss, mean, etc.) convergence is also required. For the proof of this in the case of surfaces refer to* [8].

2.4.1 Partition of Unity

Smoothing will be obtained by means of a $\mathcal{C}^\infty$ smoothing convolution kernel. Before introducing this kernel, we recall the notion of partition of unity, which represents the core of the smoothing process:

Lemma 2.25 *For every $0 < \epsilon < 1$ there exists a $\mathcal{C}^\infty$ function $\psi_1 : \mathbb{R} \to [0, 1]$, such that, $\psi_1 \equiv 0$ for $|x| \geq 1$ and $\psi_1 = 1$ for $|x| \leq (1 - \epsilon)$. Such a function is called* partition of unity (*see Fig.* 5).

Let $c^n(\epsilon)$ be the ϵ-cube around the origin in $\mathbb{R}^n$ (i.e. $X \in \mathbb{R}^n$; $-\epsilon \leq x_i \leq \epsilon, i = 1, \dots, n$). We can use the above partition of unity in order to obtain a non-negative $\mathcal{C}^\infty$-function, ψ, on $\mathbb{R}^n$, such that $\psi = 1$ on $c^n(\epsilon)$ and $\psi \equiv 0$ outside $c^n(1)$. Define $\psi(x_1, \dots, x_n) = \psi_1(x_1) \cdot \psi_1(x_2) \cdots \psi_1(x_n)$.

We now introduce the main theorem regarding smoothing of *PL*-manifolds:

Theorem 2.26 ([29]) *Let M be a $\mathcal{C}^r$ manifold, $0 \leq r < \infty$, and $f_0 : M \to \mathbb{R}^k$ a $\mathcal{C}^r$ embedding. Then, there exists a $\mathcal{C}^\infty$ embedding $f_1 : M \to \mathbb{R}^k$ which is a δ-approximation of f_0.*

The above theorem is a consequence of the following lemma concerning smoothing of maps:

Lemma 2.27 ([29]) *Let U be an open subset of $\mathbb{R}^m$. Let A be a compact subset of an open set V such that $\overline{V} \subset U$ is compact. Let $f_0 : U \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, $0 \leq r$. Let δ be a positive number. Then there exists a map $f_1 : U \to \mathbb{R}^n$ such that*

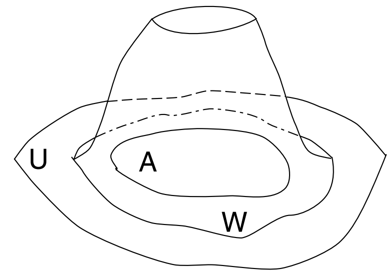

Fig. 5 Partition of unity on A

1. *f_1 is $\mathcal{C}^\infty$ on A.*
2. *$f_1 = f_0$ outside V.*
3. *f_1 is a δ-approximation of f_0.*
4. *f_1 is $\mathcal{C}^r$-homotopic to f_0 via a homotopy f_t satisfying* (2) *and* (3) *above. i.e. f_0 can be continuously deformed to f_1.*

Proof Let W be an open set containing A such that $\overline{W} \subset V$. We use partition of unity in order to obtain the following maps,

1. $\psi : \mathbb{R}^m \to \mathbb{R}_+$ so that, it is $\mathcal{C}^\infty$, and $\psi = 1$ on A and $\psi \equiv 0$ outside W.
2. $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function which is positive on $int(c^m(\epsilon))$ and vanishes outside $\overline{c^m(\epsilon)}$. ϵ is some positive number yet to be defined. Further assume that $\int_{\mathbb{R}^m} \varphi = 1$.

Define $g = \psi \cdot f$. Then, $g : \mathbb{R}^m \to \mathbb{R}^n$ and satisfies $g = f$ on A and $g \equiv 0$ outside W. Inside A, g is of the same differentiability class as f, whereas outside W it is $\mathcal{C}^\infty$.

For $x \in \mathbb{R}^m$, define

$$h(x) = \int_{c^m(\epsilon)} \varphi(y) g(x + y) dy. \tag{2.4}$$

Choose ϵ so that $\sqrt{m}\epsilon < d(W, \mathcal{R}^m \setminus V)$, then $h \equiv 0$ outside V.

Let

$$f_1(x) = f_0(x) \cdot (1 - \psi(x)) + h(x).$$

Since ψ and h vanish outside V, conclusion (2) of the lemma is fulfilled.

Inside A we have $f_1(x) = h(x)$. Since

$$h = \int_{c^m(\epsilon)} \varphi(y) g(x + y) dy = \int_{W + c^m(\epsilon)} \varphi(z - x) g(z) dz$$
$$= \int_{\mathbb{R}^m} \varphi(z - x) g(z) dz;$$

and since φ is $\mathcal{C}^\infty$, h is also $\mathcal{C}^\infty$ inside W, and in particular on A, thus fulfilling conclusion (1).

By its definition $f_1 = f_0 + (h - g)$, so we have to choose ϵ small enough so that h is a δ-approximation to g. By the mean value theorem we have:

$$h^i(x) = g^i(x + y^i);$$
$$\frac{\partial h^i}{\partial x^j} = \frac{\partial g^i(x + y^{ij})}{\partial x^j};$$

where y^i and y^{ij} are points in $c^m(\epsilon)$. We only have to take care that ϵ is so small that

$$|g^i(x) - g^i(x')| < \delta;$$

and

$$\left|\frac{\partial g^i}{\partial x^j}(x) - \frac{\partial g^i}{\partial x^j}(x')\right| < \delta;$$

for

$$|x - x'| < \epsilon;$$

this completes part (3).

Finally, let $\alpha(t)$ be a monotonic $\mathcal{C}^\infty$ function such that: $\alpha = 0$ for $0 \le t \le 1/3$; and $\alpha = 1$ for $2/3 \le t \le 1$. Define

$$f_t(x) = \alpha(t) f_1(x) + (1 - \alpha(t)) f_0(x). \tag{2.5}$$

Then, $f_t \equiv f_0$ outside V and f_t is the desired $\mathcal{C}^r$ homotopy between f_0 and f_1. This completes the proof. □

Remark 2.28 In the proof of the isometric embedding theorem, J. Nash [30] used a modified version of the smoothing process presented herein. Nash's idea was to define a radially symmetric convolution kernel φ, by taking its Fourier transform, $\hat{\varphi}$, to be a radially symmetric partition of unity. In so doing one can use a scaling process where for each N the smoothing operator of g is defined to be

$$h_N g(x) = \int_{\mathbb{R}^m} \varphi(z) g(x + z/N) dz$$
$$= \int_{\mathbb{R}^m} \varphi_N(z - x) g(z) dz;$$

where $\varphi_N(z) = N^m \varphi(Nz)$. Thus we have that the Fourier transform of φ_N satisfies

$$\hat{\varphi}_N(\omega) = \hat{\varphi}(\omega/N).$$

Note that this results in a higher degree of smoothing for small N (the partition of unity being taken over a larger neighbourhood), while for large N we have less smoothing yielding a better approximation. In this case the approximation is faithful not only to the signal and its first derivative as in the classical approach, but also to higher order derivatives, if such exist.

3 Fat Triangulation

3.1 Theorems

In this section we review, in chronological order, existence theorems dealing with fat triangulations on manifolds. (For detailed proofs see the original papers.)

Theorem 3.1 (Cairns, [13]) *Every compact $\mathcal{C}^2$ Riemannian manifold admits a fat triangulation.*

Remark 3.2 For a similar result, the proof of which does not generalize to open manifolds, see [11, 12].

