Special Topics on Optimal Control and Learning

ISYE 8803 VAN

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1. Pontryagin’s Maximum Principle

2. Shooting & Multiple Shooting

3. LQR, Riccati, QP viewpoint (finite / infinite horizon)

Topic 1 — Pontryagin's Maximum Principle (PMP)

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1.Problem Setting

We want to solve an optimal control problem:

$$\min_{u(\cdot)} J = \phi(x(T)) + \int_0^T \ell(x(t), u(t), t)\, dt$$

subject to the system dynamics:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \quad u(t) \in \mathcal{U}.$$

- $x(t)$: state trajectory.
- $u(t)$: control input.
- ℓ: stage cost.
- ϕ: terminal cost.
- f: dynamics of the system.

This is an infinite-dimensional optimization problem because the decision variable is the entire function

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2. Introducing the Costate and Hamiltonian

To enforce the dynamics constraint $\dot{x} = f(x, u, t)$, we introduce **time-varying multipliers** $\lambda(t)$, called the **costates**.

Define the **Hamiltonian**:

$$H(x, u, \lambda, t) = \ell(x, u, t) + \lambda^\top f(x, u, t).$$

This combines both the cost and the dynamics into a single function.

3.Pontryagin's Maximum (Minimum) Principle

If $(x^*(t), u^*(t))$ is an optimal solution, then there exists a costate trajectory $\lambda^*(t)$ such that:

1. State equation (forward in time):

$$\dot{x}^*(t) = \frac{\partial H}{\partial \lambda}(x^*, u^*, \lambda^*, t) = f(x^*, u^*, t), \quad x^*(0) = x_0.$$

2. Costate equation (backward in time):

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial x}(x^*, u^*, \lambda^*, t), \quad \lambda^*(T) = \nabla_x \phi(x^*(T)).$$

3. Optimality condition (pointwise in time):

$$u^*(t) \in \arg\min_{u \in \mathcal{U}} H(x^*(t), u, \lambda^*(t), t).$$

4.Transversality Conditions

If the final state or time is free, additional conditions hold:

Free final state:

$$\lambda(T) = \nabla\phi(x(T))$$

Free final time:

$$H(x(T), u(T), \lambda(T), T) = 0$$

If the system is time-invariant (no explicit t), the Hamiltonian is constant along the optimal trajectory.

5. Control Constraints and Switching Functions

If controls are bounded, eg. $u \in [u_{\min}, u_{\max}]$

When H is linear in u, the solution is often bang-bang (jumping between bounds).

Define the switching function:

$$\sigma(t) = \frac{\partial H}{\partial u}(x^*(t), u, \lambda^*(t), t).$$

- If $\sigma(t) > 0$, $u^*(t) = u_{\min}$.
- If $\sigma(t) < 0$, $u^*(t) = u_{\max}$.
- If $\sigma(t) \equiv 0$, we are on a **singular arc** $\rightarrow$ need higher-order conditions.

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6. Example (see code)

- Dynamics: $\dot{x} = u$
- Cost: $J = \frac{1}{2}\int_0^T \left(q\,x(t)^2 + r\,u(t)^2\right) dt$, with $q > 0, r > 0$
- Boundary: $x(0) = x_0,\ x(T) = 0$ (fixed terminal state; no terminal cost)

Hamiltonian

$$H(x, u, \lambda, t) = \tfrac{1}{2}(qx^2 + ru^2) + \lambda\, u.$$

PMP conditions

$$\begin{aligned}
\dot{x} &= \partial_\lambda H = u, \\
\dot{\lambda} &= -\partial_x H = -q\,x, \\
\partial_u H &= ru + \lambda = 0 \Rightarrow u^{*}(t) = -\lambda(t)/r.
\end{aligned}$$

7. Intuition

- State trajectory evolves forward (from X(0))

- Costate trajectory evolves backward (from terminal condition).

- Optimal control is chosen pointwise to minimize the Hamiltonian.

Together, these three equations form a two-point boundary value problem (BVP), which is why numerical methods (like shooting) are needed.

Topic 2 — Shooting & Multiple Shooting Methods

1. Motivation: Why Shooting Methods?

PMP gives necessary conditions → two-point BVP:

$$\dot{x} = f(x, u), \quad \dot{\lambda} = -\frac{\partial H}{\partial x}, \quad u^{*}(t) = \arg\min H(x, u, \lambda)$$

Known: $x(0) = x_0$.

