**Christian-Albrechts-Universität zu Kiel** **SS 2016**
Mathematisches Seminar
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Mark Feodoria Sheet 05

# Computational Finance

Exercises for participants of mathematical programmes

**C-Exercise 16 (Black-Scholes formula)**
Recall that for the fair price $C(t,S(t),r,\sigma,T,K)$ of a European call option with strike $K>0$ and maturity $T>0$ in the Black-Scholes model with parameters $r>0$ (interest rate) and $\sigma>0$ (volatility), given the stock price $S(t)$ at time $0\le t\le T$, we have

$$C(t,S(t),r,\sigma,T,K)=S(t)\Phi(d_1)-Ke^{-r(T-t)}\Phi(d_2).$$

Here, $\Phi$ denotes the cumulative distribution function of the standard normal distribution, and

$$d_1:=\frac{\log\left(\frac{S(t)}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},\quad d_2:=d_1-\sigma\sqrt{T-t}.$$

For the position in the stock $\varphi_1(t,S(t),r,\sigma,T,K)$ of the hedge for the call, given the stock price $S(t)$ at time $0\le t\le T$, we have $\varphi_1(t,S(t),r,\sigma,T,K)=\Phi(d_1)$.

Write a scilab function

```
[C, phi] = EuCall_BlackScholes (t, S_t, r, sigma, T, K)
```

that computes and returns the price and the hedge position for the passed parameters.

For $t=0$, $r=0.05$, $\sigma=0.2$, $T=\frac{1}{12}$ and $K=100$, plot $\varphi_1$ as function of $S(t)$ in the range $[80,\dots,120]$.

*Useful scilab commands:* `cdfnor`, `log`, `sqrt`, `plot`

**C-Exercise 17 (European option in the CRR model)**
Write a scilab function

```
[V_0, phi_1] = EuOpt_BinMod (S_0, r, sigma, T, K, OptType, M)
```

that computes and returns an approximation to the initial price $V(S(0),0)$ and the initial hedge $\varphi(1)=(\varphi_0(1),\varphi_1(1))$ of a European call or put option with strike $K>0$ and maturity $T>0$ in the Black-Scholes model with initial stock price $S_0>0$, interest rate $r>0$ and volatility $\sigma>0$. If `OptType` $=0$, the quantities shall be computed for a call option, and if `OptType` $=1$ for a put option. Use the binomial method as presented in the course with $M\in\mathbb{N}$ time steps.

Test your algorithm for a call option with

$$S(0)=100,\ r=0.05,\ \sigma=0.2,\ T=1,\ K=100,\ M=500.$$

*Useful scilab commands:* `exp`, `sqrt`, `max`, `for`, `if`, `elseif`
*Hint:* section 2.4 of the lecture notes

Please turn over.

**C-Exercise 18 (American put option in the CRR model)**
Write a scilab function

```
V_0 = AmPut_BinMod (S_0, r, sigma, T, K, M)
```

that computes and returns an approximation to the price of an American put option with strike $K > 0$ and maturity $T > 0$ in the Black-Scholes model with initial stock price $S(0) > 0$, interest rate $r > 0$ and volatility $\sigma > 0$. Use the binomial method as presented in the course with $M \in \mathbb{N}$ time steps.

Test your algorithm with

$$S(0) = 100,\ r = 0.05,\ \sigma = 0.2,\ T = 1,\ K = 100,\ M = 500.$$

*Useful scilab commands:* `exp`, `sqrt`, `max`, `for`
*Hint:* section 2.4 of the lecture notes

**T-Exercise 19 (Convergence of the CRR model to the Black-Scholes model)**
For $M \in \mathbb{N}$, denote by $\left(S^M(t)\right)_{t\in\{t_0,\ldots,t_M\}}$ the stock price process in a binomial model with $M$ timesteps for fixed parameters $S_0 > 0$, $r > 0$, $\sigma > 0$ and $T > 0$.

(a) Show that $\log\left(S^M(t_M)\right) \xrightarrow[M\to\infty]{} Z$ in law relative to the martingale measure $Q$, where the random variable $Z$ is normally distributed with mean $\log(S(0)) + \left(r - \frac{1}{2}\sigma^2\right)T$ and variance $\sigma^2 T$.

(b) Conclude that the initial prices of European put and call options with maturity $T$ and strike $K > 0$ in the binomial model with $M$ timesteps converge to the corresponding Black-Scholes prices as $M \to \infty$.

You can use without proof

**Slutsky's Theorem**: Let $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ be sequences of random variables such that $A_n \xrightarrow[n\to\infty]{} A$, $B_n \xrightarrow[n\to\infty]{} B$ in probability and $X_n \xrightarrow[n\to\infty]{} X$ in law. Then $A_nX_n + B_n \xrightarrow[n\to\infty]{} AX + B$ in law.

Please save your solution of each C-Exercise in a file named `Exercise_##.sce`, where ## denotes the number of the exercise. Please include your name(s) as comment in the beginning of the file.

**Submit until:** Thursday, 26.05.2016, 08:30
**Discussion:** in tutorials on Mon, 30.05.2016