Christian-Albrechts-Universität zu Kiel **Winter term 2015/16**
Mathematisches Seminar
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Exercise sheet 10

supporting the lecture Mathematical Statistics

(Submission of Solutions: 1. February 2016, 12:00 o'clock; Discussion: 3. February 2016)

Exercise 1. (4 points)

Let $X_1, \dots, X_n$ be independent identically $\mathcal{N}(\mu, \sigma^2)$ distributed with known mean $\mu \in \mathbb{R}$ and unknown variance $\sigma^2 > 0$. Construct an UMP test with level $\alpha \in (0, 1)$ for the hypotheses

$$H \ : \ \sigma^2 \leq \sigma_0^2 \quad \text{against} \quad K \ : \ \sigma^2 > \sigma_0^2.$$

Exercise 2. (4 points)

Proof the following extension of example 6.18 from the lecture notes. Let $X_1, \dots, X_n$ be iid $\mathcal{U}(0, \vartheta)$ distributed, $X = (X_1, \dots, X_n)$ and $\vartheta_0 > 0$.

a) For testing the hypothesis $H \ : \ \vartheta \leq \vartheta_0$ against the alternative $K \ : \ \vartheta > \vartheta_0$ every test φ, which fulfills the following properties

   (i) $\mathbb{E}_{\vartheta_0}[\varphi(X)] = \alpha$

   (ii) $\mathbb{E}_{\vartheta}[\varphi(X)] \leq \alpha$ for all $\vartheta \leq \vartheta_0$

   (iii) $\varphi(x) = 1$, if $x_{(n)} > \vartheta_0$

   is an UMP test with level α.

b) For testing the hypothesis $H \ : \ \vartheta = \vartheta_0$ against the alternative $K \ : \ \vartheta \neq \vartheta_0$ there exists a P_{ϑ_0}-a.s. unique UMP test with level α. It is P_{ϑ_0}-a.s. of the form:

$$\varphi(x) = \begin{cases} 1, & \text{if } x_{(n)} > \vartheta_0 \text{ or } x_{(n)} < \vartheta_0 \sqrt[n]{\alpha} \\ 0, & \text{otherwise} \end{cases}$$

Hint: Regarding part b): Derive the form of an UMP test with level α for testing the hypothesis $H \ : \ \vartheta = \vartheta_0$ against the alternative $K \ : \ \vartheta < \vartheta_0$ and combine this result with the previous results.

Exercise 3. (4 points)

Have a look again at the situation from exercise 1 of in-tutorial exercise sheet 10: Let $X_1, \ldots, X_n$ be independent exponentially distributed with parameter $\vartheta > 0$, i.e. the density of X_i is given by

$$f_\vartheta(x) = \vartheta \exp(-\vartheta x) 1_{[0,\infty)}(x).$$

a) We consider the hypotheses

$$H \;:\; \vartheta = \vartheta_0 \quad \text{against} \quad K \;:\; \vartheta \neq \vartheta_0.$$

Sketch the construction of a confidence region $[\overline{X}_n - a, \overline{X}_n + a]$, $a > 0$ of level α for $\gamma(\vartheta) = 1/\vartheta$ which is symmetrical with respect to $\overline{X}_n$. How could one use this confidence region to construct a test for the hypotheses above?

b) Use a normal approximation to derive a symmetrical confidence region in the situation from part a) for $n = 100$ and $\alpha = 0.1$.

Hint: A *confidence region of level* $\alpha$for a function $\gamma : \Theta \to \Gamma$ is a random set $c(X) \subset \Gamma$, such that

$$P_\vartheta\left(\gamma(\vartheta) \in c(X)\right) \geq 1 - \alpha$$

holds for all $\vartheta \in \Theta$.

Exercise 4. (4 points)

We further consider the situation from exercise 1 of in-tutorial exercise sheet 10.

a) Set $n = 1$. Develope an UMPU test with level $\alpha = 0.1$ for the hypotheses

$$H \;:\; \vartheta \in [1, 2] \quad \text{against} \quad K \;:\; \vartheta \notin [1, 2].$$

Hint: Define $a := \exp(-c_1)$, $b := \exp(-c_2)$ and find the values for a and b first.

b) Theoretically we can apply theorem 6.26 from the lecture notes in a special case to derive an UMPU test for the hypotheses

$$H \;:\; \vartheta = \vartheta_0 \quad \text{against} \quad K \;:\; \vartheta \neq \vartheta_0.$$

What problems do occur? How could one deal with them?