# Cartesian closed categories

### Andreas Abel

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#### Overview:

- 1. Categories
- 2. Functors and natural transformations
- 3. Products
- 4. Exponentials

# 1. Categories

### 1.1. Definition and examples

**Definition 1** (Category). A category  $\mathcal{C}$  is given by the following data:

- 1. Types:
  - a) A type Ob of objects.
  - b) For each pair of objects  $A, B : \mathsf{Ob}$ , a type  $\mathsf{Hom}(A, B)$  of (homo)morphisms  $f : A \longrightarrow B$ .
  - c) For each pair of objects  $A,B:\mathsf{Ob},$  an equivalence relation  $\mathsf{Eq}(A,B)$  on  $\mathsf{Hom}(A,B).$  Given  $f,g:\mathsf{Hom}(A,B),$  we write f=g for  $\mathsf{Eq}(A,B)(f,g).$

## 2. Operations:

- a) For each object  $A : \mathsf{Ob}$  an automorphism  $\mathsf{id}_A : A \longrightarrow A$  (identity).
- b) For each pair  $f:A\longrightarrow B$  and  $g:B\longrightarrow C$  of morphisms a morphism  $g\circ f:A\longrightarrow C$  (composition).

#### 3. Laws:

- a) For each morphism  $f:A\longrightarrow B$  we have  $\mathsf{id}_B\circ f=f$  (left identity) and  $f\circ \mathsf{id}_A=f$  (right identity).
- b) For all morphisms  $f:A\longrightarrow B$  and  $g:B\longrightarrow C$  and  $h:B\longrightarrow C$  we have  $(h\circ g)\circ f=h\circ (g\circ f)$  (associativity).

The arrow  $A \longrightarrow B$  is just a nice notation for  $\mathsf{Hom}(A,B)$ . It is also common to write  $\mathcal{C}(A,B)$  to clarify that we mean the type  $\mathsf{Hom}_{\mathcal{C}}(A,B)$  of morphisms of category  $\mathcal{C}$ . Also  $A:\mathcal{C}$  is short for  $A:\mathsf{Ob}_{\mathcal{C}}$ .

**Remark 1** (Homsetoid). Since a type with an equivalence relation is called a *setoid* we could just ask for a family  $\mathsf{Hom} : \mathsf{Ob} \to \mathsf{Ob} \to \mathsf{Setoid}$ .

The prime example for categories are collections of algebraic structures and their structure-preserving homomorphisms.

**Example 1** (Groups). Grp is the category of groups and group homomorphisms. More precisely, the objects of Grp are groups, and an element f: Grp(A, B) is a function  $f: A \to B$  mapping the unit of A to the unit of B and the A-composition of two elements of A to the B-composition of their images under f.

Less abstractly, a group morphism  $f:(A,0,+,-)\longrightarrow (B,1,\times,^{-1})$  has to satisfy f(0)=1 and  $f(a+a')=f(b)\times f(b')$ .

#### Exercise 1 (Groups).

- 1. Give an example for a group morphism f.
- 2. Show that a group morphism automatically preserves inverses, i.e.,  $f(-a) = (f(a))^{-1}$ .

Analogously to groups, other algebraic structures can be organized as categories as well (monoids, rings, fields). We exhibit the most basic examples:

**Example 2** (Sets). Set is the category of types A and functions  $f: A \to B$ .

**Example 3** (Setoids). Setoid is the category of setoids  $(A, \approx_A)$  and  $\approx$ -preserving functions, i.e.,  $f: A \longrightarrow B$  must satisfy  $f(a) \approx_B f(a')$  whenever  $a \approx_A a'$ .

Besides organizing algebraic structures, categories can also implement structures.

**Example 4** (Monoid). Each monoid  $(M, e, \cdot)$  can be presented as category  $\mathcal{C}_M$  with a single object 1 and  $\mathsf{Hom}(1,1) = M$ . Then  $\mathsf{id}_1 = e$  and  $f \circ g = f \cdot g$ .

**Exercise 2** (Partial monoid). Can any partial monoid be represented as category as well? If yes, how? If no, give a counterexample!

**Example 5** (Preorder). Any preorder  $(A, \leq)$  can be presented as a thin category with  $\mathsf{Ob} = A$  and  $\mathsf{Hom}(a, b) = \{0 \mid a \leq b\}$ . Identity is reflexivity and composition is transitivity.

A category is called *thin* if each homset has at least one inhabitant.

**Example 6** (Relations). The category Rel has types as objects and binary relations as morphisms:  $Rel(A, B) = \mathcal{P}(A \times B)$ .

