Christian-Albrechts-Universität zu Kiel **Winter term 2015/16**
Mathematisches Seminar
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Exercise sheet 4

supporting the lecture Mathematical Statistics

(Submission of Solutions: 30. November 2015, 12:00 o'clock; Discussion: 2. December 2015)

Exercise 1. (4 points)

Investigate the following distributions and decide which of them form an exponential family. State if applicable the parameter k, the statistics $T_1, \dots, T_k$ and the natural parameter space.

a) $f_\vartheta(x) = \frac{b^{\vartheta_1}}{\Gamma(\vartheta_1)} x^{\vartheta_1 - 1} e^{-\vartheta_2 x};\ x \geq 0;\ \vartheta_1, \vartheta_2 > 0$

b) $f_\vartheta(x) = 1_{(0,\vartheta)}(x) \exp(-2\log\vartheta + \log(2x));\ x \in \mathbb{R};\ \vartheta > 0$

c) $f_\vartheta(x) = \vartheta^{x-1}(1-\vartheta);\ x \in \mathbb{N}_0;\ \vartheta \in (0,1)$

Exercise 2. (4 points)

Let $X_1, \dots, X_n$ be i.i.d $\sim \mathcal{N}(\mu, \sigma^2)$ for $\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_{>0}$ and $X = (X_1, \dots, X_n)^T$. Show that

$$g(X) := \left(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i \right)^2 \right)^T = (\overline{X}_n, \hat{s}_n^2(X))^T$$

is the maximum-likelihood estimator for the parameter $\vartheta = (\mu, \sigma^2)^T$.

Exercise 3. (4 points)

For $\theta \in (0, \infty)$ let $X_1, \dots, X_n$ be i.i.d. $\sim U[0, \theta]$ (uniform distribution on the interval $[0, \theta]$) and $X = (X_1, \dots, X_n)^T$. Our goal is to estimate θ. Therefore we are looking at the estimator

$$g(X) := \max\{X_1, \dots, X_n\}.$$

a) Show that g is the maximum-likelihood estimator for this statistic experiment.

b) Compute the expected value and the variance and show

$$\mathrm{Var}_\theta(g(X)) = O\left(\frac{1}{n^2}\right).$$

Exercise 4. (4 points)

We are working with the same statistical model as in exercise 3).

a) Determine an estimator $\hat{g}$ for the parameter θ using the method of moments. Compute its expected value and variance.

b) Compute the Cramer-Rao-bound for an unbiased estimator for θ. Is the Cramer-Rao-inequality applicable in this setting?

c) Compare the variances of $\frac{n+1}{n}g$ and $\hat{g}$ with the Cramer-Rao-bound.