Probabilistic Machine Learning
Lecture 21
Hidden Markov Models

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Reminder: Goal for this week – *Time Series* as a problem class.

- Application Layer: Data arriving as a stream
- Model Structure Layer: Markov Chains / Hidden Markov Models
- Concrete Model Layer: Gauss–Markov Models
- Algorithm Layer: Kalman Filter & RTS Smoother

Today:

Theory: What is the connection between Gaussian processes and Gauss–Markov Processes?

Parameters: Can we learn the parameters of a Gauss–Markov model?

Generalization: What if the world isn't Gaussian?

Recap: Markov Chains

Processes with the *state* as a local memory

$$p(f) = \mathcal{GP}(f; 0, k) \qquad p\left(\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}\right) = \mathcal{N}\left(0, \begin{bmatrix} K^{-1}_{11} & K^{-1}_{12} & 0 & 0 \\ K^{-1}_{12} & K^{-1}_{22} & K^{-1}_{23} & 0 \\ 0 & K^{-1}_{23} & K^{-1}_{33} & K^{-1}_{34} \\ 0 & 0 & K^{-1}_{34} & K^{-1}_{44} \end{bmatrix}^{-1}\right) \qquad p(\boldsymbol{y} \mid \boldsymbol{f}) = \prod_i \mathcal{N}(y_i; f_i, \sigma^2)$$

$$\begin{tikzcd} f(x_1) \arrow[r, "K^{-1}_{12}"] \arrow[d] & f(x_2) \arrow[r, "K^{-1}_{23}"] \arrow[d] & f(x_3) \arrow[r, "K^{-1}_{34}"] \arrow[d] & f(x_4) \arrow[d] \\ y_1 & y_2 & y_3 & y_4 \end{tikzcd}$$

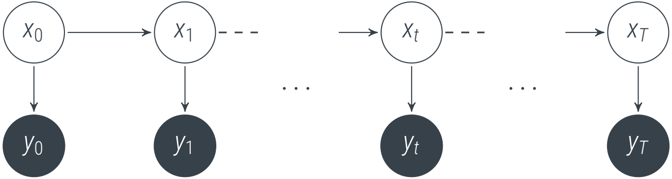
Recap: Markov Chains

Endow the algorithmic structure of filtering and smoothing

Assume:

$$p(x_t \mid X_{0:t-1}) = p(x_t \mid x_{t-1})$$

and $\quad p(y_t \mid X) = p(y_t \mid x_t)$



Filtering: $\mathcal{O}(T)$

predict: $$p(x_t \mid Y_{0:t-1}) = \int p(x_t \mid x_{t-1}) p(x_{t-1} \mid Y_{0:t-1})\, dx_{t-1}$$ (Chapman-Kolmogorov Eq.)

update: $$p(x_t \mid Y_{0:t}) = \frac{p(y_t \mid x_t) p(x_t \mid Y_{0:t-1})}{p(y_t)}$$ (Bayes' Theorem)

Smoothing: $\mathcal{O}(T)$

smooth: $$p(x_t \mid Y) = p(x_t \mid Y_{0:t}) \int p(x_{t+1} \mid x_t) \frac{p(x_{t+1} \mid Y)}{p(x_{t+1} \mid Y_{1:t})}\, dx_{t+1}$$ (backward pass)

Time Series:

- **Markov Chains** formalize the notion of a stochastic process with a *local finite memory*
- Inference over Markov Chains separates into three operations, that can be performed in *linear* time:

Filtering: $\mathcal{O}(T)$

predict: $$p(x_t \mid Y_{0:t-1}) = \int p(x_t \mid x_{t-1}) p(x_{t-1} \mid Y_{0:t-1})\, dx_{t-1}$$ (Chapman-Kolmogorov Eq.)

update: $$p(x_t \mid Y_{0:t}) = \frac{p(y_t \mid x_t) p(x_t \mid Y_{0:t-1})}{p(y_t)}$$ (Bayes' Theorem)

