# STAT 509: Statistics for Engineers

## Chapter 8: Statistical Intervals for a Single Sample

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Learning Objectives:

1. Construct confidence intervals on the mean of a normal distribution, using either the normal distribution or the $t$ distribution method
2. Construct confidence intervals on the variance and standard deviation of a normal distribution
3. Construct confidence intervals on a population proportion
4. Construct a prediction interval for a future observation

# An interval estimator

Estimating an unknown parameter $\theta$ by a point estimator $\widehat{\Theta}_n$ is useful. However, it is like shooting a bird with a pistol. Often $\widehat{\Theta}_n$ has a continuous distribution, and if so, from Chapter 4, $P(\widehat{\Theta}_n = \theta) = 0$; i.e., we never capture the true parameter by using a point estimator even though it is an unbiased estimator.

Why not shoot a bird using a shotgun or capture it using a net? Translating to statistical language, why not use an interval to capture the true parameter? This motivates the consideration of interval estimators.

We estimate $\theta$ by an interval $[L_n, U_n]$, where $L_n$ and $U_n$ are two statistics computed from a random sample of size $n$ such that

$$P[L_n \leq \theta \leq U_n] = 1 - \alpha,$$

for some pre-specified $\alpha \in (0, 1)$. We call $[L_n, U_n]$ a $100(1-\alpha)\%$ **confidence interval estimator** of $\theta$ and $100(1-\alpha)\%$ the **confidence level** of this interval estimator.

# Confidence Interval on the Mean of a Normal Distribution, Variance Known

A confidence interval estimator is often built from a point estimator and the sampling distribution of the point estimator.

We consider building a confidence interval estimator of $\mu$ in the normal distribution $N(\mu, \sigma^2)$ where $\sigma^2$ is known (based on historical data).

From Chapter 7, a good point estimator of $\mu$ is $\bar{X}_n$, and the sampling distribution of $\bar{X}_n$ is $N(\mu, \sigma^2/n)$. Thus

$$Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

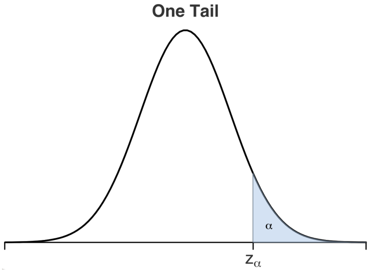
# Confidence Interval on the Mean of a Normal Distribution, Variance Known

For $a \in (0,1)$, define $z_a$ to be a quantile value of $Z \sim N(0,1)$ such that

$$P(Z > z_a) = a$$

Using StatEngine:

$$z_a = normal.quantile(0, 1, 1 - a).$$




Often used $z_a$ values:

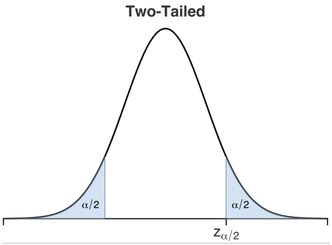
$$z_{0.005} = 2.5758$$
$$z_{0.025} = 1.96$$
$$z_{0.05} = 1.6449$$
$$z_{0.1} = 1.2816.$$

# Confidence Interval on the Mean of a Normal Distribution, Variance Known

For any $\alpha < 0.5$, we know that

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha.$$

Thus




$$\begin{aligned}
1 - \alpha &= P\left(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}\right) \\
&= P\left(-z_{\alpha/2} \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) \\
&= P\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n - \mu \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)
\end{aligned}$$

# Confidence Interval on the Mean of a Normal distribution, variance known

## Two-sided confidence interval

If $\bar{X}_n$ is the sample mean of size $n$ from a **normal** population with **known variance** $\sigma^2$, a $100(1-\alpha)\%$ (two-sided) confidence interval estimator on $\mu$ is given by

$$\left[L_n = \bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, U_n = \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$

where $z_{\alpha/2} = normal.quantile(0, 1, 1-\alpha/2)$.

Based on an observed sample $x_1, \ldots, x_n$, a $100(1-\alpha)\%$ (two-sided) confidence interval estimate on $\mu$ is

$$\left[\bar{x}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

Its length is $2 \times z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$, which becomes smaller if either $n$ is larger (more data) or $\alpha$ is larger (less confidence level).

