Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
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Summer term 2016

Exercise sheet 4

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: May 23rd 2016, 12:15 p.m.; Discussion: May 26th 2016)

Exercise 13. (4 points)

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. For $a > 0$ define $\tau_a = \inf\{t \geq 0 : B_t = a\}$ and $M_t = B_t^{\tau_a} = B_{t\wedge\tau_a}$. Prove

a) $\sup_{t\geq 0} \mathbb{E}[|M_t|] < \infty$ and $M_t \to a$ a.s. as $t \to \infty$.
Hint: $|M_t| = 2(M_t)^+ - M_t$.

b) $(M_t)_{t\geq 0}$ does not converge to a in L^1.

Exercise 14. (4 points)

Let $(X_n)_{n\in\mathbb{N}_0}$ denote a sequence of i.i.d. random variables with values in $\{-1, 0, 1\}$ and $\mathbb{P}(X_n = -1) = \mathbb{P}(X_n = 1) = 1/2n$, $\mathbb{P}(X_n = 0) = 1 - 1/n$. Define a process via $Y_0 = X_0$ and

$$Y_n = \begin{cases} X_n & \text{if } Y_{n-1} = 0, \\ nY_{n-1}|X_n| & \text{if } Y_{n-1} \neq 0 \end{cases}, \; n \geq 1.$$

a) Prove that $(Y_n)_{n\in\mathbb{N}_0}$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(Y_m : m \leq n)$.

b) Prove that $(Y_n)_{n\in\mathbb{N}_0}$ converges in probability, but does not converge almost surely.
Hint: Use the Borel-Cantelli lemma (see below) to show $\mathbb{P}(Y_n \neq 0$ for infinitely many $n \in \mathbb{N}) = 1$.

Exercise 15. (4 points)

Prove the strong law of large numbers: Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\mu := \mathbb{E}[X_1] < \infty$. Then it holds

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_n \to \mu$$

almost surely.

Hint: Show that $S_n = \overline{X}_{-n}$ is a backward martingale w.r.t. the filtration generated by $(S_n)_{n\leq 0}$. Show that the limit $S_{-\infty}$ of the backward martingale S_n is almost surely constant using Exercise P.10.

Exercise 16. (4 points)

Let $(X_t)_{t\in\mathbb{N}}$ be a martingale with $\mathbb{E}[X_0] = 0$ and $\mathbb{E}[(X_t)^2] < \infty$ for all $t \in \mathbb{N}$. Prove

$$\mathbb{P}\Big(\max_{1\leq s\leq t} X_s > x\Big) \leq \frac{\mathbb{E}[(X_t)^2]}{\mathbb{E}[(X_t)^2] + x^2}, \quad x > 0.$$

Hint: Consider the submartingale $(X_t + c)^2$ for a suitable $c > 0$.

Remark: The *Borel-Cantelli lemma* as stated in the lecture 'probability theory' (summer term 2015): Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events in $(\Omega, \mathcal{A}, \mathbb{P})$.

(i) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, it holds

$$\mathbb{P}(\{\omega | \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = \mathbb{P}(\limsup_{n\to\infty} A_n) = 0.$$

(ii) If $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, it holds

$$\mathbb{P}(\{\omega | \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = \mathbb{P}(\limsup_{n\to\infty} A_n) = 1.$$