The Simply Typed Lambda Calculus

and Variants Thereof

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On today's menu

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Part I STLC

The plain, old simply typed lambda calculus

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Part II Explicit substitutions

Adding explicit substitutions and their equational theory

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Part III De Bruijn indices

Doing away with variables…

On today's menu

Part I STLC

The plain, old simply typed lambda calculus

Part II Explicit substitutions

Adding explicit substitutions and their equational theory

Part III De Bruijn indices

Doing away with variables…

Part IV Correspondences

Everything is connected.

Syntax

$$
\begin{array}{lcll}
\mathcal{V} & \ni & x, y, z, \ldots & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= x \mid \lambda x.\, t \mid s\, t & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \cdot \mid \Gamma, x : A & \text{(Context)}
\end{array}
$$

Term typing $\boxed{\Gamma \vdash t : A}$

$$x_n : A_n, \ldots, x_0 : A_0 \vdash x_i : A_i \quad \text{(VAR)}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B} \quad \text{(ABS)}$$

$$\frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B} \quad \text{(APP)}$$

Substitutions

Substitutions are finite maps $\sigma : \mathcal{V} \to \mathcal{T}$ from variables to terms.

Application of substitutions

$$
\begin{aligned}
x[\sigma] &\coloneqq \sigma(x) \\
(\lambda x.\, t)[\sigma] &\coloneqq \lambda x.\, t[\sigma\{x \coloneqq x\}] \\
(t\, u)[\sigma] &\coloneqq t[\sigma]\, u[\sigma]
\end{aligned}
$$

Equational theory $\boxed{\Gamma \vdash t = u : A}$

$$\Gamma \vdash t = t : A \quad \text{(EREFL)}$$

$$\frac{\Gamma \vdash t_1 = t_2 : A \quad \Gamma \vdash t_2 = t_3 : A}{\Gamma \vdash t_1 = t_3 : A} \quad \text{(ETRANS)}$$

$$\frac{\Gamma \vdash t_1 = t_2 : A}{\Gamma \vdash t_2 = t_1 : A} \quad \text{(ESYM)}$$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.\, t)\, u = t[x := u] : B} \quad \text{(E-}\beta\text{)}$$

$$\frac{\Gamma \vdash t : A \to B \quad x \notin \mathsf{fv}(t)}{\Gamma \vdash t = \lambda x.\, t\, x : A \to B} \quad \text{(E-}\eta\text{)}$$

$$\frac{\Gamma, x : A \vdash t = t' : B}{\Gamma \vdash \lambda x.\, t = \lambda x.\, t' : A \to B} \quad \text{(EABS)}$$

$$\frac{\Gamma \vdash t = t' : A \to B \quad \Gamma \vdash u = u' : A}{\Gamma \vdash t\, u = t'\, u' : B} \quad \text{(EAPP)}$$

Explicit Substitutions

Substitutions are finite maps $\sigma : \mathcal{V} \to \mathcal{T}$ from variables to terms.

But so far, they were meta-theoretic.

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Substitutions are finite maps $\sigma : \mathcal{V} \to \mathcal{T}$ from variables to terms.

But so far, they were meta-theoretic.

Let's make them syntactic!

$$\mathcal{S} \quad \ni \quad \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma \qquad \text{(Substitution)}$$

Update: term typing

$\boxed{\Gamma \vdash t : A}$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta \vdash t : A}{\Gamma \vdash t[\sigma] : A} \qquad (\textsc{Sub})$$

Substitution typing

$\boxed{\Gamma \vdash \sigma : \Delta}$

$$\Gamma \vdash \langle\rangle : \cdot \qquad \text{(EMPTY)}$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, x := t \rangle : \Delta, x : A} \qquad \text{(PAIR)}$$

$$\Gamma, x : A \vdash \uparrow : \Gamma \qquad \text{(WEAKEN)}$$

$$\Gamma \vdash \mathrm{id} : \Gamma \qquad \text{(ID)}$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta \vdash \rho : E}{\Gamma \vdash \rho \circ \sigma : E} \qquad \text{(COMP)}$$

Update: term equality

$$\boxed{\Gamma \vdash t = u : A}$$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.\, t)\, u = t[\langle \mathrm{id}, x := u \rangle] : B} \quad (\text{E-}\beta)$$