## One-sided confidence bounds for the Mean of a Normal distribution, variance known

Similarly, we have $1 - \alpha = P(Z \leq z_\alpha)$ and $1 - \alpha = P(-z_\alpha \leq Z)$. Plugging $Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$, we obtain

$$1 - \alpha = P\left(\mu \leq \bar{X}_n + z_\alpha \frac{\sigma}{\sqrt{n}}\right) = P\left(\bar{X}_n - z_\alpha \frac{\sigma}{\sqrt{n}} \leq \mu\right)$$

### One-sided confidence bound

A $100(1-\alpha)\%$ **upper-confidence bound** for $\mu$ is

$$\bar{x}_n + z_\alpha \frac{\sigma}{\sqrt{n}}$$

and a $100(1-\alpha)\%$ **lower-confidence bound** for $\mu$ is

$$\bar{x}_n - z_\alpha \frac{\sigma}{\sqrt{n}}.$$

# Example

ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at $60^\circ C$ are as follows: $64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, 64.3$. Assume that impact energy is normally distributed with $\sigma = 1$J. We want to find 95% and 99% CIs for $\mu$, the mean impact energy.

```
x=c(64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, 64.3)
Zinterval(level=0.95,sigma=1,sample=x)
Zinterval(level=0.99,sigma=1,sample=x)
```

A 95% CI for $\mu$ is $[63.8402, 65.0798]$
A 99% CI for $\mu$ is $[63.6455, 65.2746]$
When confidence level increases (or $\alpha$ decreases), confidence interval becomes wider! (A larger net captures a bird more easily.)

# Interpreting a confidence interval

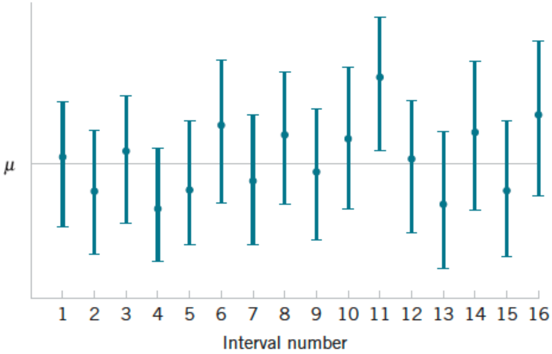
How does one interpret a confidence interval? In the previous example, a 95% CI is $63.8402 \leq \mu \leq 65.0798$, so it is tempting to conclude that $\mu$ is within this interval with probability $0.95$.

However, with a little reflection, it is easy to see that this **cannot be correct**; the true value of $\mu$ is unknown, and the statement $63.8402 \leq \mu \leq 65.0798$ is either correct (true with probability 1) or incorrect (false with probability 1).

The correct interpretation lies in the realization that a CI is a random interval because in the probability statement defining the end-points of the interval, both $L_n$ and $U_n$ are random variables. Consequently, the correct interpretation of a $100(1-\alpha)\%$ CI depends on **the relative frequency view of probability**. Specifically, if an infinite number of random samples are collected and a $100(1-\alpha)\%$ confidence interval for $\mu$ is computed from each sample, $100(1-\alpha)\%$ of these intervals will contain the true value of $\mu$.

# Interpreting a confidence interval

The situation is illustrated in the following figure, which shows several $100(1-\alpha)\%$ confidence intervals for the mean $\mu$ of a normal distribution. The dots at the center of the intervals indicate the point estimate of $\mu$ (that is, $\bar{x}_n$). Notice that one of the intervals fails to contain the true value of $\mu$. If this were a 95% confidence interval, in the long run only 5% of the intervals would fail to contain $\mu$.



# Choice of sample size

A $100(1-\alpha)\%$ CI takes the form of

$$\left[\bar{x}_n - \underbrace{z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{\text{margin of error}}, \bar{x}_n + \underbrace{z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{\text{margin of error}}\right].$$

When $\alpha$ or the confidence level is fixed, the margin of error becomes smaller if $n$ increases. It means if we have more data, the $100(1-\alpha)\%$ CI has more precision of estimation. Suppose we specify the margin of error to be $E$, what is the smallest amount sample to collect?

$$E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \implies n = \left\lceil \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \right\rceil,$$

where $\lceil a \rceil$ means the ceiling of $a$.

