Exercise sheet 1

supporting the lecture Mathematical Statistics

(Submission of Solutions: 9. November 2015, 12:00 Uhr; Discussion: 11. November 2015)

Exercise 1. (4 points)

Let X be a finitely integrable real-valued random variable on a probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{F} \subset \mathcal{A}$ be a sub-σ-algebra.

a) Proof the *conditional Jensen-inequality*: For every convex function $\varphi : \mathbb{R} \to \mathbb{R}$ we have

$$\varphi\left(\mathbb{E}\left[X|\mathcal{F}\right]\right) \leq \mathbb{E}\left[\varphi\left(X\right)|\mathcal{F}\right] \quad \text{(P-a.s.)}.$$

Hint: For every convex function $\varphi : \mathbb{R} \to \mathbb{R}$ there exists a series $(a_n, b_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$, such that

$$\varphi(x) = \sup_{(a_n, b_n)} (a_n x + b_n)$$

holds for all $x \in \mathbb{R}$.

b) Conclude that with $\|Y\|_p := (\mathbb{E}[|Y|^p])^{1/p}$ the inequality

$$\|\mathbb{E}[X|\mathcal{F}]\|_p \leq \|X\|_p$$

holds for $p \geq 1$.

Exercise 2. (4 points)

Let X be a real-valued random variable on a probability space $(\Omega, \mathcal{A}, P)$ with $\mathbb{E}[X^2] < \infty$ and $\mathcal{F} \subset \mathcal{A}$ a sub-σ-algebra. The *conditional variance* is defined by

$$\mathrm{Var}[X|\mathcal{F}] := \mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{F}])^2 \,\middle|\mathcal{F}\right].$$

a) Show

$$\mathrm{Var}[X] = \mathbb{E}\left[\mathrm{Var}[X|\mathcal{F}]\right] + \mathrm{Var}\left[\mathbb{E}[X|\mathcal{F}]\right].$$

b) How can the identity from a) be interpreted? Remember that $\mathbb{E}[X]$ is the best L^2-approximation to X by a constant random variable.

Exercise 3. (4 points)

Proof part (ii) of lemma 1.8 from the lecture: Suppose we have real-valued integrable random variables X,Y on a probability space $(\Omega, \mathcal{A}, P)$ and an additional random variable $Z : (\Omega, \mathcal{A}, P) \to (\widetilde{\Omega}, \widetilde{A})$. Than we have

$$X \leq Y \ \text{ P-f.s.} \quad \Rightarrow \quad \mathbb{E}[X|Z] \leq \mathbb{E}[Y|Z] \ \text{ P-f.s.}$$

Exercise 4. (4 points)

Let $X : (\mathbb{R}, \mathcal{B}, P) \to (\mathbb{R}, \mathcal{B})$ be a finitely integrable random variable and for P has a Lebesgue-density f, which is positive on $\mathbb{R}$. $\mathcal{C} := \{C \in \mathcal{B} | C = -C\}$ (with $-C := \{-x | x \in C\}$) is the σ-algebra, which contains all the Borel-sets, that are symmetrical with respect to 0.

a) Show that
$$Y(\omega) := \frac{f(\omega)X(\omega) + f(-\omega)X(-\omega)}{f(\omega) + f(-\omega)}$$
is a version of the conditional expectation $\mathbb{E}[X|\mathcal{C}]$.

b) What does the expression from a) look like for symmetrical f? Give an interpretation for this representation of this conditional expectation.