The Simply Typed Lambda Calculus

and Variants Thereof

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On today's menu

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Part I STLC

The plain, old simply typed lambda calculus

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Part II Explicit substitutions

Adding explicit substitutions and their equational theory

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Part I STLC

The plain, old simply typed lambda calculus

Part II Explicit substitutions

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Part III De Bruijn indices

Doing away with variables…

On today's menu

Part I	STLC
	The plain, old simply typed lambda calculus
Part II	Explicit substitutions
	Adding explicit substitutions and their equational theory
Part III	De Bruijn indices
	Doing away with variables…
Part IV	Correspondences
	Everything is connected.

Syntax

$$
\begin{array}{lcll}
\mathcal{V} & \ni & x, y, z, \ldots & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= x \mid \lambda x.\, t \mid s\, t & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \cdot \mid \Gamma, x : A & \text{(Context)}
\end{array}
$$

Term typing

$\boxed{\Gamma \vdash t : A}$

$$\frac{}{x_n : A_n, \ldots, x_0 : A_0 \vdash x_i : A_i} \quad (\textsc{Var})$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B} \quad (\textsc{Abs})$$

$$\frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B} \quad (\textsc{App})$$

Substitutions

Substitutions are finite maps $\sigma : \mathcal{V} \to \mathcal{T}$ from variables to terms.

Application of substitutions

$$\begin{aligned} x[\sigma] &:= \sigma(x) \\ (\lambda x.\, t)[\sigma] &:= \lambda x.\, t[\sigma\{x \mapsto x\}] && \text{if } x \notin \mathsf{fv}(\sigma(\mathcal{V})) \\ (t\, u)[\sigma] &:= t[\sigma]\, u[\sigma] \end{aligned}$$

NB. The hygiene condition $x \notin \mathsf{fv}(\sigma(\mathcal{V}))$ is necessary to avoid variable capture. It can be avoided through systematic α-renaming (no “shadowing”).

Equational theory

$$\boxed{\Gamma \vdash t = u : A}$$

$$\frac{}{\Gamma \vdash t = t : A}\ (\text{ERefl}) \qquad \frac{\Gamma \vdash t_1 = t_2 : A \quad \Gamma \vdash t_2 = t_3 : A}{\Gamma \vdash t_1 = t_3 : A}\ (\text{ETrans})$$

$$\frac{\Gamma \vdash t_1 = t_2 : A}{\Gamma \vdash t_2 = t_1 : A}\ (\text{ESym})$$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.\, t)\, u = t[x \mapsto u] : B}\ (\text{E-}\beta) \qquad \frac{\Gamma \vdash t : A \to B \quad x \notin \mathsf{fv}(t)}{\Gamma \vdash t = \lambda x.\, t\, x : A \to B}\ (\text{E-}\eta)$$

$$\frac{\Gamma, x : A \vdash t = t' : B}{\Gamma \vdash \lambda x.\, t = \lambda x.\, t' : A \to B}\ (\text{EAbs}) \qquad \frac{\Gamma \vdash t = t' : A \to B \quad \Gamma \vdash u = u' : A}{\Gamma \vdash t\, u = t'\, u' : B}\ (\text{EApp})$$

Explicit Substitutions

Substitutions are finite maps $\sigma : \mathcal{V} \to \mathcal{T}$ from variables to terms.

But so far, they were meta-theoretic.

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Let's make them syntactic!

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Substitutions are finite maps $\sigma : \mathcal{V} \to \mathcal{T}$ from variables to terms.

But so far, they were meta-theoretic.

Let's make them syntactic!

$$\mathcal{S} \quad \ni \quad \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma \qquad \text{(Substitution)}$$

Update: term typing

$\boxed{\Gamma \vdash t : A}$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta \vdash t : A}{\Gamma \vdash t[\sigma] : A} \quad (\textsc{Sub})$$

Substitution typing

$\boxed{\Gamma \vdash \sigma : \Delta}$

$$\frac{}{\Gamma \vdash \langle\rangle : \cdot} \quad (\textsc{Empty})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, x \mapsto t \rangle : \Delta, x : A} \quad (\textsc{Pair})$$

$$\frac{}{\Gamma, x : A \vdash \uparrow : \Gamma} \quad (\textsc{Weaken})$$

$$\frac{}{\Gamma \vdash \mathrm{id} : \Gamma} \quad (\textsc{Id})$$

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E}{\Gamma \vdash \sigma \circ \rho : E} \quad (\textsc{Comp})$$

Update: term equality

$\boxed{\Gamma \vdash t = u : A}$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.\, t)\, u = t[\langle \mathrm{id}, x \mapsto u \rangle] : B} \quad (\text{E-}\beta)$$