Unknown: $\lambda(0)$, with boundary condition at T:

$$\psi(x(T), \lambda(T)) = 0$$

For nonlinear dynamics, closed-form solutions rarely exist → need numerical shooting methods.

2. Single Shooting

Treat $\lambda(0)$ as unknown vector p

Define forward integration operator:

$$F(p) = x(T;p) - x_T$$

Solve nonlinear equation:

$$F(p) = 0$$

Methods:

Root-finding: Newton, Broyden

$$p^{(k+1)} = p^{(k)} - \left(\frac{\partial F}{\partial p}\right)^{-1} F(p^{(k)})$$

Optimization:

$$\min_p \tfrac{1}{2}||F(p)||^2$$

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3. Example: Single Shooting-Double Integrate Cost (See code)

Dynamics: $\dot{x}_1 = x_2,\ \dot{x}_2 = u.$

Cost: $J = \frac{1}{2}\int_0^T u(t)^2 dt.$

PMP gives control law: $u(t) = c_1 t - c_2.$

Unknowns: $c_1, c_2.$

Shooting method: find (c_1, c_2) so that $x(T) = [0, 0].$

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4. Limitations of Single Shooting

Jacobian $\partial F / \partial p$ may be **ill-conditioned**.

Error in $p \rightarrow$ exponential growth in $x(T)$.

Long horizon $\rightarrow$ unstable.

Path constraints $(g(x, u) \leq 0)$ are difficult to enforce.

5. Multiple Shooting

Partition horizon:

$$[0, T] = [t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{M-1}, T]$$

Introduce decision variables:

- **States at nodes**: $z_k \approx x(t_k)$.
- **Controls per interval**: u_k.

Define segment flow map (numerical integration):

$$\tilde{x}_{k+1} = \Phi_{\Delta t}(z_k, u_k)$$

Enforce continuity constraints:

$$c_k(z_k, u_k, z_{k+1}) = \tilde{x}_{k+1} - z_{k+1} = 0.$$

6. Multiple Shooting: NLP Formulation

- Variables: $z_0, \ldots, z_M,\ u_0, \ldots, u_{M-1}$.
- Objective:

$$\min \sum_{k=0}^{M-1} \ell(z_k, u_k)\Delta t + \phi(z_M)$$

- Constraints:
  - Dynamics continuity: $c_k = 0$.
  - Boundary: $z_0 = x_0,\ z_M = x_T$ (hard) or penalty.
  - Path constraints: $g(z_k, u_k) \leq 0$.

Jacobian structure (block-tridiagonal):

$$\frac{\partial c_k}{\partial(z_k, u_k, z_{k+1})} = \left[\frac{\partial \Phi}{\partial z_k},\ \frac{\partial \Phi}{\partial u_k},\ -I\right]$$

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7. Multiple Shooting Methods (Variants)

Partition horizon:

$$[0, T] = [t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{M-1}, T]$$

Introduce decision variables:

- **States at nodes**: $z_k \approx x(t_k)$.
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Define segment flow map (numerical integration):

$$\tilde{x}_{k+1} = \Phi_{\Delta t}(z_k, u_k)$$

Enforce continuity constraints:

$$c_k(z_k, u_k, z_{k+1}) = \tilde{x}_{k+1} - z_{k+1} = 0.$$

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8. Multiple Shooting Methods (Variants)

1. Direct multiple shooting (optimization):

Use NLP solvers (SQP, interior-point).

2. Equation-based multiple shooting:

Collect all continuity equations, solve as nonlinear system.

3. Hybrid with collocation:

Within each interval, approximate with polynomials (e.g., Hermite-Simpson).

4. Parallel multiple shooting:

Solve sub-intervals independently, then enforce continuity.

