

Cartesian closed categories

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Overview:

1. Categories
2. Functors and natural transformations
3. Products
4. Exponentials

1. Categories

1.1. Definition and examples

Definition 1 (Category). A category \mathcal{C} is given by the following data:

1. Types:
 - a) A type \mathbf{Ob} of *objects*.
 - b) For each pair of objects $A, B : \mathbf{Ob}$, a type $\mathbf{Hom}(A, B)$ of (homo)morphisms $f : A \longrightarrow B$.
 - c) For each pair of objects $A, B : \mathbf{Ob}$, an equivalence relation $\mathbf{Eq}(A, B)$ on $\mathbf{Hom}(A, B)$. Given $f, g : \mathbf{Hom}(A, B)$, we write $f = g$ for $\mathbf{Eq}(A, B)(f, g)$.
2. Operations:
 - a) For each object $A : \mathbf{Ob}$ an automorphism $\mathbf{id}_A : A \longrightarrow A$ (identity).
 - b) For each pair $f : A \longrightarrow B$ and $g : B \longrightarrow C$ of morphisms a morphism $g \circ f : A \longrightarrow C$ (composition).
3. Laws:
 - a) For each morphism $f : A \longrightarrow B$ we have $\mathbf{id}_B \circ f = f$ (left identity) and $f \circ \mathbf{id}_A = f$ (right identity).
 - b) For all morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ and $h : C \longrightarrow D$ we have $(h \circ g) \circ f = h \circ (g \circ f)$ (associativity).

- c) For all morphisms $f, f' : A \longrightarrow B$ such that $f = f'$ and $g, g' : B \longrightarrow C$ such that $g = g'$ we have $g \circ f = g' \circ f'$ (congruence).

The arrow $A \longrightarrow B$ is just a nice notation for $\text{Hom}(A, B)$. It is also common to write $\mathcal{C}(A, B)$ to clarify that we mean the type $\text{Hom}_{\mathcal{C}}(A, B)$ of morphisms of category \mathcal{C} . Also $A : \mathcal{C}$ is short for $A : \text{Ob}_{\mathcal{C}}$.

Remark 1 (Homsetoid). Since a type with an equivalence relation is called a *setoid* we could just ask for a family $\text{Hom} : \text{Ob} \rightarrow \text{Ob} \rightarrow \text{Setoid}$.

The prime example for categories are collections of algebraic structures and their structure-preserving homomorphisms.

Example 1 (Groups). Grp is the category of groups and group homomorphisms. More precisely, the objects of Grp are groups, and an element $f : \text{Grp}(A, B)$ is a function $f : A \rightarrow B$ mapping the unit of A to the unit of B and the A -composition of two elements of A to the B -composition of their images under f .

Less abstractly, a group morphism $f : (A, 0, +, -) \longrightarrow (B, 1, \times, ^{-1})$ has to satisfy $f(0) = 1$ and $f(a + a') = f(a) \times f(a')$.

Exercise 1 (Groups).

1. Give an example for a group morphism f .
2. Show that a group morphism automatically preserves inverses, i.e., $f(-a) = (f(a))^{-1}$.

Analogously to groups, other algebraic structures can be organized as categories as well (monoids, rings, fields). We exhibit the most basic examples:

Example 2 (Sets). Set is the category of types A and functions $f : A \rightarrow B$.

Example 3 (Setoids). Setoid is the category of setoids (A, \approx_A) and \approx -preserving functions, i.e., $f : A \longrightarrow B$ must satisfy $f(a) \approx_B f(a')$ whenever $a \approx_A a'$.

Besides organizing algebraic structures, categories can also *implement* structures.

Example 4 (Monoid). Each monoid (M, e, \cdot) can be presented as category \mathcal{C}_M with a single object 1 and $\text{Hom}(1, 1) = M$. Then $\text{id}_1 = e$ and $f \circ g = f \cdot g$.