# Choice of sample size

Consider the previous example and suppose that we want to determine how many specimens must be tested to ensure that the 95% CI on $\mu$ for A238 steel cut at $60^\circ C$ has a length of at most 1.0 J.

**Solution:** The length is at most 1J, meaning the margin of error $E$ is at most 0.5J. Thus ($\alpha = 0.05$, $\sigma = 1$)

$$n = \left\lceil \left( \frac{z_{\alpha/2}\sigma}{E} \right)^2 \right\rceil = \left\lceil \left( \frac{z_{0.025} \times 1}{0.5} \right)^2 \right\rceil = 16.$$

```
sample.size.Zinterval(level=0.95,sigma=1,E=0.5)
```

CI for $\mu$ when $\sigma$ is **known** and the distribution is **normal**:

```
Zinterval(level=?,sigma=?,sample=?)
Zinterval(level=?,sigma=?,n=?,barx=?)
sample.size.Zinterval(level=?,sigma=?,E=?)
```

What if we do not know the variance $\sigma^2$?
What if it is not normal?

# Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$. We know that

$$Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

When $\sigma$ is unknown, we replace $\sigma$ by its estimator $S_n$ and obtain

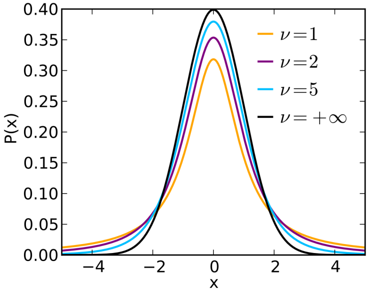
$$T_{n-1} = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t(n-1)$$

where $t(n-1)$ stands for the **student $t$ distribution** with degree of freedom $n-1$.

# Student $t(\nu)$ distribution

It is a continuous distribution with one parameter $\nu$, the degree of freedom. Its pdf is

$$f(x) = \frac{\Gamma(\{\nu+1\}/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)} \frac{1}{[(x^2/\nu)+1]^{(\nu+1)/2}}, -\infty < x < \infty.$$




- Similar to normal distributions: bell shape, symmetric with respect to 0
- Heavier tails than $N(0,1)$ when $\nu$ is small
- When $v \to \infty$, $t(\nu)$ converges to $N(0,1)$

# Confidence Interval on the Mean of a Normal Distribution, Variance unknown

For $a \in (0,1)$, define $t_{n-1,a}$ to be a quantile value of $t(n-1)$ such that

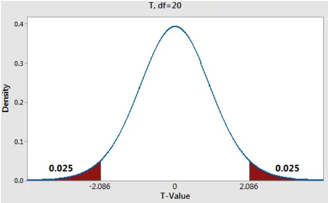
$$P(T_{n-1} > t_{n-1,a}) = a$$

Using StatEngine:

$$t_{n-1,a} = t.quantile(df = n-1, 1-a).$$

For any $\alpha < 0.5$, we know that

$$P(-t_{n-1,\alpha/2} \le T_{n-1} \le t_{n-1,\alpha/2}) = 1-\alpha$$




$$= P\left(-t_{n-1,\alpha/2} \le \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \le t_{n-1,\alpha/2}\right)$$
$$= P\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}} \le \mu \le \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}\right)$$

# Confidence Interval on the Mean of a Normal distribution, variance unknown

## Two-sided confidence interval

If $\bar{X}_n$ is the sample mean of size $n$ from a **normal** population with **unknown variance** $\sigma^2$, a $100(1-\alpha)\%$ (two-sided) confidence interval estimator on $\mu$ is given by

$$\left[L_n = \bar{X}_n - t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}, U_n = \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}\right]$$

where $t_{n-1,\alpha/2} = t.quantile(df = n-1, 1-\alpha/2)$.

