STAT 509: Statistics for Engineers

Chapter 8: Statistical Intervals for a Single Sample

Dr. Dewei Wang
Associate Professor
Department of Statistics
University of South Carolina
deweiwang@stat.sc.edu

Fall 2020

Learning Objectives:

1. Construct confidence intervals on the mean of a normal distribution, using either the normal distribution or the t distribution method
2. Construct confidence intervals on the variance and standard deviation of a normal distribution
3. Construct confidence intervals on a population proportion
4. Construct a prediction interval for a future observation

An interval estimator

Estimating an unknown parameter θ by a point estimator $\widehat{\Theta}_n$ is useful. However, it is like shooting a bird with a pistol. Often $\widehat{\Theta}_n$ has a continuous distribution, and if so, from Chapter 4, $P(\widehat{\Theta}_n = \theta) = 0$; i.e., we never capture the true parameter by using a point estimator even though it is an unbiased estimator.

Why not shoot a bird using a shotgun or capture it using a net? Translating to statistical language, why not use an interval to capture the true parameter? This motivates the consideration of interval estimators.

We estimate θ by an interval $[L_n, U_n]$, where L_n and U_n are two statistics computed from a random sample of size n such that

$$P[L_n \leq \theta \leq U_n] = 1 - \alpha,$$

for some pre-specified $\alpha \in (0, 1)$. We call $[L_n, U_n]$ a $100(1-\alpha)\%$ **confidence interval estimator** of θ and $100(1-\alpha)\%$ the **confidence level** of this interval estimator.

Confidence Interval on the Mean of a Normal Distribution, Variance Known

A confidence interval estimator is often built from a point estimator and the sampling distribution of the point estimator.

We consider building a confidence interval estimator of μ in the normal distribution $N(\mu, \sigma^2)$ where σ^2 is known (based on historical data).

From Chapter 7, a good point estimator of μ is $\bar{X}_n$, and the sampling distribution of $\bar{X}_n$ is $N(\mu, \sigma^2/n)$. Thus

$$Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

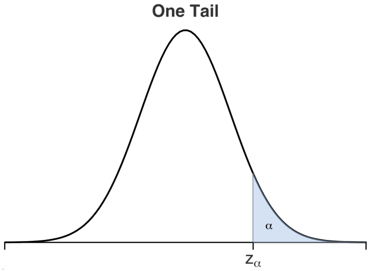
Confidence Interval on the Mean of a Normal Distribution, Variance Known

For $a \in (0,1)$, define z_a to be a quantile value of $Z \sim N(0,1)$ such that

$$P(Z > z_a) = a$$

Using StatEngine:

$$z_a = normal.quantile(0, 1, 1 - a).$$



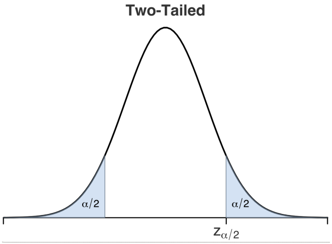
Often used z_a values:

$$z_{0.005} = 2.5758$$
$$z_{0.025} = 1.96$$
$$z_{0.05} = 1.6449$$
$$z_{0.1} = 1.2816.$$

Confidence Interval on the Mean of a Normal Distribution, Variance Known

For any $\alpha < 0.5$, we know that

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha.$$



Thus

$$\begin{aligned} 1 - \alpha &= P\left(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}\right) \\ &= P\left(-z_{\alpha/2} \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) \\ &= P\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n - \mu \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\ &= P\left(\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \end{aligned}$$

Confidence Interval on the Mean of a Normal distribution, variance known

Two-sided confidence interval

If $\bar{X}_n$ is the sample mean of size n from a **normal** population with **known variance** σ^2, a $100(1-\alpha)\%$ (two-sided) confidence interval estimator on μ is given by

$$\left[L_n = \bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, U_n = \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$

where $z_{\alpha/2} = normal.quantile(0,1,1-\alpha/2)$.