Theorem 3.3 (Peltonen, [33]) *Every open (unbounded) $\mathcal{C}^\infty$ Riemannian manifold admits a fat triangulation.*

Theorem 3.4 (Saucan, [36]) *Let M^n be an n-dimensional $\mathcal{C}^1$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then, any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Remark 3.5 The compactness condition on the boundary components in Theorem 3.4, can be replaced by the following condition: ∂M^n is endowed with a fat triangulation $\mathcal{T}$ such that $\inf_{\sigma \in \mathcal{T}} diam\sigma > 0$ ([36]). In fact, Theorem 3.4 holds even without the finiteness and compactness conditions imposed on the boundary components (see [37]).

Corollary 3.6 *If M^n is as above, then it admits a fat triangulation.*

Corollary 3.7 *Let M^n be an n-dimensional, $n \le 4$ (resp. $n \le 3$), PL (resp. topological) connected manifold with boundary, having a finite number of compact boundary components. Then, any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

3.2 Methods

3.2.1 Background

Let M^n denote an n-dimensional complete Riemannian manifold, and let M^n be isometrically embedded into $\mathbb{R}^\nu$ ("ν"-s existence is guaranteed by Nash's Theorem (see, e.g. [33, 41]).

Let $\mathbb{B}^\nu(x, r) = \{y \in \mathbb{R}^\nu | d_{eucl} < r\}$; $\partial\mathbb{B}^\nu(x, r) = \mathbb{S}^{\nu-1}(x, r)$. If $x \in M^n$, let $\sigma^n(x, r) = M^n \cap \mathbb{B}^\nu(x, r)$, $\beta^n(x, r) = \exp_x(\mathbb{B}^n(0, r))$, where: $\exp_x$ denotes the exponential map: $\exp_x : T_x(M^n) \to M^n$ and where $\mathbb{B}^n(0, r) \subset T_x(M^n)$, $\mathbb{B}^n(0, r) = \{y \in \mathbb{R}^n | d_{eucl}(y, 0) < r\}$.

Remark 3.8 Neither of the following (homeomorphisms) is guaranteed:

1. $\sigma^n(x,r) \simeq \mathbb{B}^n(0,r)$
2. $\beta^n(x,r) \simeq \mathbb{B}^n(0,r)$.

The following definitions generalize in a straightforward manner classical ones used for surfaces in $\mathbb{R}^3$:

Definition 3.9

1. $\mathbb{S}^{\nu-1}(x,r)$ is *tangent* to M^n at $x \in M^n$ iff there exists $\mathbb{S}^n(x,r) \subset \mathbb{S}^{\nu-1}(x,r)$, s.t. $T_x(\mathbb{S}^n(x,r)) \equiv T_x(M^n)$.
2. Let $l \subset \mathbb{R}^\nu$ be a line, then l is *secant* to $X \subset M^n$ iff $|l \cap X| \geq 2$.

Definition 3.10

1. $\mathbb{S}^{\nu-1}(x,\rho)$ is an osculatory sphere at $x \in M^n$ iff:
   (a) $\mathbb{S}^{\nu-1}(x,\rho)$ is tangent at x;
   and
   (b) $\mathbb{B}^n(x,\rho) \cap M^n = \emptyset$.
2. Let $X \subset M^n$. The number $\omega = \omega_X = \sup\{\rho > 0 | \mathbb{S}^{\nu-1}(x,\rho)$ osculatory at any $x \in X\}$ is called the *maximal osculatory radius* at X.

Remark 3.11

1. There exists an osculatory sphere at any point of M^n (see [13]).
2. If X is compact, then $\omega_X > 0$.

3.3 The Classical Case

In the compact case the method is to produce a point set $A \subseteq M^n$, that is maximal with respect to the following density condition:

$$d(a_1,a_2) \geq \eta, \quad \text{for all } a_1,a_2 \in A; \tag{3.1}$$

where

$$\eta < \omega_M. \tag{3.2}$$

One makes use of the fact that for a compact manifold M^n we have $|A| < \aleph_0$, to construct the finite cell complex "cut out of M" by the ν-dimensional Dirichlet complex (see Fig. 6), whose (closed) cells are given by:

$$\begin{aligned}\bar{c}_k &= \bar{c}_k^\nu \\ &= \{x \in \mathbb{R}^\nu | d_{eucl}(a_k,x) \leq d_{eucl}(a_i,x), \\ &\qquad a_i \in A, a_i \neq a_k\},\end{aligned} \tag{3.3}$$

i.e. the (closed) cell complex $\{\bar{\gamma}_k^n\}$, where:

$$\{\bar{\gamma}_k^n\} = \bar{\gamma}_k = \bar{c}_k \cap M^n \tag{3.4}$$

(see [13, 33] (for details)).

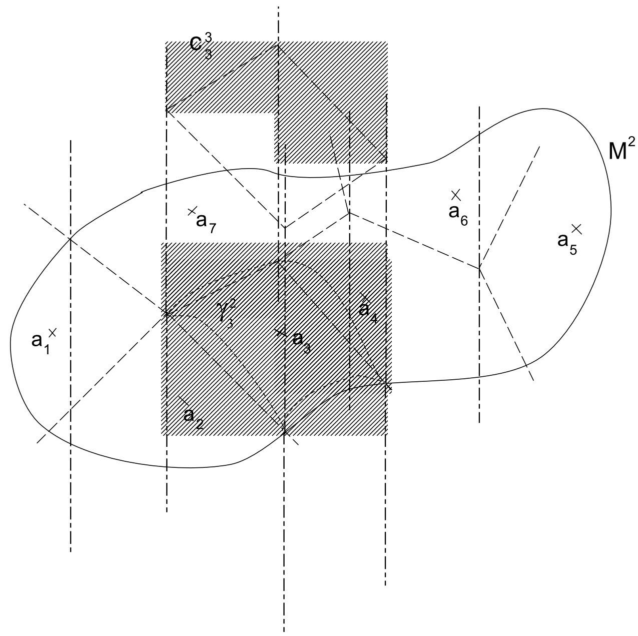

Fig. 6 Dirichlet (Voronoi) cells—the compact surface case

Remark 3.12 A result equivalent to Theorem 3.1 is attempted in [3], using basically the same method as Cairns' original one. However, the proof given in [3] is more technical and less fitted for generalization in higher dimensions than the original proof given in [13]. Moreover, the seminal papers of Cairns are not referenced therein.

Remark 3.13 Voronoi cell partitioning is also employed in "classical" sampling theory (see [40]).

3.4 Open Riemannian Manifolds

In adapting Cairns' method to the non-compact case, one has to allow for some (obviously-required) modifications. We proceed to present below the construction devised by Peltonen, which consists of two parts:
Part 1
Step A
Construct an exhaustive set $\{E_i\}$ of M^n, generated by the pair (U_i, η_i), where:

(1) U_i is the relatively compact set $E_i \setminus \bar{E}_{i-1}$ and
(2) η_i is a number that controls the fatness of the simplices of the triangulation of E_i, constructed in Part 2, such that it will not differ to much on adjacent simplices, i.e.:
   (i) The sequence $(\eta_i)_{i\geq 1}$ descends to 0;
   (ii) $2\eta_i \geq \eta_{i-1}$.

