Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
Prof. Dr. M. Vetter, O. Martin/A. Theopold

Winter term 2015/16

Exercise sheet 3

supporting the lecture Mathematical Statistics

(Submission of Solutions: 23. November 2015, 12:00 o'clock; Discussion: 25. November 2015)

Exercise 1. (4 points)

Let Z be a χ_n^2-distributed random variable. Show that Z has the Lebesque densitiy

$$f(z) = \frac{z^{n/2-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)} 1_{[0,\infty)}(z)$$

In order to do so follow these steps:

a) Calculate the density of a χ_1^2-distributed random variable using the transformation theorem.

b) Calculate the density of a χ_2^2-distributed random variable by the convolution formula for densities.

c) Calculate the density of a χ_n^2-distributed random variable using a suitable induction.

Exercise 2. (4 points)

Prove the multi-dimensional Cramér-Rao inequality from the lecture.

a) State conditions analogous to the one-dimensional case.

b) Prove the theorem under the conditions you stated in a).

Hint: Let $Y \in \mathbb{R}^p, Z \in \mathbb{R}^q$ and $\mathbb{E}[ZZ^T]$ be invertible matrices. Then the multi-dimensional Cauchy-Schwarz inequality holds:

$$\mathbb{E}[YY^T] \geq \mathbb{E}[YZ^T] \left(\mathbb{E}[ZZ^T]\right)^{-1} \mathbb{E}[ZY^T].$$

Exercise 3. (4 points)

Let $X_1, \ldots, X_n$ be i.i.d. exponentially distributed random variables with parameter $\lambda > 0$ and $X = (X_1, \ldots, X_n)^T$.
Show that $g(X) := \overline{X}_n$ is an UVMU estimator for λ^{-1}.

Hint: You may use, that in the statistical model at hand the preliminaries of the Cramér-Rao inequality are always fulfilled by any estimator $g(X) \in \mathcal{L}^2$.

Exercise 4. (4 points)

Let $X = (X_1, \ldots, X_n)^T$ be a vector of i.i.d. random variables with $X_i \sim N(\mu, \sigma^2)$ where σ is known. We search for an unbiased estimator of $\gamma(\mu) = \mu^2$.

a) Show that
$$g(X) := \left(\overline{X}_n\right)^2 - \frac{\sigma^2}{n}$$
is an unbiased estimator for μ^2.

b) Calculate the quadratic risk of $g(X)$.

c) Determine the Cramér-Rao bound of $g(X)$.

d) What is your conclusion regarding the results from b) and c)?