Smoothing: $\mathcal{O}(T)$

smooth: $$p(x_t \mid Y) = p(x_t \mid Y_{0:t}) \int p(x_{t+1} \mid x_t) \frac{p(x_{t+1} \mid Y)}{p(x_{t+1} \mid Y_{0:t})}\, dx_{t+1}$$ (backward pass)

procedure $\textsc{Inference}(Y, p(x_0), p(x_t \mid x_{t-1})\ \forall t, p(y_t \mid x_t)\ \forall t)$

  for i=1,…,n **do** // Filtering

    $p(x_t \mid y_{1:t-1}) = \int p(x_t \mid x_{t-1}) p(x_{t-1} \mid Y_{0:t-1})\, dx_{t-1}$ // Chapman-Kolmogorov eq.

    $p(x_t \mid y_{1:t}) = p(y_t \mid x_t) p(x_t \mid Y_{0:t-1}) / p(y_t)$ // Update

  end for

  for i=n-1,…,0 **do** // Smoothing

    $p(x_t \mid Y) = p(x_t \mid Y_{0:t}) \int p(x_{t+1} \mid x_t) p(x_{t+1} \mid Y) p(x_{t+1} \mid Y_{1:t})\, dx_{t+1}$

  end for

  return $p(x_t \mid Y)\ \forall t = 0, \dots, n$ // return all marginals

end procedure

Gauss–Markov Models

Local structure for univariate Gaussian models

$$p(x(t_{i+1}) \mid X_{1:i}) = \mathcal{N}(x_{i+1}; Ax_i, Q) \quad \text{and} \quad p(x_0) = \mathcal{N}(x_0; m_0, P_0) \quad \text{and} \quad p(y_i \mid X) = \mathcal{N}(y_i; Hx_i, R)$$

procedure FILTER(m_0, P_0, A, Q, H, R, y)
  for $t = 1, 2, \dots, T$ **do** // $\mathcal{O}(N)$
    $m_t^- = Am_{t-1}$ // predictive mean
    $P_t^- = AP_{t-1}A^\intercal + Q$ // predictive covariance. $\mathcal{O}(|X|^3)$
    $z = y - Hm_t^-$ // residual
    $S = HP_t^- H^\intercal + R$ // innovation covariance
    $K = P_t^- H^\intercal S^{-1}$ // gain. $\mathcal{O}(|y|^3)$ Note you probably don't want to compute S^{-1} explicitly...
    $m_t = m_t^- + Kz$ // updated mean
    $P_t = (I - KH)P_t^-$ // updated covariance
  end for
  return $(m_t, P_t), (m_t^-, P_t^-)$
end procedure

The entire filtering pass through N time steps has complexity $\mathcal{O}(N \cdot (|X|^3 + |y|^3))$.