Update: term equality $\boxed{\Gamma \vdash t = u : A}$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.\, t)\, u = t[\langle \mathrm{id}, x := u \rangle] : B} \quad (\text{E-}\beta)$$

New rules:

$$\frac{\begin{array}{c}\Gamma \vdash \sigma = \sigma' : \Delta \\ \Delta \vdash t = t' : A\end{array}}{\Gamma \vdash t[\sigma] = t'[\sigma'] : A} \quad (\textsc{ESub})$$

$$\frac{\begin{array}{c}\Gamma \vdash \rho : \Delta \\ \Delta \vdash \sigma : E \quad E \vdash t : A\end{array}}{\Gamma \vdash t[\sigma][\rho] = t[\sigma \circ \rho] : A} \quad (\textsc{EAssoc})$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t[\mathrm{id}] = t : A} \quad (\textsc{EId})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash x[\langle \sigma, x := t \rangle] = t : A} \quad (\textsc{EVarS})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta \vdash t : A \to B \quad \Delta \vdash u : A}{\Gamma \vdash (t\, u)[\sigma] = t[\sigma]\, u[\sigma] : B} \quad (\textsc{EAppS})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta, x : A \vdash t : B}{\Gamma \vdash (\lambda x.\, t)[\sigma] = \lambda x.\, t[\langle \sigma \circ \uparrow, x := x \rangle] : A \to B} \quad (\textsc{EAbsS})$$

Typing the RHS of β-contraction

$$
\dfrac{\Gamma, x : A \vdash t : B \qquad \dfrac{\dfrac{}{\Gamma \vdash \mathrm{id} : \Gamma}\,(\textsc{Id}) \qquad \Gamma \vdash u : A}{\Gamma \vdash \langle \mathrm{id}, x := u \rangle : \Delta, x : A}\,(\textsc{Pair})}{\Gamma \vdash t[\langle \mathrm{id}, x := u \rangle] : B}\,(\textsc{Sub})
$$

Substitution equality (1/2)

$\boxed{\Gamma \vdash \sigma = \rho : \Delta}$

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E \quad E \vdash \tau : \Phi}{\Gamma \vdash (\tau \circ \sigma) \circ \rho = \tau \circ (\sigma \circ \rho) : \Phi} \quad \text{(SEAssoc)}$$

$$\frac{\Gamma \vdash \sigma : \Delta}{\Gamma \vdash \mathrm{id} \circ \sigma = \sigma : \Delta} \text{(SEIdL)} \qquad \frac{\Gamma \vdash \sigma : \Delta}{\Gamma \vdash \sigma \circ \mathrm{id} = \sigma : \Delta} \text{(SEIdR)} \qquad \Gamma \vdash \sigma = \sigma : \Delta \ \text{(SERefl)}$$

$$\frac{\Gamma \vdash \sigma_1 = \sigma_2 : \Delta}{\Gamma \vdash \sigma_2 = \sigma_1 : \Delta} \text{(SESym)} \qquad \frac{\Gamma \vdash \sigma_1 = \sigma_2 : \Delta \quad \Gamma \vdash \sigma_2 = \sigma_3 : \Delta}{\Gamma \vdash \sigma_1 = \sigma_3 : \Delta} \text{(SETrans)}$$

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E \quad \Delta \vdash t : A}{\Gamma \vdash \langle \sigma, x := t \rangle \circ \rho = \langle \sigma \circ \rho, x := t[\rho] \rangle : E, x : A} \quad \text{(SEPairS)}$$

Substitution equality (2/2)

$$\boxed{\Gamma \vdash \sigma = \rho : \Delta}$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \uparrow \circ \langle \sigma, x := t \rangle = \sigma : \Delta} \quad (\text{SE-}\beta)$$

$$\Gamma, x : A \vdash \mathbf{id} = \langle \uparrow, x := x \rangle : \Gamma, x : A \quad (\text{SE-}\eta)$$

$$\frac{\Gamma \vdash \sigma : \cdot \quad \Gamma \vdash \sigma' : \cdot}{\Gamma \vdash \sigma = \sigma' : \cdot} \quad (\text{SEEmpty})$$

$$\frac{\Gamma \vdash \rho = \rho' : \Delta \quad \Delta \vdash \sigma = \sigma' : E}{\Gamma \vdash \sigma \circ \rho = \sigma' \circ \rho' : E} \quad (\text{SEComp})$$

$$\frac{\Gamma \vdash \sigma = \sigma' : \Delta \quad \Gamma \vdash t = t' : A}{\Gamma \vdash \langle \sigma, x := t \rangle = \langle \sigma, x := t' \rangle : \Delta, x : A} \quad (\text{SEPair})$$

