Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
Prof. Dr. M. Vetter, A. Theopold

In-tutorial exercise sheet 0

supporting the lecture interest rate models

(Discussion in the tutorial on 8. November 2016, 14:15 Uhr)

Let $b : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be Prog $\otimes \mathcal{B}(\mathbb{R}^n)$-measurable functions. Let ξ be some $\mathcal{F}_0$-measurable initial value. A process X is said to be a solution[6] of the stochastic differential equation

$$\begin{aligned} dX(t) &= b(t, X(t))\,dt + \sigma(t, X(t))\,dW(t), \\ X(0) &= \xi \end{aligned} \tag{4.3}$$

if X is an Itô process satisfying

$$X(t) = \xi + \int_0^t b(s, X(s))\,ds + \int_0^t \sigma(s, X(s))\,dW(s).$$

We say that X is unique if any other solution X' of (4.3) is indistinguishable from X, that is, $X(t) = X'(t)$ for all $t \geq 0$ a.s.

If $b(\omega, t, x) = b(t, x)$ and $\sigma(\omega, t, x) = \sigma(t, x)$ are deterministic functions, a solution X of (4.3) is also called a (time-inhomogeneous) diffusion with diffusion matrix $a(t, x) = \sigma(t, x)\sigma(t, x)^\top$ and drift $b(t, x)$.

Here is a basic existence and uniqueness theorem for diffusions.

Theorem 4.4 *Suppose $b(t, x)$ and $\sigma(t, x)$ satisfy the Lipschitz and linear growth conditions*

$$\begin{aligned} \|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| &\leq K\|x - y\|, \\ \|b(t, x)\|^2 + \|\sigma(t, x)\|^2 &\leq K^2(1 + \|x\|^2), \end{aligned}$$

for all $t \geq 0$ and $x, y \in \mathbb{R}^n$, where K is some finite constant. Then, for every time–space initial point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a unique solution $X = X^{(t_0,x_0)}$ of the stochastic differential equation

$$\begin{aligned} dX(t) &= b(t_0 + t, X(t))\,dt + \sigma(t_0 + t, X(t))\,dW(t), \\ X(0) &= x_0. \end{aligned} \tag{4.4}$$

Theorem 4.5 *Suppose $b(t,x)$ and $a(t,x) = \sigma(t,x)\sigma(t,x)^\top$ are continuous in (t,x), and assume that for every time–space initial point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a unique solution $X = X^{(t_0,x_0)}$ of the stochastic differential equation (4.4). Then X has the Markov property. That is, for every bounded measurable function f on $\mathbb{R}^n$, there exists a measurable function F on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ such that*

$$\mathbb{E}[f(X(T)) \mid \mathcal{F}_t] = F(t, T, X(t)), \quad t \leq T.$$

In words, the $\mathcal{F}_t$-conditional distribution[7] of $X(T)$ is a function of t, T and $X(t)$ only.

Exercise P.1.

Let $a, \mu, \theta, \sigma \in \mathbb{R}$, a stochastic process $(X(t))_{t\geq 0}$ is called *Ornstein-Uhlenbeck-Process* with starting point a, if it solves the SDE:

$$\begin{aligned} &dX(t) = \theta(\mu - X(t))dt + \sigma dW(t) \\ &X(0) = a, \end{aligned}$$

where $(W(t))_{t\geq 0}$ is a standard Brownian motion.
Show that

$$X(t) = a\exp(-\theta t) + \mu(1 - \exp(-\theta t)) + \int_0^t \sigma \exp(\theta(s-t))dW(s)$$

is the unique solution to the SDE above.