Stochastics II
Stochastic Processes
Winterterm 2016/17

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Sheet 4

Let X be a Markov chain with state space E and transition matrix P.
For $x, y \in E$ we define: $\varrho_{xy} := \mathbb{P}_x(\tau_y < \infty)$.
Let $y \in E$ and $N(y) := |\{n \in \mathbb{N} | X_n = y\}|$ be the „number of visits in y “.

Exercise 1
With the definitions above show that for $x \in E$

$$\mathbb{E}_x(N(y)) = \frac{\varrho_{xy}}{1 - \varrho_{yy}} = \sum_{n \in \mathbb{N}} P^n(x, y)$$

holds.
Hint: Without further proof you may use that $\mathbb{P}_x(\tau_y^k < \infty) = \varrho_{xy}\varrho_{yy}^{k-1}$.

Exercise 2 (Sheet 2, Ex. 2 continued)
Let X be a Markov chain with state space $E := \{AA, Aa, aa\}$ and transition probabilities:

$$\begin{aligned}
&\mathbb{P}(X_{n+1} = AA | X_n = Aa) = 1/4 = \mathbb{P}(X_{n+1} = aa | X_n = Aa),\\
&\mathbb{P}(X_{n+1} = Aa | X_n = Aa) = 1/2,\\
&\mathbb{P}(X_{n+1} = AA | X_n = AA) = 1 = \mathbb{P}(X_{n+1} = aa | X_n = aa).
\end{aligned}$$

(a) Check the states of the Markov chain for transience and recurrence.

(b) Calculate the mean number of descendants of type Aa, i.e. $\mathbb{E}_{Aa}(N(Aa))$.

Exercise 3 (distribution of τ_x for two states)
Let X be a Markov chain with state space $E := \{0, 1\}$ and transition matrix $\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$ for $a, b \in (0, 1)$.
Calculate the distribution of the return time to 0 with respect to $\mathbb{P}_0$, i.e. $\mathbb{P}_0(\tau_0 = n), n \in \mathbb{N}$ and the expected return time $\mathbb{E}_0\tau_0$.

Exercise 4 (transience and recurrence of random walks)
Let X be a 1-dim. random walk on $\mathbb{Z}$. Show that for a symmetric random walk, i.e. $p = 1/2$, all states are recurrent, otherwise ($p \neq 1/2$) all states are transient.

Hint: Without further proof you may use the Stirling-formula

$$n! \sim \sqrt{2\pi n} n^n e^{-n}.$$

Hand in until friday, 25.11.2016, 12:00