1 The Momentum & Continuity Equation

Convective-Diffusive transport of momentum in steady state is described by the equation:

$$\nabla \cdot (\mathbf{u}u) = -\frac{1}{\rho}\nabla \mathbf{P} + \nabla \cdot (\nu \nabla u) \tag{1}$$

Or, together with the continuity equation in 2D:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{2}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \tag{3}$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \tag{4}$$

where ν is the kinematic viscosity, ρ is the density, U is the x-directional velocity and V is the y-directional velocity. To solve this equation, it must be discretized and written as a system of linear equations of the form

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + S. \tag{5}$$

Where ϕ is the transported field (in this case, U and V respectively).

2 Discretization

Discretization of the diffusive part was achieved similarly as in task 1 & 2 and yielded the coefficients

$$D_e = \rho\nu\frac{\Delta y}{\delta x_{P,E}} \qquad D_w = \rho\nu\frac{\Delta y}{\delta x_{P,W}} \qquad D_n = \rho\nu\frac{\Delta x}{\delta y_{P,N}} \qquad D_s = \rho\nu\frac{\Delta x}{\delta y_{P,S}}. \tag{6}$$

These will be reused for the convective-diffusive problem. Discretization of the convective part was achieved similarly to task 2:

$$\int_{\Delta V} \nabla \cdot \left(\rho\vec{u}\phi\right)\, dV = (\Delta y\rho U)_e\phi_e - (\Delta y\rho U)_w\phi_w + (\Delta y\rho V)_n\phi_n - (\Delta y\rho V)_s\phi_s \tag{7}$$

Where U_e, U_w, V_n, V_s are the Rhie-chow approximations (see section 2.3) of the face velocities, see section 2.3, which yields

$$F_e = \Delta y\rho U_e \qquad F_w = \Delta y\rho U_w \qquad F_n = \Delta x\rho V_n \qquad F_s = \Delta x\rho V_s, \tag{8}$$

with the caveat that these are evaluated on the faces of ΔV rather than in the nodes as in the diffusive case. The discrete equation now reads

$$F_e\phi_e - F_w\phi_w + F_n\phi_n - F_s\phi_e = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) + D_n(\phi_N - \phi_P) - D_s(\phi_P - \phi_S) + S^\phi \tag{9}$$

Where the values of ϕ multiplying the convective coefficients are the values at the faces and depend on the discretization scheme of choice, and S^ϕ is the pressure dependant source term, which is not yet discretized. For determining the relationship between the D and F coefficients, and the coefficients in equation 5 we will use a hybrid scheme based on a combination of the central difference and upwind schemes (see section 2.1). For the implementation of the continuity equation, we will also define the continuity error ΔF

$$\Delta F =: F_e - F_w + F_n - F_s \tag{10}$$

Integrating the components of the pressure gradient over ΔV and assuming that the gradient is constant on the control volume we get

$$\int_{\Delta V} -\frac{\partial P}{\partial x} dV = -\left(\delta V \frac{\partial P}{\partial x}\right)_P \qquad \int_{\Delta V} -\frac{\partial P}{\partial y} dV = -\left(\delta V \frac{\partial P}{\partial y}\right)_P, \tag{11}$$

and use a central difference quotient to get the following expressions for the momentum source terms

$$S^u = -\frac{\Delta x \Delta y}{\delta x_E + \delta x_W}(P_E - P_W) \qquad S^v = -\frac{\Delta x \Delta y}{\delta y_N + \delta y_S}(P_N - P_S). \tag{12}$$

2.1 Hybrid Scheme

Using a hybrid scheme which uses upwind or central differencing depending on the relative magnitude of convection and diffusion and which can be summarized as:

$$\begin{aligned} a_E^{uv} &= max\big[0, \quad -F_e, \quad D_e - \tfrac{F_e}{2}\big] \\ a_W^{uv} &= max\big[0, \quad F_w, \quad D_w + \tfrac{F_w}{2}\big] \\ a_N^{uv} &= max\big[0, \quad -F_n, \quad D_n - \tfrac{F_n}{2}\big] \\ a_S^{uv} &= max\big[0, \quad F_s, \quad D_s + \tfrac{F_s}{2}\big] \end{aligned} \tag{13}$$

with

$$a_P^{uv} = a_E^{uv} + a_W^{uv} + a_N^{uv} + a_S^{uv} + \Delta F. \tag{14}$$

2.2 Pressure correction on a co-located mesh

The U and V -momentum equations with the pressure source terms not yet discretized are

$$a_P^{uv} U_P = \sum_{nb} a_{nb} U_{nb} - \left(\delta V \frac{\partial P}{\partial x}\right)_P \qquad a_P^{uv} V_P = \sum_{nb} a_{nb} V_{nb} - \left(\delta V \frac{\partial P}{\partial y}\right)_P. \tag{15}$$