Exercise 2 (Partial monoid). Can any partial monoid be represented as category as well? If yes, how? If no, give a counterexample!

Recall that a set M with a distinguished element $e : M$ and a *partial* binary operation $_ \circ _ : M \times M \rightarrow M$ is called a *partial* monoid if 1. $e \circ x = x = x \circ e$ for each element $x : M$, and 2. $(x \times y) \times z = x \times (y \times z)$ if $y \times z$ and $x \times (y \times z)$, or $x \times y$ and $(x \times y) \times z$ are defined.

Example 5 (Preorder). Any preorder (A, \leq) can be presented as a thin category with $\text{Ob} = A$ and $\text{Hom}(a, b) = \{0 \mid a \leq b\}$. Identity is reflexivity and composition is transitivity.

A category \mathcal{C} is called *thin* if each homset $\text{Hom}_{\mathcal{C}}(A, B)$ has *at most one* inhabitant, that is for all pairs of parallel morphisms $f, f' : A \longrightarrow B$ in \mathcal{C} we have $f = f'$.

Example 6 (Relations). The category Rel has types as objects and binary relations as morphisms: $\text{Rel}(A, B) = \mathcal{P}(A \times B)$.

Example 7 (Contexts and substitutions). Take the typing contexts Γ of simply-typed lambda-calculus as objects, $\text{Ob} = \text{Cxt}$, and the set of substitutions $\text{Sub } \Gamma \Delta$ as morphisms from Γ to Δ .

Definition 2 (Subcategory). A category \mathcal{D} is a *subcategory* of \mathcal{C} if $\text{Ob}_{\mathcal{D}} \subseteq \text{Ob}_{\mathcal{C}}$, $\text{Hom}_{\mathcal{D}}(A, B) \subseteq \text{Hom}_{\mathcal{C}}(A, B)$ for all $A, B : \text{Ob}_{\mathcal{D}}$, $\text{id}_{\mathcal{D}, A} = \text{id}_{\mathcal{C}, A}$ for all $A : \text{Ob}_{\mathcal{D}}$, and $g \circ_{\mathcal{D}} f = g \circ_{\mathcal{C}} f$ for all $f : \text{Hom}_{\mathcal{D}}(A, B)$ and $g : \text{Hom}_{\mathcal{D}}(B, C)$.

If $\text{Ob}_{\mathcal{D}} = \text{Ob}_{\mathcal{C}}$, the subcategory is *wide*.

If $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$ for all $A, B : \text{Ob}_{\mathcal{D}}$, the subcategory is *full*.

In other words, a subcategory \mathcal{D} of \mathcal{C} is a selection of objects and morphisms from \mathcal{C} that still forms a category, i.e., is closed under identity and composition.

1.2. On the equality of objects

Our definition of category does not include an equivalence relation on Ob . This is by intention, speaking about object equality is not considered pure category-theoretic spirit. All category-theoretic notions should respect isomorphic objects.

Definition 3 (Isomorphism). An *isomorphism* (short *iso*) between two objects A and B is a pair of morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow A$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$. The existence of an isomorphism is written $A \cong B$, and the set of isomorphisms is denoted by $\text{Iso}(A, B)$.

Lemma 1 (Inverse). *Fixing f , the inverse g is uniquely determined and denoted by f^{-1} .*

In other words, being an isomorphism is not a property of a pair of morphisms but a single morphism, that is for each pair of objects A and B the function

$$\text{Iso}(A, B) \rightarrow \{f \in \text{Hom}(A, B) \mid \exists g. g \circ f = \text{id} \wedge f \circ g = \text{id}\} \quad (f, g) \mapsto f$$

is a bijection of sets with inverse $f \mapsto (f, f^{-1})$.

Exercise 3. Prove this!

Exercise 4 (Subcategory of isomorphisms). Show that the isomorphisms of a category constitute a wide subcategory.

