JOSH ROBERT OBAOB

MATHWIZZ 2023 Elims Problems (Part 1)

Problems of the Week (2023 Edition)

April 10, 2023

After many months of hiatus, I have finally exploit the opportunity to begin another edition of my **Problems of the Week** edition for this year! The following problems are from the recent **MATHWIZZ 2023 Elimination Round**. THe first ten problems of the round are presented here with my solutions while the second and last ten questions will be on next week. If you see any typographical errors, wrong calculations, feel free to contact me at `jrdobaob@addu.edu.ph` ↗

Note. Some of the problems are rephrased either for fun or for saving time on typing these stuff.

1. Jim spent $\frac{1}{4}$ of his starting pocket money on food and 50 pesos on school supplies. As a result, he has $\frac{5}{12}$ of his starting pocket money remaining. By how much did Jim spend on food alone?

    Answer. $\boxed{\mathbb{P}\ 37.5}$.

    Solution. Let x be Jim's pocket money. So,

    $$x - \frac{1}{4}x - 50 = \frac{5}{12}x \iff \frac{3x}{4} - \frac{5x}{12} = 50 \iff \frac{x}{3} = 50.$$

    We see that Jim's money is 150 pesos. Therefore, Jim spent $\boxed{150/4 = \mathbb{P}\,37.5}$ on his food alone.

2. Find all values of x, $0° \leq x° < 360°$, such that $\sin x = \sin 2x$.

    Answer. $\boxed{60,\ 300}$.

    Solution. By double-angle identity, $\sin 2x = 2 \sin x \cos x$. Hence,

    $$
    \begin{aligned}
    \sin x &= \sin 2x \\
    \sin x &= 2 \sin x \cos x \\
    1 &= 2 \cos x \\
    \tfrac{1}{2} &= \cos x
    \end{aligned}
    $$

    We see that the possible values of x are $\boxed{60 \text{ and } 300}$.

3. Paula and Pilar have money with ratio $1 : 9$ respectively. After Pilar gave Paula 450 pesos, the ratio is $3 : 2$. How much money did Pilar have from the beginning?

    Answer. $\boxed{\mathbb{P}\,810}$.

    Solution. Mathematically, that is just

    $$\frac{a + 450}{9a - 450} = \frac{3a}{2a} \iff 2250 = 25a \iff a = 90$$

    Therefore, Pilar have $9 \times 90 = \boxed{\mathbb{P}\,810}$ pesos at the beginning

4. If 2.4 wids = 4.2 wads, and 1.6 wids = 3.5 wods, how many wads are there in 10 wods?

Answer. $\boxed{64/7 \text{ wads}}$.

Solution. We see that 1.0 wad is equivalent to 0.5 wid, and $\frac{16}{35}$ wid $= 1$ wod. Therefore,

$$\implies 10 \text{ wods} \times \frac{\frac{16}{35} \text{ wid}}{1 \text{ wod}} \times \frac{2 \text{ wads}}{1 \text{ wid}} = \boxed{64/7 \text{ wads}}$$

5. Determine the value of $\int_0^2 (x-2)^5 (x+1) \; dx$.

Answer. $\boxed{-96/7}$.

Solution. Let $u = x - 2$ and $du = dx$. Therefore,

$$
\begin{aligned}
\int_0^2 (x-2)^5 (x+1) \; dx &= \int_0^2 u^5 (u+3) \; du \\[2mm]
&= \int_0^2 u^6 + 3u^5 \; du \\[2mm]
&= \frac{u^7}{7} + \frac{u^6}{2} \Big|_0^2 \\[2mm]
&= \frac{(x-2)^7}{7} + \frac{(x-2)^6}{2} \Big|_0^2 \\[2mm]
&= \left(\frac{(2-2)^7}{7} + \frac{(2-2)^6}{2} \right) - \left(\frac{(0-2)^7}{7} + \frac{(0-2)^6}{2} \right) \\[2mm]
&= \frac{128}{7} - 32 \\[2mm]
&= \boxed{-96/7}
\end{aligned}
$$

6. How many whole numbers from 1 to 1,000 are divisible by 4, but not divisible by 3?

Answer. $\boxed{167}$.

Solution. Let N be the number of whole integers that are divisible by 4 but not by 3. By the principle of exclusion-inclusion, this is just

$$
\begin{aligned}
N &= P(3|N) - P(12|N) \\
N &= \left\lfloor \frac{1000}{4} \right\rfloor - \left\lfloor \frac{1000}{12} \right\rfloor \\
&= 167
\end{aligned}
$$

7. Compute: $i^{1^2} + i^{2^2} + \ldots + i^{2022^2} + i^{2023^2}$. $(i = \sqrt{-1})$

Answer. $\boxed{1011 + 1012i}$.

