**Christian-Albrechts-Universität zu Kiel** **Summer term 2016**
**Mathematisches Seminar**
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# Exercise sheet 10

**supporting the lecture Mathematical Finance and Stochastic Integration**

(Submission of Solutions: July 4th 2016, 12:15 p.m.; Discussion: July 7th 2016)

**Exercise 37.** (4 points)

a) Let $(B_t)_{t\geq 0}$ be a Brownian motion and $f \in C^1([0,\infty),\mathbb{R})$. Use integration by parts to find a semimartingale representation $X = M + A$ for $X = (f(t)B_t)_{t\geq 0}$.

b) Show that the definition of the Paley-Wiener integral (compare Exercise 25) is consistent with the definition of the stochastic integral, i.e. show

$$\int_0^1 f(t) * dB_t = \int_0^1 f(t)dB_t$$

for all $f \in C^1([0,1])$ with $f(1) = 0$.

**Exercise 38.** (4 points)

Let $N = (N_t)_{t\in[0,T]}$ be a Poisson process with intensity 1 (compare Exercise 27) under the probability measure $\mathbb{P}$. For $\lambda > 0$ we define a probability measure $\mathbb{Q}$ via the $\mathbb{P}$-density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp((1-\lambda)T + N_T \log(\lambda)).$$

Show, that the above defines a valid density process and that under $\mathbb{Q}$ $N$ is a Poisson process with intensity $\lambda$.

*Hint:* The moment generating function of a random variable $Z \sim Poisson(\lambda)$ is given by

$$\mathbb{E}[\exp(uZ)] = \exp(\lambda(e^u - 1)), \quad u \in \mathbb{R}.$$

The moment generating function characterizes a probability distribution uniquely just like the characteristic function, if the moment generating function exits for all $u \in \mathbb{R}$.

**Exercise 39.** (4 points)

Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent measures on $\mathcal{F}_T$ with density process $L = (L_t)_{t\in[0,T]}$.

a) Prove, that if $X = (X_t)_{t\in[0,1]}$ is a semimartingale with respect to $\mathbb{P}$, then $X$ is also a semimartingale with respect to $\mathbb{Q}$.

b) Prove, that the stochastic integral is invariant under equivalent measure transformations i.e. if $H$ is in $\mathcal{L}(X)$ under $\mathbb{P}$ (notation: $H \in \mathcal{L}_{\mathbb{P}}(X)$) it is also in $\mathcal{L}(X)$ under $\mathbb{Q}$ (i.e $H \in \mathcal{L}_{\mathbb{Q}}(X)$) and the integrals

$$\int H d_{\mathbb{P}} X, \int H d_{\mathbb{Q}} X$$

defined under $\mathbb{P}$ and $\mathbb{Q}$ are $\mathbb{P}/\mathbb{Q}$-a.s. identical for continuous $H$.

*Hint:* Use $\int H^m d_{\mathbb{P}} M \xrightarrow{\mathbb{P}} \int H d_{\mathbb{P}} M$ as $m \to \infty$ for $L^2$-martingales $M$, $H \in \mathcal{L}(M)$ and suitable $(H^m)_{m\in\mathbb{N}}$, which was used in the proof of Theorem 6.4.

**Exercise 40.** (4 points)

Consider the market model $S = (S^0, S^1)^*$ given by

$$\begin{aligned} S_t^0 &= \mathcal{E}(Z^0)_t, & Z_t^0 &= \rho t, \\ S_t^1 &= s_0 \mathcal{E}(Z^1)_t, & Z_t^1 &= \mu t + \sigma B_t, \end{aligned}$$

with $\rho, \sigma, s_0 > 0$, $\mu \in \mathbb{R}$ and $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. For a self-financing strategy $\varphi = (\varphi^0, \varphi^1)$ with value process $V = V(\varphi)$ it holds $V > 0$. Show that with

$$\pi_t = \frac{\varphi_t^1 S_t^1}{V_t}$$

it holds

$$V_t = V_0 \mathcal{E}\Big( \int (1-\pi) dZ^0 + \int \pi dZ^1 \Big)_t.$$