Stochastics II
Stochastic Processes
Winterterm 2016/17

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Sheet 6

Reminder: A distribution α on E is called *stationary distribution* with respect to the Markov chain X represented by P, if

$$\sum_{x\in E}\alpha(x)P(x,y)=\alpha(y)$$

holds true for all $y\in E$.

Exercise 1
Let $x\in E$ be positive recurrent and define

$$\mu(y):=\sum_{n\in\mathbb{N}_0}\mathbb{P}_x(X_n=y,\tau_x>n),$$
$$\alpha(y):=\frac{\mu(y)}{\mathbb{E}_x(\tau_x)}\ \text{ for all } y\in E.$$

Show that α is a stationary distribution.

Theorem / Definition
Let X be a Markov chain. For $y\in E, n\in\mathbb{N}$ we define

$$N_n(y):=|\{k\in\{1,\dots,n\}\,|X_k=y\}|=\sum_{k=1}^{n}\mathbb{1}_{\{X_k=y\}}$$

„the number of visits in y up to time n “.
Now let X be irreducible. Then for all $y\in E$

$$\frac{N_n(y)}{n}\xrightarrow{a.s.}\frac{1}{\mathbb{E}_y(\tau_y)}$$

holds true.

Exercise 2

Let X be an irreducible, aperiodic Markov chain with stationary distribution α. For all $y \in E$

$$\alpha(y) = \frac{1}{\mathbb{E}_y(\tau_y)}$$

holds true.

Hint: Use **Theorem 40** from the lecture to show that $P^n(x, y) \xrightarrow{n\to\infty} \alpha(y)$.

Exercise 3 (Ehrenfest urn model)

Consider an urn with two separate chambers and a total of N balls. In each step one ball from the urn is chosen at random and placed in the „other“ chamber. Let X_n: the number of balls in the left chamber after the n-th pick.

The process X can be described as a Markov chain with transition matrix

$$P(i,j) = \begin{cases} \frac{i}{N}, & j = i-1 \\ \frac{N-i}{N}, & j = i+1 \\ 0, & \text{else.} \end{cases}$$

(i) Show that P has stationary distribution $\alpha = \texttt{Bin}(N, \frac{1}{2})$.

Now consider the modified transition matrix $Q := \frac{1}{2}(P + I_E)$ with I_E the identity matrix.

(ii) Show that Q has stationary distribution α and $\lim_{n\to\infty} Q^n(y,y) = \alpha(y)$.

(iii) Show that $\mathbb{E}_0(\tau_0) = 2^N$.

Hand in until friday, 09.12.2016, 12:00