Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
Prof. Dr. M. Vetter, O. Martin **Summer term 2016**

Exercise sheet 1

supporting the lecture Mathematical Finance and Stochastic Integration

(Submission of Solutions: 25. April 2016, 12:15; Discussion: 28. April 2016)

Exercise 1. (4 points)

Let τ be a stopping time w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$. Prove that

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$

is a σ-algebra.

Exercise 2. (4 points)

Let $(X_t)_{t\geq 0}$ be a real-valued stochastic process with càdlàg paths and $X_0 = 0$ almost surely, which is adapted to the right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. $B \subset \mathbb{R}$ is an open set with $0 \notin B$. Prove that

$$\tau_B = \inf\{t \geq 0 : X_t \in B\}$$

is a stopping time.

Exercise 3. (4 points)

Show that the paths of a Brownian motion are almost surely nowhere monotone without using Theorem 1.20 from the lecture notes.

a) Prove $P(A) = 0$ for
$$A = \{t \mapsto B_t \text{ is monotone increasing on } [0,1]\}.$$
Hint: Use $A \subset \{0 = B_0 \leq B_{\frac{1}{n}} \leq \ldots \leq B_{\frac{n-1}{n}} \leq B_1\}$.

b) Prove
$$P(\{\exists 0 \leq a < b : t \mapsto B_t \text{ is monotone increasing on } [a,b]\}) = 0.$$

Exercise 4. (4 points)

A function $f : [0, \infty) \to \mathbb{R}$ is called *locally Lipschitz continuous* in $s \in [0, \infty)$ if there exist constants $L, \delta > 0$ with

$$|f(t) - f(s)| \leq L|t - s| \quad \forall t \in [\min\{0, s - \delta\}, s + \delta].$$

Prove that the paths of a Brownian motion $(B_t)_{t\geq 0}$ are almost surely nowhere locally Lipschitz continuous by showing a)-c).

a) For any $m \in \mathbb{N}_0$ and $L, \delta > 0$ prove

$$A_{L,\delta,m} \subset \bigcup_{i=1}^{n} B_{L,\delta,m}(i, n), \quad n \geq \frac{4}{\delta},$$

with

$$A_{L,\delta,m} = \Big\{\exists s \in [m, m + 1) : |B_t - B_s| \leq L|t - s| \ \ \forall t \in [\min\{0, s - \delta\}, s + \delta]\Big\},$$
$$B_{L,\delta,m}(i, n) = \Big\{|B_{m+\frac{i+j}{n}} - B_{m+\frac{i+j-1}{n}}| \leq \frac{8L}{n},\ j = 1, 2, 3\Big\}, \quad n \in \mathbb{N}_0, i \in \{1, \ldots, n\}.$$

b) Use the scaling property from Exercise P.4 to derive

$$P\left(B_{L,\delta,m}(i, n)\right) \leq \left(\frac{2}{\sqrt{2\pi}} \frac{8L}{\sqrt{n}}\right)^3, \quad n \geq \frac{4}{\delta},$$

and conclude

$$P\left(A_{L,\delta,m}\right) = 0.$$

c) From part b) conclude that the paths of $(B_t)_{t\geq 0}$ are almost surely nowhere locally Lipschitz continuous i.e. $P(A) = 0$ with

$$A = \bigcup_{m\in\mathbb{N}_0} \bigcup_{L,\delta>0} A_{L,\delta,m}.$$

Hint: Argue that it is sufficient to consider $L, \delta \in \mathbb{Q}$.

Comment: If a function $f : [0, \infty) \to \mathbb{R}$ is differentiable in s, it is locally Lipschitz continuous in s. Hence you have shown that the paths of a Brownian motion are almost surely nowhere differentiable!