The geometric feature that controls the sets E_i, U_i and the numbers η_i is the *maximal connectivity radius*:

Definition 3.14 Let $U \subset M^n, U \neq \emptyset$, be a relatively compact set, and let $T = \bigcup_{x\in\bar{U}} \sigma(x,\omega_U)$. The number $\kappa_U =$

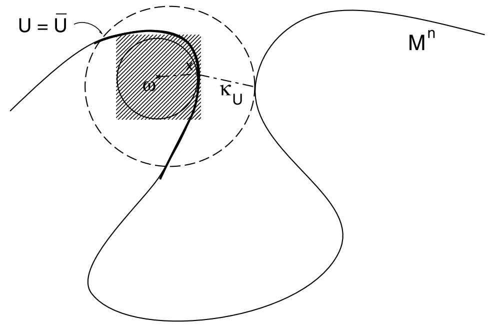

Fig. 7 Maximal connectivity radius at U

$\max\{r|\sigma^n(x,r)$ is connected for all $s \le \omega_U, x \in \bar{T}\}$, is called the *maximal connectivity radius* at U, (see Fig. 7):

The maximal connectivity radius and the maximal osculatory radius are interconnected by the following inequality:

Lemma 3.15

$$\omega_U \le \frac{\sqrt{3}}{3}\kappa_U. \tag{3.5}$$

Proof See Lemma 3.1, [33]. □

The numbers η_i are chosen such that they satisfy the following bounds:

$$\eta_i \le \frac{1}{4}\min_{i\ge 1}\{\omega_{\bar{U}_{i-1}}, \omega_{\bar{U}_i}, \omega_{\bar{U}_{i+1}}\}.$$

Step B

(1) Produce a maximal set A, $|A| \le \aleph_0$, s.t. $A \cap U_i$ satisfies:

(i) a density condition, namely:

$$d(a,b) \ge \eta_i/2, \quad \text{for all } i \ge 1;$$

and

(ii) a "gluing" condition for U_i, U_{i+1}, i.e. their intersection is large enough.

Note that according to the density condition (i), the following holds:

For any i and for any $x \in \bar{U}_i$, there exists $a \in A$ such that $d(x,a) \le \eta_i/2$.

(2) Prove that the Dirichlet complex $\{\bar{\gamma}_i\}$ defined by the sets A_i is a cell complex and every cell has a finite number of faces (so that it can be triangulated in a standard manner).

Part 2

Consider first the dual complex Γ, and prove that it is a Euclidean simplicial complex with a "good" (i.e. proper) density. Project then Γ on M^n (using the normal map). Finally, prove that the resulting complex $\widetilde{\Gamma}$ can be triangulated by fat simplices. Indeed, the fatness of any n-dimensional simplex $\gamma \in \widetilde{\Gamma}$, contained in the set U_i is given by the following bound:

$$\frac{r_\gamma}{R_\gamma} \ge \frac{1}{2^{5n+1}}\frac{(n+2)^{\frac{n+1}{2}}}{(n+1)^{n+1}}. \tag{3.6}$$

Remark 3.16 In the course of Peltonen's construction M^n is presumed to be isometrically embedded in some $\mathbb{R}^{N_1}$, where the existence of N_1 is guaranteed by Nash's Theorem (see [33, 41]).

3.5 Manifolds With Boundary of Low-Differentiability

The idea of the proof of Theorem 3.4 is to build first two fat triangulations: $\mathcal{T}_1$ of a product neighbourhood N of ∂M^n in M^n and $\mathcal{T}_2$ of $int\, M^n$ (its existence follows from Peltonen's result), and then to "mash" the two triangulations into a new triangulation $\mathcal{T}$, while retaining their fatness. While the mashing procedure of the two triangulations is basically the one developed in the original proof of Munkres' theorem, the triangulation of $\mathcal{T}_1$ has been modified, in order to ensure the fatness of the simplices of $\mathcal{T}_1$. More precisely we prove the following Theorem (see [36]):

Theorem 3.17 *Let M^n be a $\mathcal{C}^r$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then any fat $\mathcal{C}^r$-triangulation of ∂M^n can be extended to a $\mathcal{C}^r$-triangulation $\mathcal{T}$ of M^n, $1 \le r \le \infty$, the restriction of which to a product neighbourhood $\widetilde{K}_0 = \partial M^n \times I_0$ of ∂M^n in M^n is fat.*

In the general case we employ a method for fattening triangulations developed in [14]. The core of this methods resides in the following result:

Lemma 3.18 ([14], Lemma 6.3.) *Let $\mathcal{T}_1, \mathcal{T}_2$ be two fat triangulations of open sets $U_1, U_2 \subset \mathbb{R}^n$, $B_r(0) \subseteq U_1 \cap U_2$, having common fatness $\ge \varphi_0$ and such that $d_1 = \inf_{\sigma_1\in\mathcal{T}_1} diam\,\sigma_1 \le d_2 = \inf_{\sigma_2\in\mathcal{T}_2} diam\,\sigma_2$. Then there exist φ_0^*-fat triangulations $\mathcal{T}_1', \mathcal{T}_2'$, $\varphi_0^* = \varphi_0^*(\varphi_0)$, of open sets $V_1, V_2 \subseteq B_r(0)$, such that*

(1) $\mathcal{T}_i'\big|_{B_{r-8d_2}(0)} = \mathcal{T}_i\big|_{B_{r-8d_2}(0)}$, $i = 1, 2$;

(2) *$\mathcal{T}_1'$ and $\mathcal{T}_2'$ agree near their common boundary.*

Moreover:

(3) $\inf_{\sigma_1'\in\mathcal{T}_1'} diam\,\sigma_1' \le 3d_1/2$, $\inf_{\sigma_2'\in\mathcal{T}_2'} diam\,\sigma_2' \le d_2$.

Remark 3.19 A more elementary, geometric approach in two and three dimensions was developed in [35].

Remark 3.20 For the treatment of the same problem in the context of Computational Geometry, see e.g. [16, 31].

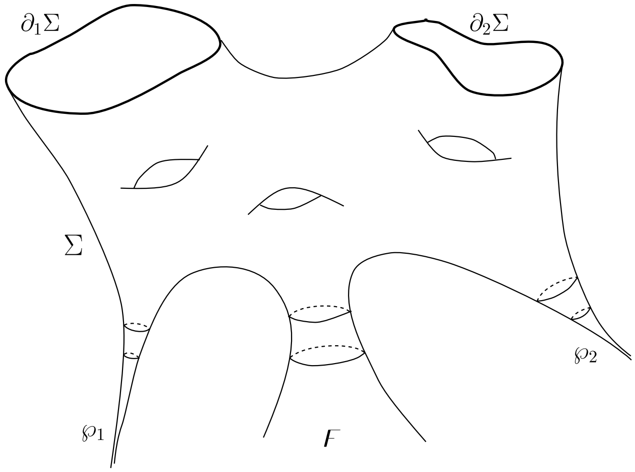

Fig. 8 A non-compact surface Σ, with two boundary components $\partial_1\Sigma$ and $\partial_2\Sigma$. Observe the cusps $\wp_1$ and $\wp_2$ and the funnel $\mathcal{F}$

Classical smoothing results are applied to derive Corollary 3.7 (see Sect. 2.2.3 and [42]).