Gauss–Markov Models

Local structure for univariate Gaussian models

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```
procedure SMOOTHER(m_0, P_0, A, Q, H, R, y)
    (m_t, P_t), (m_t^-, P_t^-) ⇐ FILTER(m_0, P_0, A, Q, H, R, y)
    for t = T − 1, T − 2, ..., 1 do
        G_t = P_t A^⊺ (P_{t+1}^-)^{-1}                      // RTS gain. Complexity O(|X|^3)
        m_t^s = m_t + G_t(m_{t+1}^s − m_{t+1}^-)             // smoothed mean
        P_t^s = P_t + G_t(P_{t+1}^s − P_{t+1}^-)G^⊺          // smoothed covariance
    end for
    return (m_t^s, P_t^s)
end procedure
```

Time Series:

- **Markov Chains** formalize the notion of a stochastic process with a *local finite memory*
- Inference over Markov Chains separates into three operations, that can be performed in *linear* time.
- If all relationships are *linear* and *Gaussian*,

$$p(x(t_i) \mid x(t_{i-1})) = \mathcal{N}(x_i; Ax_{i-1}, Q) \qquad p(y_t \mid x_t) = \mathcal{N}(y_t; Hx_t, R)$$

  then inference is analytic and given by the **Kalman Filter** and the **Rauch–Tung–Striebel Smoother**.

Question 1:

Is there a continuous time limit?

Continuous Time

Taking one specific limit of one specific discrete model

$$p(x_{i+1} \mid x_i) \sim \mathcal{N}(x_{i+1}; x_i, Q)$$

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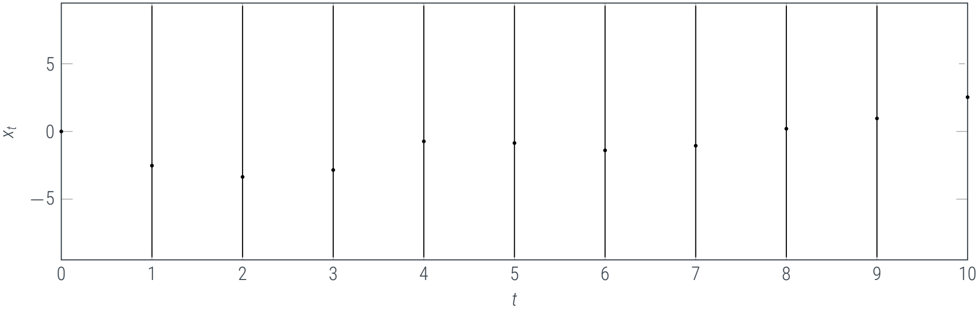
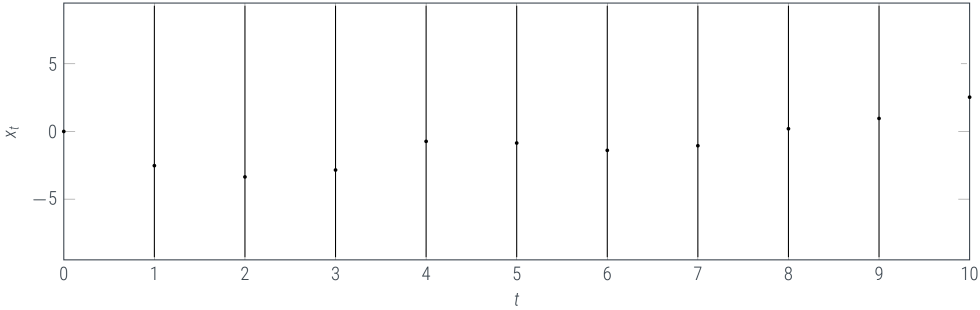


Continuous Time

Taking one specific limit of one specific discrete model

$$p(x(t+\delta t) \mid x(t)) \sim \mathcal{N}\left(x(t+\delta t); x(t), Q\right)$$



$$\delta t = 1 \qquad Q = 1$$

Continuous Time

Taking one specific limit of one specific discrete model

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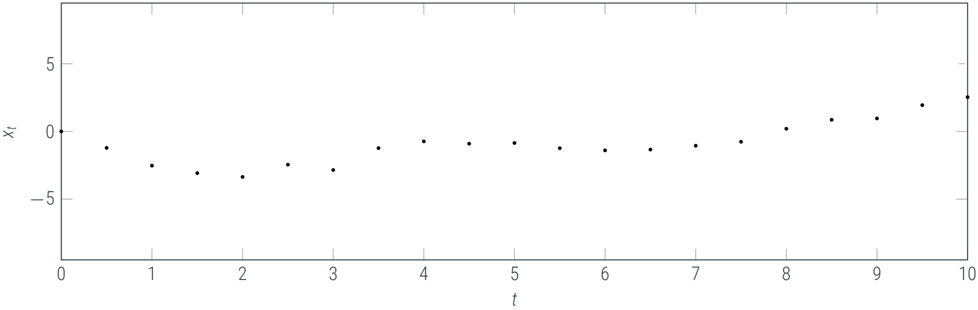
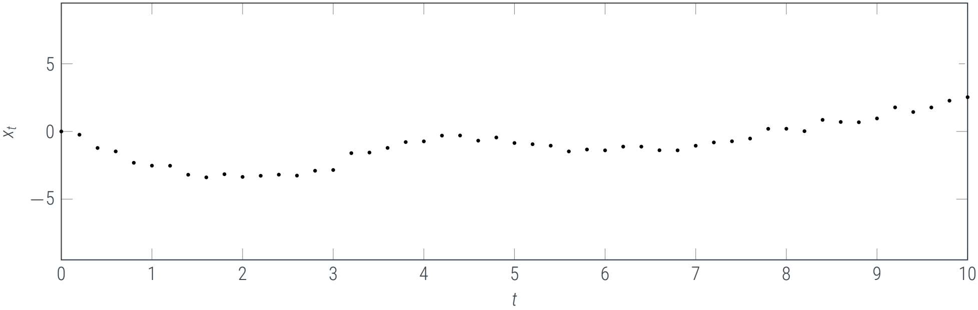


$$\delta t = 1/2 \qquad Q = 1/2$$

Continuous Time

Taking one specific limit of one specific discrete model

$$p(x(t + \delta t) \mid x(t)) \sim \mathcal{N}(x(t + \delta t); x(t), Q)$$

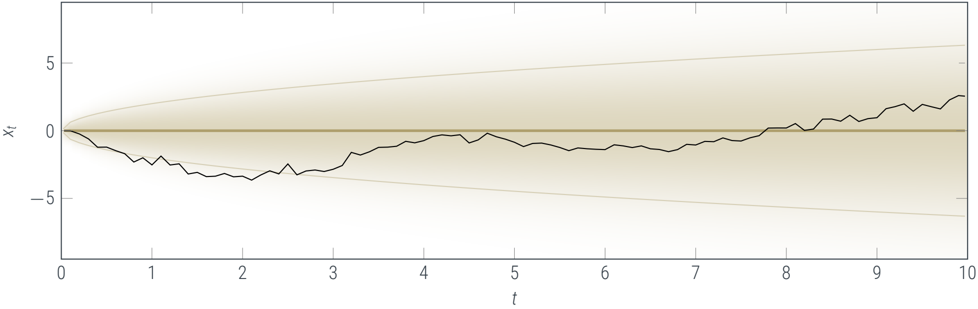


$$\delta t = 1/4 \qquad Q = \delta t$$

Continuous Time

Taking one specific limit of one specific discrete model

$$p(x(t+\delta t) \mid x(t)) \sim \mathcal{N}\left(x(t+\delta t); x(t), Q\right)$$



$$\delta t \rightarrow 0 \qquad Q_{\delta t} = ???$$

Continuous Time

The Wiener process: One specific limit of one specific discrete model

A different way to write things

$$x(t + \delta t) = x(t) + \Delta\omega(t), \qquad \text{with not-really-defined } \Delta\omega,$$

