Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
Prof. Dr. M. Vetter, O. Martin

Exercise sheet 8

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: June 20th 2016, 12:15 p.m.; Discussion: June 23th 2016)

Exercise 29. (4 points)

Let $(B_t)_{t\geq 0}$ be a Brownian motion and τ a stopping time. Compute $\mu_B([0,\tau])$.

Hint: Approximate τ by bounded stopping times τ_m and use the isometry from Theorem 4.27 for $\mathbb{1}_{[0,\tau_m]}$.

Exercise 30. (4 points)

Let $(X_t)_{t\geq 0}$ be a (right-continuous) local L^2-martingale and $(H_t)_{t\geq 0}$ a locally bounded $\mathcal{P}$-measurable process. Prove

$$H \in \mathcal{L}(X).$$

Exercise 31. (4 points)

Let $(X_t)_{t\geq 0}$ be a (right-continuous) local L^2-martingale with localizing sequence $(\tau_n)_{n\in\mathbb{N}}$. Let

$$((X_{t\wedge\tau_n})^2)_{n\in\mathbb{N}}$$

be uniformly integrable for any $t \geq 0$. Prove, that X is a L^2-martingale.

Hint: 1. You may use without proof, that Theorem 2.24 also holds with L^1 replaced by L^2 and $(X_n)_{n\in\mathbb{N}}$ uniformly integrable replaced by $((X_n)^2)_{n\in\mathbb{N}}$ uniformly integrable.
2. The conditional expectation $\mathbb{E}[Y|\mathcal{F}]$ is the best L^2-approximation to Y within the set
$\mathcal{M} = \{Z|Z \text{ is } \mathcal{F}\text{-measurable}\}$, i.e.

$$\mathbb{E}[(Y - \mathbb{E}[Y|\mathcal{F}])^2] = \min_{Z\in\mathcal{M}} \mathbb{E}[(Y-Z)^2].$$

Exercise 32. (4 points)

a) Prove, that every non-negative local martingale is a super martingale.
   Hint: Fatou's lemma also holds for conditional expectations.
b) Prove that a non-negative local martingale $(X_t)_{t\geq 0}$ with $X_0 = 0$ is almost surely identical to zero ($X_t = 0$ a.s. for all $t \geq 0$).