Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
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Exercise sheet 3

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: May 9th 2016, 12:15 p.m.; Discussion: May 12th 2016)

Exercise 9. (4 points)

For each $n \in \mathbb{N}_0$ let $(X_{n,i})_{i\in\mathbb{N}}$ denote i.i.d. random variables with $\mathbb{P}(X_{1,1} \in \mathbb{N}) = 1$ and $m = \mathbb{E}[X_{1,1}] < \infty$. The *Galton-Watson process* is defined via

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}.$$

a) Prove that $M_n = \frac{Z_n}{m^n}$ is a martingale.

b) Let $\sigma^2 = Var(X_{1,1}) < \infty$. Find the Doob decomposition of $(M_n)^2$.

Exercise 10. (4 points)

In a cloakroom there are K different cloaks which belong to K different people. In the first round each person picks a cloak at random. If it is her/his cloak she/he keeps the cloak and leaves the building. In the second round each remaining person again picks a random cloak from those that are still left and so on. Denote by N the (random) number of rounds until every person received her/his cloak and left the building.

a) Let X_n denote the number of cloaks picked in the n-th round by its rightful owner and M_n defined via
$$M_0 = K, \quad M_{n+1} = M_n - X_{n+1}$$
denote the number of cloaks left after the n-th round. Prove
$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] = \mathbb{1}_{\{M_{n-1}>0\}}.$$

b) Prove that the process defined via $Y_0 = K$, $Y_n = (M_n + n)\mathbb{1}_{\{M_{n-1}>0\}} + Y_{n-1}\mathbb{1}_{\{M_{n-1}=0\}}$, $n \geq 1$, is a martingale and conclude
$$\mathbb{E}[N] = K.$$

Hint: Use the identity $\mathbb{E}[\tau] = \sum_{n=1}^{\infty} \mathbb{P}(\tau \geq n)$ for a stopping time τ with values in $\mathbb{N}_0$ (you don't have to prove this) and Exercise 8 from the previous exercise sheet.

Exercise 11. (4 points)

At each time $n = 1, 2, \ldots$ a monkey types a capital letter at random, the sequence of letters typed forms an i.i.d. sequence of random variables, each chosen uniformly amongst the 26 possible capital letters.
Just before each time $n = 1, 2, \ldots$ a new gambler arrives and bets 1€ that

the n-th letter will be an A.

If he loses, he leaves. If he wins, he receives 26€ all of which he bets on the event that

the $(n+1)$-th letter will be a B.

If he loses, he leaves. If he wins, he receives 26^2€ all of which he bets on the event that

the $(n+2)$-th letter will be an R

and so on through the whole sequence $ABRACADABRA$. Let T denote the first time, by which the monkey has produced the consecutive sequence $ABRACADABRA$. Use martingale theory to prove

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$

Hint: Use Exercise P.9 and Exercise 8 from the previous exercise sheet.

Exercise 12. (4 points)

Prove the claim from Remark 2.28: Let $(X_t)_{t\in\mathbb{N}_0}$ be a martingale with $\mathbb{E}[(X_t)^2] > \infty$ for all $t \in \mathbb{N}_0$. Then the following two statements are equivalent:

(i) $\sup_{t\in\mathbb{N}_0} \mathbb{E}[(X_t)^2] < \infty$.

(ii) There exists $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ with $X_t \to X$ almost surely and $X_t \to X$ in L^2.

Hint: Have a look at Exercise P.8. For $(i) \Rightarrow (ii)$ first show that $(X_t)_{t\in\mathbb{N}_0}$ is uniformly integrable and use the L^1-limes X_∞.