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9. Multiple Shooting: Example (Double Integrate Cost) (See code)

Dynamics: $\dot{x}_1 = x_2,\ \dot{x}_2 = u.$

Cost: $J = \frac{1}{2}\int_0^T u(t)^2 dt.$

PMP gives control law: $u(t) = c_1 t - c_2.$

Unknowns: $c_1, c_2.$

Shooting method: find (c_1, c_2) so that $x(T) = [0, 0].$

9. Multiple Shooting: Example (Pendulum) (See code)

Dynamics:

$\dot{\theta} = \omega, \ \dot{\omega} = u - \sin\theta$

Cost:

$$J = \sum_{k=0}^{M-1} \tfrac{1}{2}\Delta t(q_\theta \theta_k^2 + q_\omega \omega_k^2 + r u_k^2) + \tfrac{1}{2}(Q_{f\theta}\theta_M^2 + Q_{f\omega}\omega_M^2)$$

Constraints:

$\theta_{k+1} = \theta_k + \Delta t\, \omega_k$

$\omega_{k+1} = \omega_k + \Delta t\, (u_k - \sin\theta_k)$

10. Comparison

Feature	Single Shooting	Multiple Shooting
Variables	Few (parameters)	Many (states+controls)
Numerical stability	Poor (error blow-up)	Good (local integration)
Path constraints	Hard	Natural
Solver type	Root-finding	NLP (SQP, IP, etc.)
Applications	Small/simple	Large-scale, nonlinear OCP

Topic 3 — LQR, Riccati, QP viewpoint (finite / infinite horizon)

3.1 Linear Quadratic Regulation (LQR)

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1. Deterministic Linear Quadratic Regulation

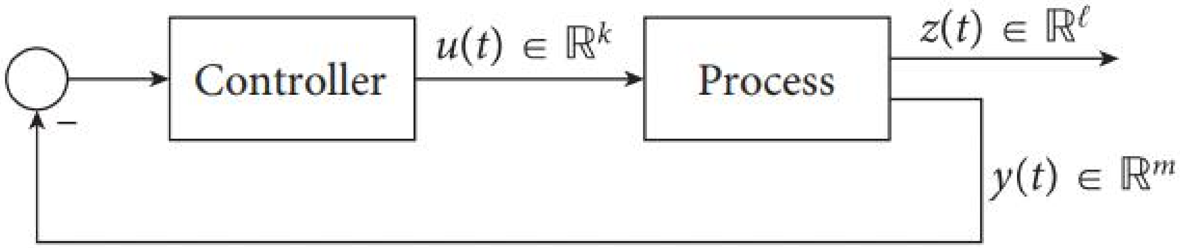
- For continuous-time LTI system of the form

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^k,$$
$$y = Cx, \quad y \in \mathbb{R}^m,$$
$$z = Gx + Hu, \quad z \in \mathbb{R}^\ell,$$

that has two distinct outputs:

1. The measured output y(t) corresponds to the signal(s) that can be measure and are therefore available for control.

2. The controlled output z(t) corresponds to the signal(s) that one would like to make as small as possible in the shortest possible time.



Sometimes z(t) = y(t), which means that our control objective is simply to make the measured output very small. At other times one may have

$$z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

which means that we want to make both the measured output y(t) and its derivative $\dot{y}$(t) very small. Many other options are possible.

2 Optimal Regulation

The LQR problem is defined as follows. Find the control input u(t), $t \in [0,\infty)$ that makes the following criterion as small as possible:

$$J_{\mathrm{LQR}} := \int_0^\infty \|z(t)\|^2 + \rho \, \|u(t)\|^2 dt,$$

where ρ is a positive constant. The term

$$\int_0^\infty \|z(t)\|^2 dt$$

corresponds to the energy of the controlled output, and the term

$$\int_0^\infty \|u(t)\|^2 dt$$

Corresponds to the energy of the control input. In LQR one seeks a controller that minimizes both energies. However, decreasing the energy of the controlled output will require a large control input, and a small control input will lead to large controlled outputs. The role of the constant ρ is to establish a trade-off between these conflicting goals.

1. When we choose ρ very large, the most effective way to decrease J of LQR(cost) is to employ a small control input, at the expense of a large controlled output.

2. When we choose ρ very small, the most effective way to decrease J of LQR(cost) is toobtain a very small controlled output, even if this is achieved at the expense of employing a large control input.

