ELSEVIER

Measurement 12 (1994) 337–344

Measurement

General solution for Tsebyshev approximation of form elements in coordinate measurement

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Abstract

The paper deals with evaluation of co-ordinate measuring data by means of best-fit algorithms according to Tsebyshev for determining adjacent profiles as defined in ISO 1101. The new solution is based on linearisation similar to the Gaussian approximation and solving the normalised linear programming problem by the Simplex method. It allows to implement algorithms for any two- and three-dimensional geometric elements in CMM software. As an important conclusion from experimental and theoretical research work it follows that the Tsebyshev approximation is to be linked with digital data filtering for averaging of random errors and also for elimination of outlyers in order to improve the accuracy. Furthermore international testing data and reference software are urgently needed for approving Tsebyshev algorithms as well as the belonging algorithms for digital data filtering.

Key words: Coordinate measurement; Best-fit-approximation; Tsebyshev approximation; Geometric elements

1. Coordinate measuring technique, ISO 1101 and tolerancing

The most important step in evaluation of measuring points taken by co-ordinate measuring machines (CMM) is the calculation of position, orientation and size of substitute geometric elements by means of best-fit routines. Until now only Gaussian best-fit routines are widely used for calculation of substitute geometric elements. There also exist approved test data and reference software for testing the accuracy by national metrological institutions [1,2]. As a higher level of best-fit routines the calculation of adjacent profiles as defined in ISO 1101 [3,4] is in the focus of research projects and new CMM software (mainly discussed as a Tsebyshev approximation of geometric elements).

However, if we look at the details of international tolerancing rules according to ISO 1101, 5459, 8015 [3,5,6] we find still some contradictions such as:

- The existing standards are based on conventional models and rules in design, manufacturing and quality control.
- Calculations of substitute geometric elements are limited to single geometric elements as the lowest level of geometric modelling of workpieces.

A few interesting and important new directions and theoretical results in tolerancing and engineering metrology are pointing at future ways such as

- new tolerancing models based on vectorial tolerancing [7], on mating geometry according to Taylor's principle [8], etc.;
- calculation of substitute compound geometric elements;
- calculation of constrained geometric elements [9];
- soft gauging according to Taylor's principle in accordance with assembly requirements [8]

In summary we may distinguish a variety of best-fit and testing algorithms for geometric quality control of workpieces as shown in Fig. 1.

SSDI 0263-2241(93)E0021-J

From this point of view CMM software available solves this wide range of workpiece testing problems only partly by means of simple Gaussian best-fit routines for geometric elements. In many cases wrong models are used and solutions give rise to wrong results and contradictions with conventional workpiece testing.

As an example of testing different types of geometric elements by means of more complicated models the drawing of Fig. 2 is assumed (only simple positional tolerances without mmc etc. are used) [9]. The datum system of the workpiece is defined by the primary datum plane A and the secondary datum cylinder B perpendicular to the primary plane. The positional tolerances of the four holes are threefold defined:

(a) positional tolerances in reference to an independent nominal bore pattern (without explicit datum element);
(b) positional tolerances in reference to the nominal bore pattern perpendicular to datum plane A;
(c) positional tolerances in reference to nominal bore pattern aligned to the datum elements A and B.

The plane A as the primary datum element is to be evaluated as a simple geometric element whereas the cylinder B ∅30 as secondary element must be evaluated perpendicular to the primary plane. Thus this cylinder is a constrained geometric element related to the plane.

The holes ∅10 can be evaluated as four single and independent goemetric elements (by means of conventional CMM software) as well as one compound geometric element by means of a best-fit method for a set of four parallel cylinders linked to each other by constraints. Each cylinder has its own location and diameter. In order to consider the adjacent profiles, best-fit routines based on the Tsebyshev approximation must be used [4,10–12]. Evaluation software available on CMM does not satisfy such models and requirements until now. Usually only single and simple (unconstrained) geometric elements according to the Gaussian principle are implemented in CMM software and sophisticated calculations are necessary for testing the given positional tolerances.

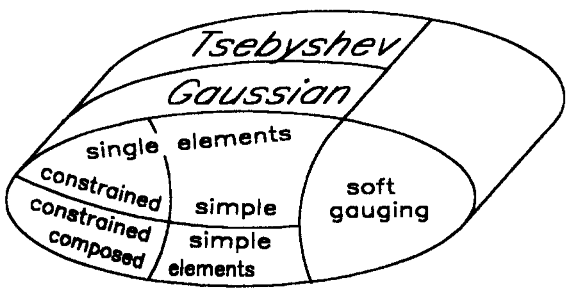

Fig. 1. Types of geometric features and inspection models in coordinate measurement.

