Christian-Albrechts-Universität zu Kiel **Winter term 2015/16**
Mathematisches Seminar
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Exercise sheet 11

supporting the lecture Mathematical Statistics

(Submission of Solutions: 8. February 2016, 12:00 o'clock; Discussion: 10. February 2016)

Exercise 1. (4 points)

Let $(X_i)_{i\in\mathbb{N}}$ a sequence of i.i.d. distributed random variables with unknown density p. We want to test

$$H \ : \ p = p_0 \quad \text{versus} \quad K \ : \ p = p_1,$$

where $\{x \in \mathbb{R} | p_1(x) > 0\} \subset \{x \in \mathbb{R} | p_0(x) > 0\}$. Using the Neyman-Pearson Lemma a UMP Test with level α is then defined via the critical region $\prod_{i=1}^n r(X_i) \geq C_n(\alpha)$ with $r(x) := p_1(x)/p_0(x)$, where $C_n(\alpha)$ is a constant depending on n and α.

a) Proof that the critical region from above can also be written as

$$\frac{1}{\sqrt{n}} \left(\sum_{i=1}^n \log r(X_i) - \mathbb{E}_0 \left[\log r(X_i)\right] \right) \geq k_n(\alpha)$$

with a suitable $k_n(\alpha)$.

b) Assume that it holds $\sigma_0^2 = Var_0 \left[\log r(X_i)\right] < \infty$. Show that

$$k_n \to \sigma_0 u_{1-\alpha},$$

where u_α is the α quantile of a $\mathcal{N}(0,1)$ distribution.

c) Show that the sequence of tests is consistent for $p_1 \neq p_0$. Use Lemma 5.16.

Comment: *Comment:*

- $\mathbb{E}_0, Var_0$ denote the expected value respectively the variance under the density p_0.
- The exercise shows that each Neyman-Person test already exhibits a built-in asymptotic.

Exercise 2. (4 points)

We look at the setting stated in Example 7.6 from the lecture notes.

a) Determine the maximum likelihood estimator in the sets Δ and Θ.

b) Determine $T = -2\log(\lambda(Z))$ with $Z = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$.

Exercise 3. (4 points)

Proof the following claim from Theorem 7.7: Let $A \sim \chi^2_d$, $B \sim \chi^2_c$ with $d > c$, $A - B$ and B be independent. Then it holds that

$$A - B \sim \chi^2_{d-c}$$

Hint: As a first step determine the characteristic function of a χ^2_n distributed random variable.

Exercise 4. (4 points)

Let $Y_1, \ldots, Y_n$ be independent with $Y_i = ax_i + \epsilon_i$ where a is an unknown parameter, $x_1, \ldots, x_n$ fixed and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. $\mathcal{N}(0,1)$ distributed.

a) Determine the maximum likelihood estimator $\hat{a}$ for the parameter a.

b) Derive the likelihood quotient test for the hypotheses

$$H \ : \ a = 0 \quad \text{versus} \quad K \ : \ a \neq 0.$$