Based on an observed sample $x_1, \ldots, x_n$, a $100(1-\alpha)\%$ (two-sided) confidence interval estimate on μ is

$$\left[\bar{x}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

Its length is $2 \times z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$, which becomes smaller if either n is larger (more data) or α is larger (less confidence level).

One-sided confidence bounds for the Mean of a Normal distribution, variance known

Similarly, we have $1-\alpha = P(Z \leq z_\alpha)$ and $1-\alpha = P(-z_\alpha \leq Z)$. Plugging $Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$, we obtain

$$1-\alpha = P\left(\mu \leq \bar{X}_n + z_\alpha \frac{\sigma}{\sqrt{n}}\right) = P\left(\bar{X}_n - z_\alpha \frac{\sigma}{\sqrt{n}} \leq \mu\right)$$

One-sided confidence bound

A $100(1-\alpha)\%$ **upper-confidence bound** for μ is

$$\bar{x}_n + z_\alpha \frac{\sigma}{\sqrt{n}}$$

and a $100(1-\alpha)\%$ **lower-confidence bound** for μ is

$$\bar{x}_n - z_\alpha \frac{\sigma}{\sqrt{n}}.$$

Example

ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at $60^\circ C$ are as follows: $64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, 64.3$. Assume that impact energy is normally distributed with $\sigma = 1$J. We want to find 95% and 99% CIs for μ, the mean impact energy.

```
x=c(64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, 64.3)
Zinterval(level=0.95,sigma=1,sample=x)
Zinterval(level=0.99,sigma=1,sample=x)
```

A 95% CI for μ is $[63.8402, 65.0798]$
A 99% CI for μ is $[63.6455, 65.2746]$
When confidence level increases (or α decreases), confidence interval becomes wider! (A larger net captures a bird more easily.)

Interpreting a confidence interval

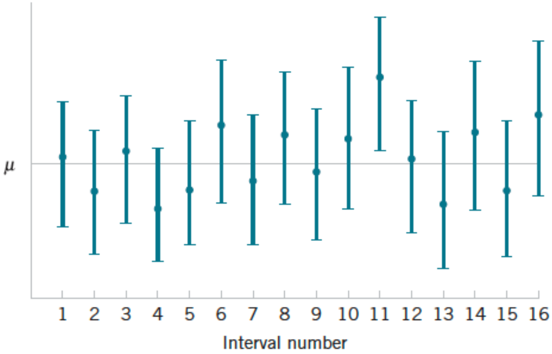
How does one interpret a confidence interval? In the previous example, a 95% CI is $63.8402 \leq \mu \leq 65.0798$, so it is tempting to conclude that μ is within this interval with probability 0.95.

However, with a little reflection, it is easy to see that this **cannot be correct**; the true value of μ is unknown, and the statement $63.8402 \leq \mu \leq 65.0798$ is either correct (true with probability 1) or incorrect (false with probability 1).

The correct interpretation lies in the realization that a CI is a random interval because in the probability statement defining the end-points of the interval, both L_n and U_n are random variables. Consequently, the correct interpretation of a $100(1-\alpha)\%$ CI depends on **the relative frequency view of probability**. Specifically, if an infinite number of random samples are collected and a $100(1-\alpha)\%$ confidence interval for μ is computed from each sample, $100(1-\alpha)\%$ of these intervals will contain the true value of μ.

Interpreting a confidence interval

The situation is illustrated in the following figure, which shows several $100(1-\alpha)\%$ confidence intervals for the mean μ of a normal distribution. The dots at the center of the intervals indicate the point estimate of μ (that is, $\bar{x}_n$). Notice that one of the intervals fails to contain the true value of μ. If this were a 95% confidence interval, in the long run only 5% of the intervals would fail to contain μ.



