Christian-Albrechts-Universität zu Kiel **Winter term 2015/16**
Mathematisches Seminar
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Exercise sheet 9

supporting the lecture Mathematical Statistics

(Submission of Solutions: 25. January 2016, 12:00 o'clock; Discussion: 27. January 2016)

Exercise 1. (4 points)

A dice should be tested for the probability ϑ with which its top face shows a six. We throw the dice until a six occurs for the first time.

a) State the propability with which the six occurs for the first time in the k-th throw.

b) Construct an UMP test φ_α with level $\alpha = 11/36$ for the test problem

$$H \;:\; \vartheta = 1/6 \quad \text{against} \quad K \;:\; \vartheta = \vartheta_1$$

with fixed $\vartheta_1 \in (1/6, 1)$.

c) Is φ_α also an UMP test with level $\alpha = 11/36$ for

$$H \;:\; \vartheta = 1/6 \quad \text{against} \quad K \;:\; \vartheta > 1/6 \text{ ?}$$

Explain your answer.

Exercise 2. (4 points)

a) Let X be an observation in $(0, 1)$. Derive an UMP test with level α for the hypothesis

$$H : \text{X has density } f(x) = 4x\mathbb{1}_{(0,\frac{1}{2})}(x) + (4-4x)\mathbb{1}_{[\frac{1}{2},1)}(x)$$

against the alternative

$$K : \text{X is uniformly distributed } \sim \mathcal{U}(0,1)$$

b) Let X be a random variable with density $f_\vartheta(x) = \frac{2(\vartheta - x)}{\vartheta^2}\mathbb{1}_{(0,\vartheta)}(x)$. Derive an UMP test with level α for the hypothesis $H : \vartheta = \vartheta_0$ against the alternative $K : \vartheta = \vartheta_1$ for given $\vartheta_1 < \vartheta_0$.

Exercise 3. (4 points)

Let $\Phi = \{\varphi | \varphi : \mathcal{X} \to [0,1]\}$ be the set of all tests on $\mathcal{X}$ for the simple hypotheses $H : \vartheta = \vartheta_0$ against $K : \vartheta = \vartheta_1$. By $\beta_\varphi : \vartheta \mapsto \mathbb{E}_\vartheta[\phi]$ we denote the power function of a test $\varphi \in \Phi$.

a) Prove that the set

$$\mathcal{G} = \{(\beta_\varphi(\vartheta_0), 1 - \beta_\varphi(\vartheta_1)) | \varphi \in \Phi\}$$

is convex and point-symmetric with respect to $(1/2, 1/2)$ and contains the elements $(0,1)$ and $(1,0)$.

b) Consider now $X \sim Exp(\vartheta)$, $\vartheta_0 = 1$ and $\vartheta_1 = 2$. Sketch $\mathcal{G}$ with help of the Neyman-Pearson lemma.

Exercise 4. (4 points)

Consider the situation from exercise 1 of in-tutorial exercise sheet 9.

a) Derive an UMP Test with level α for the one-sided hypothesis

$$H : p \leq p_0 \text{ against } K : p > p_0.$$

b) A recycling company guarantees a soft drink producer that 99,9% of all reused bottles are completely clean after a special cleaning procedure. To ensure this the company has a quality control before the bottles delivery, in which 10 from a delivery of 100.000 bottles randomly taken bottles are tested. The recycling company suggests to perform a test for the hypotheses

$$H : p \leq 0.001 \text{ against } K : p > 0.001$$

at level $\alpha = 0.05$.
First argue that the experiment can be adequately modelled by an iid-model of n Bernoulli distributed random variables.

c) Whats the test decision, if among 10 tested bottles there is no dirty bottle?

d) The soft drink producer is skeptical. Is his skepticism justified? How should an appropriate test problem be formulated from his perspective? What would be the test decision for that test?

e) How big has the sample size n of the tested bottles to be chosen, such that the guarantee of the recycling company can be verified at the level $\alpha = 0.05$, if within the n tested bottles there is one dirty bottle?