Getting rid of variables: de Bruijn indices

$$\begin{array}{lcll}
\mathcal{V} & \ni & 0, 1, 2, \ldots & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= i \mid \lambda t \mid s\,t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \cdot \mid \Gamma, A & \text{(Context)} \\
\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}$$

Typing with de Bruijn indices

$$A_n, \ldots, A_0 \vdash i : A_i \qquad \text{(Var)}$$

$$\frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda t : A \to B} \qquad \text{(Abs)}$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta, A} \qquad \text{(Pair)}$$

$$\frac{\Gamma, A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda t)\, u = t[\langle \mathrm{id}, u \rangle] : B} \qquad \text{(E-}\beta\text{)}$$

Getting rid of indices: use just 0 + weakening

Observation: we can express natural numbers in Peano-style.

$$
\begin{aligned}
1 &= \mathrm{suc}(0) \\
2 &= \mathrm{suc}(\mathrm{suc}(0)) \\
3 &= \mathrm{suc}(\mathrm{suc}(\mathrm{suc}(0))) \\
&\vdots
\end{aligned}
$$

Getting rid of indices: use just 0 + weakening

Observation: we can express natural numbers in Peano-style.

$$
\begin{aligned}
1 &= 0[\uparrow] \\
2 &= 0[\uparrow][\uparrow] \\
3 &= 0[\uparrow][\uparrow][\uparrow] \\
&\vdots
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3 &= 0[\uparrow][\uparrow][\uparrow] \\
&\vdots
\end{aligned}
$$

$$
\begin{array}{lcll}
\mathcal{V} & \ni & 0 & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= 0 \mid \lambda t \mid s\, t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \cdot \mid \Gamma, A & \text{(Context)} \\
\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Typing with de Bruijn indices

$$\Gamma, A \vdash 0 : A \qquad \text{(Var)}$$

$$\frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda t : A \to B} \qquad \text{(Abs)}$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta, A} \qquad \text{(Pair)}$$

$$\frac{\Gamma, A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda t)\, u = t[\langle \mathrm{id}, u \rangle] : B} \qquad \text{(E-}\beta\text{)}$$

Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$$
\begin{array}{lcrl}
\mathcal{V} & \ni & 0 & \text{(Variable)} \\
\mathcal{T} & \ni & s,t,u ::= 0 \mid \lambda t \mid s\,t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A,B,C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma,\Delta,E ::= \cdot \mid \Gamma, A & \text{(Context)} \\
\mathcal{S} & \ni & \rho,\sigma,\tau ::= \langle\rangle \mid \langle\sigma, t\rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$$
\begin{array}{lcll}
\mathcal{V} & \ni & \pi_2 & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= \pi_2 \mid \lambda t \mid s\,t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \mathbf{1} \mid \Gamma \times A & \text{(Context)} \\
\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \pi_1 \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$$
\begin{array}{lcrr}
\mathcal{V} & \ni & \pi_2 & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= \pi_2 \mid \lambda t \mid s\, t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C, \Gamma, \Delta, E ::= \mathrm{b} \mid A \to B \mid \mathbf{1} \mid \Gamma \times A & \text{(Type)} \\
\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \pi_1 \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$$\mathcal{V} \ni \quad \pi_2 \qquad \text{(Variable)}$$

$$\mathcal{T} \ni \quad s, t, u, \rho, \sigma, \tau ::= \pi_2 \mid \lambda t \mid s\, t \mid \langle\rangle \mid \langle \sigma, t \rangle \mid \pi_1 \mid \mathrm{id} \mid \rho \circ \sigma \qquad \text{(Term)}$$

$$\mathcal{A} \ni A, B, C, \Gamma, \Delta, E ::= \mathrm{b} \mid A \to B \mid \mathbf{1} \mid \Gamma \times A \qquad \text{(Type)}$$

