Types + Subtyping

$$A ::= A \to B \mid b \mid \top \mid \bot$$

$\boxed{A <: B}$

$$\frac{}{A <: A}\,(\text{Refl}) \qquad \frac{A <: B \quad B <: C}{A <: C}\,(\text{Trans}) \qquad \frac{}{A <: \text{Top}}\,(\text{Top})$$

$$\frac{}{\bot <: A}\,(\text{Bot}) \qquad \frac{A_2 <: A_1 \quad B_1 <: B_2}{A_1 \to B_1 <: A_2 \to B_2}\,(\to\text{-s})$$

STLC + $\{\top, \bot\}$
+ subtyping

Terms

$$t ::= \lambda x.t \mid t\,u \mid x \mid (t : A)$$

$$\Gamma ::= \cdot \mid \Gamma, x : A$$

$\boxed{\Gamma \vdash t : A}$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\,(\to\text{-i}) \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B}\,(\to\text{-e})$$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\,(\text{Var})$$

$$\frac{\Gamma \vdash t : A \quad A <: B}{\Gamma \vdash t : B}\,(\text{sub})$$

← this is new!
... and it causes lots of problems.

Subject reduction: If $t \longrightarrow^* u$ and $\Gamma \vdash t : A$ then $\Gamma \vdash u : A$.

Pf. sketch. Pf by induction on typing derivations and analysis of $\rightarrow$-deriv.

Just look at one case: $(\rightarrow\text{-e})$ + β-contraction

Case $(\rightarrow\text{-e})$ + β-step. Have: $t = (\lambda x.t_1)t_2$, $u = t_1\{t_2/x\}$

$\Gamma \vdash \lambda x.t_2 : A_1 \rightarrow A_2$ $\qquad \Gamma \vdash t_2 : A_1$

Problem: We don't know how $\Gamma \vdash \lambda x.t_2 : A_1 \rightarrow A_2$ was derived.
The subsumption rule applies (in addition to $(\rightarrow\text{-i})$).

$$\frac{\Gamma \vdash \lambda x.t_1 : ? \qquad ? <: A_1 \rightarrow A_2}{\Gamma \vdash \lambda x.t_1 : A_1 \longrightarrow A_2}$$

Need a helper lemma!

generation lemma. Let t st. $\Gamma \vdash t : A$, then

1) if $t = x$ then $\Gamma(x) <: A$
2) if $t = \lambda x.t$ then there is A_1, A_2 st. $\Gamma, x{:}A_1 \vdash t : A_2$ and $A_1 \rightarrow A_2 <: A$

⋮

Pf. By ind. on typing derivations (exercise).

Back to the S.R. proof...

We have $t = (\lambda x.t_1 . t_2)$ $\Gamma \vdash \lambda x.t_1 : A_1 \to B_1$ $\Gamma \vdash t_2 : A_1$

Want to show that $\Gamma \vdash t_1\{^{t_2}/x\} : B_1$. Want to apply subst. lemma

$$\frac{\Gamma, x : A_1 \vdash t_1 : C \qquad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\{^{t_2}/x\} : C} \text{(Subst. lemma)}$$

Use the generation lemma: $\Gamma, x : C_1 \vdash t_1 : C_2$ and $C_1 \to C_2 <: A_1 \to A_2$

We need another helper lemma.

<u>Subtyping inversion.</u> If $A <: B$ then...

- if $B = B_1 \to B_2$ then there are A_1, A_2 s.t. $A = A_1 \to A_2$ and $B_1 <: A_2$, $A_2 <: B_2$
- ...

Pf. (by induction on the derivation of $A <: B$). Exercise (TAPL).

Now the $(\to\text{-e})$ case with β-contraction goes through because we can apply the substitution lemma (modulo (Sub))

Bidirectional typing.

Problem: subsumption throws a spanner in the works when we want to implement type checking. Why? Not syntax directed.

$$\frac{\Gamma \vdash s : A \to B \quad \Gamma \vdash t : A}{\Gamma \vdash s\,t : B} \qquad \frac{\Gamma \vdash t : \underline{B} \quad \underline{B} <: A}{\Gamma \vdash t : A} \leftarrow \text{How to guess } B\text{ ?}$$

Solution: bidirectional typing.

Inference rules (neutrals) $\boxed{\Gamma \vdash t \Rightarrow A}$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \Rightarrow A} \qquad \frac{\Gamma \vdash s \Rightarrow A \to B \quad \Gamma \vdash t \Leftarrow A}{\Gamma \vdash s\,t : \Rightarrow B} \qquad \frac{\Gamma \vdash t \Leftarrow A}{\Gamma \vdash (t : A) \Rightarrow A}$$

— Checking rules (introduction forms) $\boxed{\Gamma \vdash t \quad A}$

$$\frac{\Gamma, x : A \vdash t \Leftarrow B}{\Gamma \vdash \lambda x.t \Leftarrow A \to B} \qquad \frac{\Gamma \vdash t \Rightarrow A \quad A <: B}{\Gamma \vdash t \Leftarrow B}$$

Coercive subtyping vs. CCCs

$\boxed{\Gamma \vdash t : A}$ $\quad \Gamma \xrightarrow{t} A$

$\boxed{\Delta \vdash \sigma : \Gamma}$ $\quad \Delta \xrightarrow{\sigma} \Gamma$

$\boxed{f : A <: B}$ $\quad A \xrightarrow{f} B$

Objects : $b, b', \ldots$

Arrows : $c, c', c'' \ldots$

Coercions : $\forall b, b'$ at most one coercion $f : b \to b'$

$$\frac{}{id : A <: A}(\text{Refl}) \qquad \frac{f : A <: B \quad g : B <: C}{g \circ f : A <: C}(\text{Trans})$$

$$\frac{f : A_2 <: A_1 \quad g : B_1 <: B_2}{g^f : A_1 \to B_1 <: A_2 \to B_2}$$

$\hom(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \cancel{\mathcal{C}}\ Set$
$\hom(A,B) = B^A$
$\hom(f,g) = h \mapsto g \circ h \circ f$