Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
Prof. Dr. M. Vetter, O. Martin/A. Theopold **Winter term 2015/16**

Exercise sheet 8

supporting the lecture Mathematical Statistics

(Submission of Solutions: 18. January 2016, 12:00 o'clock; Discussion: 20. January 2016)

Exercise 1. (4 points)

The *coefficient of variation* of a probability distribution P is defined as

$$c_v(P) := \frac{(Var[X])^{1/2}}{\mathbb{E}[X]}, \quad X \sim P;$$

if the needed moments exist and $\mathbb{E}[X] \neq 0$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. $\mathcal{N}(\mu, \sigma^2)$ distributed random variables with $\mu \in \mathbb{R} \setminus \{0\}, \sigma^2 > 0$.

a) Determine the method of moments estimator $\hat{c}_n(X)$ for $c_v(X)$.

b) Derive a central limit theorem for the estimator in a).

Hint: In many cases the coefficient of variation is a more suitable measure for uncertainty than the standard deviation. If for example one wants to build objects with length 1000m, a standard deviation of 0.01m might be negligible, whereas for objects of a much smaller length like 0.001m this is probably not the case.

Exercise 2. (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. $\mathcal{U}[0, \theta]$ distributed random variable. Show that $\left(\prod_{i=1}^n X_i\right)^{\frac{1}{n}}$ is a consistent estimator for $\frac{\theta}{\exp(1)}$.

Hint: One possible way is to prove the convergence in distribution

$$\sqrt{n}\left(\left(\prod_{i=1}^n X_i\right)^{\frac{1}{n}} - \frac{\theta}{\exp(1)}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\theta^2}{\exp(2)}\right).$$

Exercise 3. (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. $\mathcal{N}(\mu, 1)$ distributed random variables.

a) Let

$$\hat{\mu}_a = \begin{cases} \overline{X}_n & \text{für } \left|\overline{X}_n\right| \geq n^{-1/4} \\ a\overline{X}_n & \text{für } \left|\overline{X}_n\right| < n^{-1/4} \end{cases}, \quad a > 0.$$

Show that

$$\sqrt{n}(\hat{\mu}_a - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, v(\mu))$$

with $v(\mu) = 1$ for $\mu \neq 0$ and $v(\mu) = a^2$ in the case of $\mu = 0$.

b) For which a is $\hat{\mu}_a$ efficient?

c) Show that there exist cases in which it holds that

$$v(\mu) \leq I_1^{-1}(\mu).$$

Exercise 4. (4 points)

Let X be a $Bin(n,p)$ distributed random variable. We define the hypotheses $H : p = 0.8$ and $K : p = 0.6$.

a) Let $n = 10$. We consider the test

$$\phi(x) = 1_{\{0,\ldots,6\}}(x).$$

Determine the type 1 and type 2 error.

b) As before $n = 10$. Construct an unbiased test with level $\alpha = 0.1$.

c) Let now be $n = 100$. Determine approximately the type 1 and type 2 error for the test

$$\phi(x) = 1_{\{0,\ldots,69\}}(x).$$