Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
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Exercise sheet 5

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: May 30th 2016, 12:15 p.m.; Discussion: June 2nd 2016)

Exercise 17. (4 points)

Prove the following statement, which was used in the proof of Wald's second lemma: Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $\sigma \leq \tau$ stopping times with $\mathbb{E}[\tau] < \infty$. Then it holds

$$\mathbb{E}[(B_\tau)^2] = \mathbb{E}[(B_\sigma)^2] + \mathbb{E}[(B_\tau - B_\sigma)^2].$$

Remark: This yields the identity $\mathbb{E}[(B_\tau)^2] \geq \mathbb{E}[(B_\sigma)^2]$ from the optional sampling theorem for the submartingale $((B_t)^2)_{t\geq 0}$ for stopping times with finite expectation.

Exercise 18. (4 points)

In this exercise you are supposed to explicitly compute the stopping time τ from the Skorokhod embedding theorem for a simple random variable X which is uniformly distributed on $\{-4, -2, 2, 4\}$ (i.e. $\mathbb{P}(X = -4) = \mathbb{P}(X = -2) = \mathbb{P}(X = 2) = \mathbb{P}(X = 4) = 1/4$).

a) Construct a binary splitting martingale $(X_n)_{n\in\mathbb{N}_0}$ which converges to X following the proof of Lemma 3.7.

   Hint: It suffices to compute X_1, X_2 because X only takes on $4 = 2^2$ different values.

c) Define the stopping time τ recursively as in the proof of Theorem 3.8.

Exercise 19. (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $Var[X_1] = 1$ and $S_n = \sum_{i=1}^n X_i$. $(B_t)_{t\geq 0}$ is a standard Brownian motion. Prove for $a \geq 0$

$$\mathbb{P}\Big(\sup_{k\leq n} \frac{S_k}{\sqrt{n}} \leq a\Big) \to \mathbb{P}\Big(\sup_{t\in[0,1]} B_t \leq a\Big) \quad (n \to \infty).$$

Hint: Show that the function $h : (C[0,1], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|), f \mapsto \sup_{x\in[0,1]} f(x)$ is continuous and apply the continuous mapping theorem.

Exercise 20. (4 points)

Let M^* be defined via $B_{M^*} = \sup_{t\in[0,1]} B_t$. M^* equals the time where the Brownian motion B attains it's maximum. Prove the *arcsine-law*

$$\mathbb{P}(M^* < x) = \frac{2}{\pi}\arcsin(\sqrt{x}),\ \ x \in [0,1].$$

Proceed as follows:

a) Show

$$\mathbb{P}(M^* < s) = \mathbb{P}(M_s^{(1)} < M_{1-s}^{(2)})$$

with $M_s^{(i)} = \sup_{u\le s} B_u^{(i)}$, $i = 1, 2$, for independent Brownian motions $B^{(1)}, B^{(2)}$.

Hint: $X_t = B_{s-t} - B_s$, $t \in [0, s]$, is a Brownian motion on $[0, s]$.

b) Show

$$\mathbb{P}(M_s^{(1)} < M_{1-s}^{(2)}) = \frac{2}{\pi}\arcsin(\sqrt{s}).$$

Hint: You can use that for independent standard normally distributed random variables Z_1, Z_2 it holds

$$\mathbb{P}\Big(\frac{|Z_2|}{\sqrt{|Z_1|^2 + |Z_2|^2}} < x\Big) = \frac{2}{\pi}\arcsin(x).$$

c) Plot the density of M^* on $[0,1]$ and describe what you see!