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Often the optimal LQR problem is defined more generally and consists of finding the control input that minimizes

$$J_{\mathrm{LQR}} \coloneqq \int_0^\infty z(t)'\bar{Q}z(t) + \rho\, u(t)'\bar{R}u(t)\, dt,$$

where Q and R are symmetric positive-define matrices and ρ is a postive constant

We shall consider the most general form for a quadratic criterion, which is

$$J_{\mathrm{LQR}} \coloneqq \int_0^\infty x(t)'Qx(t) + u(t)'Ru(t) + 2x(t)'Nu(t)dt. \qquad \text{(J-LQR)}$$

$$J = \int_0^{T/\infty} \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\begin{bmatrix} Q & N \\ N^\top & R \end{bmatrix}}_{\succeq 0,\ R \succ 0} \begin{bmatrix} x \\ u \end{bmatrix} dt.$$

Proof:

$$z = Gx + Hu$$

as

$$J = \int_0^{T/\infty} \left(\|z(t)\|_{\bar{Q}}^2 + \rho \, \|u(t)\|_{\bar{R}}^2 \right) dt \qquad \left(\|v\|_M^2 := v^\top M v \right),$$

then by **expanding** $\|z\|_{\bar{Q}}^2$ you recover the standard LQR quadratic form with the same matrices Q, N, R:

$$\begin{aligned} \|z\|_{\bar{Q}}^2 &= (Gx + Hu)^\top \bar{Q} \, (Gx + Hu) \\ &= x^\top \underbrace{(G^\top \bar{Q} G)}_{Q} \, x \; + \; 2 \, x^\top \underbrace{(G^\top \bar{Q} H)}_{N} \, u \; + \; u^\top \underbrace{(H^\top \bar{Q} H)}_{R_z} \, u. \end{aligned}$$

Adding the separate control penalty $\rho \, \|u\|_{\bar{R}}^2 = \rho \, u^\top \bar{R} u$ gives

$$u^\top (R_z + \rho \, \bar{R}) \, u.$$

$$x^\top \underbrace{(G^\top \bar{Q} G)}_{Q} x \;+\; 2\,x^\top \underbrace{(G^\top \bar{Q} H)}_{N} u \;+\; u^\top \underbrace{(H^\top \bar{Q} H + \rho\,\bar{R})}_{R} u.$$

$$\boxed{Q = G^\top \bar{Q} G, \qquad N = G^\top \bar{Q} H, \qquad R = H^\top \bar{Q} H + \rho\,\bar{R}}$$

$$J = \int_0^{T/\infty} \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\begin{bmatrix} Q & N \\ N^\top & R \end{bmatrix}}_{\succeq 0,\; R \succ 0} \begin{bmatrix} x \\ u \end{bmatrix} dt.$$

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3. Feedback Invariants in optimal control

Given a continuous-time LTI system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^k \tag{AB-CLTI}$$

we say that a functional

$$H\big(x(\cdot); u(\cdot)\big)$$

that involves the system's input and state is a *feedback invariant* for the system (AB-CLTI) if, when computed along a solution to the system, its value depends only on the initial condition $x(0)$ and not on the specific input signal $u(\cdot)$.

Proposition

for every symmetric matrix P , the functional

$$H\big(x(\cdot);u(\cdot)\big) := -\int_0^{\infty} \big(Ax(t)+Bu(t)\big)'Px(t) + x(t)'P\big(Ax(t)+Bu(t)\big)\,dt$$

is a feedback invariant for the system (AB-CLTI), as long as lim x(t) = 0 when t→∞.

Suppose that we are able to express a criterion J to be minimized by an appropriate choice of the input $u(\cdot)$ in the form

$$J = H\big(x(\cdot); u(\cdot)\big) + \int_0^\infty \Lambda\big(x(t), u(t)\big)dt,$$

where H is a feedback invariant and the function $\Lambda(x, u)$ has the property that for every $x \in \mathbb{R}^n$

$$\min_{u\in\mathbb{R}^k} \Lambda(x, u) = 0.$$

In this case, the control

$$u(t) = \arg\min_{u\in\mathbb{R}^k} \Lambda(x, u),$$

minimizes the criterion J, and the optimal value of J is equal to the feedback invariant

$$J = H\big(x(\cdot); u(\cdot)\big).$$

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4. Optimal State Feedback

Introduce a symmetric matrix P and add/subtract the feedback invariant:

$$
\begin{aligned}
J_{\text{LQR}} &:= \int_0^\infty x'Qx + u'Ru + 2x'Nu\,dt \\
&= H\big(x(\cdot);u(\cdot)\big) + \int_0^\infty x'Qx + u'Ru + 2x'Nu + (Ax+Bu)'Px \\
&\quad + x'P(Ax+Bu)\,dt \\
&= H\big(x(\cdot);u(\cdot)\big) + \int_0^\infty x'(A'P+PA+Q)x + u'Ru + 2u'(B'P+N')x\,dt.
\end{aligned}
$$

