Christian-Albrechts-Universität zu Kiel **Winter term 2015/16**
Mathematisches Seminar
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Exercise sheet 1

supporting the lecture Mathematical Statistics

(Submission of Solutions: 16. November 2015, 12:00 Uhr; Discussion: 18. November 2015)

Exercise 1. (4 points)

Let $X = (X_1, X_2)^T \in \mathbb{R}^2$ be a 2-dimensional normally distributed random variable $X \sim N_2(\mu, \Sigma)$ wih $\mu = (0,0)^T$ and covariance matrix $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$. Show that:

$$\mathbb{E}[X_1|X_2] = \rho\frac{\sigma_1}{\sigma_2}X_2$$

Hint: Show that the common density of X_1, X_2 can be written as

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sqrt{\det(\Sigma)}}\exp\left(-\frac{(1-\rho^2)\sigma_1^2}{\det(\Sigma)}x_2^2\right)\exp\left(-\frac{1}{2}\left(\frac{x_1-\rho\frac{\sigma_1}{\sigma_2}x_2}{\sqrt{\det(\Sigma)\sigma_2^{-1}}}\right)^2\right).$$

Exercise 2. (4 points)

Let X, Y be two real valued random variables with common density

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda x}1_{(0,\infty)}(x-y)1_{(0,\infty)}(y).$$

a) Determine the marginal laws of f_X and f_Y. How are X and Y distributed?

b) Calculate $\mathbb{E}[Y|X]$.

c) Use your knowledge from the previous parts to verify the identity $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$.

Exercise 3. (4 points)

Let X and Y be two real valued random variables on a common probability space $(\Omega, \mathcal{A}, P)$ and $\varphi : \Omega\times\Omega \to \mathbb{R}$ a $\mathcal{A}\times\mathcal{A}$-measurable function with $\mathbb{E}[|\varphi(X,Y)|] < \infty$. Show that with the function $g : x \mapsto \mathbb{E}[\varphi(x,Y)]$ it holds:

$$g(X) = \mathbb{E}[\varphi(X,Y)|X] \quad P-a.s.$$

Hint: Use Fubini's theorem.

Exercise 4. (4 points)

Let $X = (X_1, ..., X_n)$ be a vector of random variables with identical expectation $\mathbb{E}[X_i] = \vartheta, i = 1, .., n$ and finite second moment $\mathbb{E}[X_i^2] < \infty, i = 1, .., n$. The parameter to be estimated is ϑ.

a) Determine the quadratic risk of the estimator

$$g_b(X) := n^{-1} \sum_{k=1}^{n} X_k + b$$

for ϑ and show that the estimator $g_0(X) = n^{-1} \sum_{k=1}^{n} X_k$ is better than every estimator $g_b(X)$ for $b \neq 0$, i.e.

$$R_\vartheta(g_0(X), \vartheta) < R_\vartheta(g_b(X), \vartheta) \quad \forall \vartheta \in \mathbb{R}$$

b) Show that using the quadratic risk no estimator of the form

$$g_{a,b}(X) := an^{-1} \sum_{k=1}^{n} X_k + b$$

for $a > 1$ is admissible.