Christian-Albrechts-Universität zu Kiel **Winter term 2015/16**
Mathematisches Seminar
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Exercise sheet 7

supporting the lecture Mathematical Statistics

(Submission of Solutions: 11. January 2016, 12:00 o'clock; Discussion: 13. January 2016)

Exercise 1. (2 points)

Let X be a random variable with values in $\mathcal{X}$ and distribution $P_\vartheta, \vartheta \in \Theta$. A statistic $S : \mathcal{X} \to \mathcal{T}$ is called *distribution free* if its distribution does not depend on ϑ. It is called *distribution free of first order* if its expectation $\mathbb{E}_\theta[S(X)]$ is independent of ϑ. Show that:
A statistic $T : \mathcal{X} \to \mathcal{T}$ is complete if and only if there exists no nonconstant function $f : \mathcal{T} \to \tilde{\mathcal{T}}$ such that $f(T)$ is distribution free of first order.

Exercise 2. (2 points)

Let $P = \{\mathcal{P}_\vartheta \mid \vartheta \in \Theta\}$ be a family of probability distributions and $T : \mathcal{X} \to \mathbb{R}^d$ a sufficient and complete statistic for ϑ. Show that T is minimal sufficient if a minimal sufficient statistic exists.

Exercise 3. (4 points)

Prove the following statement: Let $P_\vartheta,\ \vartheta \in \Theta$ be a family of equivalent measures with μ-densities f_ϑ. Let $T : (\mathcal{X}, \mathcal{B}) \to (\mathcal{T}, \mathcal{D})$ be a statistic with the property that the fraction

$$\frac{f_\vartheta(x)}{f_\vartheta(y)}$$

does not depend on ϑ if and only if $T(x) = T(y)$. Than T is minimal sufficient.
Use the following steps:

a) First show that T is sufficient. Consider

$$f_\vartheta(x) = \frac{f_\vartheta(x)}{f_\vartheta(y)} f_\vartheta(y)$$

for an adequate y.

b) Prove that for every sufficient statistic S the implication

$$S(x) = S(y) \Rightarrow T(x) = T(y)$$

holds and conclude the claim.

Exercise 4. (4 points)

Let $X_1, \dots, X_n$ be i.i.d $N(\mu, \sigma^2)$-distributed and $Y_1, \dots, Y_m$ be i.i.d $N(\eta, \tau^2)$-distributed. In addition $X = (X_1, \dots X_n)^T$ and $Y = (Y_1, \dots Y_n)^T$ are independent. Find minimal sufficient statistics for the parameter $(\mu, \eta, \sigma, \tau)$ of the distribution of the vector

$$Z := (X_1, \dots, X_n, Y_1, \dots, Y_m)^T$$

in the following three cases:

a) μ, η, σ, τ are arbitrary with $\mu, \eta \in \mathbb{R}; \sigma, \tau > 0$,

b) $\sigma = \tau$ and μ, η, σ arbitrary and as in a),

c) $\mu = \eta$ and μ, σ, τ arbitrary and as in a).

Exercise 5. (4 points)

Prove the following variant of Slutsky's theorem: Let (X_n) and (Y_n) be series of real-valued random variables with

$$X_n \xrightarrow{\mathcal{L}} X \text{ and } Y_n \xrightarrow{\mathbb{P}} c$$

for a real-valued random variable X and a constant $c \in \mathbb{R}$. It holds:

$$X_n + Y_n \xrightarrow{\mathcal{L}} X + c$$

Hint: You may use the following characterization of convergence in distribution: A series (Z_n) of real-valued random variables converges in distribution to a random variable Z if and only if

$$\mathbb{E}[f(Z_n)] \to \mathbb{E}[f(Z)]$$

for all uniformally continuous and bounded functions $f : \mathbb{R} \to \mathbb{R}$

Exercise 6. (4 extra - points)

Create a *Mind-Map* in which you link the following 25 terms with each other using arrows and short explaining commentaries.

1) A posteriori distribution
2) A priori distribution/prior distribution
3) Bayes estimator
4) Conditional expectation
5) Cramér-Rao inequality
6) Dominating measure
7) Efficient estimator
8) Unbiased estimator
9) Exponential family
10) Factorization Theorem
11) Consistent estimator
12) Maximum-Likelihood estimator
13) Minimal sufficient statistic
14) Minimax estimator
15) Method of Moments
16) Neyman criterion
17) Risk
18) Lehmann-Scheffé Theorem
19) Radon-Nikodým Theorem
20) Rao-Blackwell Theorem
21) Sufficient statistic
22) UMVU estimator
23) Least favorable prior distribution
24) Loss function
25) Complete statistic