Choice of sample size

A $100(1-\alpha)\%$ CI takes the form of

$$\left[\bar{x}_n - \underbrace{z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{\text{margin of error}}, \bar{x}_n + \underbrace{z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{\text{margin of error}}\right].$$

When α or the confidence level is fixed, the margin of error becomes smaller if n increases. It means if we have more data, the $100(1-\alpha)\%$ CI has more precision of estimation. Suppose we specify the margin of error to be E, what is the smallest amount sample to collect?

$$E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \implies n = \left\lceil \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \right\rceil,$$

where $\lceil a \rceil$ means the ceiling of a.

Choice of sample size

Consider the previous example and suppose that we want to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at $60^\circ C$ has a length of at most 1.0 J.

Solution: The length is at most 1J, meaning the margin of error E is at most 0.5J. Thus ($\alpha = 0.05$, $\sigma = 1$)

$$n = \left\lceil \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \right\rceil = \left\lceil \left(\frac{z_{0.025} \times 1}{0.5}\right)^2 \right\rceil = 16.$$

```
sample.size.Zinterval(level=0.95,sigma=1,E=0.5)
```

CI for μ when σ is **known** and the distribution is **normal**:

```
Zinterval(level=?,sigma=?,sample=?)
Zinterval(level=?,sigma=?,n=?,barx=?)
sample.size.Zinterval(level=?,sigma=?,E=?)
```

What if we do not know the variance σ^2?
What if it is not normal?

Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. We know that

$$Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

When σ is unknown, we replace σ by its estimator S_n and obtain

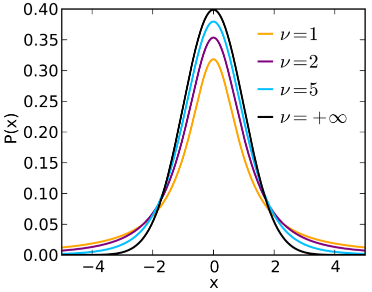
$$T_{n-1} = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t(n-1)$$

where $t(n-1)$ stands for the **student t distribution** with degree of freedom $n-1$.

Student $t(\nu)$ distribution

It is a continuous distribution with one parameter ν, the degree of freedom. Its pdf is

$$f(x) = \frac{\Gamma(\{\nu+1\}/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)} \frac{1}{[(x^2/\nu)+1]^{(\nu+1)/2}}, -\infty < x < \infty.$$



- Similar to normal distributions: bell shape, symmetric with respect to 0
- Heavier tails than $N(0,1)$ when ν is small
- When $v \to \infty$, $t(\nu)$ converges to $N(0,1)$

Confidence Interval on the Mean of a Normal Distribution, Variance unknown

For $a \in (0,1)$, define $t_{n-1,a}$ to be a quantile value of $t(n-1)$ such that

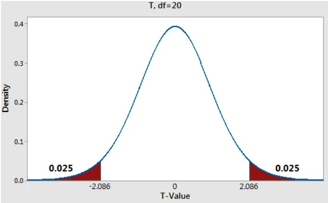
$$P(T_{n-1} > t_{n-1,a}) = a$$

Using StatEngine:

$$t_{n-1,a} = t.quantile(df = n-1, 1-a).$$

For any $\alpha < 0.5$, we know that

$$P(-t_{n-1,\alpha/2} \leq T_{n-1} \leq t_{n-1,\alpha/2}) = 1-\alpha$$



$$= P\left(-t_{n-1,\alpha/2} \leq \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \leq t_{n-1,\alpha/2}\right)$$

$$= P\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}} \leq \mu \leq \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}\right)$$

Confidence Interval on the Mean of a Normal distribution, variance unknown

Two-sided confidence interval

If $\bar{X}_n$ is the sample mean of size n from a **normal** population with **unknown variance** σ^2, a $100(1-\alpha)\%$ (two-sided) confidence interval estimator on μ is given by

$$\left[L_n = \bar{X}_n - t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}, U_n = \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}\right]$$

where $t_{n-1,\alpha/2} = t.quantile(df = n-1, 1-\alpha/2)$.

