M1P1 Analysis I Problem Sheet 1

1. For real numbers x, y, z, consider the following inequalities.

(a) $|x+y| \leq |x|+|y|$

(b) $|x+y| \geq |x|-|y|$

(c) $|x+y| \geq |y|-|x|$

(d) $|x-y| \geq \big||x|-|y|\big|$

(e) $|x| \leq |y|+|x-y|$

(f) $|x| \geq |y|-|x-y|$

(g) $|x-y| \leq |x-z|+|y-z|$

Prove (a) from first principles. Why is it called the "triangle inequality" ?

Deduce (b,c,d,e,f,g) from (a).

2.* Fix nonempty $S \subset \mathbb{R}$ with an upper bound. Give proofs or counterexamples to the following statements.

(a) If $S \subset \mathbb{Q}$ then $\sup S \in \mathbb{Q}$.

(b) If $S \subset \mathbb{R}\backslash\mathbb{Q}$ then $\sup S \in \mathbb{R}\backslash\mathbb{Q}$.

(c) If $S \subset \mathbb{Z}$ then $\sup S \in \mathbb{Z}$.

(d) There exists a $\max S$ if and only if $\sup S \in S$.

(e) $\sup S = \inf(\mathbb{R}\backslash S)$.

(f) $\sup S = \inf(\mathbb{R}\backslash S)$ if and only if S is an interval of the form $(-\infty, a)$ or $(-\infty, a]$.

3. Suppose that u is an upper bound for the nonempty subset $S \subset \mathbb{R}$. Prove that $u = \sup S$ if and only if $\forall \epsilon > 0$, $\exists s \in S$ such that $s > u - \epsilon$.

4. Without looking at your notes, say out loud (ideally to a friend) the definition of $a_n \to a$ *in English* (not maths!).
Pass back and forwards between maths and English (e.g. $\forall\, \epsilon > 0 \quad \longleftrightarrow$ "However close I want to get", etc.).
Write down your definition. Now check your notes. Are there any subtle differences (things in a different order, $\forall$ replaced by $\exists$, etc ?) If so they're VERY important. Is your definition still correct ? There are many correct – and incorrect – ways of writing the same definition.
If it's only nearly correct, it's very wrong – can you find a counterexample ?

5. Give *without proof* examples of sequences (a_n), (b_n) with the following properties.

(i) Neither of a_n, b_n is convergent, but $a_n + b_n$, $a_n b_n$ and a_n/b_n all converge.

(ii) a_n converges, b_n is *un*bounded, but $a_n b_n$ converges.

(iii) a_n converges, b_n bounded, but $a_n b_n$ diverges.

6. Let $a_n = \frac{n}{n+2}$. Does the sequence $(a_n)_{n\geq 1}$ converge or not? Prove your answer. It will be a mess first time, but now prove it again. Keep iterating until you converge to a short, clean, easy-to-understand logical proof that the precise definition given in lectures is satisfied (or is not satisfied).

*You should prepare starred questions * to discuss with your personal tutor.*