By completing the square, we can group the quadratic term in u with the cross-term in u times x:

$$(u' + x'K')R(u + Kx) = u'Ru + x'(PB + N)R^{-1}(B'P + N')x + 2u'(B'P + N')x,$$

where

$$K := R^{-1}(B'P + N'),$$

from which we conclude that

$$J_{\text{LQR}} = H\big(x(\cdot); u(\cdot)\big) + \int_0^\infty x'\big(A'P + PA + Q - (PB + N)R^{-1}(B'P + N')\big)x + (u' + x'K')R(u + Kx)\,dt.$$

If we are able to select the matrix P so that

$$A'P + PA + Q - (PB + N)R^{-1}(B'P + N') = 0,$$

we obtain precisely an expression such as (3.1.3) with

$$\Lambda(x, u) := (u' + x'K')R(u + Kx),$$

which has a minimum equal to zero for

$$u = -Kx, \quad K := R^{-1}(B'P + N'),$$

leading to the closed-loop system

$$\dot{x} = \big(A - BR^{-1}(B'P + N')\big)x.$$

The following has been proved.

Theorem

Assume that there exists a symmetric solution P to the algebraic Riccati equation (3.1.5) for which $(A - BR^{-1}(B'P + N'))$ is a stability matrix.

Then the feedback law

$$u(t) := -Kx(t), \quad \forall t \geq 0, \quad K := R^{-1}(B'P + N')$$

minimizes the J-LQR and leads to

$$J_{\mathrm{LQR}} := \int_0^\infty x'Qx + u'Ru + 2x'Nu \, dt = x'(0)Px(0).$$

3.2 The Algebraic Riccati Equation (ARE)

1. Statement and equivalent forms

We seek a symmetric matrix P satisfying

$$A^\top P + PA + Q - (PB + N)R^{-1}(B^\top P + N^\top) = 0,$$

with the **stabilizing** requirement that the closed-loop

$$A_c \;:=\; A - B\,R^{-1}(B^\top P + N^\top)$$

is a stability matrix (all eigenvalues in the open left half-plane).

Expanding the cross-term gives the equivalent quadratic form

$$(A - BR^{-1}N^\top)^\top P + P(A - BR^{-1}N^\top) + (Q - NR^{-1}N^\top) - PBR^{-1}B^\top P = 0.$$

Define the **Hamiltonian matrix**

$$\mathcal{H} = \begin{bmatrix} A - BR^{-1}N^\top & -BR^{-1}B^\top \\ -(Q - NR^{-1}N^\top) & -(A - BR^{-1}N^\top)^\top \end{bmatrix}.$$

Then the ARE is equivalent to the **graph identity**

$$\begin{bmatrix} P & -I \end{bmatrix} \mathcal{H} \begin{bmatrix} I \\ P \end{bmatrix} = 0.$$

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2. Domain of the Riccati operator

We say $\mathcal{H}$ lies in the domain of the Riccati operator if there exist matrices

$$M = \begin{bmatrix} I \\ P \end{bmatrix}, \qquad H_- \text{ (stability matrix)}$$

such that

$$\mathcal{H}\, M = M\, H_- .$$

This means the **graph** of P spans an invariant subspace of $\mathcal{H}$ associated with eigenvalues in the open left half-plane.

Consequences. If the relation above holds, then:

1. P solves the ARE.
2. The closed-loop $A_c = A - BR^{-1}(B^\top P + N^\top) = H_-$ is a stability matrix.
3. P is symmetric.

3. Stable invariant subspace of H

Let V_- denote the **stable subspace** of $\mathcal{H}$ (span of generalized eigenvectors with $\Re\lambda < 0$). Key facts:

- The spectrum of $\mathcal{H}$ is symmetric with respect to the imaginary axis: if λ is an eigenvalue, so is $-\bar{\lambda}$.
- If $\mathcal{H}$ has **no eigenvalues on** $j\mathbb{R}$, then it has exactly n eigenvalues in $\Re s < 0$ and n in $\Re s > 0$.
- If $\dim V_- = n$ and a basis of V_- is written as

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \qquad V_1 \in \mathbb{R}^{n\times n} \text{ nonsingular},$$

then the matrix

$$P := V_2 V_1^{-1}$$

is well-defined and the columns of $\begin{bmatrix} I \\ P \end{bmatrix} = V V_1^{-1}$ span V_-.