Based on an observed sample $x_1, \ldots, x_n$, a $100(1-\alpha)\%$ (two-sided) confidence interval estimate on μ is

$$\left[\bar{x}_n - t_{n-1,\alpha/2}\frac{s_n}{\sqrt{n}}, \bar{x}_n + t_{n-1,\alpha/2}\frac{s_n}{\sqrt{n}}\right].$$

Its length is $2 \times t_{n-1,\alpha/2}\frac{s_n}{\sqrt{n}}$, which becomes smaller if either n is larger (more data) or α is larger (less confidence level).

One-sided confidence bounds for the Mean of a Normal distribution, variance unknown

One-sided confidence bound

A $100(1-\alpha)\%$ **upper-confidence bound** for μ is

$$\bar{x}_n + t_{n-1,\alpha}\frac{s_n}{\sqrt{n}}$$

and a $100(1-\alpha)\%$ **lower-confidence bound** for μ is

$$\bar{x}_n - t_{n-1,\alpha}\frac{s_n}{\sqrt{n}}.$$

StatEngine:

```
Tinterval(level=?,sample=?)
Tinterval(level=?,n=?,barx=?,s=?)
```

Remark: T-intervals are quite robust to the normality assumption when n is small. Thus, in practice, even if we do not have normality, one can still use T-inteverals.

Large-Sample Confidence Interval on the Mean of a population

Two-sided confidence interval

If $\bar{X}_n$ is the sample mean of size n from a population with mean μ and a finite variance σ^2. When n is large ($n \geq 25$), the CLT (plus the Slutsky theorem) tells

$$\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim AN(0,1).$$

Thus, a large-sample confidence interval estimator for μ with confidence level of **approximately** $100(1-\alpha)\%$ is given by

$$\left[L_n = \bar{X}_n - z_{\alpha/2}\frac{s_n}{\sqrt{n}}, U_n = \bar{X}_n + z_{\alpha/2}\frac{s_n}{\sqrt{n}}\right].$$

One-sided confidence bounds for the Mean of a Normal distribution, variance known

Based on an observed sample $x_1, \dots, x_n$, a large-sample confidence interval estimator for μ with confidence level of approximately $100(1-\alpha)\%$ is given by

$$\left[\bar{x}_n - z_{\alpha/2}\frac{s_n}{\sqrt{n}}, \bar{x}_n + z_{\alpha/2}\frac{s_n}{\sqrt{n}}\right].$$

One-sided large-sample confidence bound

A $100(1-\alpha)\%$ **large-sample upper-confidence bound** for μ is

$$\bar{x}_n + z_\alpha \frac{\sigma}{\sqrt{n}}$$

and a $100(1-\alpha)\%$ **large-sample lower-confidence bound** for μ is

$$\bar{x}_n - z_\alpha \frac{\sigma}{\sqrt{n}}.$$

StatEngine summary of CIs on the population mean

CI for μ when σ is **known** and the distribution is **normal**:

```
Zinterval(level=?,sigma=?,sample=?)
Zinterval(level=?,sigma=?,n=?,barx=?)
sample.size.Zinterval(level=?,sigma=?,E=?)
```

CI for μ when σ is **unknown** and the distribution is **normal** (or for any distribution, but in this case, it provides approximated CIs):

```
Tinterval(level=?,sample=?)
Tinterval(level=?,n=?,barx=?,s=?)
```

Large-sample CI ($n \geq 25$) for μ under any distribution:

```
AZinterval(level=?,sample=?)
AZinterval(level=?,n=?,barx=?,s=?)
```

Both T-interval and AZ-interval are approximated CIs when normality does not hold. The T-intervals are more conservative (wider).

Practice 1

An article in the Journal of Materials Engineering ["Instrumented Tensile Adhesion Tests on Plasma Sprayed Thermal Barrier Coatings" (1989, Vol. 11(4), pp. 275–282)] describes the results of tensile adhesion tests on 22 U-700 alloy specimens. The load at specimen failure is as follows (in megapascals):

```
x=c(19.8, 10.1, 14.9, 7.5, 15.4, 15.4, 15.4, 18.5, 7.9, 12.7, 11.9, 11.4, 11.4, 14.1,
17.6, 16.7, 15.8, 19.5, 8.8, 13.6, 11.9, 11.4)
```

Find a 95% CI on μ, the population mean load at specimen failure.

