Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
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Exercise sheet 7

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: June 13th 2016, 12:15 p.m.; Discussion: June 16th 2016)

Exercise 25. (4 points)

For deterministic functions $f \in C^1([0,1])$ with $f(1) = 0$ the *Paley-Wiener integral* is defined via

$$\int_0^1 f(t) * dB_t := -\int_0^1 f'(t) B_t dt$$

where the integral on the right hand side is pathwise a classical Riemann integral.

a) Prove the identity

$$\mathbb{E}\Big[\Big(\int_0^1 f(t) * dB_t\Big)^2\Big] = \int_0^1 (f(t))^2 dt.$$

Hint: Use Fubini to interchange integration with respect to ω and integration with respect to t.

b) Use part a) to define the Paley-Wiener integral for all $f \in L^2([0,1], \lambda_{[0,1]})$, where $\lambda_{[0,1]}$ denotes the Lebesgue measure on $[0,1]$.

Hint: You may use without proof, that the set $\{f \in C^1([0,1]) : f(1) = 0\}$ is dense in $L^2([0,1], \lambda_{[0,1]})$.

Exercise 26. (4 points)

Let $(B_s)_{s \geq 0}$ be a Brownian motion, $H \in \mathcal{L}^2(B)$ and $t > 0$.

a) Show

$$\mathbb{E}\Big[\int_0^t H dB\Big] = 0, \quad Var\Big[\int_0^t H dB\Big] = \int_0^t \mathbb{E}[(H_s)^2] ds.$$

b) Let H be deterministic and left-continuous. Show

$$\int_0^t H dB \sim \mathcal{N}(0, \int_0^t (H_s)^2 ds).$$

Hint: Find a sequence $(H^n)_n \subset \mathcal{E}$ of processes H^n which approximate H and use Lévy's continuity theorem.

Exercise 27. (4 points)

A process $(N_t)_{t\geq 0}$ is called *Poisson-Process with parameter* $\lambda > 0$, if $N_0 = 0$ and the process has independent increments with $N_{t+h} - N_t \sim Poisson(\lambda h)$ for all $t \geq 0, h > 0$.

a) Show that $M_t = N_t - \lambda t$ defines a martingale with respect to its natural filtration.

b) Determine the Doléans measure μ_M.
Hint: Start by computing $\mu_M(R)$ for $R \in \mathcal{R}$.

Exercise 28. (4 points)

Prove Lemma 4.32 from the lecture notes: Let X be a (right-continuous) L^2-martingale, $H \in \mathcal{L}^2(X)$, $s < t$ and Z bounded, $\mathcal{F}_s$-measurable. Then it holds

$$\int Z \mathbb{1}_{(s,t]} H dX = Z \int \mathbb{1}_{(s,t]} H dX.$$

Hint: First, consider the statement for $Z = \mathbb{1}_F$, $F \in \mathcal{F}_s$, and $H \in \mathcal{E}$. Extend the result for $Z = \mathbb{1}_F$ to all $H \in \mathcal{L}^2(X)$ and finish the proof by using measure-theoretic induction to show the result for arbitrary Z.