Update: term equality $\boxed{\Gamma \vdash t = u : A}$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.\, t)\, u = t[\langle \mathrm{id}, x \mapsto u \rangle] : B} \;(\text{E-}\beta)$$

New rules:

$$\frac{\Gamma \vdash \sigma = \sigma' : \Delta \quad \Delta \vdash t = t' : A}{\Gamma \vdash t[\sigma] = t'[\sigma'] : A} \;(\textsc{ESub})$$

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E \quad E \vdash t : A}{\Gamma \vdash t[\sigma][\rho] = t[\sigma \circ \rho] : A} \;(\textsc{EAssoc})$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t[\mathrm{id}] = t : A} \;(\textsc{EId})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash x[\langle \sigma, x \mapsto t \rangle] = t : A} \;(\textsc{EVarS})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta \vdash t : A \to B \quad \Delta \vdash u : A}{\Gamma \vdash (t\, u)[\sigma] = t[\sigma]\, u[\sigma] : B} \;(\textsc{EAppS})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta, x : A \vdash t : B}{\Gamma \vdash (\lambda x.\, t)[\sigma] = \lambda x.\, t[\langle \sigma \circ \uparrow, x \mapsto x \rangle] : A \to B} \;(\textsc{EAbsS})$$

Typing the RHS of β-contraction

$$
\dfrac{\Gamma, x : A \vdash t : B \qquad \dfrac{\dfrac{}{\Gamma \vdash \mathrm{id} : \Gamma}\,(\textsc{Id}) \qquad \Gamma \vdash u : A}{\Gamma \vdash \langle \mathrm{id}, x \mapsto u \rangle : \Delta, x : A}\,(\textsc{Pair})}{\Gamma \vdash t[\langle \mathrm{id}, x \mapsto u \rangle] : B}\,(\textsc{Sub})
$$

Substitution equality (1/2)

$\boxed{\Gamma \vdash \sigma = \rho : \Delta}$

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E \quad E \vdash \tau : \Phi}{\Gamma \vdash (\tau \circ \sigma) \circ \rho = \tau \circ (\sigma \circ \rho) : \Phi} \text{ (SEAssoc)}$$

$$\frac{\Gamma \vdash \sigma : \Delta}{\Gamma \vdash \mathrm{id} \circ \sigma = \sigma : \Delta} \text{ (SEIdL)} \qquad \frac{\Gamma \vdash \sigma : \Delta}{\Gamma \vdash \sigma \circ \mathrm{id} = \sigma : \Delta} \text{ (SEIdR)} \qquad \frac{}{\Gamma \vdash \sigma = \sigma : \Delta} \text{ (SERefl)}$$

$$\frac{\Gamma \vdash \sigma_1 = \sigma_2 : \Delta}{\Gamma \vdash \sigma_2 = \sigma_1 : \Delta} \text{ (SESym)} \qquad \frac{\Gamma \vdash \sigma_1 = \sigma_2 : \Delta \quad \Gamma \vdash \sigma_2 = \sigma_3 : \Delta}{\Gamma \vdash \sigma_1 = \sigma_3 : \Delta} \text{ (SETrans)}$$

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E \quad \Delta \vdash t : A}{\Gamma \vdash \langle \sigma, x \mapsto t \rangle \circ \rho = \langle \sigma \circ \rho, x \mapsto t[\rho] \rangle : E, x : A} \text{ (SEPairS)}$$

Substitution equality (2/2)

$$\boxed{\Gamma \vdash \sigma = \rho : \Delta}$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash {\uparrow} \circ \langle \sigma, x \mapsto t \rangle = \sigma : \Delta} \quad (\text{SE-}\beta)$$

$$\frac{}{\Gamma, x : A \vdash \mathbf{id} = \langle {\uparrow}, x \mapsto x \rangle : \Gamma, x : A} \quad (\text{SE-}\eta)$$

$$\frac{\Gamma \vdash \sigma : \cdot}{\Gamma \vdash \sigma = \langle\rangle : \cdot} \quad (\text{SEEmpty})$$

$$\frac{\Gamma \vdash \rho = \rho' : \Delta \quad \Delta \vdash \sigma = \sigma' : E}{\Gamma \vdash \sigma \circ \rho = \sigma' \circ \rho' : E} \quad (\text{SEComp})$$

$$\frac{\Gamma \vdash \sigma = \sigma' : \Delta \quad \Gamma \vdash t = t' : A}{\Gamma \vdash \langle \sigma, x \mapsto t \rangle = \langle \sigma, x \mapsto t' \rangle : \Delta, x : A} \quad (\text{SEPair})$$