**Example 7** (Contexts and substitutions). Take the typing contexts  $\Gamma$  of simply-typed lambda-calculus as objects,  $\mathsf{Ob} = \mathsf{Cxt}$ , and the set of substitutions  $\mathsf{Sub}\,\Gamma\,\Delta$  as morphisms from  $\Gamma$  to  $\Delta$ .

**Definition 2** (Subcategory). A category  $\mathcal{D}$  is a *subcategory* of  $\mathcal{C}$  if  $\mathsf{Ob}_{\mathcal{D}} \subseteq \mathsf{Ob}_{\mathcal{C}}$  and  $\mathsf{Hom}_{\mathcal{D}}(A,B) \subseteq \mathsf{Hom}_{\mathcal{C}}(A,B)$  for all  $A,B : \mathsf{Ob}_{\mathcal{D}}$ .

If  $\mathsf{Ob}_{\mathcal{D}} = \mathsf{Ob}_{\mathcal{C}}$ , the subcategory is wide.

If  $\mathsf{Hom}_{\mathcal{D}}(A,B) = \mathsf{Hom}_{\mathcal{C}}(A,B)$  for all  $A,B : \mathsf{Ob}_{\mathcal{D}}$ , the subcategory is full.

In other words, a subcategory  $\mathcal{D}$  of  $\mathcal{C}$  is a selection of objects and morphisms from  $\mathcal{C}$  that still forms a category, i.e., is closed under identity and composition.

#### 1.2. On the equality of objects

Our definition of category does not include an equivalence relation on Ob. This is by intention, speaking about object equality is not considered pure category-theoretic spirit. All category-theoretic notions should respect isomorphic objects.

**Definition 3** (Isomorphism). An *isomorphim* (short *iso*) between two objects A and B is a pair of morphisms  $f: A \longrightarrow B$  and  $g: B \longrightarrow A$  such that  $g \circ f = id_A$  and  $f \circ g = id_B$ . The existence of an isomorphism is written  $A \cong B$ .

**Lemma 1** (Inverse). Fixing f, the inverse g is uniquely determined and denoted by  $f^{-1}$ .

Exercise 3. Prove this!

Exercise 4 (Subcategory of isomorphisms). Show that the isomorphisms of a category constitute a wide subcategory.

**Exercise 5.** Does the concept *subcategory* (Definition 2) honor the ideal that no category-theoretic concept should distinguish between isomorphic objects?

If not, suggest a modification of the definition, or defend the current definition against the ideal.

#### 1.3. Operations on categories

Some operations on the object types can be lifted to categories.

- 1. The product  $\mathcal{C} \times \mathcal{D}$  of two categories forms again a category with  $\mathsf{Ob}_{\mathcal{C} \times \mathcal{D}} = \mathsf{Ob}_{\mathcal{C}} \times \mathsf{Ob}_{\mathcal{D}}$ .
- 2. The latter can be generalized to nullary, finite, and even infinite products.
- 3. Any type can be turned into a discrete category where the identities are the only morphisms.

**Definition 4** (Opposite category). Given a category C, its *opposite*  $C^{op}$  has the same objects but flipped morphisms,  $C^{op}(A, B) = C(B, A)$ , and thus flipped composition:  $f \circ_{C^{op}} g = g \circ_{C} f$ .

**Remark 2.** The opposite category is really just the original category with morphisms relabeled so that source and target are formally exchanged.

**Exercise 6.** Show that  $C^{op}$  is indeed a category. Show that  $(C^{op})^{op} = C$ .

### 2. Functors and Natural Transformations

A functor  $F : [\mathcal{C}, \mathcal{D}]$  is a category morphism:

**Definition 5** (Functor). Given categories  $\mathcal{C}$  and  $\mathcal{D}$  a functor  $F : [\mathcal{C}, \mathcal{D}]$  is given by the following data:

- 1. Maps:
  - a) A function  $F_0: \mathsf{Ob}_{\mathcal{C}} \to \mathsf{Ob}_{\mathcal{D}}$ .
  - b) For any pair of objects  $A, B : \mathcal{C}$ , a function  $F_1 : \mathsf{Hom}_{\mathcal{C}}(A, B) \to \mathsf{Hom}_{\mathcal{D}}(F_0A, F_0B)$ .
- 2. Laws:
  - a) For any object  $A : \mathcal{C}$  we have  $F_1(\mathsf{id}_A) = \mathsf{id}_{F_0A}$ .
  - b) For any pair of morphisms  $f: \mathcal{C}(A,B)$  and  $g: \mathcal{C}(B,C)$  we have  $F_1(g \circ_{\mathcal{C}} f) = F_1 g \circ_{\mathcal{D}} F_1 f$ .