A different way to write things

$$
\begin{aligned}
x(t+\delta t) &= x(t) + \Delta\omega(t), \qquad \text{with not-really-defined } \Delta\omega, \\
\Leftrightarrow \qquad \frac{x(t+\delta t) - x(t)}{\delta t} &= \frac{\Delta\omega(t)}{\delta t}.
\end{aligned}
$$

Continuous Time

The Wiener process: One specific limit of one specific discrete model

A different way to write things

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& x(t+\delta t) = x(t) + \Delta\omega(t), \qquad \text{with not-really-defined } \Delta\omega, \\
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What about the limits?

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What about the limits?

$$\lim_{\delta t \to 0} \frac{x(t + \delta t) - x(t)}{\delta t} = \frac{dx(t)}{dt},$$

Continuous Time

The Wiener process: One specific limit of one specific discrete model

A different way to write things

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What about the limits?

$$\lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} = \frac{dx(t)}{dt},$$

$$\lim_{\delta t \to 0} \frac{\Delta\omega(t)}{\delta t} = ???$$

Continuous Time

The Wiener process: One specific limit of one specific discrete model

A different way to write things

$$x(t+\delta t) = x(t) + \Delta\omega(t), \qquad \text{with not-really-defined } \Delta\omega,$$

$$\Leftrightarrow \qquad \frac{x(t+\delta t) - x(t)}{\delta t} = \frac{\Delta\omega(t)}{\delta t}.$$

What about the limits?

$$\lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} = \frac{dx(t)}{dt},$$

$$\lim_{\delta t \to 0} \frac{\Delta\omega(t)}{\delta t} = w(t), \quad \text{where} \quad w(t) \sim \mathcal{N}(0,1). \qquad \text{(“\textit{weak}” derivative)}$$

Continuous Time

The Wiener process: One specific limit of one specific discrete model

A different way to write things

$$x(t+\delta t) = x(t) + \Delta\omega(t), \qquad \text{with not-really-defined } \Delta\omega,$$
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What about the limits?

$$\lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} = \frac{dx(t)}{dt},$$
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This is one of the key properties of the *Wiener process* (aka. *Brownian motion*).

Note that: *This is not a proper definition of the Wiener process!* This would go beyond the scope of this course. See [Särkkä & Solin, *Applied Stochastic Differential Equations*, 2019] for a thorough introduction.

Linear Time-Invariant Stochastic Differential Equations

A *very* pragmatic introduction to SDEs