Based on an observed sample $x_1, \ldots, x_n$, a $100(1-\alpha)\%$ (two-sided) confidence interval estimate on $\mu$ is

$$\left[\bar{x}_n - t_{n-1,\alpha/2}\frac{s_n}{\sqrt{n}}, \bar{x}_n + t_{n-1,\alpha/2}\frac{s_n}{\sqrt{n}}\right].$$

Its length is $2 \times t_{n-1,\alpha/2}\frac{s_n}{\sqrt{n}}$, which becomes smaller if either $n$ is larger (more data) or $\alpha$ is larger (less confidence level).

# One-sided confidence bounds for the Mean of a Normal distribution, variance unknown

## One-sided confidence bound

A $100(1-\alpha)\%$ **upper-confidence bound** for $\mu$ is

$$\bar{x}_n + t_{n-1,\alpha}\frac{s_n}{\sqrt{n}}$$

and a $100(1-\alpha)\%$ **lower-confidence bound** for $\mu$ is

$$\bar{x}_n - t_{n-1,\alpha}\frac{s_n}{\sqrt{n}}.$$

StatEngine:

```
Tinterval(level=?,sample=?)
Tinterval(level=?,n=?,barx=?,s=?)
```

Remark: T-intervals are quite robust to the normality assumption when $n$ is small. Thus, in practice, even if we do not have normality, one can still use T-inteverals.

# Large-Sample Confidence Interval on the Mean of a population

## Two-sided confidence interval

If $\bar{X}_n$ is the sample mean of size $n$ from a population with mean $\mu$ and a finite variance $\sigma^2$. When $n$ is large ($n \geq 25$), the CLT (plus the Slutsky theorem) tells

$$\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim AN(0, 1).$$

Thus, a large-sample confidence interval estimator for $\mu$ with confidence level of **approximately** $100(1-\alpha)\%$ is given by

$$\left[L_n = \bar{X}_n - z_{\alpha/2}\frac{s_n}{\sqrt{n}}, U_n = \bar{X}_n + z_{\alpha/2}\frac{s_n}{\sqrt{n}}\right].$$

# One-sided confidence bounds for the Mean of a Normal distribution, variance known

Based on an observed sample $x_1, \ldots, x_n$, a large-sample confidence interval estimator for $\mu$ with confidence level of approximately $100(1-\alpha)\%$ is given by

$$\left[\bar{x}_n - z_{\alpha/2}\frac{s_n}{\sqrt{n}}, \bar{x}_n + z_{\alpha/2}\frac{s_n}{\sqrt{n}}\right].$$

## One-sided large-sample confidence bound

A $100(1-\alpha)\%$ **large-sample upper-confidence bound** for $\mu$ is

$$\bar{x}_n + z_\alpha \frac{\sigma}{\sqrt{n}}$$

and a $100(1-\alpha)\%$ **large-sample lower-confidence bound** for $\mu$ is

$$\bar{x}_n - z_\alpha \frac{\sigma}{\sqrt{n}}.$$

# StatEngine summary of CIs on the population mean

CI for $\mu$ when $\sigma$ is **known** and the distribution is **normal**:

```
Zinterval(level=?,sigma=?,sample=?)
Zinterval(level=?,sigma=?,n=?,barx=?)
sample.size.Zinterval(level=?,sigma=?,E=?)
```

CI for $\mu$ when $\sigma$ is **unknown** and the distribution is **normal** (or for any distribution, but in this case, it provides approximated CIs):

```
Tinterval(level=?,sample=?)
Tinterval(level=?,n=?,barx=?,s=?)
```

Large-sample CI ($n \geq 25$) for $\mu$ under any distribution:

```
AZinterval(level=?,sample=?)
AZinterval(level=?,n=?,barx=?,s=?)
```

Both T-interval and AZ-interval are approximated CIs when normality does not hold. The T-intervals are more conservative (wider).

