Christian-Albrechts-Universität zu Kiel **Summer term 2017**
Mathematisches Seminar
Prof. Dr. M. Vetter

Exercise sheet 3

supporting the lecture on Malliavin Calculus

(Submission of the solutions: June 2, 2017, 10:15 a.m.)

Exercise 7.

Let $H = L^2((0,\tau], \mathcal{B}_{(0,\tau]}, \lambda)$, let W be the corresponding Brownian motion on $(0,\tau]$, and let $\mathcal{G}$ denote the σ-algebra generated by W. Prove with the aid of the martingale representation theorem and Theorem 2.16 that any $F \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ satisfies

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

for appropriate functions $(f_n)_{n \in N_0}$, where $f_0 = \mathbb{E}[F]$ and $I_0(x) = x$.

Remark: Just prove

$$F = \sum_{n=0}^{\infty} I_n(f_n) + R$$

with $R \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ and $\mathbb{E}[I_n(f_n)R] = 0$ for all $n \in \mathbb{N}$. The final step follows with a similar argument on totality as in the proof of Theorem 1.14.

Hint: Once you have established that the adapted process $u(s)$ in the martingale representation

$$F = \mathbb{E}[F] + \int_0^\tau u(s) dW(s)$$

satisfies $u(s)^2 < \infty$ almost everywhere on $(0,\tau)$, you may assume without loss of generality that $u(s)^2 < \infty$ for all $0 < s < \tau$.

Exercise 8.

Let $\{W(h) \mid h \in H\}$ be an isonormal Gaussian process and $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Prove that

$$\langle Df(W(h_1), W(h_2)), h \rangle_H = \lim_{\varepsilon \to 0} \frac{f(W(h_1) + \varepsilon \langle h_1, h \rangle_H, W(h_2) + \varepsilon \langle h_2, h \rangle_H) - f(W(h_1), W(h_2))}{\varepsilon}$$

for any $h_1, h_2, h \in H$.

Exercise 9.

Prove the following product rules:

(a) Let X be in $\mathbb{D}^{1,p}$ and Y be in $\mathbb{D}^{1,q}$ where $\frac{1}{p}+\frac{1}{q}=1$. Prove that $XY \in \mathbb{D}^{1,1}$ and show that the product rule

$$D(XY) = DXY + DYX$$

is true.

(b) Let X, Y be elements of $\mathbb{D}^{1,2}$ such that X and $||DX||_H$ are bounded. Prove that $XY \in \mathbb{D}^{1,2}$ holds as well and show that the product rule

$$D(XY) = DXY + DYX$$

is true.

Hint: For part (b) it is helpful to work with approximating sequences X_n and Y_n for X and Y which converge almost surely as well and are uniformly bounded. Why is this possible?