Christian-Albrechts-Universität zu Kiel **Winter term 2016/2017**
Mathematisches Seminar
Prof. Dr. M. Vetter, A. Theopold

Exercise sheet 3

supporting the lecture interest rate models

(Submission of Solutions: 25. November 2016, 12:00; Discussion: 29. November 2016)

In all exercises $W(t)$ is a Brownian motion with respect to the real world measure $\mathbb{P}$ and $\overline{W}(t)$ is a Brownian motion with respect to an EMM $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_\infty(\gamma \bullet W)$.

Exercise 7. (4 points)

Consider the Hull-White extended Vasiček short-rate dynamics und the EMM $\mathbb{Q} \sim \mathbb{P}$

$$dr(t) = (b(t) + \beta r(t))dt + \sigma d\overline{W}(t).$$

where $\overline{W}(t)$ is standard real-valued $\mathbb{Q}$-Brownian motion, β and $\sigma > 0$ are constants and $b(t)$ is a deterministic continuous function.

a) Use the results derived on the in-tutorial exercise sheet 1 to find the HJM forward rate dynamics that correspond to $r(t)$ under the real world measure $\mathbb{P}$

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^T \sigma(s,T)dW(s)$$

by determining the processes $\alpha(s,T), \sigma(s,T)$ and $f(0,T)$.

b) Verify your findings in a) by checking whether $\alpha(s,T)$ and $\sigma(s,T)$ satisfy the HJM drift condition.

c) Which components of the forward curve rely on $b(t)$? What does this imply for the non extended Vasiček model ($b(t) \equiv b$)?

Exercise 8. (4 points)

An *HJM forward curve evolution by parallel shifts* is a forward curve of the form

$$f(t,T) = h(T-t) + Z(t)$$

for some deterministic initial curve $f(0,T) = h(T)$ and some Itô process $dZ(t) = b(t)dt + \rho(t)d\overline{W}(t)$ with $Z(0) = 0$. Note that this is a special case of the standard HJM model where the coefficients don't depend on T.

a) Show that the HJM drift condition implies $b(t) \equiv b, \rho(t)^2 \equiv a$ and

$$h(x) = -\frac{a}{2}x^2 + bx + c$$

for some constants $a \geq 0$ and $b, c \in \mathbb{R}$.

b) How does this relate to the Ho-Lee model?

c) Use b) to show that all non-trivial forward curve evolutions by parallel shifts are excluded by the HJM drift condition.

Exercise 9. (4 points)

We look at a HJM-model where the volatility is proportional to the forward rate, i.e.

$$\sigma(t,T) = \sigma f(t,T)$$

for some constant $\sigma > 0$.
Show that the forward rate must then satisfy

$$f(t,T) = f(0,T)\exp(\sigma^2 \int_0^t \int_s^T f(s,u)du\ ds + \sigma \overline{W}(t) - \frac{\sigma^2}{2}t).$$