# Practice 1

An article in the Journal of Materials Engineering ["Instrumented Tensile Adhesion Tests on Plasma Sprayed Thermal Barrier Coatings" (1989, Vol. 11(4), pp. 275–282)] describes the results of tensile adhesion tests on 22 U-700 alloy specimens. The load at specimen failure is as follows (in megapascals):

```
x=c(19.8, 10.1, 14.9, 7.5, 15.4, 15.4, 15.4, 18.5, 7.9, 12.7, 11.9, 11.4, 11.4, 14.1,
17.6, 16.7, 15.8, 19.5, 8.8, 13.6, 11.9, 11.4)
```

Find a 95% CI on $\mu$, the population mean load at specimen failure.

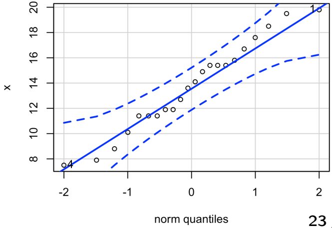
**Solution:** The sample size is $n = 22 < 25$, large-sample inference might not work. We do not know the population variance $\sigma$ nor the type of the population distribution (did not say normal). But we can check normality first using the QQ plot.

```
data.summary(x)
```

It appears that the sample follows a normal distribution. Thus use the T-interval.

```
Tinterval(level=0.95,sample=x)
```

Conclusion: based on the data, we are 95% confident that the population mean load at specimen failure falls between 12.1381 and 15.2892.



# Practice 2

An article in the 1993 volume of the Transactions of the American Fisheries Society reports the results of a study to investigate the mercury contamination in large mouth bass. A sample of fish was selected from 53 Florida lakes, and mercury concentration in the muscle tissue was measured (ppm). The mercury concentration values were

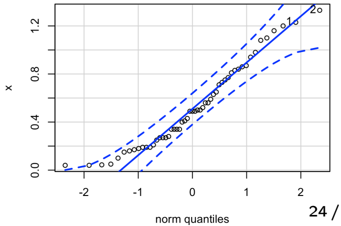
```
x=scan("https://raw.githubusercontent.com/Harrindy/StatEngine/master/Data/Mercury.csv")
```

Find a 95% confidence interval estimate for $\mu$, the population mean mercury concentration.

**Solution:** The dashed lines do not cover all dots; i.e., the sample might not be from a normal distribution. But $n = 53 > 25$, we could use a large-sample CI.

```
data.summary(x)
AZinterval(level=0.95,sample=x)
Tinterval(level=0.95,sample=x) #Try this!
```

Conclusion: based on the data, we are 95% confident that the population mean mercury concentration falls between 0.4311 and 0.6188.



# Practice 3

Past experience has indicated that the breaking strength of yarn used in manufacturing drapery material is normally distributed and that $\sigma = 2$ psi. A random sample of nine specimens is tested, and the average breaking strength is found to be 98 psi. Find a 95% two-sided confidence interval on the true mean breaking strength.

**Solution:** Normality and known $\sigma = 2$.

```
Zinterval(level=0.95,sigma=2,n=9,barx=98)
```

Conclusion: based on the data, we are 95% confident that the the true mean breaking strength falls between 96.6934 and 99.3066.

# Practice 4

A confidence interval estimate is desired for the gain in a circuit on a semiconductor device. Assume that gain is normally distributed. Consider the following cases where we suppose the sample standard deviation $s_n$ is always $20$.

(a) Find a 95% CI for $\mu$ when $n = 10$ and $\bar{x}_n = 1000$.
(b) Find a 95% CI for $\mu$ when $n = 25$ and $\bar{x}_n = 1000$.
(c) Find a 99% CI for $\mu$ when $n = 10$ and $\bar{x}_n = 1000$.
(d) Find a 99% CI for $\mu$ when $n = 25$ and $\bar{x}_n = 1000$.

**Solution:** Normality and $\sigma$ unknown.

```
Tinterval(level=0.95,n=10,barx=1000,s=20)
[ 985.6929 , 1014.307 ]
Tinterval(level=0.95,n=25,barx=1000,s=20)
[ 991.7444 , 1008.256 ]
Tinterval(level=0.99,n=10,barx=1000,s=20)
[ 979.4462 , 1020.554 ]
Tinterval(level=0.99,n=25,barx=1000,s=20)
[ 988.8122 , 1011.188 ]
```

# Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

StatEngine:

```
Chi2interval(level=?,sample=?)
Chi2interval(level=?,n=?,s=?)
```

Reasoning: Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$, and let $S_n^2$ be the sample variance. Then the random variable

$$\boldsymbol{X}_{n-1}^2 = \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1)$$

where $\chi^2(n-1)$ stands for the chi-square distribution with $n-1$ degrees of freedom.

# Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Let $\chi^2_{n-1,\alpha/2}$ and $\chi^2_{n-1,1-\alpha/2}$ be the value such that

$P(\boldsymbol{X}^2_{n-1} > \chi^2_{n-1,\alpha/2}) = \alpha/2,$ and $P(\boldsymbol{X}^2_{n-1} > \chi^2_{n-1,1-\alpha/2}) = 1-\alpha/2,$

respectively, where $\chi^2_{n-1,\alpha/2} = Chi2.quantile(df = n-1, 1-\alpha/2)$ and $\chi^2_{n-1,1-\alpha/2} = Chi2.quantile(df = n-1, \alpha/2)$.

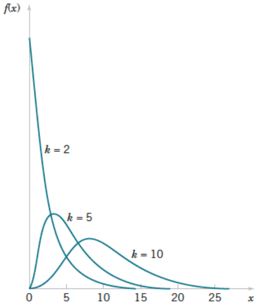



**FIGURE 8.8**

**Probability density functions of several $\chi^2$ distributions.**

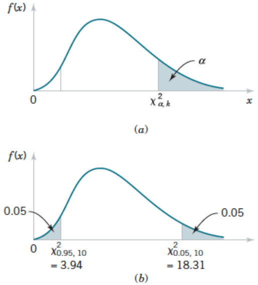



**FIGURE 8.9**

**Percentage point of the $\chi^2$ distribution. (a) The percentage point $\chi^2_{\alpha,k}$. (b) The upper percentage point $\chi^2_{0.05,10} = 18.31$ and the lower percentage point $\chi^2_{0.95,10} = 3.94$.**

# Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

$$
\begin{aligned}
1-\alpha &= P(\chi^2_{n-1,1-\alpha/2} \le \boldsymbol{X}^2_{n-1} \le \chi^2_{n-1,\alpha/2}) \\
&= P\left(\chi^2_{n-1,1-\alpha/2} \le \frac{(n-1)S_n^2}{\sigma^2} \le \chi^2_{n-1,\alpha/2}\right) \\
&= P\left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}}\right)
\end{aligned}
$$

If $s_n^2$ is the sample variance from a random sample of $n$ observations from a **normal** distribution with unknown variance $\sigma^2$, then a $100(1-\alpha)\%$ confidence interval on $\sigma^2$ is

$$
\frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha/2}}
$$

# One-sided confidence bounds on the Variance and Standard Deviation of a Normal Distribution

The $100(1-\alpha)\%$ lower and upper confidence bounds on $\sigma^2$ are

$$\sigma^2 \geq \frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha}}, \text{ and } \sigma^2 \leq \frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha}}.$$

respectively. If the parameter of interest is the population standard deviation $\sigma$ instead of the population variance $\sigma^2$, one can take square root of the above results:

$$\sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha/2}}} \leq \sigma \leq \sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha/2}}},$$

$$\sigma \geq \sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha}}}, \text{ and } \sigma \leq \sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha}}}.$$

Remark: Chi2-intervals are **not** robust to the normality assumption.

# Example

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s_n^2 = 0.01532^2$ (fluid ounce). If the variance of fill volume is too large, an unacceptable proportion of bottles will be under- or overfilled. We will assume that the fill volume is approximately **normally distributed**. Find a 95% upper confidence bound for $\sigma$, the population standard deviation of fill volume.

