Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
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Summer term 2016

Exercise sheet 6

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: June 6th 2016, 12:15 p.m.; Discussion: June 9th 2016)

Exercise 21. (4 points)

Prove the claim made in Remark 4.2 from the lecture notes for deterministic functions: Let $(\pi_n)_n$ be a sequence of nested partitions and $f : [a,b] \to \mathbb{R}$ a right-continuous function. Then it holds

$$V^f_{[a,b]} = \lim_{n\to\infty} \sum_{t_i \in \pi_n} |f(t_{i+1}) - f(t_i)|$$

if $\max\{t_{i+1} - t_i | t_i \in \pi_n\} \to 0$ as $n \to \infty$.

Exercise 22. (4 points)

Let $(\pi_n)_n$ be a sequence of partitions of $[0,t]$ with $\max\{t_{i+1} - t_i | t_i \in \pi_n\} \to 0$ as $n \to \infty$ and $s \mapsto H_s$, $s \mapsto A_s$ be functions on $[0,\infty)$. The *Riemann-Stieltjes integral* is defined by

$$\int_0^t H_s dA_s := \lim_{n\to\infty} \sum_{t_i \in \pi_n} H_{s_i}(A_{t_{i+1}} - A_{t_i})$$

with $s_i \in [t_i, t_{i+1}]$ if this limit exists and does not depend on the choice of $(\pi_n)_n$ and the s_i.

In this exercise you are supposed to examine differences between the Lebesgue-Stieltjes and the Riemann-Stieltjes integral. Let $A_t = \mathbb{1}_{[1,\infty)}(t)$.

a) Is the Riemann-Stieltjes integral
$$\int_0^2 A_s dA_s$$
well defined? If it is, compute its value!

b) What is the minimal condition on the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that the Riemann-Stieltjes integral
$$\int_0^2 f(s) dA_s$$
is well defined?

c) Determine the measure μ_A which is induced by A. Compute the Lebesgue-Stieltjes integrals
$$\int_0^2 A_s dA_s, \ \int_0^2 A_{s-} dA_s,$$
with $A_{s-} := \lim_{t \nearrow s} A_t$.

Exercise 23. (4 points)

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and define

$$L_n = \sum_{t_i \in \pi_n} B_{t_i}(B_{t_{i+1}} - B_{t_i})$$
$$R_n = \sum_{t_i \in \pi_n} B_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})$$

for a sequence of nested partitions $(\pi_n)_n$ of $[0,t]$ with $\max\{t_{i+1} - t_i | t_i \in \pi_n\} \to 0$ as $n \to \infty$.

a) Compute $\lim_{n\to\infty} R_n$ and $\lim_{n\to\infty} L_n$.
Hint: Compute first $\lim_{n\to\infty}(R_n + L_n)$ and $\lim_{n\to\infty}(R_n - L_n)$.

b) Use part a) and Remark 4.8(ii) to argue, that

$$\int_0^t B_s dB_s$$

can't be well defined as a Stieltjes integral.

Exercise 24. (4 points)

a) Prove that the predictable σ-algebra $\mathcal{P}$ is also generated by $\{F \times \{0\} | F \in \mathcal{F}_0\}$ together with the set of random intervals $\{(\tau, \infty) : \tau \text{ is stopping time}\}$, i.e.

$$\mathcal{P} = \sigma(\{F \times \{0\} | F \in \mathcal{F}_0\}, \{(\tau, \infty) : \tau \text{ is stopping time}\}).$$

Hint: It might be useful to consider stopping times which are similar to those used in the proof of Theorem 2.12.

b) The *optional σ-algebra* $\mathcal{O}$ is generated by random intervals of the form $[\tau, \infty)$, where τ is a stopping time. Prove

$$\mathcal{P} \subset \mathcal{O}.$$