Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
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Winter term 2016/2017

Exercise sheet 4

supporting the lecture interest rate models

(Submission of Solutions: 2. Dezember 2016, 12:00; Discussion: 5. Dezember 2016)

Exercise 10. (4 points)

Let $0 \leq u \leq t \leq S \wedge T$ for $T, S \geq 0$.

a) Prove equation (4.1) from the lecture:

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} Y \big| \mathcal{F}_u\right] = \frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_u} \mathbb{E}_{\mathbb{P}}[Y|\mathcal{F}_u].$$

b) Complete the last step in the proof of Lemma 4.3 by showing:

$$\frac{\mathcal{E}_t(v(\cdot, S) \bullet \overline{W})}{\mathcal{E}_t(v(\cdot, T) \bullet \overline{W})} = \mathcal{E}_t(\sigma_{S,T} \bullet W^T).$$

Exercise 11. (4 points)

The aim of this exercise is to verify the Dybvig-Ingersoll-Ross theorem for specific models.

a) Show that the Vasiček short-rate model admitts a long rate $R_\infty(t)$ if $\beta \leq 0$. Verify that it is nondecreasing.
b) Same for the CIR model, without restrictions on β.
c) Show that the HJM model with $d = 1$ and $\sigma(t, T) = (1 + T - t)^{-1/2}$ admits a strictly increasing long rate $R_\infty(t)$.

Exercise 12. (4 points)

Let $F(t; T, S)$ be the simple forward rate for the time interval $[T, S]$ prevailing at time t ($t \leq T \leq S$). Prove that $(F(t; T, S))_{t \leq T}$ is a martingale under the forward measure Q^S via two different ways:

a) By using exercise 10.a).
b) By making the connection between a Forward Rate Agreement over the time points t, T, S and an S-Forward where the good exchanged at time S are $\frac{1}{P(T,S)}$ S-Bonds and then using remark 4.16.