It follows that $\mathcal{H} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$ with H_- stable, hence P is a stabilizing ARE solution.

4. Conditions ensuring existence of the stabilizing solution

Assume:

- $Q - NR^{-1}N^\top \succeq 0$,
- (A, B) is **stabilizable**,
- $\left(A - BR^{-1}N^\top,\ Q - NR^{-1}N^\top\right)$ is **detectable**.

Then:

1. $\mathcal{H}$ has **no purely imaginary eigenvalues**.
2. Its stable subspace V_- has **dimension** n.
3. There exists a **unique stabilizing** solution $P = P^\top \succeq 0$ to the ARE.
4. The closed loop $A_c = A - BR^{-1}(B^\top P + N^\top)$ is Hurwitz.
5. Under observability of $\left(A - BR^{-1}N^\top,\ Q - NR^{-1}N^\top\right)$, one further gets $P \succ 0$.

Interpretation with performance output $z = Gx + Hu$:
If the cost is $\int (z^\top \bar{Q} z + \rho\, u^\top \bar{R} u)\, dt$ with $Q = G^\top \bar{Q} G,\ R = H^\top \bar{Q} H + \rho\, \bar{R},\ N = G^\top \bar{Q} H$, then when $N = 0$ the detectability condition reduces to detectability of (A, G). Intuitively, unstable modes invisible at z would keep the cost small while the state diverges, which is inadmissible.

5. Lyapunov characterization and positivity

For the stabilizing solution P, the closed-loop satisfies the Lyapunov identity

$$A_c^\top P + P A_c = -\left[(Q - N R^{-1} N^\top) + P B R^{-1} B^\top P\right].$$

The right-hand side is negative semidefinite (and negative definite under the observability condition above), which yields $P \succeq 0$ (and $P \succ 0$ in the strict case).

6. Numerical computation (structure-preserving)

Hamiltonian/Schur method. Compute the real Schur form of $\mathcal{H}$, separate the stable n-dimensional subspace, partition it as $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ with X_1 invertible, and set $P = X_2 X_1^{-1}$.

Doubling (SDA). A structure-preserving iteration on a symplectic pencil; globally convergent and robust for ill-conditioned cases.

Newton–Kleinman. Iterate Lyapunov solves starting from a stabilizing gain K_0; locally quadratically convergent.

Backward DRE integration. Integrate the finite-horizon DRE backward from a large terminal weight until convergence to steady state (simple but slower near the limit).

3. Stable invariant subspace of H

Let V_- denote the **stable subspace** of $\mathcal{H}$ (span of generalized eigenvectors with $\Re\lambda < 0$). Key facts:

- The spectrum of $\mathcal{H}$ is symmetric with respect to the imaginary axis: if λ is an eigenvalue, so is $-\bar{\lambda}$.
- If $\mathcal{H}$ has **no eigenvalues on** $j\mathbb{R}$, then it has exactly n eigenvalues in $\Re s < 0$ and n in $\Re s > 0$.
- If $\dim V_- = n$ and a basis of V_- is written as

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \qquad V_1 \in \mathbb{R}^{n\times n} \text{ nonsingular},$$

  then the matrix

$$P := V_2 V_1^{-1}$$

  is well-defined and the columns of $\begin{bmatrix} I \\ P \end{bmatrix} = V V_1^{-1}$ span V_-.

  It follows that $\mathcal{H} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$ with H_- stable, hence P is a stabilizing ARE solution.

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3.3 QP viewpoint (finite / infinite horizon)

1. From LQR to a condensed finite-horizon QP

$$x_{k+1} = Ax_k + Bu_k, \quad z_k = Gx_k + Hu_k, \qquad k = 0, \ldots, N-1$$

Cost:

$$J = \sum_{k=0}^{N-1} (x_k^\top Q x_k + 2x_k^\top N u_k + u_k^\top R u_k) + x_N^\top Q_f x_N,$$

with $Q = G^\top G,\ N = G^\top H,\ R = H^\top H + \rho I$ (this is the standard expansion of $\|z\|^2 + \rho\|u\|^2$ used throughout the frequency lecture, under the canonical case $N = G^\top H = 0$ when needed).