4 Sampling Theorems

4.1 Surfaces

4.1.1 Smooth Surfaces

Theorem 4.1 *Let Σ be a connected, non-necessarily compact smooth surface (i.e. of class $\mathcal{C}^k, k \geq 2$), with finitely many boundary components. Then, there exists a sampling scheme of Σ, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}(\frac{1}{k(p)})$, where $k(p) = \max\{|k_1|, |k_2|\}$, and k_1, k_2 are the principal curvatures of Σ, at the point $p \in \Sigma$.*

Proof The existence of the sampling scheme follows immediately from Corollary 3.6, where the sampling points are the vertices of the triangulation. The fact that the density is a function solely of $k = \max\{|k_1|, |k_2|\}$ follows from the proof of Theorem 3.3 and from the fact that the osculatory radius $\omega_\gamma(p)$ at a point p of a curve γ equals $1/k_\gamma(p)$, where $k_\gamma(p)$ is the curvature of γ at p; hence that the maximal osculatory radius (of Σ) at p is: $\omega(p) = \max\{|k_1|, |k_2|\} = \max\{\frac{1}{\omega_1}, \frac{1}{\omega_2}\}$. (Here ω_1, ω_2 denote the minimal, respective maximal sectional osculatory radii at p.) □

Remark 4.2 Since for unbounded surfaces (see Fig. 8) it may well be that $\kappa \to \infty$, it follows that an infinite density of the sampling is possible. However, for practical implementations, where such cases are excluded, we have the following corollary:

Corollary 4.3 *Let $\Sigma, \mathcal{D}$ be as above. Assume that there exists $k_0 > 0$, such that $k_0 \geq k(p)$, where for all $p \in \Sigma$. Then there exists a sampling of Σ having uniformly bounded density.*

Proof The proof is deduced immediately from Theorem 4.1 above. □

Corollary 4.4 *In the following cases there exist k_0 as in Corollary 4.3 above*:

(1) *Σ is compact.*
(2) *There exist H_1, H_2, K_1, K_2, such that $H_1 \leq H(p) \leq H_2$ and $K_1 \leq K(p) \leq K_2$, for any $p \in \Sigma$, where H, K denote the mean, respective Gauss curvature*. (*That is both mean and Gauss curvatures are* pinched.)
(3) *The Willmore integrand $W(p) = H^2(p) - K(p)$ and K (or H) are pinched.*

Proof

(1) It follows immediately from a compactness argument and from the continuity of the principal curvature functions.
(2) Since $K = k_1k_2$, $H = \frac{1}{2}(k_1 + k_2)$, the bounds for K and H imply the desired one for k.
(3) Reasoning analogous to that of (ii), applies in the case of $W = \frac{1}{4}(k_1 - k_2)^2$.

This concludes the proof of the theorem. □

Remark 4.5 Condition (iii) on W is not only compact, it has the additional advantage that the Willmore energy $\int_\Sigma W dA$ (where dA represents the area element of Σ) is a conformal invariant of Σ. See [38] for its importance in quasiconformal mappings and their applications to imaging.

4.1.2 Non-Smooth Surfaces

We begin by proposing the following definition:

Definition 4.6 Let Σ^2 be a (connected) surface of class $\mathcal{C}^1$, and let Σ^2_δ be a smooth δ-approximation to Σ^2. A sampling of Σ^2_δ is called a *δ-sampling of* Σ^2.

Theorem 4.7 *Let Σ^2 be a connected, non-necessarily compact surface of class $\mathcal{C}^0$. Then, for any $\delta > 0$, there exists a δ-sampling of Σ^2, such that if $\Sigma^2_\delta \to \Sigma^2$ uniformly, and $\mathcal{D}_\delta \to \mathcal{D}$ in the sense of measures, where $\mathcal{D}_\delta$ denote the densities of Σ^2_δ and $\mathcal{D}$ is the density of the smoothing $\widetilde{\Sigma}^2$ of Σ^2.*

Proof The proof is an immediate consequence of Theorem 3.4 and its proof and the methods exposed in Sect. 2.4. We adopt the sampling of some smooth δ-approximation of Σ. □

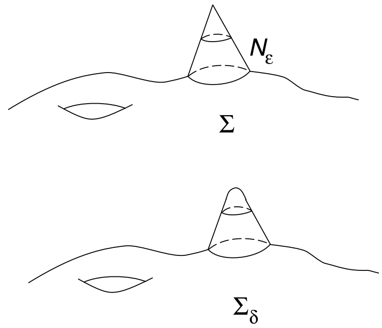

Fig. 9 A neighbourhood $\mathcal{N}_\varepsilon$ such that $\Sigma \equiv \Sigma_\delta$ outside $\mathcal{N}_\varepsilon$

Corollary 4.8 *Let Σ^2 be a $\mathcal{C}^0$ surface having only a finite number of points $\{p_1, \dots, p_k\}$ at which Σ^2 fails to be smooth. Then every δ-sampling $\mathcal{S}^2_\delta$ of a smooth δ-approximation $\mathcal{S}$ of Σ^2 is, in fact, a sampling of Σ^2, apart from ε-neighborhoods N_i of the points p_i, $i = 1, \dots, k$.*

Proof From Lemma 2.27 and Theorem 2.26 it follows that any such δ-approximation, Σ_δ, coincides with Σ outside of finitely many such small neighborhoods (see Fig. 9). □

Remark 4.9 Even in the case where $\Sigma^2_\delta \in \mathcal{C}^2$, and curvature measures exist for Σ^2 (e.g. if Σ^2 is a *PL*-surface), it does not follow that the curvature measures converge punctually to the curvatures of Σ^2 (see [8] and the discussion in Sect. 2.3.1). However, if Σ^2 is compact and with empty boundary, the desired convergence property holds ([8]).

Remark 4.10 We use the secant map as defined in Definition 2.17 in order to reproduce a PL-surface as a δ-approximation for the sampled surface. As said in the beginning of Sect. 2.3 we may now use smoothing in order to obtain a $\mathcal{C}^\infty$ approximation.

4.2 Higher Dimensional Manifolds

Theorem 4.1 and Corollary 4.3 have straightforward generalizations to any dimension:

Theorem 4.11 *Let $\Sigma^n \subset \mathbb{R}^{n+1}, n \geq 2$ be a connected, not necessarily compact, smooth hypersurface, with finitely many compact boundary components. Then there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}(\frac{1}{k(p)})$, where $k(p) = \max\{|k_1|, \dots, |k_n|\}$, and where $k_1, \dots, k_n$ are the principal curvatures of Σ^n, at the point $p \in \Sigma^n$.*

Corollary 4.12 *Let $\Sigma^n, \mathcal{D}$ be as above. If there exists $k_0 > 0$, such that $k(p) \leq k_0$, for all $p \in \Sigma^n$, then there exists a sampling of Σ^n of finite density everywhere.*

Some of the conclusions of Corollary 4.4 also generalize. In particular we have:

Corollary 4.13 *If Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density.*

Remark 4.14 Obviously, Theorem 4.11 above is of little relevance for the *space forms* $(\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n)$. Indeed, as noted above, this method is relevant for manifolds considered (by the Nash embedding theorem [30]) as submanifolds of $\mathbb{R}^N$, for some N large enough.

However, more geometric conditions, such as those given in Corollary 4.4 are hard to impose in higher dimension, hence the study of such precise geometric constraints is left for further study.

The definition of δ-samplings and Theorem 4.7 and its corollary also admit immediate generalizations:

Definition 4.15 Let $\Sigma^n, n \geq 2$ be a (connected) manifold of class $\mathcal{C}^1$, and let Σ^n_δ be a smooth δ-approximation to Σ^n. A sampling of Σ^n_δ is called a *δ-sampling of Σ^n*.