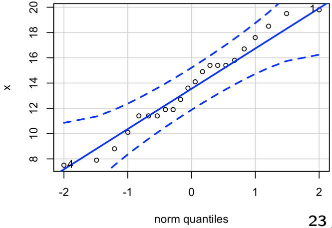
Solution: The sample size is $n = 22 < 25$, large-sample inference might not work. We do not know the population variance σ nor the type of the population distribution (did not say normal). But we can check normality first using the QQ plot.

```
data.summary(x)
```

It appears that the sample follows a normal distribution. Thus use the T-interval.

```
Tinterval(level=0.95,sample=x)
```

Conclusion: based on the data, we are 95% confident that the population mean load at specimen failure falls between 12.1381 and 15.2892.



Practice 2

An article in the 1993 volume of the Transactions of the American Fisheries Society reports the results of a study to investigate the mercury contamination in large mouth bass. A sample of fish was selected from 53 Florida lakes, and mercury concentration in the muscle tissue was measured (ppm). The mercury concentration values were

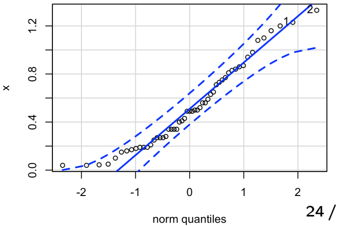
```
x=scan("https://raw.githubusercontent.com/Harrindy/StatEngine/master/Data/Mercury.csv")
```

Find a 95% confidence interval estimate for μ, the population mean mercury concentration.

Solution: The dashed lines do not cover all dots; i.e., the sample might not be from a normal distribution. But $n = 53 > 25$, we could use a large-sample CI.

```
data.summary(x)
AZinterval(level=0.95,sample=x)
Tinterval(level=0.95,sample=x) #Try this!
```

Conclusion: based on the data, we are 95% confident that the population mean mercury concentration falls between 0.4311 and 0.6188.



Practice 3

Past experience has indicated that the breaking strength of yarn used in manufacturing drapery material is normally distributed and that $\sigma = 2$ psi. A random sample of nine specimens is tested, and the average breaking strength is found to be 98 psi. Find a 95% two-sided confidence interval on the true mean breaking strength.

Solution: Normality and known $\sigma = 2$.

```
Zinterval(level=0.95,sigma=2,n=9,barx=98)
```

Conclusion: based on the data, we are 95% confident that the the true mean breaking strength falls between 96.6934 and 99.3066.

Practice 4

A confidence interval estimate is desired for the gain in a circuit on a semiconductor device. Assume that gain is normally distributed. Consider the following cases where we suppose the sample standard deviation s_n is always 20.

(a) Find a 95% CI for μ when $n = 10$ and $\bar{x}_n = 1000$.
(b) Find a 95% CI for μ when $n = 25$ and $\bar{x}_n = 1000$.
(c) Find a 99% CI for μ when $n = 10$ and $\bar{x}_n = 1000$.
(d) Find a 99% CI for μ when $n = 25$ and $\bar{x}_n = 1000$.

Solution: Normality and σ unknown.

```
Tinterval(level=0.95,n=10,barx=1000,s=20)
[ 985.6929 , 1014.307 ]
Tinterval(level=0.95,n=25,barx=1000,s=20)
[ 991.7444 , 1008.256 ]
Tinterval(level=0.99,n=10,barx=1000,s=20)
[ 979.4462 , 1020.554 ]
Tinterval(level=0.99,n=25,barx=1000,s=20)
[ 988.8122 , 1011.188 ]
```

Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

StatEngine:

```
Chi2interval(level=?,sample=?)
Chi2interval(level=?,n=?,s=?)
```

Reasoning: Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, and let S_n^2 be the sample variance. Then the random variable

$$\boldsymbol{X}_{n-1}^2 = \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1)$$

where $\chi^2(n-1)$ stands for the chi-square distribution with $n-1$ degrees of freedom.

Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Let $\chi^2_{n-1,\alpha/2}$ and $\chi^2_{n-1,1-\alpha/2}$ be the value such that

$$P(\boldsymbol{X}^2_{n-1} > \chi^2_{n-1,\alpha/2}) = \alpha/2, \text{ and } P(\boldsymbol{X}^2_{n-1} > \chi^2_{n-1,1-\alpha/2}) = 1-\alpha/2,$$

respectively, where $\chi^2_{n-1,\alpha/2} = Chi2.quantile(df = n-1, 1-\alpha/2)$ and $\chi^2_{n-1,1-\alpha/2} = Chi2.quantile(df = n-1, \alpha/2)$.

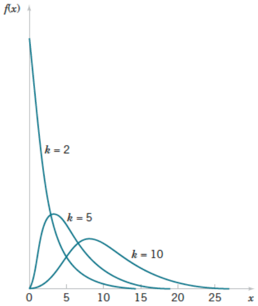


FIGURE 8.8

Probability density functions of several χ^2 distributions.

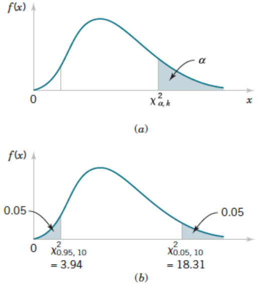


FIGURE 8.9

Percentage point of the χ^2 distribution. (a) The percentage point $\chi^2_{\alpha,k}$. (b) The upper percentage point $\chi^2_{0.05,10} = 18.31$ and the lower percentage point $\chi^2_{0.95,10} = 3.94$.

Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

$$
\begin{aligned}
1-\alpha &= P(\chi^2_{n-1,1-\alpha/2} \le \boldsymbol{X}^2_{n-1} \le \chi^2_{n-1,\alpha/2}) \\
&= P\left(\chi^2_{n-1,1-\alpha/2} \le \frac{(n-1)S_n^2}{\sigma^2} \le \chi^2_{n-1,\alpha/2}\right) \\
&= P\left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}}\right)
\end{aligned}
$$

If s_n^2 is the sample variance from a random sample of n observations from a **normal** distribution with unknown variance σ^2, then a $100(1-\alpha)\%$ confidence interval on σ^2 is

$$
\frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha/2}}
$$

One-sided confidence bounds on the Variance and Standard Deviation of a Normal Distribution

The $100(1-\alpha)\%$ lower and upper confidence bounds on σ^2 are

$$\sigma^2 \geq \frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha}}, \text{ and } \sigma^2 \leq \frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha}}.$$

respectively. If the parameter of interest is the population standard deviation σ instead of the population variance σ^2, one can take square root of the above results:

$$\sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha/2}}} \leq \sigma \leq \sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha/2}}},$$

$$\sigma \geq \sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,\alpha}}}, \text{ and } \sigma \leq \sqrt{\frac{(n-1)s_n^2}{\chi^2_{n-1,1-\alpha}}}.$$

Remark: Chi2-intervals are **not** robust to the normality assumption.

Example

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s_n^2 = 0.01532^2$ (fluid ounce). If the variance of fill volume is too large, an unacceptable proportion of bottles will be under- or overfilled. We will assume that the fill volume is approximately **normally distributed**. Find a 95% upper confidence bound for σ, the population standard deviation of fill volume.