Getting rid of variables: de Bruijn indices

$$
\begin{array}{lcll}
\mathcal{V} & \ni & 0, 1, 2, \ldots & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= i \mid \lambda t \mid s\, t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \cdot \mid \Gamma, A & \text{(Context)} \\
\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Typing with de Bruijn indices

$$\frac{}{A_n, \ldots, A_0 \vdash i : A_i} \quad \text{(Var)}$$

$$\frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda t : A \to B} \quad \text{(Abs)}$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta, A} \quad \text{(Pair)}$$

$$\frac{\Gamma, A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda t)\, u = t[\langle \mathrm{id}, u \rangle] : B} \quad \text{(E-}\beta\text{)}$$

Getting rid of indices: use just 0 + weakening

Observation: we can express natural numbers in Peano-style.

$$\begin{aligned}
1 &= \mathrm{suc}(0) \\
2 &= \mathrm{suc}(\mathrm{suc}(0)) \\
3 &= \mathrm{suc}(\mathrm{suc}(\mathrm{suc}(0))) \\
&\vdots
\end{aligned}$$

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$$\frac{}{\Gamma, A \vdash {\uparrow} : \Gamma} \quad (\textsc{Weaken})$$

Getting rid of indices: use just 0 + weakening

Observation: we can express natural numbers in Peano-style.

$$
\begin{aligned}
1 &= 0[\uparrow] \\
2 &= 0[\uparrow][\uparrow] \\
3 &= 0[\uparrow][\uparrow][\uparrow] \\
&\vdots
\end{aligned}
$$

$$\frac{}{\Gamma, A \vdash \uparrow : \Gamma} \quad (\textsc{Weaken})$$

Getting rid of indices: use just 0 + weakening

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$$1 = 0[\uparrow]$$
$$2 = 0[\uparrow][\uparrow]$$
$$3 = 0[\uparrow][\uparrow][\uparrow]$$
$$\vdots$$

$$\frac{}{\Gamma, A \vdash \uparrow : \Gamma} \quad (\textsc{Weaken})$$

$$
\begin{array}{llll}
\mathcal{V} & \ni & 0 & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= 0 \mid \lambda t \mid s\, t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
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\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Typing with de Bruijn indices

$$\frac{}{\Gamma, A \vdash 0 : A} \quad (\textsc{Var})$$

$$\frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda t : A \to B} \quad (\textsc{Abs})$$

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta, A} \quad (\textsc{Pair})$$

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Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$$
\begin{array}{llll}
\mathcal{V} & \ni & 0 & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= 0 \mid \lambda t \mid s\, t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \cdot \mid \Gamma, A & \text{(Context)} \\
\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \uparrow \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$$
\begin{array}{lcll}
\mathcal{V} & \ni & \pi_2 & \text{(Variable)} \\
\mathcal{T} & \ni & s, t, u ::= \pi_2 \mid \lambda t \mid s\, t \mid t[\sigma] & \text{(Term)} \\
\mathcal{A} & \ni & A, B, C ::= \mathrm{b} \mid A \to B & \text{(Type)} \\
\mathcal{C} & \ni & \Gamma, \Delta, E ::= \mathbf{1} \mid \Gamma \times A & \text{(Context)} \\
\mathcal{S} & \ni & \rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \pi_1 \mid \mathrm{id} \mid \rho \circ \sigma & \text{(Substitution)}
\end{array}
$$

Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$\mathcal{V}$	$\ni$	π_2	(Variable)
$\mathcal{T}$	$\ni$	$s, t, u ::= \pi_2 \mid \lambda t \mid s\,t \mid t[\sigma]$	(Term)
$\mathcal{A}$	$\ni$	$A, B, C, \Gamma, \Delta, E ::= \mathrm{b} \mid A \to B \mid \mathbf{1} \mid \Gamma \times A$	(Type)
$\mathcal{S}$	$\ni$	$\rho, \sigma, \tau ::= \langle\rangle \mid \langle \sigma, t \rangle \mid \pi_1 \mid \mathrm{id} \mid \rho \circ \sigma$	(Substitution)

Getting rid of contexts: products

Observation: we can express contexts as (cartesian) products.

$$\mathcal{V} \ni \quad \pi_2 \qquad \text{(Variable)}$$

$$\mathcal{T} \ni \quad s, t, u, \rho, \sigma, \tau ::= \pi_2 \mid \lambda t \mid s\, t \mid \langle\rangle \mid \langle \sigma, t \rangle \mid \pi_1 \mid \mathbf{id} \mid \rho \circ \sigma \qquad \text{(Term)}$$

$$\mathcal{A} \ni A, B, C, \Gamma, \Delta, E ::= \mathrm{b} \mid A \to B \mid \mathbf{1} \mid \Gamma \times A \qquad \text{(Type)}$$