For our purposes the (linear, time-invariant) **Stochastic Differential Equation (SDE)**

$$dx(t) = Fx(t)\,dt + L\,d\omega(t),$$

together with $x(0) = x_0$, describes the local behaviour of the (unique) Gaussian process with

$$\mathbb{E}(x(t)) =: m(t) = e^{Ft}x_0 \qquad \operatorname{cov}(x(t), x(t')) = \int_{\min(t,t')}^{\max(t,t')} e^{F(\max(t,t')-\tau)} L L^\intercal e^{F^\intercal(\max(t,t')-\tau)}\,d\tau$$

This GP is known as the **solution** of the SDE. It gives rise to the discrete-time stochastic recurrence relation $p(x(t_{i+1}) \mid x(t_i)) = \mathcal{N}(x(t_{i+1}); A_i x(t_i), Q_i)$ with $(\Delta t_i := t_{i+1} - t_i)$

$$A_i = e^{F\Delta t_i} \quad \text{and} \quad Q_i = \int_0^{\Delta t_i} e^{F(\Delta t_i-\tau)} L L^\intercal e^{F^\intercal(\Delta t_i-\tau)}\,d\tau.$$

Matrix exponential: $e^X := \sum_{i=0}^{\infty} \frac{X^i}{i!}$. *Thus :* $\quad e^0 = I, \quad (e^X)^{-1} = e^{-X}, \quad X = VDV^{-1} \Rightarrow Ve^DV^{-1}, \quad e^{\operatorname{diag}_i d_i} = \operatorname{diag}_i e^{d_i}, \quad \det e^X = e^{\operatorname{tr} X}.$

What this means:

- LTI-SDEs have a correspondence to GPs (so-called *Gauss–Markov processes*)
- LTI-SDEs can be discretized *exactly* to get discrete, linear Gaussian transition models

$\Rightarrow$ Gauss–Markov process inference can be done *in linear time* via filtering and smoothing!

The Connection to GPs

Some well-studied examples

$$dx(t) = Fx(t)\,dt + L\,d\omega_t$$

$$\mathbb{E}(x(t)) =: m(t) = e^{Ft}x_0 \qquad \operatorname{cov}(x(t), x(t')) =: k(t,t') = \int_{\min(t,t')}^{\max(t,t')} e^{F(\max(t,t')-\tau)}LL^\intercal e^{F^\intercal(\max(t,t')-\tau)}\,d\tau$$

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The (scaled) Wiener process

$$F = 0, L = \theta \qquad \Rightarrow \qquad m(t) = x_0 \qquad k(t,t') = \theta^2 \min(t,t')$$

$$A_i = I \qquad Q_i = \theta^2 (t_{i+1} - t_i)$$

The Connection to GPs

Some well-studied examples

$$dx(t) = Fx(t)\,dt + L\,d\omega_t$$

$$\mathbb{E}(x(t)) =: m(t) = e^{Ft}x_0 \qquad \operatorname{cov}(x(t), x(t')) =: k(t,t') = \int_{\min(t,t')}^{\max(t,t')} e^{F(\max(t,t')-\tau)} L L^\intercal e^{F^\intercal(\max(t,t')-\tau)}\,d\tau$$

The Ornstein-Uhlenbeck process

$$F = -\frac{1}{\lambda}, L = \frac{2\theta}{\sqrt{\lambda}} \qquad \Rightarrow \qquad m(t) = x_0 e^{-\frac{t}{\lambda}} \qquad k(t,t') = \theta^2\left(e^{-\frac{|t-t'|}{\lambda}} - e^{-\frac{t+t'}{\lambda}}\right)$$

$$A_i = e^{-\Delta t_i/\lambda} \qquad Q_i = \theta^2\left(1 - e^{-2\Delta t_i/\lambda}\right)$$

