These exercises are meant to accompany the talk 2018–02–08 at the Initial Types Club of Chalmers university of technology and Gothenburg university. They mostly constitute, and are meant to illustrate, basic facts of ZFC which are principally taken from the books [2] and [1] also referred to in the notes of the talk. As such solutions may almost always be found in the notes or their references.

Exercise 1 (Cartesian products)**.** *Using the Kuratowski pairing operation*

$$[x = (y, z)] := [x = \{\{y\}, \{y, z\}\}]$$

construct a class function $[x = A \times B]$ *and verify the following:*

1. $[x = (y, z)]$ *is a class function in* ZFC*.*
2. $[x = A \times B]$ *is a class function in* ZFC*.*
3. $\mathrm{ZFC} \vdash \forall x, y, u, v((x, y) = (u, v) \to x = u \wedge y = v)$.
4. $\mathrm{ZFC} \vdash \forall A, B, p(p \in A \times B \leftrightarrow \exists x \in A, y \in B(p = (x, y)))$.

Hint: to construct $A \times B$*, first find a set containing all pairs from it and use separation.*

Exercise 2 (Relations and functions)**.** *In* ZFC *we identify relations and functions with their graphs, i.e. subsets of the Cartesian product. A consequence is that the equalities between relations and functions, respectively, are extensional. With a pairing operation which satisfies 3 and 4 from the previous exercise:*

1. *Construct a formula* $\mathrm{Eq}(r, A)$ *saying that* $r \subset A \times A$ *is an equivalence relation on* A*.*
2. *Construct a formula* $f : A \longrightarrow B$ *saying that* f *is a function from* A *to* B*.*
3. *Construct a class function* $[x = {}^A B]$ *such that* $\mathrm{ZFC} \vdash \forall A, B \forall f(f \in {}^A B \leftrightarrow f : A \longrightarrow B)$*.*
4. *Assuming we have a set* $\mathbb{N} = (N, 0, S)$ *of natural numbers (see Exercise 5), verify that the axiom of foundation is equivalent to the sentence*

   $$\neg \exists A \exists f : N \longrightarrow A \forall n \in N(f(S(n)) \in f(n))$$

   relative to the other axioms of ZFC*.*

Exercise 3 (Quotients)**.** *One important feature of* ZFC *is the ease with which it handles quotient sets. Construct a class-function* $[x = A/\sim]$ *and prove*

$$\begin{aligned} \mathrm{ZFC} \vdash & \forall A, \sim (\mathrm{Eq}(\sim, A) \to (\forall x \in A/\sim (x \neq \varnothing) \wedge \\ & \left(\bigcup A/\sim\right) = A \wedge \forall x, y \in A/\sim (x \neq y \to x \cap y = \varnothing) \\ & \wedge \forall a, b \in A(a \sim b \leftrightarrow \exists x \in A/\sim (a \in x \wedge b \in x))). \end{aligned}$$

Thus for a set A *and an equivalence relation* $\sim$ *on* A*, the set* $A/\sim$ *is the quotient of* A *by* $\sim$*, and its members are the equivalence classes of the elements of* A*. Note that as a consequence the equality on* $A/\sim$ *is the ordinary equality.*

Exercise 4 (Ordinals)**.** *A central notion in classical set theory is that of ordinal (number). An ordinal is a specific representative of an "isomorphism class" of well-ordered sets. They are defined as follows: Let*

$$\mathrm{Trans}(x) := \forall y \in x \forall z \in y (z \in x),$$

and

$$\begin{aligned}\mathrm{Ord}(\alpha) := \mathrm{Trans}(\alpha) \wedge \forall x, y \in \alpha (x \notin x \wedge (x \in y \vee y \in x \vee x = y)) \\ \wedge \forall A \in \mathcal{P}(\alpha)(A \neq \varnothing \rightarrow \exists x \in A \forall y \in A (x \in y \vee x = y)).\end{aligned}$$

Thus, an ordinal α is a transitive set well-ordered by (the restriction of) $\in$ (to α).

1. *Verify that the empty set is an ordinal.*
2. *Show that every element of an ordinal is an ordinal.*
3. *Show that, if $\alpha \subset \beta$ are ordinals, then $\alpha \in \beta$. Hint: consider $\beta \setminus \alpha = \{x \in \beta \mid x \notin \alpha\}$.*
4. *Show that if α and β are ordinals, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. Use the above to conclude that $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$. Hint: consider $\alpha \cap \beta$.*
5. *Let $\varphi(x, \bar{p})$ be a formula with the shown free variables. Show that for all $\bar{p}$, if $\varphi(x, \bar{p})$ holds for at least one x and only if x is an ordinal, then there is an ordinal α such that $\varphi(\alpha, \bar{p})$ and for all ordinals β with $\varphi(\beta, \bar{p})$ do we have $\alpha \subseteq \beta$. Conclude that the ordinals are "well-ordered" by $\in$. Hint: define α to satisfy the last requirement and verify the first.*
6. *Show that, for each ordinal α, $\alpha \cup \{\alpha\}$ is an ordinal and the successor of α in the order from the previous exercise ($\alpha \cup \{\alpha\}$ is then called a successor ordinal).*
7. *Show that if α is an ordinal which is not a successor ordinal, then $\alpha = \bigcup \alpha (= \bigcup_{\beta \in \alpha} \beta)$ (α is then called a limit ordinal).*
8. *Show that for each well-ordered set there is a unique ordinal isomorphic to it. Hint: uses replacement.*