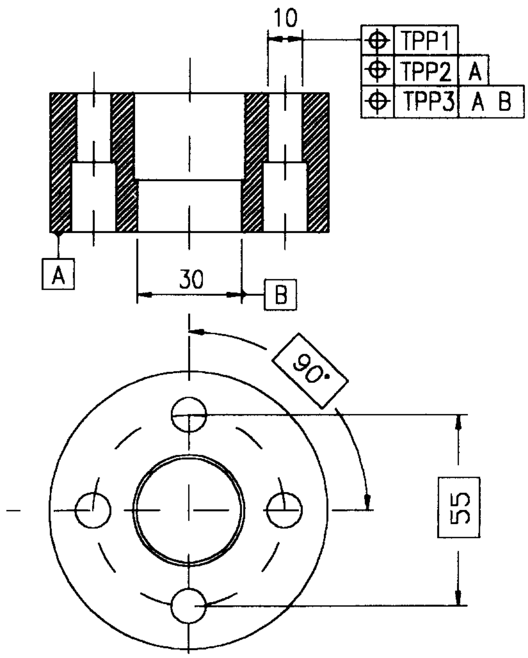

Fig. 2. Workpiece consisting of simple geometric element A, constrained simple element B and compound geometric element consisting of four holes.

The first step to overcome this stage and to satisfy ISO 1101 is to introduce Tsebyshev algorithms in CMM software. Thus adjacent profiles as reference elements for testing of form departure and in some cases as datum elements for testing positional departures must be included as

- equidistant geometric elements with minimised distance or minimum zone (used for determining departure of form);
- circumscribed elements with minimised size (such as the circumscribed circle of a shaft);

- inscribed elements with maximised size (such as the inscribed circle in a hole).

From the mathematical point of view these tasks are in general non-linear optimising problems. The first one is also called the best-fit approximation according to Tsebyshev. These routines should work with high accuracy and stability, as known from Gaussian routines, and they should be approved by international testing data and reference algorithms too. Several papers are dealing with this problem and even an EC working group is engaged.

2. Fundamentals of best-fit substitute geometric elements

For calculation of best-fit geometric elements the following axioms are valid for the equations of geometric elements [13]:

- invariance in respect to co-ordinate systems used for mathematical representation and measurement;
- invariance with respect to the size of the geometric element;
- length as unique dimension of defining equations;
- point distances are defined as Euclidean distances perpendicular to the substitute geometric element.

As the simplest mathematical expression for defining a geometrical element implicit equations can be used [14]:

$$F(x, y, z; a_1, a_2, \ldots, a_m) = F(\boldsymbol{x}; \boldsymbol{a}) = 0, \tag{1}$$

with co-ordinate vector $\boldsymbol{x} = (x\ y\ z)^{\mathrm{T}}$ and parameter vector $\boldsymbol{a} = (a_1\ a_2 \ldots a_m)^{\mathrm{T}}$.

If this equation fulfils the condition

$$|\mathrm{grad}\, F| = 1, \tag{2}$$

the distance f_i of a measuring point $P_i(x_i, y_i, z_i)$ follows in first order as

$$f_i = F(\boldsymbol{x}_i; \boldsymbol{a}). \tag{3}$$

There are other types of mathematical equations, such as parametric equations used for complicated geometric elements (gear flanks etc.), but the implicit equation is very suitable for demonstration of the general solution.

The Gaussian approximation of a geometric element measured by a sample of n measuring points is based on the well-known objective function for the squares of the point distances f_i with parameters $\boldsymbol{a} = (a_1\ a_2 \ldots a_m)^{\mathrm{T}}$ to be optimised [12–16].

$$Q(\boldsymbol{a}) = \sum_{i=1}^{n} f_i^2. \tag{4}$$

From this equation follows also the expression for the variance σ^2 of the measuring points in reference to the best-fit geometric element

$$\sigma^2 = Q/(n - m). \tag{5}$$

Furthermore the covariance matrix $\boldsymbol{S}_a$ of the optimised parameter $\boldsymbol{a}$ can be calculated for determining the confidence range of the substitute geometric element [14].