Stacked dynamics (block-Toeplitz reachability matrix):

$$X := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{\mathcal{A}} x_0 + \underbrace{\begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & \cdots & AB & B \end{bmatrix}}_{\mathcal{B}} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{U}.$$

Stacked weights

$$\bar{Q} = \mathrm{blkdiag}(Q, \ldots, Q, Q_f), \quad \bar{R} = \mathrm{blkdiag}(R, \ldots, R), \quad \bar{N} = \mathrm{blkdiag}(N, \ldots, N, 0).$$

Condense (eliminate X) ⇒ a strict convex QP in U:

$$\boxed{\min_U \ \tfrac{1}{2} U^\top H U + f^\top U \quad H = \mathcal{B}^\top \bar{Q} \mathcal{B} + (\mathcal{B}^\top \bar{N} + \bar{N}^\top \mathcal{B}) + \bar{R}, \quad f = (\mathcal{B}^\top \bar{Q} \mathcal{A} + \bar{N}^\top \mathcal{A}) x_0}$$

Since $R = H^\top H + \rho I \succ 0$ (for $\rho > 0$), one gets $H \succ 0 \Rightarrow$ a **strictly convex** QP.

2. Sparse (uncondensed) QP and KKT ⇒ Riccati

Instead of eliminating X, keep **both** X, U and impose dynamics as **equality constraints**:

$$\min_{X,U} \ \frac{1}{2}\begin{bmatrix} X \\ U \end{bmatrix}^\top \underbrace{\begin{bmatrix} \bar{Q} & \bar{N} \\ \bar{N}^\top & \bar{R} \end{bmatrix}}_{\text{block diagonal in time}} \begin{bmatrix} X \\ U \end{bmatrix} \quad \text{s.t.} \quad X - \mathcal{A}x_0 - \mathcal{B}U = 0.$$

Form the Lagrangian with multiplier Λ for the dynamics:

$$\mathcal{L} = \frac{1}{2}\begin{bmatrix} X \\ U \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{N} \\ \bar{N}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} + \Lambda^\top (X - \mathcal{A}x_0 - \mathcal{B}U).$$

KKT conditions give the **block bidiagonal** optimality system

$$\begin{aligned} \bar{Q}X + \bar{N}U + \Lambda &= 0, && (\text{stationarity w.r.t. } X) \\ \bar{N}^\top X + \bar{R}U - \mathcal{B}^\top \Lambda &= 0, && (\text{stationarity w.r.t. } U) \\ X - \mathcal{A}x_0 - \mathcal{B}U &= 0, && (\text{primal feasibility}) \end{aligned}$$

which, when **unstacked along time**, is exactly the **costate/state recursion** you see in PMP/KKT form in your Lecture-7 notes, and reduces algebraically to the familiar **(discrete) Riccati backward recursion** + $u_k = -K_k x_k$. (The notes present PMP/KKT as the special case of discrete optimal

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3. Infinite-horizon limit of the QP ⇒ steady-state feedback

Let $N \to \infty$ under stabilizability/detectability. The finite-horizon Riccati $P_k \to P$, the gains $K_k \to K$, and the QP policy converges to the **stationary** law

$$u = -Kx, \qquad K = R^{-1}B^\top P \quad (N = 0 \text{ case}),$$

where P solves the **ARE**

$$A^\top P + PA + G^\top G - PBR^{-1}B^\top P = 0, \quad R := H^\top H + \rho I.$$

(This is exactly the steady-state LQR result stated in the frequency lecture before developing the frequency properties.)

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4. Example

Plant: 4-mass–spring–damper chain (8 states, 1 input, 2 outputs)

States: $x = [q_1,\ q_2,\ q_3,\ q_4,\ \dot{q}_1,\ \dot{q}_2,\ \dot{q}_3,\ \dot{q}_4]^\top$

Input: force u applied to **mass 1**

Outputs (for visualization): positions of masses 1 and 4

5. Intuition

1. What the QP is doing:
Imagine planning all inputs at once over a short “preview” window so the sum of squared performance is smallest, while respecting how states evolve. That plan’s first input is applied, then you re-plan—this is MPC.

2. Why it’s convex and reliable:
The input penalty $R \succ 0$ makes the problem strictly convex—no local minima, stable numerics, clear sensitivity to weights.

3. Finite vs. infinite horizon:
Short horizon = look-ahead planner; long horizon → behavior settles to a time-invariant law (LQR). So LQR is the steady personality your short-horizon planner is converging to.

4. QP = natural engine for constraints:
Squared penalties fit physics and yield convexity; linear dynamics/constraints keep the math solvable fast with structure.

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Thanks !

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