Exercise 5 (Natural numbers)**.** *The natural numbers consists of a triple $(N, 0, S)$ satisfying the Peano axioms:*

$$\begin{aligned}&0 \in N, \\ &S : N \longrightarrow N, \\ &\forall x, y \in N (S(x) = S(y) \rightarrow x = y), \\ &\forall x \in N (S(x) \neq 0), \\ &\forall A \in \mathcal{P}(N)(0 \in A \wedge \forall x \in A (S(x) \in A) \rightarrow A = N).\end{aligned}$$

In ZFC *one often takes as* $(N, 0, S)$ *above the triple* $(\omega, \varnothing, \_^+)$*, where* ω *is the smallest nonzero (i.e. nonempty) limit ordinal, which can be defined by the (constant) class-function*

$$[x = \omega] := \forall z(z \in x \leftrightarrow \forall A(\mathrm{Ind}(A) \rightarrow z \in A)). \tag{1}$$

and $[x = \alpha^+] := \alpha \cup \{\alpha\}$ *is the ordinal successor class-function.*

1. *Show that* ω *is the least nonzero limit ordinal and that* $(\omega, \varnothing, \_^+)$ *satisfies the Peano axioms, for example in the following way:*
   a) *Show that this is a class function (that is,* ω *exists and is unique with the property in* (1)*).*
   b) *Show that* ω *is inductive. Conclude that* ω *is nonempty and that the first, second and forth Peano axioms hold for* $(\omega, \varnothing, \_^+)$*.*
   c) *Verify the induction axiom for* $(\omega, \varnothing, \_^+)$*.*
   d) *Show by induction that* ω *is transitive.*
   e) *Show by induction that every element of* ω *is an ordinal.*
   f) *Use the properties from Exercise 4 to conclude that* ω *is an ordinal, and the least nonzero limit ordinal.*
   g) *Use the properties from Exercise 4 (or the foundation axiom) to verify the third Peano axiom for* $(\omega, \varnothing, \_^+)$*.*

2. *Having verified there is a natural choice for a triple satisfying the Peano axioms, let* $\mathbb{N} = (N, 0, S)$ *be any such triple. Verify that functions on* N *can be defined by recursion in the sense that if* A *is a set,* $a \in A$ *and* $g : A \longrightarrow A$*, then there is a unique* $f : N \longrightarrow A$ *such that*

   $$f(0) = a \qquad \text{and} \qquad f(S(n)) = g(f(n))$$

   for all $n \in N$*. Construct* f *as follows:*
   a) *Let*

      $$D = \{d \in \mathcal{P}(N) \mid 0 \in d \wedge \forall n \in N(S(n) \in d \rightarrow n \in d)\} \text{ and}$$
      $$F = \{p \in \mathcal{P}(N \times A) \mid \exists d \in D(p : d \longrightarrow A \wedge p(0) = a \wedge p(S(n)) = g(p(n)))\}.$$

      Show that, given $d, e \in D$ *and* $p, q \in F$ *with* $p : d \longrightarrow A$ *and* $q : e \longrightarrow A$*, for all* $n \in N$*, we have* $n \in d \cap e$ *only if* $p(n) = q(n)$*.*
   b) *Verify that* $\bigcup F$ *is a function from* N *to* A *which satisfies the recursive equations and that it is unique with this property.*

3. *Use recursion to show that any two triples satisfying the Peano axioms are uniquely isomorphic.*

Exercise 6 (Implementation of constructive sets in Agda[1])**.** *A constructive notion of set can be implemented in Agda in the following way:*

```
data cset : Set_1 where
  node : (I : Set_1) -> (f : I -> cset) -> cset
```

where the intuition is that `I` *is an index set and* `f` *a function enumerating the constructive set, possibly with repetitions. Define equality, elementhood and subsethood as functions* `cset -> cset -> Set_1` *in Agda. Hint: these will have to be defined mutually.*

References

[1] Thomas J. Jech. *Set Theory.* Springer Monographs in Mathematics. Springer, third millennium edition, 2002.

[2] Yiannis Moschovakis. *Notes on Set Theory.* Undergraduate Texts in Mathematics. Springer, second edition, 2006.

[1] Andreas