As an important property of the Gaussian approximation method the averaging effect of the belonging algorithms must be mentioned. It was the wonderful idea of Gauss to minimise the influence of the random errors during any evaluation of measuring data by means of averaging.

In co-ordinate measurement every measuring point in space contains random errors which can be assumed as element of a spatial normal error distribution. Thus any measuring point error influences the result only by the factor $1/n$. The error distribution of the results will then be reduced by the factor $\sqrt{1/(n-m)}$ in the case of independent parameters and measuring points.

The Gaussian approximation always implies a strong filtering process against the individual random errors. This is finally the background for the great success and reasonable accuracy of CMM combined with software based on Gaussian principles. In contradiction to this property of Gaussian best-fit algorithms the calculation of adjacent profiles according to Tsebyshev is very sensitive for random errors and even for outlyers as shown in Fig. 3.

If the real profile is measured by a sample of points with random errors as shown by error ellipses, the adjacent profiles and even so the form

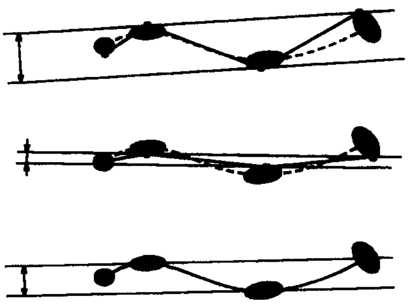

Fig. 3. Influence of random errors on adjacent profiles measured by uncertain points.

departure directly depend on the random errors of the points which define the adjacent profiles. The remaining points have no influence on the final result. Thus the uncertainty of the Tsebyshev best-fit algorithm will always be greater if no other averaging process is used. The same effect takes place in case of outlyers.

3. Best-fit evaluation according to Tsebyshev

The Tsebyshev approximation of geometric elements is to be used for determining two equidistant adjacent profiles with minimised distance (minimum zone). It is based on the fundamental objective function [11,12,17,18]

$$Q(\boldsymbol{a}) = \mathrm{Max}(|f_i|) \overset{!}{=} \mathrm{Min}. \tag{6}$$

It is also called a minimax-condition because it minimises the largest distance beween measuring points on both sides of the geometric element. The final substitute element describes the adjacent profiles as equidistant profiles with departure $\pm Q/2$.

From Eq. (1) and by means of Eq. (6) an optimisation problem results for the parameters $a_1, \ldots, a_m$. In most cases Eq. (1) is non-linear with respect to the parameters a_i. If we assume $\Gamma = \mathrm{Max}(|f_i|)$, the objective function Q and $2n$ constraints become

$$Q(\boldsymbol{a}) = \Gamma, \tag{7}$$

$$F(\boldsymbol{x}_i; \boldsymbol{a}) = f_i \leqslant \Gamma \quad \text{if } f_i > 0, \tag{8}$$

$$-F(\boldsymbol{x}_i; \boldsymbol{a}) = -f_i \leqslant \Gamma \quad \text{if } f_i < 0. \tag{9}$$

As a good approach the parameter $\boldsymbol{a}$ from the Gaussian approximation of the measuring points can be used as starting solution for solving the non-linear optimisation problem.

A very simple solution even in the case of non-linear equations is the Monte-Carlo method, but it takes time and there is no way to stop the procedure depending on a certain accuracy. An other method is based on a succession of weighted Gaussian approximation [16]. A further method used for approximation of geometric elements is the so called L_p approximation [27] according to the objective function

$$Q(\boldsymbol{a}) = \left(\sum_{i=1}^{n} f_i^p \right)^{1/p} \overset{!}{=} \mathrm{Min}. \tag{10}$$

This looks very similar to the Gaussian approximation but numerical problems arise in the case of higher-order p. Even so the solution by successive weighted Gaussian methods [14,20] is not reliable. On the other hand for application in CMM software algorithms and programming modules are required which work fast, like the Gaussian algorithms, and which can be implemented on a PC as simple as Guassian software. We have found that very reliable solutions can be found by means of linearisation and the Simplex method for solving the non-linear optimisation problem.

As mentioned before the powerful Gaussian best-fit algorithms were only found by means of linearisation of the non-linear equations for representation of geometric elements. We can use the same way to transfer the non-linear minimax function and even the non-linear constraints in a linearized form. The validity of this linearisation is guaranteed in case of very small distances f_i of the measuring points. This is the case for every real measured workpiece because a real geometric feature differs only slightly from the ideal geometric element.