The 'guessed' velocities are U^* and V^* satisfying a similar equation but with the guessed pressure P^* in place of P:

$$a_P^{uv} U_P^* = \sum_{nb} a_{nb} U_{nb}^* - \left(\delta V \frac{\partial P^*}{\partial x}\right)_P \qquad a_P^{uv} V_P^* = \sum_{nb} a_{nb} V_{nb}^* - \left(\delta V \frac{\partial P^*}{\partial y}\right)_P. \tag{16}$$

Define U', V' and P' as the difference between guessed and real fields i.e.

$$P = P^* + P' \qquad U = U^* + U' \qquad V = V^* + V'. \tag{17}$$

Subtracting the equation in the guessed fields from the actual ones yields

$$a_P^{uv}U_P' = \sum_{nb} a_{nb}U_{nb}' - \left(\delta V\frac{\partial P'}{\partial x}\right)_P \qquad a_P^{uv}V_P' = \sum_{nb} a_{nb}V_{nb}' - \left(\delta V\frac{\partial P'}{\partial y}\right)_P. \tag{18}$$

The sums over neighbouring nodes goes to zero when the guessed values have converged to the actual ones, we make the approximation that they are zero for every iteration and get

$$U_P = U_P^* - \frac{1}{a_P^{uv}}\left(\delta V\frac{\partial P'}{\partial x}\right)_P \qquad V_P = V_P^* - \frac{1}{a_P^{uv}}\left(\delta V\frac{\partial P'}{\partial y}\right)_P. \tag{19}$$

The discretized continuity equation

$$0 = F_w - F_e + F_s - F_n = (\Delta y\rho U)_w - (\Delta y\rho U)_e + (\Delta x\rho V)_s - (\Delta x\rho V)_n \tag{20}$$

contains the velocities evaluated at faces. We evaluate our expression for U and V at the east and north face using a central difference quotient in place of the pressure gradients and get

$$U_e = U_e^* - \frac{1}{(a_P^{uv})_e}\left(\delta V\frac{\partial P'}{\partial x}\right)_e = U_e^* - \left(\frac{\Delta x\Delta y}{(a_P^{uv})_e\delta x_{EP}}\right)(P'_{I+1,J} - P'_{I,J}) \tag{21}$$

$$V_n = U_n^* - \frac{1}{(a_P^{uv})_n}\left(\delta V\frac{\partial P'}{\partial y}\right)_n = U_n^* - \left(\frac{\Delta x\Delta y}{(a_P^{uv})_n\delta x_{NP}}\right)(P'_{I,J+1} - P'_{I,J}) \tag{22}$$

Inserting this into the continuity equation the factors multiplying $P'_{I+1,J}$ & $P'_{I,J+1}$ respectively are

$$a_E = \left(\frac{\Delta x\Delta y^2\rho}{(a_P^{uv})_e\delta x_{EP}}\right) \qquad a_N = \left(\frac{\Delta x^2\Delta y\rho}{(a_P^{uv})_n\delta x_{NP}}\right) \tag{23}$$

The west face of (I,J) is the east face of $(I-1,J)$ so the velocities U_w and U_s can be calculated by a shift of indices and replacement of e and n by w and s from the equations for U_e and U_n. Thus we can write a_W and a_S as

$$a_W = \left(\frac{\Delta x\Delta y^2\rho}{(a_P^{uv})_w\delta x_{WP}}\right) \qquad a_S = \left(\frac{\Delta x^2\Delta y\rho}{(a_P^{uv})_s\delta x_{SP}}\right). \tag{24}$$

We collect the terms that do not contain P' as

$$S_P = (\Delta y\rho U^*)_w - (\Delta y\rho U^*)_e + (\Delta x\rho V^*)_s - (\Delta x\rho V^*)_n \tag{25}$$

and write down the complete pressure correction equation

$$a_P P'_{I,J} = a_E P'_{I+1,J} + a_W P'_{I-1,J} + a_N P'_{I,J+1} + a_S P'_{I,J-1} + S_{I,J}. \tag{26}$$

Once the pressure correction is calculated the velocity fields must be updated. With a central difference quotient in place of the derivatives in equation 19 yields

$$U_P = U_P^* - \frac{1}{a_P^{uv}}\frac{\Delta x\Delta y}{\delta x_E + \delta x_W}(P'_E - P'_W), \qquad V_P = V_P^* - \frac{1}{a_P^{uv}}\frac{\Delta x\Delta y}{\delta y_N + \delta y_S}(P'_N - P'_S). \tag{27}$$

2.3 Rhie-Chow

The pressure correction source term and the convective coefficients in the discrete momentum equations involve the velocity fields U and V evaluated at the faces. On a collocated grid these values are not known and linear interpolation can result in oscillations in the solution, instead we introduce the Rhie-Chow interpolated face-velocities

$$U_e = \frac{U_P + U_E}{2} + \frac{\Delta y}{4a_P^{uv}}(P_{EE} - 3P_E + 3P_P - P_W), \tag{28}$$

$$V_n = \frac{V_P + V_N}{2} + \frac{\Delta x}{4a_P^{uv}}(P_{NN} - 3P_N + 3P_P - P_S). \tag{29}$$

U_w and V_s are taken as U_e in the cell to the left and V_n in the cell below respectively, in order to guarantee conservation