Solution. Recall the following properties:

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$

This gives us the motivation to use the modulo 4 in each exponent of each term. Notice that an exponent $(2k+1)^2 \equiv 4k^2 + 4k + 1 \equiv 1 \pmod 4$ and an odd exponent $(2k)^2 \equiv 0 \pmod 4$. Therefore, our answer is $1011(i^0) + 1012(i^1) = \boxed{1011 + 1012i.}$

8. Point P is outside a rectangle $ABCD$ such that $PA = 6, PB = 11$ and $PC = 13$. Find PD.

Answer. $\boxed{2\sqrt{21}}$.

Solution. (Inspired from the solution of Benedict Rodil). The figure can be configured like this:
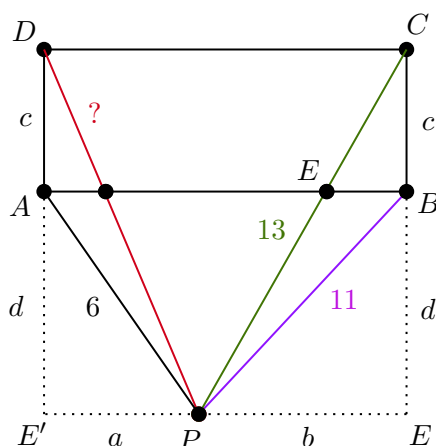
Invoking the Pythagorean theorem, we have the following system of quadratic equations

$$\begin{cases} d^2 + a^2 = 36 \\ d^2 + b^2 = 121 \\ (c+d)^2 + b^2 = 169 \\ (c+d)^2 + a^2 = PD^2 \end{cases}$$

Subtract the second equation from the first and you get $a^2 - b^2 = -85$. Add this equation to the third equation and you get

$$[(c+d)^2 + b^2] + [a^2 - b^2] = PD^2 \iff 169 + (-85) = 84 = PD^2.$$

Therefore, $\boxed{PD = 2\sqrt{21}}$.

9. Denote $f(x)$ as a polynomial function whereas $f(2x-1) = 8x^2 - 10x + 7$. Find the remainder of $f(x)$ divided by $(x-3)$.

Answer. $\boxed{19}$.

Solution. Let $u = x$:

$$f(2a - 1) = 8a^2 - 10a + 7.$$

Next, let $x = 2a - 1 \iff a = \frac{x+1}{2}$:

$$f\left(2\left(\frac{x+1}{2}\right)-1\right) = 8\left(\frac{x+1}{2}\right)^2 - 10\left(\frac{x+1}{2}\right) + 7$$
$$=$$
$$f(x) = 2\left(x^2 + 2x + 1\right) - 5(x+1) + 7$$
$$= 2x^2 + 4x + 2 - 5x - 5 + 7$$
$$= 2x^2 - x + 4$$

Now rewriting $f(x)$ in terms of $x - 3$ gives us

$$f(x) = 2(x-3)^2 + 11x - 14 \Longleftrightarrow [2(x-3)^2] + [11(x-3) + 19]$$

Therefore, the remainder of $f(x)$ is $\boxed{19.}$

10. Determine the value of $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2022^2}\right)\left(1 - \frac{1}{2023^2}\right)$

Answer. $\boxed{1012}$.

Solution. Evaluate every term and we have:

$$N = \left(\frac{(1)(3)}{2^2}\right)\left(\frac{(2)(4)}{3^2}\right)\left(\frac{(3)(5)}{4^2}\right)\cdots\left(\frac{(2021)(2023)}{2022^2}\right)\left(\frac{(2022)(2024)}{2023^2}\right)$$
$$=$$
$$= \frac{1 \times 2 \times 3^2 \times 4^2 \times 5^2 \times \ldots \times 2023 \times 2024}{(2 \times 3 \times 4 \times \ldots \times 2022 \times 2023)^2}$$

This is similar to the expression: $\dfrac{(1 \times 2 \times 3 \times \ldots \times 2023)(1 \times 2 \times 3 \times \ldots \times 2024)}{(1 \times 2 \times 3 \times \ldots \times 2022 \times 2023)^2} = \dfrac{2023!2024!}{(2023!)^2}$

So, N is just

$$N = \frac{\frac{2023!2024!}{2023!2023!}}{2}$$
$$=$$
$$N = \frac{2024}{2}$$
$$= \boxed{1012}$$

Special thanks to Engr. Isaiah James Maling for posting the problems of the contest in his Facebook page ☑! [1]