$$dx(t) = Fx(t)\,dt + L\,d\omega_t$$

- So far, we have seen examples with $x(t) \in \mathbb{R}$.
- But F and L can also be matrices. Consider the example

$$x = \begin{bmatrix} x_{(1)} \\ x_{(2)} \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is:

$$\begin{bmatrix} dx_{(1)}(t) \\ dx_{(2)}(t) \end{bmatrix} = \begin{bmatrix} x_{(2)}(t)\,dt + 0\,d\omega \\ 0\,dt + d\omega \end{bmatrix} \qquad \Rightarrow \qquad x_{(1)}(t) = \int_0^t x_{(2)}(t)\,dt + [x_0]_1$$

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$$dx(t) = Fx(t)\,dt + L\,d\omega_t$$

The Wiener velocity (aka. "once-integrated Wiener process")

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad m(t) = e^{Ft}x_0 \qquad k(t,t') = \frac{\min^3(t,t')}{3} + |t-t'|\frac{\min^2(t,t')}{2}$$

$$A_i = \begin{bmatrix} 1 & \Delta t_i \\ 0 & 1 \end{bmatrix}, \qquad Q_i = \begin{bmatrix} \frac{\Delta t_i^3}{3} & \frac{\Delta t_i^2}{2} \\ \frac{\Delta t_i^2}{2} & \Delta t_i \end{bmatrix}$$

Q1 summary:

- Certain Gaussian processes can be written as LTI-SDEs
  - (integrated) Wiener process
  - (integrated) Ornstein–Uhlenbeck process
  - Matern processes
  - Even the square-exponential kernel can be approximated by an LTI-SDE
- LTI-SDEs can be discretized *exactly* to get discrete, linear Gaussian transition models
- Inference in linear Gauss–Markov models (and thus in Gauss–Markov processes) can be done *in linear time* via filtering and smoothing

Question 2:

Can we learn the model?

Recap (Lecture 10) **Learning** the Kernel for GPs

Bayesian Hierarchical Inference

$$p(f \mid \theta) = \mathcal{GP}(f; m_\theta, k_\theta) \qquad \text{e.g.} \quad m_\theta(\bullet) = \phi(\bullet)^\intercal \boldsymbol{\theta}, \text{ or } k_\theta(\bullet, \circ) = \theta_1 \exp\left(-\frac{(\bullet - \circ)^2}{2\theta_2^2}\right).$$

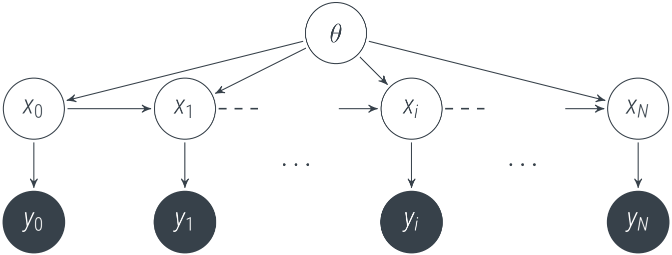
- The **evidence** in Bayes' theorem is the **marginal likelihood for the model**

$$p(f \mid \boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\theta}) = \frac{p(\boldsymbol{y} \mid f, \boldsymbol{x}, \boldsymbol{\theta}) p(f \mid, \boldsymbol{\theta})}{\int p(\boldsymbol{y} \mid f, \boldsymbol{x}, \boldsymbol{\theta}) p(f \mid, \boldsymbol{\theta})\, df} = \frac{p(\boldsymbol{y} \mid f, \boldsymbol{x}, \boldsymbol{\theta}) p(f \mid, \boldsymbol{\theta})}{p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\theta})}$$

- For Gaussians *and Gaussian processes*, the evidence has **analytic form**:

$$\underbrace{\mathcal{N}(\boldsymbol{y}; \phi_X^{\boldsymbol{\theta}\intercal} w + \boldsymbol{b}, \Lambda)}_{p(\boldsymbol{y}|f,\boldsymbol{x},\boldsymbol{\theta})} \cdot \underbrace{\mathcal{N}(w, \mu, \Sigma)}_{p(f)} = \underbrace{\mathcal{N}(w; m_{\text{post}}^{\boldsymbol{\theta}}, V_{\text{post}}^{\boldsymbol{\theta}})}_{p(f|\boldsymbol{y},\boldsymbol{x},\boldsymbol{\theta})} \cdot \underbrace{\mathcal{N}(\boldsymbol{y}; \phi_X^{\boldsymbol{\theta}\intercal} \mu + b, \phi_X^{\boldsymbol{\theta}\intercal} \Sigma \phi_X^{\boldsymbol{\theta}} + \Lambda)}_{p(\boldsymbol{y}|\boldsymbol{\theta},\boldsymbol{x})}$$

$$\mathcal{N}(\boldsymbol{y}; f^{\boldsymbol{\theta}}(X), \Lambda^{\boldsymbol{\theta}}) \cdot \mathcal{GP}(f, \mu^{\boldsymbol{\theta}}, k^{\boldsymbol{\theta}}) = \mathcal{GP}(f; m_{\text{post}}^{\boldsymbol{\theta}}, V_{\text{post}}^{\boldsymbol{\theta}}) \cdot \mathcal{N}(\boldsymbol{y}; \mu^{\boldsymbol{\theta}}(X), \Lambda^{\boldsymbol{\theta}} + k^{\boldsymbol{\theta}}(X, X))$$



For Gauss–Markov Models, is there a way to compute the model evidence in $\mathcal{O}(N)$?

- Assume unknown **model hyper-parameters** θ (define $x_{-1} = \varnothing$):

$$p(\mathbf{y}, \mathbf{x}, \theta) = p(\theta) \cdot p(\mathbf{y}, \mathbf{x} \mid \theta) = p(\theta) \prod_{i=0}^{N} p(x_i \mid x_{i-1}, \theta) p(y_i \mid x_i, \theta)$$

- to learn θ, we need the **evidence** (aka. **marginal/type-II likelihood**) (define $y_{-1} = \varnothing$)

$$p(\mathbf{y} \mid \theta) = \prod_{i=0}^{N} p(y_i \mid y_{0:i-1}, \theta)$$

- the terms in the product decompose into local predictions:

$$p(y_i \mid y_{0:i-1}, \theta) = \int p(y_i, x_i \mid y_{0:i-1}, \theta)\, dx_i \qquad = \int p(y_i \mid x_i, y_{0:i-1}, \theta) p(x_i \mid y_{0:i-1}, \theta)\, dx_i$$

$$= \int p(y_i \mid x_i, \theta) p(x_i \mid y_{0:i-1}, \theta)\, dx_i$$ which, for linear Gaussian systems, is

$$= \int \mathcal{N}(y_i; Hx_i, R) \mathcal{N}(x_i; m_i^-, P_i^-)\, dx_i = \mathcal{N}(y_i; Hm_i^-, HP_i^- H^\intercal + R) = \mathcal{N}(z_i; 0, S_i)$$

Learning parameters of a Gauss–Markov Model

For the LTI system

$$p(x_i \mid x_{i-1}, \theta) = \mathcal{N}(x_i; Ax_{i-1}, Q), \quad \text{and} \quad p(y_i \mid x_i, \theta) = \mathcal{N}(y_i; Hx_i, R),$$

the (log) evidence is given by

$$\begin{aligned} p(\boldsymbol{y} \mid \theta) &= \prod_{i=1}^{N} p(y_i; y_{0:i-1}, \theta) \\ &= \prod_{i=0}^{N} \mathcal{N}(y_i; Hm_i^-, HP_i^- H^\intercal + R) \\ \log p(\boldsymbol{y} \mid \theta) &= -\frac{1}{2} \sum_{i=1}^{N} \left(z_i^\intercal S_i^{-1} z_i + \log |S_i| + \log 2\pi \right) \end{aligned}$$