Theorem 4.16 *Let Σ^n be a connected, non-necessarily compact manifold of class $\mathcal{C}^1$. Then, for any $\delta > 0$, there exists a δ-sampling of Σ^n, such that if $\Sigma^n_\delta \to \Sigma^n$ uniformly, and $\mathcal{D}_\delta \to \mathcal{D}$ in the sense of measures, where $\mathcal{D}_\delta$ denote the densities of Σ^n_δ and $\mathcal{D}$ is the density of a smoothing $\widetilde{\Sigma}^n$ of Σ^n.*

Corollary 4.17 *Let Σ^n be a $\mathcal{C}^0$ manifold having only a finite number of points $\{p_1, \dots, p_k\}$ at which Σ^n fails to be smooth. Then every δ-sampling $\mathcal{S}^n_\delta$ of a smooth δ-approximation $\mathcal{S}^n$ of Σ is, in fact, a sampling of Σ^n, apart from ε-neighborhoods N_i of the points p_i, $i = 1, \dots, k$.*

Remark 4.18 For image processing and computer graphics purposes it would be ideal if one could make avail of smoothing theorems for topological manifolds, and not just for those of class $\mathcal{C}^1$. Unfortunately, such results do not hold, in general, for manifolds of any dimension (see [28]). However, in low dimensions, the smoothness condition can be discarded. Indeed, every *PL* manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) smoothing (see [28, 42]), and every topological manifold of dimension $n \leq 3$ admits a *PL* structure (cf. [27, 42]). (We have used of some of these facts in formulating our sampling theorem for non-smooth surfaces.)

Remark 4.19 In order to obtain a better approximation it is advantageous, in this case, to employ Nash's method for smoothing, cf. Remark 2.28 (see [4, 30] for details).

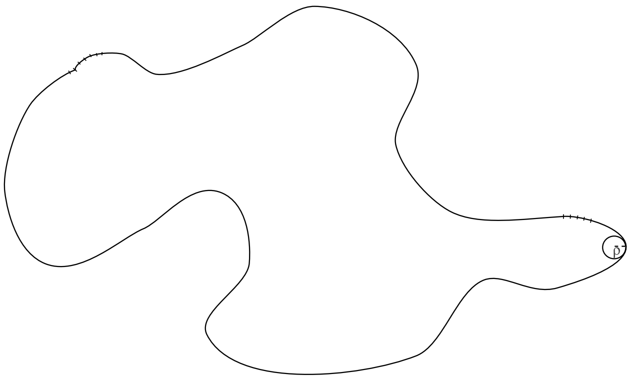

Fig. 10 Sampling of a $\mathcal{C}^2$ curve: the sampling rate is $\rho = 1/r$, where r is the minimal radius of curvature

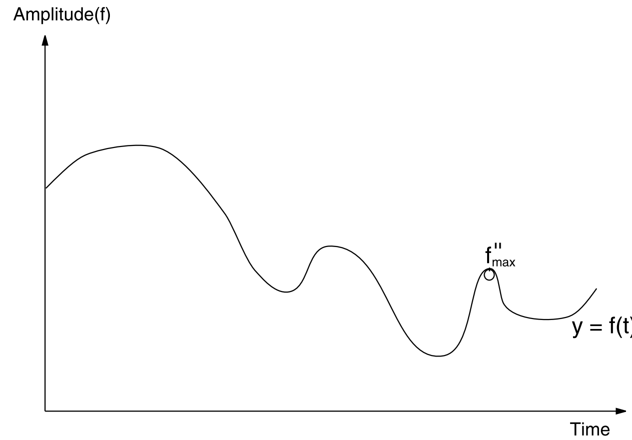

Fig. 11 A band-limited signal $y = f(t)$

5 Applications to Classical Sampling Theory

5.1 1-Dimension: The Classical Shannon Sampling Theorem

Our approach and formalism lend themselves to the derivation of a geometric sampling theorem for 1-dimensional signals. Indeed, one can think of the maximal absolute value of the second derivative as a sampling rate criterion. We show that band-limited signals considered in the context of the classical Shannon-Whittaker theorem require, indeed, a finite sampling. We first consider only smooth "intuitive" or "blackboard" signals, i.e. functions $S \in L^2$ such that their graphs are smooth ($\mathcal{C}^2$) planar curves (see also discussion in Sect. 7.1 below).

Definition 5.1 Let $S = S(t)$ be a $\mathcal{C}^2$ planar curve parameterized by arc-length and let $k(S)$ denote its maximal absolute curvature. We will call $\rho(S) = k(S)/2$ the *sampling rate* of S (see Fig. 10)

We begin by giving the one-dimensional version of Theorem 4.11:

Theorem 5.2 *Let S is a $\mathcal{C}^2$ planar curve parameterized by arc-length. Then it can be sampled in sampling rate $\eta(S)$ namely, the arc-length distance between each consecutive samples is $\leq 1/\eta(S)$. If S satisfies the condition that $\rho(S)$ is bounded, then the required sampling rate is finite.*

Corollary 5.3 *If $S(t)$ is a band-limited "blackboard" signal, then it necessitates a finite sampling rate (in any finite time interval) according to $\eta(S)$.*

Proof By Theorem 5.2 above, the proof amounts to showing that the second derivative of band-limited "blackboard" signals (see Fig. 11) is everywhere bounded.

A Taylor expansion of such signals is given for instance in [25]. In particular, for a band-limited signal $S(t)$, we have by Shannon-Whittaker:

$$S(t) = \sum_{-\infty}^{\infty} S(t_n)\operatorname{sinc}(2W(t-t_n)),$$

and it is shown that its p-th derivative is given by:

$$S^p(t) = (2W)^p \sum_{-\infty}^{\infty} S(t_n)\left(\frac{d}{dt}\right)^p \operatorname{sinc}(2W(t-t_n)).$$

By Marks and Hall ([25]) we have that:

$$\begin{aligned}\left(\frac{d}{dt}\right)^p \operatorname{sinc}(t) &= \int_{-1/2}^{1/2} (2\pi i f)^p e^{2\pi i f}\, df \\ &= \frac{(-1)^p p!}{\pi t^{p+1}}[\sin(\pi t)\cos_{p/2}(\pi t) \\ &\quad - \cos(\pi t)\sin_{(p-1)/2}(\pi t)],\end{aligned}$$

where:

$$\cos_r(t) = \sum_{n=0}^{[r]} \frac{(-1)^n t^{2n}}{(2n)!};$$

$$\sin_r(t) = \sum_{n=0}^{[r]} \frac{(-1)^n t^{2n+1}}{(2n+1)!}.$$

The above terms have the following asymptotic behavior from which the boundedness of the second derivative (even for very large values of t), is evident.

$$\left(\frac{d}{dt}\right)^p \operatorname{sinc}(t) \to \begin{cases} (-1)^{p/2}\pi^p(\operatorname{sinc}(t)); & p \text{ even} \\ (-1)^{(p-1)/2}\pi^p(\frac{\cos(\pi t)}{\pi t}); & p \text{ odd}. \end{cases}$$

