Christian-Albrechts-Universität zu Kiel **Summer term 2016**
Mathematisches Seminar
Prof. Dr. M. Vetter, O. Martin

Exercise sheet 2

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: May 2nd 2016, 12:15 p.m.; Discussion: May 5th 2016)

Exercise 5. (4 points)

Let $x > 0$, $(B_t)_{t\geq 0}$ and $(\widetilde{B}_t)_{t\geq 0}$ be Brownian motions with start in x respectively $-x$. Prove the following identity for $A \subset [0, \infty)$

$$\mathbb{P}(B_s \geq 0 \text{ for } 0 \leq s \leq t \text{ and } B_t \in A) = \mathbb{P}(B_t \in A) - \mathbb{P}(\widetilde{B}_t \in A).$$

Exercise 6. (4 points)

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $a, b > 0$. Prove the identity

$$\mathbb{P}(\exists t \geq 0 : B_t = a + bt) = \exp(-2ab)$$

by solving a) and b).

a) With the notation
$$\phi_b(a) = \mathbb{P}(\exists t \geq 0 : B_t = a + bt)$$
prove
$$\phi_b(a_1 + a_2) = \phi_b(a_1)\phi_b(a_2), \quad a_1, a_2 > 0.$$
Hint: Although the stopping time $\sigma = \inf\{t \geq 0 : B_t = a_1 + bt\}$ is not almost surely finite, the strong Markov property is still valid on $\{\sigma < \infty\}$.

b) From part a) it follows that $\phi_b(a)$ must be of the form $\phi_b(a) = \exp(-ca)$ for a constant c (you don't need to prove this). Argue that $c > 0$ and use $\mathbb{E}[\exp(-\lambda\tau_a)] = \exp(-a\sqrt{2\lambda})$, $\lambda \geq 0$, for $\tau_a = \inf\{t \geq 0 : B_t = a\}$ to determine the constant c.

Exercise 7. (4 points)

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

a) Show that for $X \sim \mathcal{N}(\mu, \sigma^2)$ it holds $\mathbb{E}[\exp(X)] = \exp(\mu + \frac{1}{2}\sigma^2)$.

b) Show that the process $(X_t)_{t\geq 0}$ defined via
$$X_t = \exp\left(B_t - \frac{1}{2}t\right)$$
is a positive martingale, sometimes called *exponential martingale.*

Exercise 8. (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a martingale in discrete time which is adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and τ be a stopping time. Prove the identity

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

from the optional stopping theorem if

a) τ is almost surely finite and there exists $K \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}_0} |X_n| < K$ almost surely.

b) $\mathbb{E}[\tau] < \infty$ and there exists $K \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}} |X_n - X_{n-1}| < K$ almost surely.