Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
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Exercise sheet 7

supporting the lecture interest rate models

(Submission of Solutions: 6. Januar 2017, 12:00; Discussion: 9. Januar 2017)

Exercise 19. (4 points)

In this exercise we look at an example of a multivariate affine process defined on state space which is not of the form $\mathbb{R}_+^m \times \mathbb{R}^n$.
Consider the space $\mathcal{X} = \{x \in \mathbb{R}^2 | x_1 \geq x_2^2\}$. Let $W = (W_1, W_2)^*$ be a two-dimensional Brownian motion. For every $y \geq 0$, there exists a unique nonnegative affine diffusion process $Y = Y^y$ satisfying

$$dY = 2\sqrt{Y} dW_1,\ Y(0) = y$$

(you don't have to prove this). For every $x \in \mathcal{X}$ we define the $\mathcal{X}$-valued diffusion process $X = X^x$ by

$$\begin{aligned} X_1(t) &= (W_2(t) + x_2)^2 + Y^y(t), \\ X_2(t) &= W_2(t) + x_2, \end{aligned}$$

where $y = y(x)$ is the unique nonnegative number with $x_1 = x_2^2 + y$.

a) Show that X satisfies

$$\begin{aligned} dX_1 &= dt + 2\sqrt{X_1 - X_2^2} dW_1 + 2X_2 dW_2, \\ dX_2 &= dW_2. \end{aligned}$$

Conclude that the drift and diffusion matrix of X are affine functions of x. Verify that the dffusion matrix is positive semi-definite on $\mathcal{X}$.

b) Verify by solving the corresponding Ricatti equations that

$$\mathbb{E}\left[\exp(u^* X(T)) | F_t\right] = \frac{1}{\sqrt{1 - 2u_1(T-t)}} \exp\left(\frac{(T-t)u_2^2 + 2u^* X(t)}{2(1 - 2u_2(T-t))}\right) \text{ for } u = (u_1, u_2)^* \in i\mathbb{R}^2.$$

Conclude that X is an affine process.

Exercise 20. (4 points)

Let B be a Brownian motion and define the $\mathbb{R}_+^2$-valued process X by $X_i(t) = (\sqrt{x_i} + B(t))^2$, for $i = 1, 2$, for some $x \in \mathbb{R}_+^2$.

a) Show that X satisfies

$$\begin{aligned} dX_1 &= dt + 2\sqrt{X_1}dW, \\ dX_2 &= dt + 2\sqrt{X_2}dW, \\ X(0) &= x, \end{aligned}$$

for some Brownian motion W. Check whether X is an affine process and justify your answer.

b) Compute the characteristic function of $X(t)$ and verify your findings on the affine property in part a).

Exercise 21. (4 points)

A random variable X is said to be noncentral χ^2-distributed with δ degrees of freedom and noncentrality parameter ζ if its characteristic function equals

$$\mathbb{E}\left[\exp(uX)\right] = \frac{\exp\left(\frac{\zeta u}{1-2u}\right)}{(1-2u)^{(\delta/2)}}, \ u \in \mathbb{C}_-.$$

Now fix $\delta \in \mathbb{N}$ and some real number $\nu_1, \dots, \nu_\delta$ and define $\zeta = \sum_{i=1}^{\delta} \nu_i^2$.

a) Let $N_1, \dots, N_\delta$ be independent standard normal distributed random variables. Define $Z = \sum_{i=1}^{\delta}(N_i + \nu_i)^2$. Show that Z is noncentral χ^2-distributed with δ degrees of freedom and noncentrality parameter ζ.

b) Now let $W_1, \dots, W_\delta$ be independent standard Brownian motions with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$, and define the process $X(t) = \sum_{i=1}^{\delta}(W_i(t) + \nu_i)^2, t \geq 0$. Using a) show that the $\mathcal{F}_t$-conditional characteristic function of $X(T)$ equals

$$\mathbb{E}\left[\exp(uX(T))|\mathcal{F}_t\right] = \frac{\exp\left(\frac{u}{1-2(T-t)u}X(t)\right)}{(1-2(T-t)u)^{\delta/2}}, u \in \mathbb{C}_-, t < T. \tag{1}$$

Conclude that the $\mathcal{F}_t$- conditional distribution of $X(T)$ is noncentral χ^2 with δ degrees of freedom and noncentrality parameter $\frac{X(t)}{(T-t)}$.

c) Similarly to exercise 4 on sheet 2 show that X satisfies the SDE

$$dX(t) = \delta dt + 2\sqrt{X(t)}dB(t), X(0) = \zeta,$$

for the Brownian motion $dB = \sum_{i=1}^{\delta} \frac{W_i + \nu_i}{\sqrt{X}} dW_i$.

d) Conclude by either b) or c) that X is an affine process with state space $\mathbb{R}_+$.

e) Verify (1) by calculating the explicit solutions of the corresponding Riccati equations

$$\partial_t \phi(t, u) = \dots, \qquad \partial_t \psi(t, u) = \dots,$$

for X.