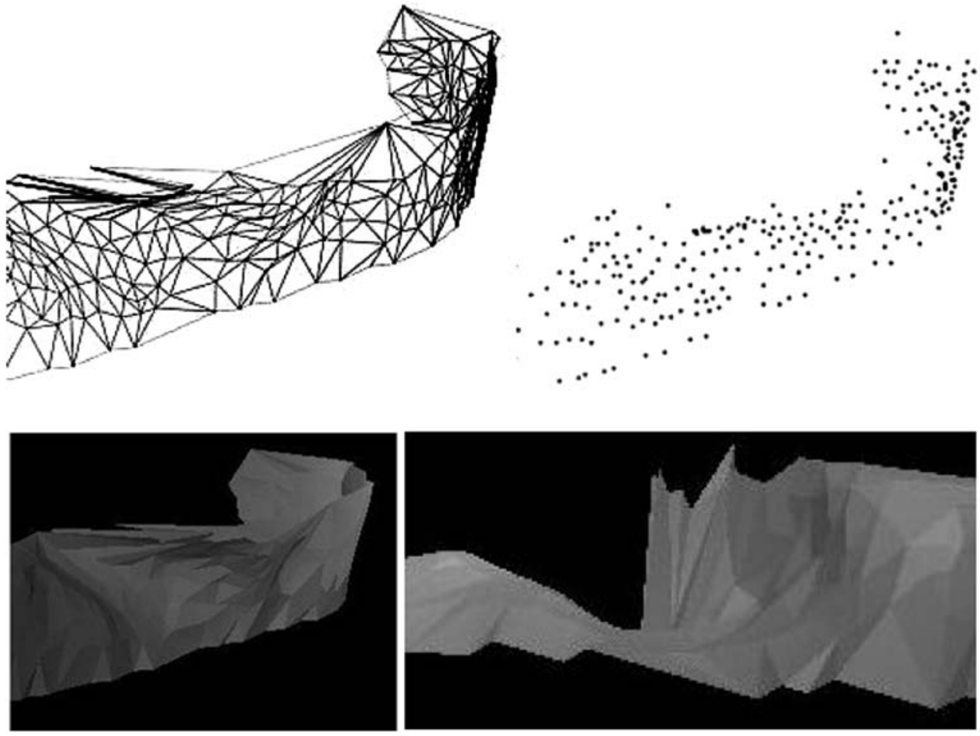

Fig. 12 The triangulation (*upper image, left*) obtained from a "naive" sampling (*upper image, right*) resulting from a CT scan of part of the back-side of the human colon (*bottom left*). Note the "flat" triangles and the uneven mesh of the triangulation. This is a result of the high, concentrated curvature, as revealed in a view obtained after a rotation of the image (*bottom right*). These and other images will be accessible through an interactive applet on the website [46]. CT-data is in courtesy of Dr. Doron Fisher from Rambam Madical Center in Haifa

From the presentation above we conclude that a band-limited signal possesses a "geometric" sampling of finite rate. □

It is important to point out that a similar weaker result was recently proved by G. Meenakshisundaram ([26]). Also, yet another theorem similar to Theorem 5.2 appeared in [34].

Remark 5.4 An approximation approach was already employed for "classical" sampling theory—see [44].

5.2 2-Dimensions: Images

Perhaps the most direct application of the sampling theorem for surfaces is to the field of images, via "inpainting" (see, e.g. [40], p. 280). In this approach, images are viewed as parametrized surfaces $S = (u, v, f(u, v))$, where $(u, v) \in R$ – a rectangle of pixels, and $f(u, v) \in [0, 1]$ represents the shade of grey associated to the pixel (u, v).

Of course, if more attributes of the image are added, such as colors, luminosity, etc., then a higher dimensional manifold is obtained, and we may make again a recourse to the fitting sampling theorem.

In a completely analogous manner one can approach the problem of image compression (see, e.g. [40], p. 280): here the samples represent the coarse pixel set and the surface the fine pixel set.

6 Some Computational Results

In this section we present quantitative estimates, obtained on some analytic surfaces (see Figs. 13, 14, 15), of the error caused by two reproducing schemes employed in this work, namely piecewise-linear reconstruction (see Table 1) and Nyquist (trigonometric approximation) reconstruction. Error assessments was estimated in three versions, yielding similar results:

(1) In the first version, four points where chosen for each of the triangles: three points on the mid-edges and one point at the triangle's barycenter. The error was computed at these points and the maximum over all these error values was taken.
(2) A larger number of points where chosen for each triangle but the number of triangles was reduced. This was done by considering only triangles at which maximal curvature was obtained, where the curvature is assessed by the normal deviation at the vertices.
(3) The control points where uniformly spread along the sampling domain. For the Nyquist approximation only this method was applied.

The error term was computed using L_1 norm difference between the reconstructed surface and its analytic expression.

As observed in the table, the approximation yielded by the secant map (PL-reconstruction) is better than the one obtained by Nyquist reconstruction, giving in general, an error which is 10 times lesser than the Nyquist reconstruction.

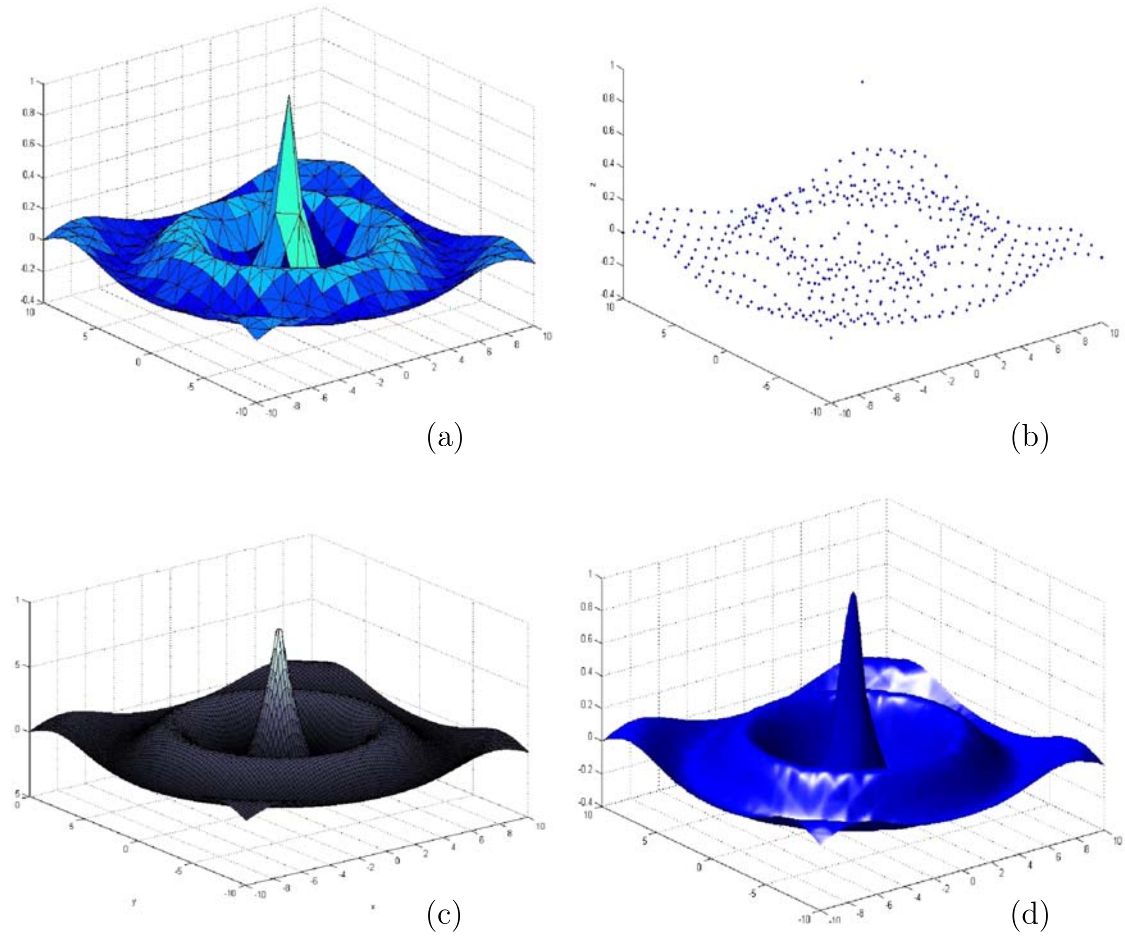

Fig. 13 The triangulation (**a**) obtained from the uniform sampling (**b**) of the surface $S = \left(x, y, \frac{\cos\sqrt{x^2+y^2}}{1+\sqrt{x^2+y^2}}\right)$ (**c**) and smoothing of the triangulation (**d**). Note the low density of sampling points in the region of high curvature

Table 1 Error estimates for the secant and Nyquist reconstructions of various surfaces. The error for the secant approximation is in general 10 times less than for Nyquist approximation

Surface	Secant Approx. 4-points	Secant Max. Curvature	Secant Uniform	Nyquist Uniform
Hyperbolic Paraboloid	$5.0397e^{-4}$	$2.2894e^{-4}$	$4.4877e^{-4}$	0.0071
Monkey Saddle	0.0012	$4.4895e^{-4}$	$9.1302e^{-4}$	0.0071
Sphere	0.0067	0.0045	0.0060	0.0065

One should compare the results above with those obtained using a "naive" sampling (see Fig. 12).