Exercise 5. Does the concept *subcategory* (Definition 2) honor the ideal that no category-theoretic concept should distinguish between isomorphic objects?

If not, suggest a modification of the definition, or defend the current definition against the ideal.

1.3. Operations on categories

Some operations on the object types can be lifted to categories.

1. The product $\mathcal{C} \times \mathcal{D}$ of two categories forms again a category with $\text{Ob}_{\mathcal{C} \times \mathcal{D}} = \text{Ob}_{\mathcal{C}} \times \text{Ob}_{\mathcal{D}}$.
2. The latter can be generalized to nullary, finite, and even infinite products.
3. Any type can be turned into a *discrete* category where the identities are the only morphisms.

Definition 4 (Opposite category). Given a category \mathcal{C} , its *opposite* \mathcal{C}^{op} has the same objects but flipped morphisms, $\mathcal{C}^{\text{op}}(A, B) = \mathcal{C}(B, A)$, and thus flipped composition: $f \circ_{\mathcal{C}^{\text{op}}} g = g \circ_{\mathcal{C}} f$.

Remark 2. The opposite category is really just the original category with morphisms relabeled so that source and target are formally exchanged.

Exercise 6. Show that \mathcal{C}^{op} is indeed a category.

Show that $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$.

2. Functors and Natural Transformations

A *functor* $F : [\mathcal{C}, \mathcal{D}]$ is a category morphism:

Definition 5 (Functor). Given categories \mathcal{C} and \mathcal{D} a functor $F : [\mathcal{C}, \mathcal{D}]$ is given by the following data:

1. Maps:
 - a) A function $F_0 : \text{Ob}_{\mathcal{C}} \rightarrow \text{Ob}_{\mathcal{D}}$.
 - b) For any pair of objects $A, B : \mathcal{C}$, a function $F_1 : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F_0 A, F_0 B)$.
2. Laws:
 - a) For any object $A : \mathcal{C}$ we have $F_1(\text{id}_A) = \text{id}_{F_0 A}$.
 - b) For any pair of morphisms $f : \mathcal{C}(A, B)$ and $g : \mathcal{C}(B, C)$ we have $F_1(g \circ_{\mathcal{C}} f) = F_1 g \circ_{\mathcal{D}} F_1 f$.
 - c) For any pair of parallel morphisms $f, f' : \mathcal{C}(A, B)$ such that $f = f'$ we have $F_1(f) = F_1(f')$.

It is common to drop the indices 0 and 1 and simply write, e.g., $Ff : FA \rightarrow FB$. Also, since there is little chance of confusion, one often writes $F : \mathcal{C} \rightarrow \mathcal{D}$ instead of $F : [\mathcal{C}, \mathcal{D}]$.

Example 8 (Forgetful functor). “Forgetting” algebraic structure gives rise to trivial functors, the so-called *forgetful functors*, often denoted by U . For example, $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ maps groups to their carriers, and group morphisms to their underlying functions on the carriers.

A forgetful functor does nothing to the “values”, only changes their “types”.

Exercise 7. Define the duplication functor $\text{Dup} : [\mathcal{C}, \mathcal{C} \times \mathcal{C}]$ from a category to its square.

Since functors are not mathematical structures (such as groups and categories) it is not obvious what the notion of morphism between two functors $F, G : [\mathcal{C}, \mathcal{D}]$ should be. The definition states that it is a family of morphisms $FA \rightarrow GA$ parametric in A :

Definition 6 (Natural transformation). Given functors $F, G : [\mathcal{C}, \mathcal{D}]$, a *natural transformation* $\eta : F \rightarrow G$ is a family of morphisms $\eta_A : FA \rightarrow GA$ indexed by $A : \mathcal{C}$ such that for all $f : A \rightarrow B$ we have $Gf \circ \eta_A = \eta_B \circ Ff$.

