Christian-Albrechts-Universität zu Kiel **Winter term 2016**
Mathematisches Seminar
Prof. Dr. M. Vetter, A. Theopold

Exercise sheet 1

supporting the lecture Zismodelle

(Submission of Solutions: 11. October 2016, 12:00; Discussion: 15. October 2016)

Exercise 1. (4 points)

Consider a FRA with current, expiry and maturity time $t < T < S$ and cash flow to the lender:

- At time T: $-K$,
- At time S: $K \exp(R^*(S-T))$

for some predetermined principal K and interest rate R^*. Show that in order for the value of the FRA to equal zero at t, the rate R^* has to equal the forward rate $R(t; S, T)$.

Hint: That means that the continuous forward rate at time t over the time interval $[T, S]$ is the rate s.t. if one charges continuously interest on a nominal $K > 0$ over the period $[T, S]$ neither the lender or the borrower has to pay for the contract.

Exercise 2. (4 points)

It is market practice to price a cap/floor according to *Black's formula*. Black's formula for the value of the ith caplet ($t < T_{i-1} < T_i, \delta = T_{i-1} - T_i$) is

$$Cpl(t; T_{i-1}, T_i) = \delta P(t, T_i)(F(t; T_{i-1}, T_i)\Phi(d_1(i; t)) - \kappa\Phi(d_2(i; t)))$$

where

$$d_{1,2}(i; t) := \frac{\log\left(\frac{F(t;T_{i-1},T_i)}{\kappa}\right) \pm \frac{1}{2}\sigma(t)^2(T_{i-1} - t)}{\sigma(t)\sqrt{T_{i-1} - t}}$$

and $\sigma(t)$ is the *cap implied volatility*, i.e. the value which yields the price of the cap if one knows the reset and settlement date (see chapter 9 of stochastic integration script). Show that assuming $\log(F(T_{i-1}, T_i)) \sim \mathcal{N}(\log(F(0; T_{i-1}, T_i)) - \frac{\sigma^2(0)}{2}T_{i-1}, \sigma^2(0)T_{i-1})$ it holds that

$$\delta P(0, T_i)(F(0; T_{i-1}, T_i)\Phi(d_1(i; 0)) - \kappa\Phi(d_2(i; 0))) = \delta P(0, T_i)\mathbb{E}[(F(T_{i-1}, T_i) - \kappa)^+].$$

Exercise 3. (4 points)

We look at a swap, cap and floor which are determined by the sequence of reset/cash flow dates $0 < T_1 < T_2 < \ldots < T_n$ (T_0 is the reset/maturity date for the swaption) such that $T_i - T_{i-1} = \delta$ and a fixed rate $\kappa > 0$. Let $t \leq T_0$.

a) Show that the cash flow of the ith caplet

$$\delta(F(T_{i-1}, T_i) - \kappa)^+$$

at Time T_i is equivalent to the cash-flow

$$(1 + \kappa\delta)\left(\frac{1}{1 + \kappa\delta} - P(T_{i-1}, T_i)\right)^+$$

at maturity T_{i-1} of a put option on a T_i-bond.

b) Prove the relations (which hold in an arbitrage free market, the index indicates "payer" or "receiver")

$$Cp(t) - Fl(t) = \Pi_p(t) \text{ and } Swpt_p(t) - Swpt_r(t) = \Pi_p(t).$$

for swaps with nominal one and cap/floor both with the same rate as the swap κ.