Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
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Summer term 2016

Exercise sheet 11

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: July 11th 2016, 12:15 p.m.; Discussion: July 14th 2016)

Exercise 41. (4 points)

Let B^1, B^2 be independent Brownian motions. We consider the market $S = (S^0, S^1, S^2)^*$ given by

$$\begin{aligned} S_t^0 &= 1, \\ S_t^1 &= \mathcal{E}(-I + B^1 - 2B^2)_t, \\ S_t^2 &= \mathcal{E}(2I - B^1 + 2B^2)_t, \end{aligned}$$

where $I = (I_t)_{t\geq 0}$ is the process with $I_t = t$.

a) Find an arbitrage strategy according to Remark 8.10.

b) Now the asset S^0 is removed from the market. Is $\widetilde{\varphi} = (\varphi^1, \varphi^2)^*$ for φ^1, φ^2 from your arbitrage strategy from part a) still an arbitrage strategy in the new market $\widetilde{S} = (S^1, S^2)^*$? What is the role of the asset S^0?

Exercise 42. (4 points)

Consider the market model $S = (S^0, S^1)^*$ where $S_t^0 = 1$ for $t \in [0, T]$ and S^1 is a continuous adapted process of bounded variation with $S_0^1 = 1$ and $\mathbb{P}(S_T^1 = 1) < 1$. Find an arbitrage strategy.
Hint: Consider a self-financing strategy with $\varphi_t^1 = 2(S_t^1 - 1)$.

Exercise 43. (4 points)

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by the Brownian motion B. Show, that the market $S = (S_0, S_1)^*$ with time horizon T given via

$$\begin{aligned} S_t^0 &= 1, \\ S_t^1 &= 1 + \mu t + \sigma B_t \end{aligned}$$

for constants $\sigma > 0$ and $\mu \in \mathbb{R}$ is arbitrage-free and complete.

Hint: Consider Exercise P.21 to find an equivalent martingale measure and use the martingale representation theorem 6.29.

Remark: A market model where the stock price follows a Brownian motion (with drift) is sometimes called *Bachelier model*.

Exercise 44. (4 points)

Consider the market model $S = (S^0, S^1)^*$ with time horizon T given by

$$S_t^0 = 1,$$
$$S_t^1 = \mathcal{E}(\sigma B)_t$$

for a standard Brownian motion $B = (B_t)_{t \geq 0}$ and a constant $\sigma > 0$. Determine the fair price process and the duplication strategy (you don't have to show that the strategy is allowed) for a *digital option with strike* $K \in (0, \infty)$ which has the pay-off

$$X = \mathbb{1}_{\{S_T^1 > K\}}.$$

Hint: Use Itô's formula on $f(t, S_t^1) = \mathbb{E}[\hat{X} | \mathcal{F}_t]$.