Optimal Control and Decision Making

Handout of Mountain Car Problem

1. Task Overview

The mountain car problem is a classical control challenge that illustrates the dynamics of a car moving on a slope. This problem is influenced by gravity, the slope's inclination, and an applied acceleration input, as depicted in Figure 1. The primary objective is to design a controller using various methods (e.g. LQR, MPC, RL) to drive the car from an initial state x_0 to a predefined terminal state x_T, adhering to constraints throughout the process. The choice of controller and specific task requirements will depend on the weekly lecture content, which will be detailed in each week's Jupyter Notebook introduction.

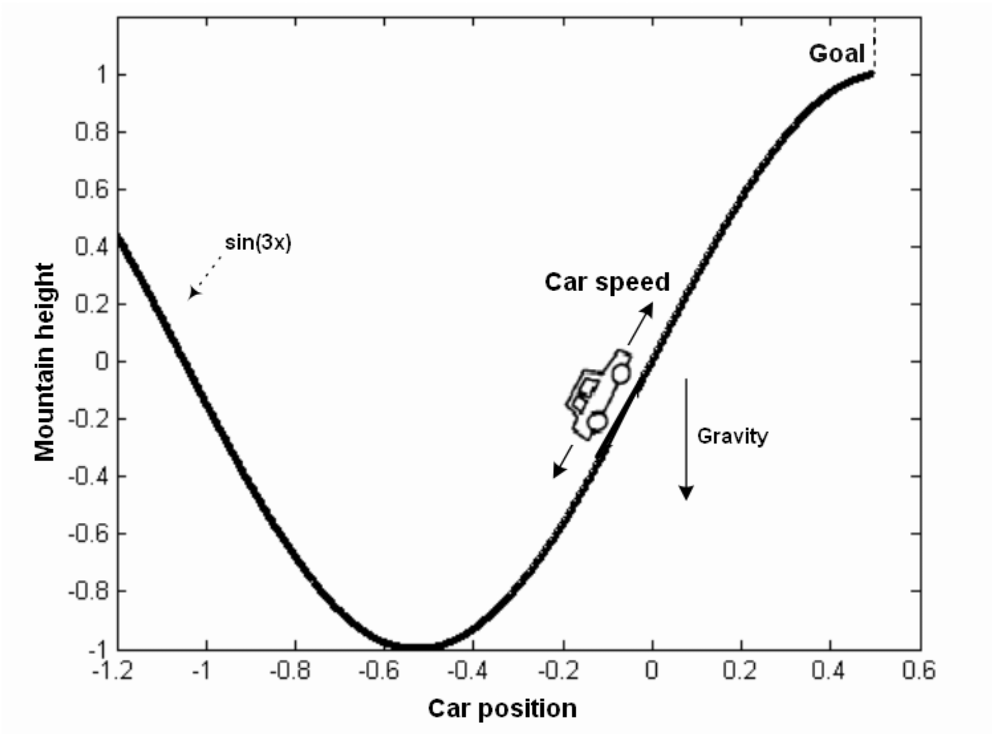

Figure 1: Illustration of a mountain car problem

2. Mountain Car Dynamics

2.1 Derivation of System Dynamics

Consider the following set of parameter that describing the motion of a mountain car:

a) p: horizontal position of the car;

b) $v = \dot{p}$: horizontal velocity of the car;

c) a: the applied acceleration input, excluding gravitational and slope-induced effects.

Figure 2 illustrates the free-body diagram of the mountain car, where its motion is influenced by the applied acceleration a, gravitational forces, and the supportive force arising from the slope.

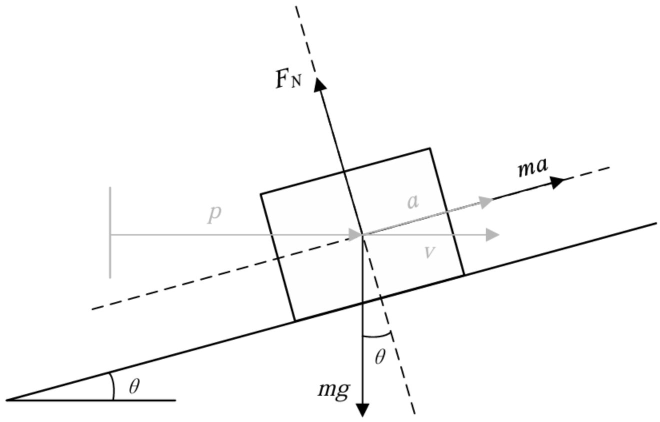

Figure 2: Free body diagram of the 1D mountain car

By applying Newton's second law along the horizontal direction, the equation of motion can be expressed as:

$$m\ddot{p} = (ma - mg\sin(\theta))\cos(\theta), \tag{1}$$

where m represents the mass of the car, g is the acceleration due to gravity, and θ denotes the inclination angle of the slope. This equation can be reformulated into a system of first-order differential equations as follows:

$$\begin{cases} \dot{p} = v, \\ \dot{v} = -g\sin(\theta)\cos(\theta) + a\cos(\theta), \end{cases} \tag{2}$$

This representation captures the car's dynamic characteristics, providing the basis for state-space modeling.

2.2 State Space Representation

To facilitate analysis and control design, the system dynamics can be expressed in a state-space representation. With the state vector x and input u defined as:

$$x = \begin{bmatrix} p \\ v \end{bmatrix}, \quad u = a, \tag{3}$$

The system dynamics can be expressed in the form of state space representation:

$$\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -g\sin(\theta)\cos(\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(\theta) \end{bmatrix} a, \tag{4}$$

where the inclination angle θ is a function of position p, i.e., $\theta = \theta(p)$. This dependency introduces nonlinearities into the system dynamics, necessitating careful consideration in controller design.

Additionally, practical applications impose constraints on the system's states and inputs. These constraints are defined as:

a) State constraints: $\mathbb{X} = \{[p, v]^T \mid p \in [\underline{p}, \overline{p}], v \in [\underline{v}, \overline{v}]\}$;

b) Input constraints: $\mathbb{U} = \{a \mid a \in [\underline{a}, \overline{a}]\}$.

These constraints ensure that the controller operates within feasible physical limits.

3. Mountain Car Environment

3.1 Slope Profiles

The slope profile $h(p)$ defines the geometry of the mountain car's path and plays a critical role in the system's dynamics. Several representative slope profiles are considered, each corresponding to different characteristics of the environment:

a) Flat slope (zero slope): The path is horizontal, with a constant height:

$$h(p) = c, \quad p \in [\underline{p}, \overline{p}]. \tag{5}$$

b) Constant slope: The height increases linearly with the horizontal position:

$$h(p) = k \cdot p, \quad p \in [\underline{p}, \overline{p}]. \tag{6}$$

c) Varying slope (cosine profile): The path exhibits periodic variations in height:

$$h(p) = k \cdot \cos(\omega p), \quad p \in [\underline{p}, \overline{p}]. \tag{7}$$

d) Varying slope (sine profile, underactuated case): Another periodic variation characterized by sinusoidal behavior:

$$h(p) = k \cdot \sin(\omega p), \quad p \in [\underline{p}, \overline{p}]. \tag{8}$$

Each profile introduces distinct challenges for controller design, as the dynamics must account for the corresponding inclination angles and their impact on the car's motion.

3.2 Mapping from Slope Profile to Inclination Angle

From formula 4 we can observe that the slope profile will not directly entry the system dynamics, but in the form of the inclination angle θ, therefore we still need to derive the relationship between the slope $h(p)$ and the inclination angle θ.

From the definition, we can know that $tan(\theta)$ equals to the gradient of slope:

$$\tan(\theta) = \frac{dh}{dp}, \tag{9}$$

which means we can compute the inclination angle θ for given slope profile $h(p)$ by:

$$\theta = \arctan(\frac{dh}{dp}), \tag{10}$$

For most of the implementation in our exercise, we will use the CasADi symbolic expression to set up the controller, which means that formula 10 can be directly applied to the controller design. But in general we still need to substitute the inclination angle θ in the system dynamics with mapping 10, then the following formulas may be useful as well:

$$\sin(\theta) = \sin(\arctan(\frac{dh}{dp})) = \frac{1}{\sqrt{1+(\frac{dh}{dp})^2}}\frac{dh}{dp}, \tag{11}$$

$$\cos(\theta) = \cos(\arctan(\frac{dh}{dp})) = \frac{1}{\sqrt{1+(\frac{dh}{dp})^2}}, \tag{12}$$