Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
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Summer term 2016

In-tutorial exercise sheet 10

supporting the lecture Mathematical Finance and Stochastic Integration

(Discussion in the tutorial on June 30th 2016, 2:15 p.m.)

Exercise P.21.

Let $(X_t)_{t\in[0,T]}$ be a *Brownian motion with drift* $a \in \mathbb{R}$ *and volatility* $\sigma > 0$ (i.e. it holds $X_t = at + \sigma B_t$ for a standard Brownian motion $(B_t)_{t\geq 0}$) on $(\Omega, \mathcal{F}, \mathbb{P})$.

a) Show, that we can define an equivalent probability measure $\mathbb{Q}$ via the $\mathbb{P}$-density

$$L_T = \exp\Big(\frac{b-a}{\sigma^2}X_T - \frac{b^2-a^2}{2\sigma^2}T\Big),\ \rho \in \mathbb{R},$$

i.e. show $L_T \geq 0$, L_T is $\mathcal{F}_T$-measurable and $\mathbb{E}_{\mathbb{P}}[L_T] = 1$.

b) Compute the density process $(L_t)_{t\geq 0} = (\mathbb{E}_{\mathbb{P}}[L_T|\mathcal{F}_t])_{t\geq 0}$.

c) Show, that X is a Brownian motion with drift b and volatility σ under $\mathbb{Q}$.
Hint: Consider the conditional characteristic functions $u \mapsto \mathbb{E}_{\mathbb{Q}}[\exp(iu(X_t - X_s))|\mathcal{F}_s]$ for $t \geq s$.

Exercise P.22.

Consider the Black-Scholes model with stock price process $A_t = A_0 \exp((\mu - \sigma^2/2)t + \sigma B_t)$ for a Brownian motion $B = (B_t)_{t\in[0,T]}$ under the probability measure $\mathbb{P}$. Then the discounted stock price process $e^{-\rho t}A_t,\ \rho > 0$, is a $\mathbb{Q}$-martingale where $\mathbb{Q}$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(\frac{\rho-\mu}{\sigma}B_T - \frac{(\rho-\mu)^2}{2\sigma^2}T\Big).$$