Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
Prof. Dr. M. Vetter, A. Theopold **Winter term 2016/2017**

Exercise sheet 2

supporting the lecture interest rate models

(Submission of Solutions: 18. November 2016, 12:00; Discussion: 22. November 2016)

Exercise 4. (4 points)

Let $W = (W_1, \ldots, W_n)$ be a d-dimensional Brownian motion and $X = (X_1, \ldots, X_n)$ the unique solution to the system of SDE's

$$dX_i(t) = cX_i(t)dt + \rho dW_i(t), X_i(0) = x_i, i = 1, \ldots, d$$

for some real coefficients c and ρ. Show that there exists a Brownian motion B such that the nonnegative process

$$Y := X_1^2 + \ldots + X_d^2$$

satisfies the SDE

$$dY(t) = (b + \beta Y(t))dt + \sigma\sqrt{Y(t)}dB(t)$$

where $b = d\rho^2, \beta = 2c$ and $\sigma = 2\rho$.

Hint: Use Lêvy's characterization theorem of a Brownian motion to show that $dB = \sum_{i=1}^n \frac{X_i}{\sqrt{Y}} dW_i$ defines a Brownian motion.

Exercise 5. (4 points)

We take a diffusion short-rate model with $\mathbb{Q}$-dynamics as specified in (2.3). Consider a T-claim $\Phi(r(T))$ and assume that $F(t,r))$ fulfills the assumptions of Theorem 2.8. Let $\Pi(t) = F(t, r(t))$ be the price process of the claim. Show that the local rate of return is equal to the short rate, i.e. it is of the form

$$\frac{d\Pi(t)}{\Pi(t)} = r(t)dt + \sigma_\Pi(t)d\overline{W}(t).$$

and try to interpret your representation of $\sigma_\Pi(t)$.

Exercise 6. (4 points)

The Hull-White model considered in this exercise is an extension of the Vasiček Model by letting the parameter b be dependent on time β and σ are still constants:

$$dr(t) = (b(t) + \beta r(t))dt + \sigma d\overline{W}(t).$$

As in the Ho-Lee Model we chose $b(t)$ such that we match the initial forward curve. The solutions of $A(t,T)$ and $B(t,T)$ in the ATS equations are give by

$$\begin{aligned} A(t,T) &= -\frac{\sigma^2}{2}\int_t^T B(s,T)^2 ds + \int_t^T b(s)B(s,T)ds, \\ B(t,T) &= \frac{1}{\beta}(e^{\beta(T-t)} - 1). \end{aligned}$$

Using $\partial_T B(s,T) = -\partial_s B(s,T)$ we have that

$$f_0(T) = \underbrace{-\frac{\sigma^2}{2\beta}(e^{\beta T} - 1)^2}_{=:g(T)} + \underbrace{\int_0^T b(s)e^{\beta(T-s)}ds + e^{\beta T}r(0)}_{=:\phi(T)}.$$

The function $\phi(t)$ satisfies

$$\partial_T \phi(T) = \beta\phi(T) + b(T), \;\; \phi(0) = r(0)$$

and therefore

$$\begin{aligned} b(T) &= \partial_T \phi(T) - \beta\phi(t) \\ &= \partial_T(f_0(T) + g(T)) - \beta(f_0(T) + g(T)). \end{aligned}$$

Show that the forward curve at time t can be written as

$$f(t,T) = f_0(T) - e^{\beta(T-t)}f_0(t) - \frac{\sigma^2}{2\beta}(e^{\beta(T-t)} - 1)(e^{\beta(T-t)} - e^{\beta(T+t)}) + e^{\beta(T-t)}r(t).$$