Christian-Albrechts-Universität zu Kiel **Winter term 2015/16**
Mathematisches Seminar
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Exercise sheet 5

supporting the lecture Mathematical Statistics

(Submission of Solutions: 7. December 2015, 12:00 o'clock; Discussion: 9. December 2015)

Exercise 1. (4 points)

Show the following statements concerning the connection between Minimax/Bayes estimators and the definition of admissibility.

a) If g^* is an admissible estimator with constant risk, then g^* is a Minimax estimator.

b) If g^* is a Bayes estimator for a prior distribution π and unique in the sense that for each other Bayes estimator $\tilde{g}$ it holds that

$$R(\vartheta, g^*) = R(\vartheta, \tilde{g}) \quad \forall \vartheta \in \Theta,$$

then g^* is admissible.

Exercise 2. (4 points)

Let $X_1, \dots, X_n$ i.i.d. $\sim N(\mu, \sigma^2)$, $X = (X_1, \dots, X_n)^T$, we look at the loss function $L(\sigma^2, d) = (d/\sigma^2 - 1)^2$. Assuming $\mu = 0$ is known, show that

$$g(X) = \frac{1}{n+2} \sum_{i=1}^{n} X_i^2$$

is a Minimax estimator for the parameter σ^2.

Hint: To ease you calculations, replace the parameter $1/\sigma^2$ by the parameter $\lambda = 1/\sigma^2$ and consider $\pi_{p,b} = \gamma(p, b)$ as a prior distribution for it. Notice that for the a posteriori risk of an estimator g it holds:

$$R^x_{\pi_{p,b}}(\sigma^2, g) = \int_\Theta L(\sigma^2, g(x)) Q^{\theta|X=x}(d(1/\sigma^2)) = \int_\Theta (g(x)\lambda - 1)^2 Q^{\theta|X=x}(d\lambda)$$

Exercise 3. (4 points)

We face the same situation as in exercise 2 and look at other estimators for the parameter σ^2.

a) Show that for $\mu = 0$ known, $g(X) = (n+2)^{-1} \sum_{i=1}^n X_i^2$ is uniformly best in the class of estimators having the form $g_c(X) = c \sum_{i=1}^n X_i^2$. In particular the ML estimator $n^{-1} \sum_{i=1}^n X_i^2$ is not admissible.

b) Show that in the case when μ is unknown the estimator

$$h(X) = \frac{1}{n+1} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2$$

is better than every estimator of the form $h_c(X) = c \sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2$. As before, the ML estimator $n^{-1} \sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2$ is not admissible.

Exercise 4. (4 points)

In the lecture notes, remark 3.6 a density is presented, given that it exists under the stated preliminaries. Instead of inserting a probability density one can also insert a nonnegative measurable function $h(\vartheta)$ which is not a density. One still gets a probability density, given that the integral in the denominator is finite. We then speak of a *generalized prior distribution* $h(\vartheta)$.

Let X be $N(\vartheta, 1)$ distributed.

a) Show that X is a *generalized Bayes estimator* for ϑ using the quadratic risk and a generalized prior distribution $h(\vartheta) = 1_{(-\infty,\infty)}(\vartheta)$.

b) Determine the generalized Bayes estimator $\hat{\vartheta}_{a,b}$ for the quadratic risk and the generalized prior distribution $h_{a,b}(\vartheta) = 1_{(a,b)}(\vartheta)$ with $a, b \in \mathbb{R}$.