Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
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In-tutorial exercise sheet 4

supporting the lecture Mathematical Finance and Stochastic Integration

(Discussion in the tutorial on May 12th 2016, 2:15 p.m.)

Remark: Two σ-algebras $\mathcal{A}, \mathcal{B}$ are called *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Exercise P.10.

Prove *Kolmogorov's zero-one law*: Let $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ be a sequence of independent σ-algebras and

$$\mathcal{F}_n = \sigma\Big(\bigcup_{m\geq n} \mathcal{A}_m\Big), n \in \mathbb{N}_0.$$

Denote by

$$\mathcal{F}_\infty = \bigcap_{n=0}^{\infty} \mathcal{F}_n$$

the *terminal σ-algebra*. Then $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{F}_\infty$. Proceed as follows:

a) Prove that $\mathcal{F}_{n+1}$ is independent from

$$\mathcal{G}_n = \sigma\Big(\bigcup_{m\leq n} \mathcal{A}_m\Big).$$

Hint: It suffices to look at sets $A = \bigcap_{k=1}^m A_{n_k}$ with $A_{n_k} \in \mathcal{A}_{n_k}$, $n_k \geq n+1$, $m \in \mathbb{N}$ and $B = \bigcap_{k=1}^l B_{n_k}$ with $B_{n_k} \in \mathcal{B}_{n_k}$, $n_k \leq n$, $l \in \mathbb{N}$.

b) Use part a) to deduce $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{F}_\infty$.

Hint: Have a look at the proof of Blumenthal's zero-one law (Theorem 1.18 from the lecture notes).