7 Discussion

7.1 Sampling

Most important, one honestly has to ask himself the following question: "What is a signal?"

If the answer to the question above is given in the classical context, i.e. if a signal is viewed as an element f of $L^2(\mathbb{R})$, such that supp $(\hat{f}) \subseteq [-\pi, \pi]$, where $\hat{f}$ denotes the Fourier transform of f, then our result does not hold. Indeed, we have the following counterexample:

Counterexample 7.1 *There exist band limited signals* (*as above*) f *such that*:

(i) $f \in L^2(\mathbb{R})$, $f'' \in L^\infty(\mathbb{R})$;
but
(ii) f'' *is not bounded*.

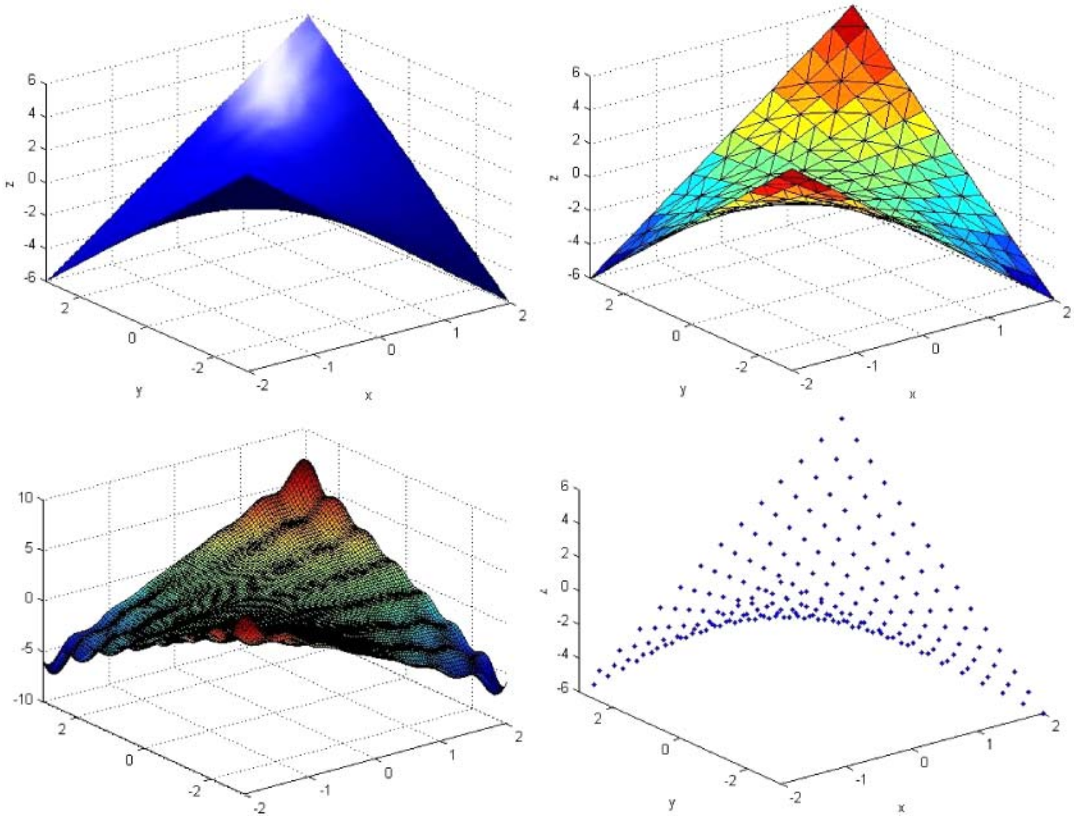

Fig. 14 Hyperbolic Paraboloid: *Top left*—analytic representation, $z = xy$. *Bottom right*—sampling according to curvature. *Top left*—PL reconstruction. *Bottom left*—Nyquist reconstruction. To appreciate the triangulation results requires a full size display of color images [46]

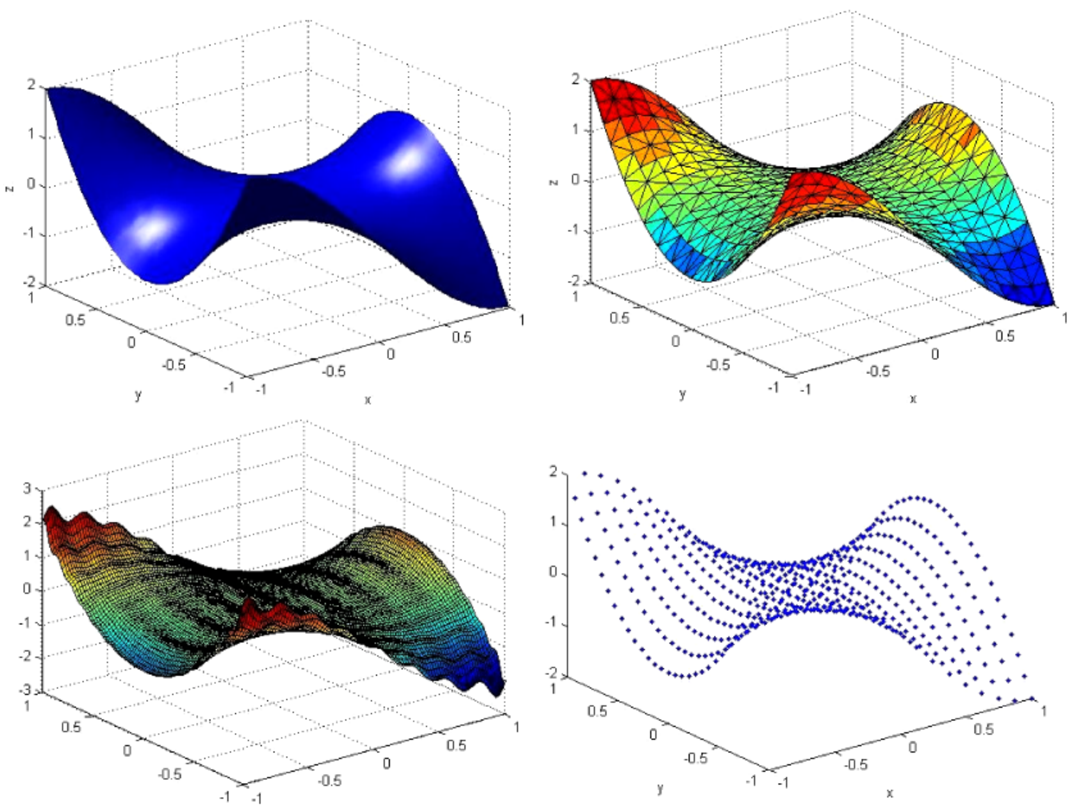

Fig. 15 Monkey Saddle: *Top left*—analytic representation, $z = x(x^2 - 3y^2)$. *Bottom right*—sampling according to curvature. *Top left*—PL reconstruction. *Bottom left*—Nyquist reconstruction. To appreciate the triangulation results requires a full size display of color images [46]