**Solution:** Normality checked (with data, use QQ plot).

```
Chi2interval(level=0.95,n=20,s=0.01532)
```

```
The sample standard variance is 0.0002347024 and sample size is 20
A 95% two-sided confidence interval for the population variance is [ 0.0001357391 , 0.0005006835 ]
A 95% upper-confidence bound for the population variance is 0.0004407769
A 95% lower-confidence bound for the population variance is 0.0001479371

The sample standard deviation is 0.01532 and sample size is 20
A 95% two-sided confidence interval for the population standard deviation is [ 0.01165071 , 0.02237596 ]
A 95% upper-confidence bound for the population standard deviation is 0.02099469
A 95% lower-confidence bound for the population standard deviation is 0.01216294
```

Conclusion: based on the data, we are 95% confident that the population standard deviation of fill volume $\sigma$ is bounded above by 0.021.

# Large-Sample Confidence Interval for a Population Proportion

It is often necessary to construct confidence intervals on a population proportion. For example, suppose that a random sample of size $n$ has been taken from a large (possibly infinite) population and that $X(\leq n)$ observations in this sample belong to a class of interest. Then $\widehat{p} = \frac{X}{n}$ is a point estimator of the proportion of the population $p$ that belongs to this class. Note that $X \sim \text{Binomial}(n, p)$.

When $n$ is large (rule of thumb : $n\widehat{p} \geq 5, n(1-\widehat{p}) \geq 5$), we have

$$Z = \frac{\widehat{p} - p}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}} \sim AN(0, 1).$$

Thus

$$1 - \alpha \approx P\left[-z_{\alpha/2} \leq \frac{\widehat{p} - p}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}} \leq z_{\alpha/2}\right]$$

# Large-Sample Confidence Interval for a Population Proportion

After some algebra and approximation, we have

## Approximate Confidence Interval on a Population Proportion

If $\widehat{p}$ is the proportion of observation in a random sample of size $n$ that belongs to a class of interest, an approximate $100(1-\alpha)\%$ confidence interval on the proportion $p$ of the population that belongs to this class is

$$\widehat{p} - z_{\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} \leq p \leq \widehat{p} + z_{\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}.$$

Approximate $100(1-\alpha)\%$ One-Sided lower and upper Confidence Bounds are

$$p \geq \widehat{p} - z_{\alpha}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}, \text{ and } p \leq \widehat{p} + z_{\alpha}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}.$$

StatEngine: $Propinterval(level =?, n =?, X =?)$.

# Choice of Sample Size

Suppose we want to choose $n$ such that $100(1-\alpha)\%$ confident that the error is less than some specified value $E$, we have

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1-p).$$

Now we have a question, if we know $p$, we can calculate $n$. However, if we know $p$, why do we need to estimate $p$? Too solutions:

1. Suppose we have an initial estimate of $p$, denoted by $\tilde{p}$:

$$n = \left\lceil \left(\frac{z_{\alpha/2}}{E}\right)^2 \tilde{p}(1-\tilde{p}) \right\rceil.$$

2. If no information about $p$ is available, then we use a conservative approach (because $p(1-p) \leq 0.25$)

$$n = \left\lceil \left(\frac{z_{\alpha/2}}{E}\right)^2 0.25 \right\rceil.$$

StatEngine:

$sample.size.Propinterval(level=?, ini.p=?, E=?)$

# Example

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Find a 95% two-sided confidence interval for $p$, the proportion of bearings in the population that exceeds the roughness specification.

**Solution:** $x = 10, n = 85, \widehat{p} = 10/85, n\widehat{p} = 10 \geq 5, n(1 - \widehat{p}) = 75 \geq 5$. Condition checked!

```
Propinterval(level=0.95,n=85,X=10)
```

Conclusion: based on the data, we are 95% confidence that $p$ falls between 0.0492 and 0.1861.

Now using $\tilde{p} = 0.12$ as an initial estimate of $p$, how large a sample is required if we want to be 95% confident that the error in using $\widehat{p}$ to estimate $p$ is less than 0.05? Then redo this problem using the conservative approach (answer: 163 and 385).

```
sample.size.Propinterval(level=0.95,ini.p=0.12,E=0.05)
sample.size.Propinterval(level=0.95,ini.p=0.5,E=0.05)
```

