Exercise sheet 4

supporting the lecture on Malliavin Calculus

(Submission of the solutions: June 23, 2017, 10:15 a.m.)

Exercise 10.

(a) Let $t_1, \dots, t_n \in [0,1]$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Show that $F : C[0,1] \to \mathbb{R}$ with

$$F(\omega) = f(\omega(t_1), \dots, \omega(t_n))$$

is Fréchet differentiable with derivative operator

$$DF_\omega(\nu) = \sum_{i=1}^{n} \partial_i f(\omega(t_1), \dots, \omega(t_n))\nu(t_i).$$

(b) Let μ be the Wiener measure on $C[0,1] \subset L^2[0,1]$ and let ω be a realization of μ. Prove that the Malliavin derivative DF of

$$F = f(\omega(t_1), \dots, \omega(t_n))$$

satisfies

$$\langle DF, h \rangle_H = DF_\omega(\bar{h})$$

with $\bar{h}(t) = \int_0^t h(s)ds$ and $H = L^2[0,1]$, and where DF_ω denotes the Fréchet derivative from part (a).

Exercise 11.

Let H be a separable Hilbert space and $X \in \mathcal{S}$. Prove that the operator $D_h X = \langle DX, h \rangle_H$ satisfies

$$D_{h_2}(D_{h_1} X) = \langle D^2 X, h_1 \otimes h_2 \rangle_{H^{\otimes 2}}$$

for all $h_1, h_2 \in H$.

Exercise 12.

Let H be a separable Hilbert space and $h_1, h_2 \in H$. Furthermore, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of elements from $\mathcal{S}$ such that $X_n \to 0$ in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $D^2 X_n \to \xi$ in $L^p(\Omega; H^{\otimes 2})$. Prove that

$$\mathbb{E}[Y \langle \xi, h_1 \otimes h_2 \rangle_{H^{\otimes 2}}] = 0$$

for $Y = \exp(-\varepsilon(W(h_1)^2 + W(h_2)^2))$, any $\varepsilon > 0$.