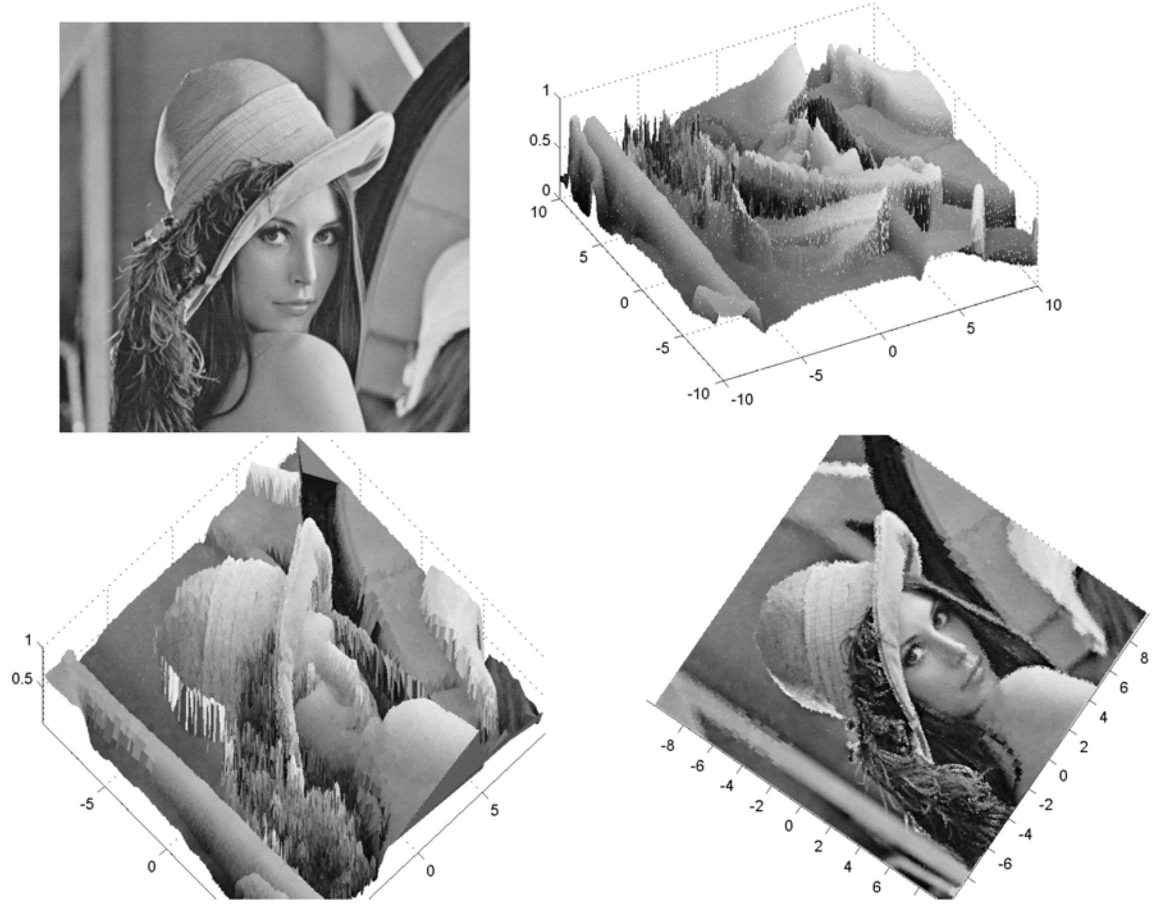

Fig. 16 The surface in $\mathbb{R}^3$ (*upper image, right*) corresponding to "Lena" (*upper image, left*). The *lower image*, on the *left*, shows the *PL* surface obtained using the sampling and geometric methods introduced in this paper. A rotated view (*lower image, right*) of this *PL* reconstruction shows how close even the linear interpolation based upon geometric sampling is to the original image

Therefore, our approach refers to a more "intuitive" or "blackboard" interpretation of signals. On the other hand, it is more broad, in the sense that it applies to any at most countable union of piece-wise smooth (not necessarily planar) curves, not only for graphs of function. (For a possible approach to defining curvature at points were a curves fails to be twice differentiable, see Sect. 7.3.)

7.2 Images as Manifolds

While viewing images as manifolds embedded in higher dimensional Riemannian manifolds (in particular in some Euclidean space) (see, e.g. [20, 23, 39, 40]) the common approach to the problem tends to ignore the intrinsic difficulties of the embedding process. In particular, when considering *isometric* embeddings, one is constrained by the necessary high-dimension of the embedding space (see [4, 18, 30]). This is even more poignant when one wishes to view images as 2-dimensional smooth surfaces isometrically embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$), as in [9, 10]. Indeed, for such surfaces the Nash-Kuiper-Gromov-Günther algorithm gives embedding dimension 10 for a generic compact surface (see, e.g. [4, 17, 19]).

However, for *gray scale images*, i.e. for surfaces $S = (x, y, g(x, y)) \subset \mathbb{R}^3$, where the function g represents the gray level (luminosity) of the image, one can apply easily the sampling and reconstruction results proved in Sect. 4. For some first results in this direction, see Fig. 16 above.

7.3 Simplex Fatness and Future Study

Since the fatness of the triangulation of $\operatorname{int} M^n$ depends, by formula (3.6), only on the dimension n of the given manifold, and since by Lemma 3.18, the fatness of the *mash* (i.e. common simplicial subdivision) of the triangulations of ∂M^n and $\operatorname{int} M^n$ is a function solely on the fatness of the given triangulation (and hence upon the dimension n), it follows that a lower bound for the fatness of any triangulations is achieved.

However, since the bound given by formula (3.6) is achieved via the specific construction of [33], the following question arises naturally: Is the lower bound of formula (3.6) the lowest possible?

The answer to the question above seems to be negative, since Peltonen's construction depends upon the specific isometric embedding employed.

More important, the diameters of the simplices obtained in our construction (i.e. the *mesh* of the triangulation) are a function of the curvature radii, hence an *extrinsic* constraint, therefore again strongly dependent upon the specific embedding in higher dimensional Euclidean space. This fact immediately generates the following problem: What are the precise restrictions the Nash embedding technique imposes upon the curvature radii? The existence of such restrictions follows from the fact that, in the Nash embedding method, the curvature of the embedding is controlled. Moreover, in the smoothing part of the Nash technique, a *star finite* partition of the embedding, obtained using curvature radii of an intermediate embedding, is considered (see [4, 30]). (For further problems related to the quality of the obtained triangulation and its relevance to the theory of quasiregular mappings, see [37].)

We conclude with the following remarks and suggestions for further study:

We have obtained in Corollary 4.4 week intrinsic condition for the existence of fat triangulation with mesh bounded from below. As already noted, one would like to find such non-extrinsic (i.e. curvature restricting) conditions (perhaps coupled with fitting topological constraints) in higher dimension, as well. Indeed, in dimensions greater or equal three, even deciding which curvature (sectional, Ricci, scalar) is most relevant, represents a highly non-trivial problem, that we defer for further study.

Another direction of study stems from the need, both in the classical signal-processing context and in that of manifold sampling, for mashing and sampling methods of geometrical objects that are not even *PL*, and hence no smoothing techniques can be applied for them. In this general setting, *metric curvatures*, represent, in our view, the most promising tool. Indeed, research in this direction is currently undertaken.

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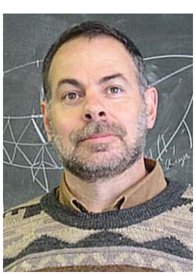

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