# Prediction Interval

Suppose that $X_1, \ldots, X_n$ is a random sample from a normal population. The sample mean and sample variance are $\bar{X}_n$ and $S_n^2$, respectively. We wish to predict the value $X_{n+1}$, a single future observation. A point prediction of $X_{n+1}$ is $\bar{X}_n$, the prediction error is $X_{n+1} - \bar{X}_n$ and the variance of the prediction error is

$$V(X_{n+1} - \bar{X}_n) = \sigma^2 + \frac{\sigma^2}{n}, \text{ and } X_{n+1} - \bar{X}_n \sim N\left(0, \sigma^2\left(1 + \frac{1}{n}\right)\right)$$

Estimate $\sigma$ by $S_n$, we have

$$T_{n-1} = \frac{X_{n+1} - \bar{X}_n}{S_n\sqrt{1 + \frac{1}{n}}} \sim t(n-1).$$

# Prediction Interval

Based on a random sample $x_1, \ldots, x_n$, A $100(1-\alpha)\%$ **prediction interval (PI)** on a single future observation from a normal distribution is given by

$$\bar{x}_n - t_{n-1,\alpha/2} s_n \sqrt{1 + \frac{1}{n}} \leq X_{n+1} \leq \bar{x}_n + t_{n-1,\alpha/2} s_n \sqrt{1 + \frac{1}{n}} \leq X_{n+1}.$$

One could also compute the upper- and lower- prediction bounded. Use StatEngin:

```
Predinterval(level=?,sample=?)
Predinterval(level=?,n=?,barx=?,s=?)
```

The prediction interval for $X_{n+1}$ will always be longer than the confidence interval for $\mu$ because more variability is associated with the prediction error than with the error of estimation.

# Example

Consider the tensile adhesion tests on specimens of U-700 alloy described in Practice 1:

```
x=c(19.8, 10.1, 14.9, 7.5, 15.4, 15.4, 15.4, 18.5,
7.9, 12.7, 11.9, 11.4, 11.4, 14.1, 17.6, 16.7, 15.8,
19.5, 8.8, 13.6, 11.9, 11.4)
```

A 95% confidence interval on $\mu$ is $Tinterval(level = 0.95, sample = x)$ which gives

$$12.1381 \leq \mu \leq 15.2892.$$

We plan to test a 23rd specimen. A 95% prediction interval on the load at failure for this specimen is
$Predinterval(level = 0.95, sample = x)$ which gives

$$6.1575 \leq X_{23} \leq 21.2698$$

Conclusion, we are 95% confident that the next observation will be between 6.1575 and 21.2698.

# StatEngine Summary of One-Sample CIs

CI for $\mu$ when $\sigma$ is **known** and the distribution is **normal**:

```
Zinterval(level=?,sigma=?,sample=?)
Zinterval(level=?,sigma=?,n=?,barx=?)
sample.size.Zinterval(level=?,sigma=?,E=?)
```

CI for $\mu$ when $\sigma$ is **unknown** and the distribution is **normal** (or for any distribution, but in this case, it provides approximated CIs):

```
Tinterval(level=?,sample=?)
Tinterval(level=?,n=?,barx=?,s=?)
```

Large-sample CI ($n \geq 25$) for $\mu$ under any distribution:

```
AZinterval(level=?,sample=?)
AZinterval(level=?,n=?,barx=?,s=?)
```

Both T-interval and AZ-interval are approximated CIs when normality does not hold. The T-intervals are more conservative (wider).

# StatEngine Summary of One-Sample CIs

CI for $\sigma^2$ (or $\sigma$) when the distribution is **normal** (does not work well if lack of nomarlity)

```
Chi2interval(level=?,sample=?)
Chi2interval(level=?,n=?,s=?)
```

Large-sample CI $(n\widehat{p}, n(1-\widehat{p}) \geq 5)$ for a population proportion $p$:

```
Propinterval(level=?,n=?,X=?)
sample.size.Propinterval(level=?,ini.p=?,E=?)
```

Prediction interval of a future observation from **normal** distribution.

```
Predinterval(level=?,sample=?)
Predinterval(level=?,n=?,barx=?,s=?)
```

Common patterns:

- Based on the same sample, an interval estimate becomes wider if its confidence level increases.
- When confidence level is fixed, a larger sample size leads to a narrower (more precise) interval estimate.