Term typing $\boxed{\Gamma \vdash t : A}$

$$\frac{}{\Gamma, A \vdash 0 : A}\ (\textsc{Var}) \qquad \frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta, A}\ (\textsc{Pair})$$

$$\frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda t : A \to B}\ (\textsc{Abs}) \qquad \frac{}{\Gamma, A \vdash \pi_1 : \Gamma}\ (\textsc{Weaken})$$

$$\frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B}\ (\textsc{App}) \qquad \frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E}{\Gamma \vdash \sigma \circ \rho : E}\ (\textsc{Comp})$$

$$\frac{}{\Gamma \vdash \langle\rangle : \cdot}\ (\textsc{Empty}) \qquad \frac{}{\Gamma \vdash \mathrm{id} : \Gamma}\ (\textsc{Id})$$

Term typing $\boxed{\Gamma \vdash t : A}$

$$\frac{}{\Gamma \times A \vdash \pi_2 : A} \ (\textsc{Var}) \qquad \frac{\Gamma \vdash \sigma : \Delta \quad \Gamma \vdash t : A}{\Gamma \vdash \langle \sigma, t \rangle : \Delta \times A} \ (\textsc{Pair})$$

$$\frac{\Gamma \times A \vdash t : B}{\Gamma \vdash \lambda t : A \to B} \ (\textsc{Abs}) \qquad \frac{}{\Gamma \times A \vdash \pi_1 : \Gamma} \ (\textsc{Weaken})$$

$$\frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B} \ (\textsc{App}) \qquad \frac{\Gamma \vdash \rho : \Delta \quad \Delta \vdash \sigma : E}{\Gamma \vdash \sigma \circ \rho : E} \ (\textsc{Comp})$$

$$\frac{}{\Gamma \vdash \langle\rangle : \mathbf{1}} \ (\textsc{Empty}) \qquad \frac{}{\Gamma \vdash \mathrm{id} : \Gamma} \ (\textsc{Id})$$

Combinator typing

$\boxed{t : \Gamma \longrightarrow A}$

$$\frac{}{\pi_2 : \Gamma \times A \longrightarrow A}\ (\textsc{Var}) \qquad \frac{\sigma : \Gamma \longrightarrow \Delta \quad t : \Gamma \longrightarrow A}{\langle \sigma, t \rangle : \Gamma \longrightarrow \Delta \times A}\ (\textsc{Pair})$$

$$\frac{t : \Gamma \times A \longrightarrow B}{\lambda t : \Gamma \longrightarrow A \to B}\ (\textsc{Abs}) \qquad \frac{}{\pi_1 : \Gamma \times A \longrightarrow \Gamma}\ (\textsc{Weaken})$$

$$\frac{t : \Gamma \longrightarrow A \to B \quad u : \Gamma \longrightarrow A}{t\,u : \Gamma \longrightarrow B}\ (\textsc{App}) \qquad \frac{\rho : \Gamma \longrightarrow \Delta \quad \sigma : \Delta \longrightarrow E}{\sigma \circ \rho : \Gamma \longrightarrow E}\ (\textsc{Comp})$$

$$\frac{}{\langle\rangle : \Gamma \longrightarrow \mathbf{1}}\ (\textsc{Empty}) \qquad \frac{}{\mathrm{id} : \Gamma \longrightarrow \Gamma}\ (\textsc{Id})$$

Intuitionistic Propositional Logic $\boxed{\Gamma \vdash A}$

$$\frac{}{\Gamma \wedge A \vdash A} \; (\textsc{Var})$$

$$\frac{\Gamma \vdash \Delta \quad \Gamma \vdash A}{\Gamma \vdash \Delta \wedge A} \; (\textsc{Pair})$$

$$\frac{\Gamma \wedge A \vdash B}{\Gamma \vdash A \supset B} \; (\textsc{Abs})$$

$$\frac{}{\Gamma \wedge A \vdash \Gamma} \; (\textsc{Weaken})$$

$$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \; (\textsc{App})$$

$$\frac{\Gamma \vdash \Delta \quad \Delta \vdash E}{\Gamma \vdash E} \; (\textsc{Comp})$$

$$\frac{}{\Gamma \vdash \top} \; (\textsc{Empty})$$

$$\frac{}{\Gamma \vdash \Gamma} \; (\textsc{Id})$$