Longitudinal MPC

This repository implement speed planning via MPC.

Basic MPC

$$
\begin{gathered}
J=\min_{x_k,u_k}\sum_{k=0}^{N-1}\left[\left(x_k-x_{k,r}\right)^T Q\left(x_k-x_{k,r}\right)+u_k^T R u_k\right]+ \\
\left(x_N-x_{N,r}\right)^T Q_N\left(x_N-x_{N,r}\right)
\end{gathered}
$$

$$
\begin{gathered}
s.t. \\
x_{k+1}=Ax_k+Bu_k \\
\begin{bmatrix} x_{min,k} \\ u_{min,k} \end{bmatrix} \leq A_{ineq}\begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq \begin{bmatrix} x_{max,k} \\ u_{max,k} \end{bmatrix}
\end{gathered}
$$

where

- x_k is state variable;
- $x_{k,r}$ is state reference;
- u_k is control variable;
- Q_N is usually solved by an algebraic discrete Riccati equation.

Longitudinal MPC

In this section, we will modify J to fit for a speed planning problem.

$$
\begin{gathered}
J=\min_{x_k,u_k}\sum_{k=0}^{N-1}\left[\left(x_k-x_{k,r}\right)^T Q\left(x_k-x_{k,r}\right)+u_k^T R u_k+\dot{u}_k^T\dot{R}\dot{u}_k\right]+ \\
\left(x_N-x_{N,r}\right)^T Q_N\left(x_N-x_{N,r}\right)+\sigma_k^T W\sigma_k
\end{gathered}
$$

where, specifically, in this equation,

- x_k represents $[s_k,\dot{s}_k]^T$;

- $x_{k,r}$ represents $[s_{k,r}, \dot{s}_{k,r}]^T$;
- u_k represents $\ddot{s}_k$;
- σ_k represents slack variables.

To a Quadratic Problem

We have a standard QP formulation as below.

$$
\begin{gathered}
J = \min_z \frac{1}{2} z^T P z + q^T z \\
s.t. \\
z_{min} \leq Az \leq z_{max}
\end{gathered}
$$

Cost

First of all, $\dot{u}_k$ could be written as

$$\dot{u}_k = \frac{u_k - u_{k-1}}{\Delta t}$$

where $k = 1, ..., N-1$.

Perticular,

$$\dot{u}_0 = \frac{u_0 - u_{-1}}{\Delta t_{-1}}$$

where -1 index means the previous control command. For simplicity, we choose Δt_{-1} equals Δt.

Thus, the control part is

$$
\begin{aligned}
u_k^T R u_k + \dot{u}_k^T \dot{R} \dot{u}_k &= u_k^T R u_k + \frac{(u_k - u_{k-1})^T}{\Delta t} \dot{R} \frac{u_k - u_{k-1}}{\Delta t} \\
&= u_k^T (R + \frac{\dot{R}}{\Delta t^2}) u_k + u_{k-1}^T \frac{\dot{R}}{\Delta t^2} u_{k-1} \\
&- u_k^T \frac{\dot{R}}{\Delta t^2} u_{k-1} - u_{k-1}^T \frac{\dot{R}}{\Delta t^2} u_k
\end{aligned}
$$

Then, we have

$$
P = \begin{bmatrix}
Q & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & Q & & & & & & & \\
& & & Q_N & & & & & & \\
& & & & R + 2\frac{\dot{R}}{\Delta t^2} & -\frac{\dot{R}}{\Delta t^2} & & & & \\
& & & & -\frac{\dot{R}}{\Delta t^2} & \ddots & \ddots & & & \\
& & & & & \ddots & R + 2\frac{\dot{R}}{\Delta t^2} & -\frac{\dot{R}}{\Delta t^2} & & \\
& & & & & & -\frac{\dot{R}}{\Delta t^2} & R + \frac{\dot{R}}{\Delta t^2} & & \\
& & & & & & & & w & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & w
\end{bmatrix}
$$

$$
q = \begin{bmatrix} -Qx_{0,ref} & \dots & -Q_N x_{N,ref} & -\frac{\dot{R}}{\Delta t^2} u_{-1} & 0 & 0 \end{bmatrix}^T
$$

Constraints

Equality Constraints

We consider an error differential equation

$$
\dot{x}(t) = A_c x(t) + B_c u(t)
$$

where $x(t)$ is state variable; $u(t)$ is control variable.

Here, we have

$$
x_k = [s_k, \dot{s}_k]^T
$$

$$u_k = [\ddot{s}_k]$$

Then,

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, the equality constraints

$$x_{k+1} = (I + A_c \Delta t) x_k + B_c \Delta t u_k$$

could be written in the following format

$$b_{eq} \le A_{eq} z \le b_{eq}$$

where

$$A_{eq} = \begin{bmatrix} -I & & & & 0 & & & \\ I + A_c \Delta t & -I & & & B_c \Delta t & & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & I + A_c \Delta t & -I & & & B_c \Delta t & 0 \end{bmatrix}$$
$$b_{eq} = \begin{bmatrix} -x_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Inequality Constraints

Generally, maximum ranges of acceleration and jerk of the ego vehicle constrain a speed planning problem. Slack variables are often applied to relax station and velocity constraints. In this section, we only support to relax s_k and $\dot{s}_k$ with slack variables, denoting as $\sigma_{x,k}$ for upper bounds and $\mu_{x,k}$ for lower bounds.

$$
A_{ineq} = \begin{bmatrix} 1_{x,k} & & -1_{\sigma_{x,k}} & \\ 1_{x,k} & & & 1_{\mu_{x,k}} \\ & 1_{u,k} & & \\ & a_{\dot{u},k} & & \\ & & 1_{\sigma_{x,k}} & \end{bmatrix}
$$

$$
b_{ineq,lower} = \begin{bmatrix} -\infty \\ x_{min,k} \\ u_{min,k} \\ b_{\dot{u},k,min} \\ 0_{\sigma_{x,k}} \end{bmatrix}
$$

$$
b_{ineq,upper} = \begin{bmatrix} x_{max,k} \\ +\infty \\ u_{max,k} \\ b_{\dot{u},k,max} \\ \sigma_{max,x,k} \end{bmatrix}
$$

where

$$
a_{\dot{u},k} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}
$$

$$
b_{\dot{u},k,max} = \begin{bmatrix} u_{-1} + j_{max}\Delta t \\ j_{max}\Delta t \\ \vdots \\ j_{max}\Delta t \end{bmatrix}
$$

$$
b_{\dot{u},k,min} = \begin{bmatrix} u_{-1} + j_{min}\Delta t \\ j_{min}\Delta t \\ \vdots \\ j_{min}\Delta t \end{bmatrix}
$$

Please Note that we do **NOT** support to constrain lower slack variables $\mu_{x,k}$ of x_k.