Solution: Normality checked (with data, use QQ plot).

```
Chi2interval(level=0.95,n=20,s=0.01532)
```

```
The sample standard variance is 0.0002347024 and sample size is 20
A 95% two-sided confidence interval for the population variance is [ 0.0001357391 , 0.0005006835 ]
A 95% upper-confidence bound for the population variance is 0.0004407769
A 95% lower-confidence bound for the population variance is 0.0001479371

The sample standard deviation is 0.01532 and sample size is 20
A 95% two-sided confidence interval for the population standard deviation is [ 0.01165071 , 0.02237596 ]
A 95% upper-confidence bound for the population standard deviation is 0.02099469
A 95% lower-confidence bound for the population standard deviation is 0.01216294
```

Conclusion: based on the data, we are 95% confident that the population standard deviation of fill volume σ is bounded above by 0.021.

Large-Sample Confidence Interval for a Population Proportion

It is often necessary to construct confidence intervals on a population proportion. For example, suppose that a random sample of size n has been taken from a large (possibly infinite) population and that $X(\leq n)$ observations in this sample belong to a class of interest. Then $\widehat{p} = \frac{X}{n}$ is a point estimator of the proportion of the population p that belongs to this class. Note that $X \sim \text{Binomial}(n, p)$.

When n is large (rule of thumb : $n\widehat{p} \geq 5, n(1-\widehat{p}) \geq 5$), we have

$$Z = \frac{\widehat{p} - p}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}} \sim AN(0,1).$$

Thus

$$1-\alpha \approx P\left[-z_{\alpha/2} \leq \frac{\widehat{p} - p}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}} \leq z_{\alpha/2}\right]$$

Large-Sample Confidence Interval for a Population Proportion

After some algebra and approximation, we have

Approximate Confidence Interval on a Population Proportion

If $\widehat{p}$ is the proportion of observation in a random sample of size n that belongs to a class of interest, an approximate $100(1-\alpha)\%$ confidence interval on the proportion p of the population that belongs to this class is

$$\widehat{p} - z_{\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} \le p \le \widehat{p} + z_{\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}.$$

Approximate $100(1-\alpha)\%$ One-Sided lower and upper Confidence Bounds are

$$p \ge \widehat{p} - z_{\alpha}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}, \text{ and } p \le \widehat{p} + z_{\alpha}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}.$$

StatEngine: $Propinterval(level=?, n=?, X=?)$.

Choice of Sample Size

Suppose we want to choose n such that $100(1-\alpha)\%$ confident that the error is less than some specified value E, we have

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1-p).$$

Now we have a question, if we know p, we can calculate n. However, if we know p, why do we need to estimate p? Too solutions:

1. Suppose we have an initial estimate of p, denoted by $\tilde{p}$:

$$n = \left\lceil \left(\frac{z_{\alpha/2}}{E}\right)^2 \tilde{p}(1-\tilde{p}) \right\rceil.$$

2. If no information about p is available, then we use a conservative approach (because $p(1-p) \leq 0.25$)

$$n = \left\lceil \left(\frac{z_{\alpha/2}}{E}\right)^2 0.25 \right\rceil.$$

StatEngine:
$sample.size.Propinterval(level=?, ini.p=?, E=?)$

Example

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Find a 95% two-sided confidence interval for p, the proportion of bearings in the population that exceeds the roughness specification.

Solution: $x = 10, n = 85, \widehat{p} = 10/85, n\widehat{p} = 10 \geq 5, n(1 - \widehat{p}) = 75 \geq 5$. Condition checked!

```
Propinterval(level=0.95,n=85,X=10)
```

Conclusion: based on the data, we are 95% confidence that p falls between 0.0492 and 0.1861.

Now using $\tilde{p} = 0.12$ as an initial estimate of p, how large a sample is required if we want to be 95% confident that the error in using $\widehat{p}$ to estimate p is less than 0.05? Then redo this problem using the conservative approach (answer: 163 and 385).

```
sample.size.Propinterval(level=0.95,ini.p=0.12,E=0.05)
sample.size.Propinterval(level=0.95,ini.p=0.5,E=0.05)
```

