Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
Prof. Dr. M. Vetter, O. Martin

Exercise sheet 9

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: June 27th 2016, 12:15 p.m.; Discussion: June 30th 2016)

Exercise 33. (4 points)

a) Let X, Y be continuous semimartingales and $H \in \mathcal{L}(X) \cap \mathcal{L}(Y)$. Prove

$$\int H d(X+Y) = \int H dX + \int H dY.$$

b) Prove the following *integration by parts rule* for continuous semimartingales

$$X_t Y_t = X_0 Y_0 + \int_0^t X dY + \int_0^t Y dX + [X,Y]_t.$$

Exercise 34. (4 points)

Let X be a continuous semimartingale and $f \in C^2(\mathbb{R}, \mathbb{R})$. Prove

$$[f(X), f(X)]_t = \int_0^t (f'(X_s))^2 d[X]_s.$$

Exercise 35. (4 points)

Let $(M_t)_{t\geq 0}$ be a local martingale with $M_0 = 0$. Prove that for any stopping time τ with $\mathbb{E}[[M]_\tau] < \infty$ it holds

$$\mathbb{E}[M_\tau] = 0, \quad \mathbb{E}[(M_\tau)^2] = \mathbb{E}[[M]_\tau].$$

Remark: With $M = B$ being a standard Brownian motion the above statement yields Wald's lemmata because of $[B]_\tau = \tau$.

Exercise 36. (4 points)

This exercise shows that any continuous local martingale can be represented as a time-shifted Brownian motion. Let $(M_t)_{t\geq 0}$ be a continuous local martingale with $M_0 = 0$ and $\lim_{t\to\infty}[M]_t = \infty$ almost surely. For $s \geq 0$ define $T_s := \inf\{t > 0 : [M]_t > s\}$, $\mathcal{G}_s := \mathcal{F}_{T_s}$ and $B_s := M_{T_s}$.

a) Prove that the mapping $s \mapsto T_s$ is almost surely increasing, right-continuous and a right inverse mapping to $t \mapsto [M]_t$, i.e. it holds

$$[M]_{T_s} = s \quad \forall s \geq 0.$$

b) Prove, that $(B_s)_{s\geq 0}$ is a local martingale with respect to the filtration $(\mathcal{G}_s)_{s\geq 0}$.
Hint: Use the optional sampling theorem (2.12) and Exercise 35.

c) Conclude, that $(B_s)_{s\geq 0}$ is a standard Brownian motion with respect to the filtration $(\mathcal{G}_s)_{s\geq 0}$.
Hint: Use Theorem 6.28.

d) Show $M_t = B_{[M]_t}$ almost surely for any $t \geq 0$.

Remark: This result is known as the *Dubins-Schwarz Theorem.*