Diagrammatically, the commutation law can be depicted as follows:

$$\begin{array}{ccc} A & & FA \xrightarrow{\eta_A} GA \\ \downarrow f & & \downarrow Ff \quad \quad \downarrow Gf \\ B & & FB \xrightarrow{\eta_B} GB \end{array}$$

Exercise 8 (Functor category). Show that functors in $[\mathcal{C}, \mathcal{D}]$ form a category with natural transformations as morphisms.

Definition 7 (Cat). Taking categories \mathcal{C} as objects themselves and functor sets $[\mathcal{C}, \mathcal{D}]$ as homsets, we arrive at the category \mathbf{Cat} of categories!

For consistency reasons $\text{Ob}_{\mathbf{Cat}}$ needs to be a large type containing categories \mathcal{C} whose $\text{Ob}_{\mathcal{C}}$ is a small type.

Exercise 9. Prove that functors are indeed closed under composition and that \mathbf{Cat} is indeed a category.

Remark 3 (2-categories). In \mathbf{Cat} , the functor types $[\mathcal{C}, \mathcal{D}]$ are only taken as sets, but they are categories themselves! Categories whose homsets are categories again are called *2-categories* or *bicategories*. These have extra structure—we’ll not dive further into this now.

3. Cartesian Categories

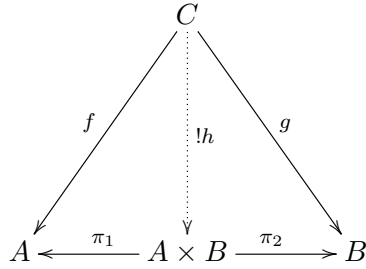
Category theory rarely studies pure categories, but usually categories with extra structure.

Definition 8 (Product). Given $A, B : \mathcal{C}$, a *product* of A and B is given by the following data:

1. An object $P : \mathcal{C}$, and
2. a pair of morphisms $\pi_1 : P \longrightarrow A$ and $\pi_2 : P \longrightarrow B$, such that
3. for each object C and morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there is a unique morphism $h : C \longrightarrow P$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

The uniqueness of h justifies the notation $h = \langle f, g \rangle$. Since P is unique up to isomorphism (see below), the notation $P = A \times B$ is justified.

The so-called *universal property* that defines the product can be diagrammatically displayed as follows:



Example 9.

1. The cartesian product is the product in **Set**, **Setoid**, **Grp** etc.
2. In **Sub**, the cartesian product is context concatenation.

Exercise 10. What is a product in a preorder? Under which conditions do preorders have all products?

Exercise 11 (Uniqueness of product). Let (P, π_1, π_2) and (Q, q_1, q_2) be both products of A and B . Show that $P \cong Q$.

Exercise 12 (Commutativity). Show that $A \times B \cong B \times A$.

Exercise 13 (Derived laws). Proof the following theorems using the universal property:

1. $\langle \pi_1, \pi_2 \rangle = \text{id}$.
2. $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$.

Exercise 14 (Morphism product). Given $f_1 : A_1 \longrightarrow B_1$ and $f_2 : A_2 \longrightarrow B_2$, define $f_1 \times f_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$.

The nullary product is called the terminal object.

Definition 9 (Terminal object). An object $T : \mathcal{C}$ is *terminal* if for any object C there is a unique morphism $h : C \longrightarrow T$.

The uniqueness of h justifies the notation $h = !_C$. Since T is unique up to isomorphism (see below), it is usually denoted by 1 .

Exercise 15. Give, if it exists, the terminal object in the categories **Set**, **Setoid**, **Grp**, **Rel**.

Exercise 16. What is a terminal object in a preorder?

Exercise 17. The terminal object is unique up to isomorphism.

Exercise 18 (Naturality of !). Show that ! is a natural transformation from **Id** to **K1** where $\text{Id} : A \mapsto A$ is the identity functor and $\text{K1} : A \mapsto 1$ the constant functor returning the terminal object.