Prediction Interval

Suppose that $X_1, \ldots, X_n$ is a random sample from a normal population. The sample mean and sample variance are $\bar{X}_n$ and S_n^2, respectively. We wish to predict the value X_{n+1}, a single future observation. A point prediction of X_{n+1} is $\bar{X}_n$, the prediction error is $X_{n+1} - \bar{X}_n$ and the variance of the prediction error is

$$V(X_{n+1} - \bar{X}_n) = \sigma^2 + \frac{\sigma^2}{n}, \text{ and } X_{n+1} - \bar{X}_n \sim N\left(0, \sigma^2\left(1 + \frac{1}{n}\right)\right)$$

Estimate σ by S_n, we have

$$T_{n-1} = \frac{X_{n+1} - \bar{X}_n}{S_n\sqrt{1 + \frac{1}{n}}} \sim t(n-1).$$

Prediction Interval

Based on a random sample $x_1, \ldots, x_n$, A $100(1-\alpha)\%$ **prediction interval (PI)** on a single future observation from a normal distribution is given by

$$\bar{x}_n - t_{n-1,\alpha/2} s_n \sqrt{1 + \frac{1}{n}} \leq X_{n+1} \leq \bar{x}_n + t_{n-1,\alpha/2} s_n \sqrt{1 + \frac{1}{n}} \leq X_{n+1}.$$

One could also compute the upper- and lower- prediction bounded. Use StatEngin:

```
Predinterval(level=?,sample=?)
Predinterval(level=?,n=?,barx=?,s=?)
```

The prediction interval for X_{n+1} will always be longer than the confidence interval for μ because more variability is associated with the prediction error than with the error of estimation.

Example

Consider the tensile adhesion tests on specimens of U-700 alloy described in Practice 1:

```
x=c(19.8, 10.1, 14.9, 7.5, 15.4, 15.4, 15.4, 18.5,
7.9, 12.7, 11.9, 11.4, 11.4, 14.1, 17.6, 16.7, 15.8,
19.5, 8.8, 13.6, 11.9, 11.4)
```

A 95% confidence interval on μ is $Tinterval(level = 0.95, sample = x)$ which gives

$$12.1381 \leq \mu \leq 15.2892.$$

We plan to test a 23rd specimen. A 95% prediction interval on the load at failure for this specimen is
$Predinterval(level = 0.95, sample = x)$ which gives

$$6.1575 \leq X_{23} \leq 21.2698$$

Conclusion, we are 95% confident that the next observation will be between 6.1575 and 21.2698.

StatEngine Summary of One-Sample CIs

CI for μ when σ is **known** and the distribution is **normal**:

```
Zinterval(level=?,sigma=?,sample=?)
Zinterval(level=?,sigma=?,n=?,barx=?)
sample.size.Zinterval(level=?,sigma=?,E=?)
```

CI for μ when σ is **unknown** and the distribution is **normal** (or for any distribution, but in this case, it provides approximated CIs):

```
Tinterval(level=?,sample=?)
Tinterval(level=?,n=?,barx=?,s=?)
```

Large-sample CI ($n \geq 25$) for μ under any distribution:

```
AZinterval(level=?,sample=?)
AZinterval(level=?,n=?,barx=?,s=?)
```

Both T-interval and AZ-interval are approximated CIs when normality does not hold. The T-intervals are more conservative (wider).

StatEngine Summary of One-Sample CIs

CI for σ^2 (or σ) when the distribution is **normal** (does not work well if lack of nomarlity)

```
Chi2interval(level=?,sample=?)
Chi2interval(level=?,n=?,s=?)
```

Large-sample CI ($n\widehat{p}, n(1-\widehat{p}) \geq 5$) for a population proportion p:

```
Propinterval(level=?,n=?,X=?)
sample.size.Propinterval(level=?,ini.p=?,E=?)
```

Prediction interval of a future observation from **normal** distribution.

```
Predinterval(level=?,sample=?)
Predinterval(level=?,n=?,barx=?,s=?)
```

Common patterns:

- Based on the same sample, an interval estimate becomes wider if its confidence level increases.
- When confidence level is fixed, a larger sample size leads to a narrower (more precise) interval estimate.