It is common to drop the indices 0 and 1 and simply write, e.g.,  $Ff: FA \longrightarrow FB$ . Also, since there is little chance of confusion, one often writes  $F: \mathcal{C} \to \mathcal{D}$  instead of  $F: [\mathcal{C}, \mathcal{D}]$ .

**Example 8** (Forgetful functor). "Forgetting" algebraic structure gives rise to trivial functors, the so-called *forgetful functors*, often denoted by U. For example,  $U: \mathsf{Grp} \to \mathsf{Set}$  maps a groups to their carriers, and group morphisms to ther underlying functions on the carriers.

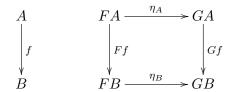
A forgetful functor does nothing to the "values", only changes their "types".

**Exercise 7.** Define the duplication functor  $Dup : [C, C \times C]$  from a category to its square.

Since functors are not mathematical structures (such as groups and categories) it is not obvious what the notion of morphism between two functors  $F, G : [\mathcal{C}, \mathcal{D}]$  should be. The definition states that it is a family of morphims  $FA \longrightarrow GA$  parametric in A:

**Definition 6** (Natural transformation). Given functors  $F, G : [\mathcal{C}, \mathcal{D}]$ , a natural transformation  $\eta : F \to G$  is a family of morphisms  $\eta_A : FA \longrightarrow GA$  indexed by  $A : \mathcal{C}$  such that for all  $f : A \longrightarrow B$  we have  $Gf \circ \eta_A = \eta_B \circ Ff$ .

Diagrammatically, the commutation law can be depicted as follows:



**Exercise 8** (Functor category). Show that functors in [C, D] form a category with natural transformations as morphisms.

**Definition 7** (Cat). Taking categories C as objects themselves and functor sets [C, D] as homsets, we arrive at the category Cat of categories!

For consistency reasons  $\mathsf{Ob}_{\mathsf{Cat}}$  needs to be a large type containing categories  $\mathcal C$  whose  $\mathsf{Ob}_{\mathcal C}$  is a small type.

**Exercise 9.** Prove that functors are indeed closed under composition and that Cat is indeed a category.

**Remark 3** (2-categories). In Cat, the functor types  $[\mathcal{C}, \mathcal{D}]$  are only taken as sets, but they are categories themselves! Categories whose homsets are categories again are called 2-categories or bicategories. These have extra structure—we'll not dive further into this now.

# 3. Cartesian Categories

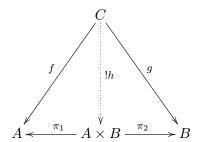
Category theory rarely studies pure categories, but usually categories with extra structure.

**Definition 8** (Product). Given  $A, B : \mathcal{C}$ , a product of A and B is given by the following data:

- 1. An object  $P: \mathcal{C}$ , and
- 2. a pair of morphisms  $\pi_1: P \longrightarrow A$  and  $\pi_2: P \longrightarrow B$ , such that
- 3. for each object C and morphisms  $f: C \longrightarrow A$  and  $g: C \longrightarrow B$  there is a unique morphism  $h: C \longrightarrow P$  such that  $\pi_1 \circ h = f$  and  $\pi_2 \circ h = g$ .

The uniqueness of h justifies the notation  $h = \langle f, g \rangle$ . Since P is unique up to isomorphism (see below), the notation  $P = A \times B$  is justified.

The so-called *universal property* that defines the product can be diagrammatically displayed as follows:



#### Example 9.

- 1. The cartesian product is the product in Set, Setoid, Grp etc.
- 2. In Sub, the cartesian product is context concatenation.

**Exercise 10.** What is a product in a preorder? Under which conditions do preorders have all products?

**Exercise 11** (Uniqueness of product). Let  $(P, \pi_1, \pi_2)$  and  $(Q, q_1, q_2)$  be both products of A and B. Show that  $P \cong Q$ .

**Exercise 12** (Commutativity). Show that  $A \times B \cong B \times A$ .

Exercise 13 (Derived laws). Proof the following theorems using the universal property:

- 1.  $\langle \pi_1, \pi_2 \rangle = id$ .
- 2.  $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$ .

**Exercise 14** (Morphism product). Given  $f_1: A_1 \longrightarrow B_1$  and  $f_2: A_2 \longrightarrow B_2$ , define  $f_1 \times f_2: A_1 \times A_2 \longrightarrow B_1 \times B_2$ .