Exercise 19 (Naturality of pairing). Let \mathcal{C} be a category that has binary products.

1. Complete the definition of the product functor $_ \times _ : [\mathcal{C} \times \mathcal{C}, \mathcal{C}]$, $_ \times _ (A, B) = A \times B$ with its action $_ \times _$ on morphisms (see Exercise 14) and prove the functor laws.
2. Formulate (if possible) a naturality statement for pairing $\langle _, _ \rangle$ and prove naturality.

Definition 10 (Cartesian (monoidal) category). A *cartesian category*, more precisely, a *cartesian monoidal category*, has finite products (including the nullary one).

Definition 11 (Lawvere theory). A *Lawvere theory* is a cartesian monoidal category T where each object is isomorphic to a power X^n of a distinguished object X , called the generic object for T .

A *model* A of T is a product-preserving functor $A : [T, \mathbf{Set}]$ in the sense that for each $n \in \mathbb{N}$ the set $A(X^n)$ together with the morphisms $A(\pi_i) : A(X^n) \rightarrow A(X)$ for $i \leq n$ is a product of n copies of the set $A(X)$.

Example 10. The Lawvere theory of groups has morphism $e : X^0 \rightarrow X$ and $op : X^2 \rightarrow X$ and $inv : X \rightarrow X$. A specific group can be represented as a model of this theory, e.g., $\text{Int}(X) = \mathbb{Z}$ and $\text{Int}(e) = 0$ and $\text{Int}(op)(i, j) = i + j$ and $\text{Int}(inv)(i) = -i$.

4. Cartesian Closed Categories

In a cartesian category, we can represent first-order functions as morphisms $f : A_1 \times \dots \times A_n \rightarrow B$. To get higher-order functions as in simply-typed lambda-calculus, we need to be able to internalize homsets as objects.

Definition 12. Given $A, B : \mathcal{C}$, an *exponential* of B to the A is given by the following data:

1. An object $E : \mathcal{C}$ with
2. a morphism $\text{eval} : E \times A \rightarrow B$, such that
3. for each C and $f : C \times A \rightarrow B$ there is a unique $h : C \rightarrow E$ such that $\text{eval} \circ (h \times \text{id}_A) = f$.

The uniqueness of h justifies the notation $h = \text{curry}(f)$ (also: $h = \Lambda(f)$ or $h = \lambda(f)$). Since E is unique up to isomorphism, the notation $E = B^A$ or $E = A \Rightarrow B$ is justified.

The universal property of exponentials is visualized as follows:

$$\begin{array}{ccc}
 C \times A & & \\
 \downarrow \text{curry}(f) \times \text{id} & \searrow f & \\
 B^A \times A & \xrightarrow{\text{eval}} & B
 \end{array}$$

Exercise 20. Explain the exponentials of Set and Setoid! Does Grp have exponentials?

Exercise 21. Give an example of a preorder that has exponentials.

Exercise 22. Show that the exponential is unique up to isomorphism!

Exercise 23 (Derived laws). Prove these laws about exponentials:

1. $\text{curry}(f) \circ h = \text{curry}(f \circ (h \times \text{id}))$.
2. $\text{curry}(\text{eval}) = \text{id}_{B^A}$.
3. $\text{curry}(\text{eval} \circ (f \times \text{id}_A)) = f : C \rightarrow B^A$.

Definition 13 (CCC). A *cartesian closed category* has finite products and exponentials.

Exercise 24. Show that Cat is cartesian closed.

A. Solutions

Solution (Exercise 2). Given a partial monoid (M, e, \cdot) let $\mathbf{Ob} = \mathcal{P}(M)$ and $m : \mathbf{Hom}(A, B)$ if and only if $a \cdot m$ is defined for all $a \in A$ and $m \cdot b$ is defined for all $b \in B$. Then we can set $\text{id}_A = e$ and $f \circ g = f \cdot g$ just as in the case for total monoids.