Term typing

$$\boxed{\Gamma \vdash t : A}$$

$$\Gamma, A \vdash 0 : A \qquad (\textsc{Var})$$

$$\frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda t : A \to B} \quad (\textsc{Abs})$$

$$\frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B} \quad (\textsc{App})$$

$$\Gamma \vdash \langle\rangle : \cdot \qquad (\textsc{Empty})$$

$$\frac{\Gamma \vdash \sigma : \Delta \qquad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta, A} \quad (\textsc{Pair})$$

$$\Gamma, A \vdash \pi_1 : \Gamma \quad (\textsc{Weaken})$$

$$\frac{\Gamma \vdash \rho : \Delta \qquad \Delta \vdash \sigma : E}{\Gamma \vdash \sigma \circ \rho : E} \quad (\textsc{Comp})$$

$$\Gamma \vdash \mathrm{id} : \Gamma \qquad (\textsc{Id})$$

Term typing $\boxed{\Gamma \vdash t : A}$

$$\Gamma \times A \vdash \pi_2 : A \quad (\textsc{Var})$$

$$\frac{\Gamma \times A \vdash t : B}{\Gamma \vdash \lambda t : A \to B} \quad (\textsc{Abs})$$

$$\frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B} \quad (\textsc{App})$$

$$\Gamma \vdash \langle\rangle : \mathbf{1} \quad (\textsc{Empty})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta \times A} \quad (\textsc{Pair})$$

$$\Gamma \times A \vdash \pi_1 : \Gamma \quad (\textsc{Weaken})$$

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E}{\Gamma \vdash \sigma \circ \rho : E} \quad (\textsc{Comp})$$

$$\Gamma \vdash \mathrm{id} : \Gamma \quad (\textsc{Id})$$

Combinator typing

$$\boxed{t : \Gamma \longrightarrow A}$$

$$\pi_2 : \Gamma \times A \longrightarrow A \quad (\textsc{Var})$$

$$\frac{\sigma : \Gamma \longrightarrow \Delta \quad t : \Gamma \longrightarrow A}{\langle \sigma, t \rangle : \Gamma \longrightarrow \Delta \times A} \; (\textsc{Pair})$$

$$\frac{t : \Gamma \times A \longrightarrow B}{\lambda t : \Gamma \longrightarrow A \to B} \; (\textsc{Abs})$$

$$\pi_1 : \Gamma \times A \longrightarrow \Gamma \; (\textsc{Weaken})$$

$$\frac{t : \Gamma \longrightarrow A \to B \quad u : \Gamma \longrightarrow A}{t\, u : \Gamma \longrightarrow B} \; (\textsc{App})$$

$$\frac{\rho : \Gamma \longrightarrow \Delta \quad \sigma : \Delta \longrightarrow E}{\sigma \circ \rho : \Gamma \longrightarrow E} \; (\textsc{Comp})$$

$$\langle\rangle : \Gamma \longrightarrow \mathbf{1} \quad (\textsc{Empty})$$

$$\mathrm{id} : \Gamma \longrightarrow \Gamma \quad (\textsc{Id})$$

Combinatory IPL

$$\boxed{\Gamma \Longrightarrow A}$$

$$\Gamma \times A \Longrightarrow A \quad (\textsc{Var})$$

$$\frac{\Gamma \times A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \quad (\textsc{Abs})$$

$$\frac{\Gamma \Longrightarrow A \supset B \quad \Gamma \Longrightarrow A}{\Gamma \Longrightarrow B} \quad (\textsc{App})$$

$$\Gamma \Longrightarrow \top \quad (\textsc{Empty})$$

$$\frac{\Gamma \Longrightarrow \Delta \quad \Gamma \Longrightarrow A}{\Gamma \Longrightarrow \Delta \times A} \quad (\textsc{Pair})$$

$$\Gamma \times A \Longrightarrow \Gamma \quad (\textsc{Weaken})$$

$$\frac{\Gamma \Longrightarrow \Delta \quad \Delta \Longrightarrow E}{\Gamma \Longrightarrow E} \quad (\textsc{Comp})$$

$$\Gamma \Longrightarrow \Gamma \quad (\textsc{Id})$$