The nullary product is called the terminal object.

**Definition 9** (Terminal object). An object  $T : \mathcal{C}$  is *terminal* if for any object C there is a unique morphism  $h : C \longrightarrow T$ .

The uniqueness of h justifies the notation  $h = !_C$ . Since T is unique up to isomorphism (see below), it is usually denoted by 1.

**Exercise 15.** Give, if it exists, the terminal object in the categories Set, Setoid, Grp, Rel.

**Exercise 16.** What is a terminal object in a preorder?

Exercise 17. The terminal object is unique up to isomorphism.

**Exercise 18** (Naturality of !). Show that ! is a natural transformation from Id to K1 where Id :  $A \mapsto A$  is the identity functor and K1 :  $A \mapsto 1$  the constant functor returning the terminal object.

**Exercise 19** (Naturality of pairing). Let  $\mathcal{C}$  be a category that has binary products.

- 1. Complete the definition of the product functor  $\_\times\_: [\mathcal{C} \times \mathcal{C}, \mathcal{C}], \, \_\times\_(A, B) = A \times B$  with its action  $\_\times\_$  on morphisms (see Exercise 14) and prove the functor laws.
- 2. Formulate (if possible) a naturality statement for pairing  $\langle -, \rangle$  and prove naturality.

**Definition 10** (Cartesian (monoidal) category). A cartesian category, more precisely, a cartesian monoidal category, has finite products (including the nullary one).

**Definition 11** (Lawvere theory). A Lawvere theory is a cartesion monoidal category T where each object is isomorphic to a power  $X^n$  of a distinguished object X, called the generic object for T.

A model of T is a product-preserving functor A : [T, Set].

**Example 10.** The Lawevere theory of groups has morphism  $e: X^0 \longrightarrow X$  and  $op: X^2 \longrightarrow X$  and  $inv: X \longrightarrow X$ . A specific group can be represented as a model of this theory, e.g.,  $Int(X) = \mathbb{Z}$  and Int(e) = 0 and Int(op)(i,j) = i+j and Int(inv)(i) = -i.

# 4. Cartesian Closed Categories

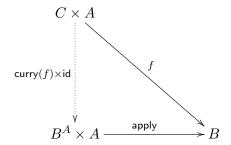
In a cartesian category, we can represent first-order functions as morphisms  $f: A_1 \times \cdots \times A_n \longrightarrow B$ . To get higher-order functions as in simply-typed lambda-calculus, we need to be able to internalize homsets as objects.

**Definition 12.** Given  $A, B : \mathcal{C}$ , an *exponential* of B to the A is given by the following data:

- 1. An object  $E : \mathcal{C}$  with
- 2. a morphism eval :  $E \times A \longrightarrow B$ , such that
- 3. for each C and  $f:C\times A\longrightarrow B$  there is a unique  $h:C\longrightarrow E$  such that  $\operatorname{eval}\circ(h\times\operatorname{id}_A)=f.$

The uniqueness of h justifies the notation  $h = \operatorname{curry}(f)$  (also:  $h = \Lambda(f)$  or  $h = \lambda(f)$ ). Since E is unique up to isomorphism, the notation  $E = B^A$  or  $E = A \Rightarrow B$  is justified.

The universal property of exponentials is visualized as follows:



**Exercise 20.** Explain the exponentials of Set and Setoid! Does Grp have exponentials?

Exercise 21. Give an example of a preorder that has exponentials.

Exercise 22. Show that the exponential is unique up to isomorphism!

Exercise 23 (Derived laws). Prove these laws about exponentials:

- 1.  $\operatorname{curry}(f) \circ h = \operatorname{curry}(f \circ (h \times \operatorname{id})).$
- 2.  $\operatorname{curry}(\operatorname{eval}) = \operatorname{id}_{BA}$ .
- 3.  $\operatorname{curry}(\operatorname{eval} \circ (f \times \operatorname{id}_A)) = f : C \to B^A$ .

**Definition 13** (CCC). A cartesian closed category has finite products and exponentials.

Exercise 24. Show that Cat is cartesian closed.

# A. Solutions

**Solution** (Exercise 2). Given a partial monoid  $(M,e,\cdot)$  let  $\mathsf{Ob} = \mathcal{P}(M)$  and  $m: \mathsf{Hom}(A,B)$  if and only if  $a\cdot m$  is defined for all  $a\in A$  and  $m\cdot b$  is defined for all  $b\in B$ . Then we can set  $\mathsf{id}_A=e$  and  $f\circ g=f\cdot g$  just as in the case for total monoids.