In principle, this could also be used to learn A, Q, R, H directly, but there's a more elegant way to do this for linear Gaussian systems. For more, cf. Ghahramani & Hinton, 1996, and Särkkä 2013.

Question 3:

What if the world is not linear Gaussian?

Hidden Markov Models

Generalization to non-Gaussian relationships

Name	Distribution	Algorithm
Markovian System:	$p(\boldsymbol{y}, \boldsymbol{x}) = \prod_{i=0}^{N} p(x_i \mid x_{i-1}) p(y_i \mid x_i)$	General Bayesian filtering and smoothing
Linear Gaussian System:	$p(\boldsymbol{y}, \boldsymbol{x}) = \prod_{i=0}^{N} \mathcal{N}(x_i; A_i x_{i-1}, Q_i) \mathcal{N}(y_i; H x_i, R)$	Kalman filter, Rauch−Tung−Striebel smoother
Nonlinear Gaussian System:	$p(\boldsymbol{y}, \boldsymbol{x}) = \prod_{i=0}^{N} \mathcal{N}(x_i; a(x_{i-1}), Q_i) \mathcal{N}(y_i; h(x_i), R)$	e.g. Extended/Unscented/Particle filter etc.
Non-Gaussian observations:	$p(\boldsymbol{y}, \boldsymbol{x}) = \prod_{i=0}^{N} \mathcal{N}(x_i; A_i x_{i-1}, Q_i) p(y_i \mid h(x_i))$	
Hidden Markov Model (e.g.):	$p(\boldsymbol{y}, \boldsymbol{x}) = \prod_{i=0}^{N} p(x_i = \Pi x_{i-1}) \mathcal{N}(y_i; h(x_i), R)$	

- For continuous systems with nonlinear dynamics and/or non-linear observations, a number of *approximately Gaussian filters* have been developed. For more see, e.g.
  - Simo Särkkä. *Bayesian Filtering and Smoothing*. Cambridge University Press, 2013 `https://users.aalto.fi/~ssarkka/pub/cup_book_online_20131111.pdf`

Because streaming data is a common data type, time series are an entire sub-field of their own, studied in a diverse range of domains. There is no time to cover them all in this course.

Summary:

Markov Chains capture **finite memory** of a time series through conditional independence

Gauss–Markov models map this state to linear algebra

Kalman filter is the name for the corresponding algorithm

SDEs (Stochastic Differential Equations) are the continuous-time limit of discrete-time stochastic recurrence relations (in particular, linear SDEs are the continuous-time generalization discrete-time linear Gaussian systems)

Parameters of the model can be learnt by optimizing the (log) evidence, which is also $\mathcal{O}(N)$.

Non-Gaussian models can be learnt by approximate inference, analogous to GP models.

Please cite this course, as

```
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Gauss–Markov models form the algorithmic scaffold for time-series models .