If we develop the equation $F(\boldsymbol{x}; \boldsymbol{a})$ as a power series of the unknown parameter corrections $\Delta\boldsymbol{a} = (\Delta a_1 \ldots \Delta a_m)^{\mathrm{T}}$, it follows with $\boldsymbol{a} = \mathring{\boldsymbol{a}} + \Delta\boldsymbol{a}$

$$F(\boldsymbol{x}; \boldsymbol{a}) = F(\boldsymbol{x}; \mathring{\boldsymbol{a}}) + \frac{\partial F}{\partial \boldsymbol{a}} \Delta\boldsymbol{a} = F(\boldsymbol{x}; \mathring{\boldsymbol{a}}) + \boldsymbol{J}\Delta\boldsymbol{a}, \tag{11}$$

with the Jacobian matrix

$$J=\left(\frac{\partial F}{\partial a_1}\ \frac{\partial F}{\partial a_2}\ \cdots\ \frac{\partial F}{\partial a_m}\right). \qquad (12)$$

By means of these linear expressions and the approach $f_i = F(\boldsymbol{x}_i; \mathring{\boldsymbol{a}})$ a linear programming problem follows for maximising $\Delta\Gamma$:

$$\Delta\Gamma = \text{Max}, \qquad (13)$$

$$f_i + \boldsymbol{J}_i\Delta\boldsymbol{a} \leqslant \Gamma \qquad \text{if } f_i > 0, \qquad (14)$$

$$-f_i - \boldsymbol{J}_i\Delta\boldsymbol{a} \leqslant \Gamma \quad \text{if } f_i < 0. \qquad (15)$$

Because of the linearisation, the solution $\Delta\boldsymbol{a}$ describes only the parameter correction in the first order. The corresponding linear equation system as the normalised linear equation system can be written for this case as a matrix equation

$$\begin{pmatrix} \boldsymbol{A} & -\boldsymbol{e} \\ -\boldsymbol{A} & -\boldsymbol{e} \\ \boldsymbol{0} & -1 \end{pmatrix}\begin{pmatrix} \Delta\boldsymbol{a} \\ \Delta\Gamma \end{pmatrix} = \begin{pmatrix} \Gamma\boldsymbol{e}-\boldsymbol{f} \\ \Gamma\boldsymbol{e}+\boldsymbol{f} \\ \Gamma \end{pmatrix}, \qquad (16)$$

with the following expressions for the variables

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{J}(\boldsymbol{x}_1; \mathring{\boldsymbol{a}}) \\ \boldsymbol{J}(\boldsymbol{x}_2; \mathring{\boldsymbol{a}}) \\ \vdots \\ \boldsymbol{J}(\boldsymbol{x}_n; \mathring{\boldsymbol{a}}) \end{pmatrix}, \qquad (17)$$

$\boldsymbol{e} = (1\ 1\ 1 \ldots 1)^{\mathrm{T}}$ is a unit vector of n elements, $\boldsymbol{0} = (0\ 0\ 0 \ldots 0)^{\mathrm{T}}$ is a zero vector of m elements, $\boldsymbol{f} = (f_1 f_2 f_3 \ldots f_n)^{\mathrm{T}}$ is a vector of the point distances, and $\Delta\boldsymbol{a} = (\Delta a_1\ \Delta a_2 \ldots \Delta a_m)^{\mathrm{T}}$ is a vector of parameter corrections.

This equation system is to be solved by the Simplex method for the corrections $\Delta\boldsymbol{a}$ and $\Delta\boldsymbol{\Gamma}$ and from these the corrected parameters $\boldsymbol{a}$ and the distance $\boldsymbol{\Gamma}$ according to

$$\boldsymbol{a} = \mathring{\boldsymbol{a}} + \Delta\boldsymbol{a}, \qquad (18)$$

$$\boldsymbol{\Gamma} = \mathring{\boldsymbol{\Gamma}} + \Delta\boldsymbol{\Gamma}. \qquad (19)$$

The parameter correction must be carried out iteratively until an accuracy limit is reached according to

$$\varepsilon = \sum_{j=1}^{m} p_j|\Delta a_j| \leqslant \epsilon \qquad (20)$$

with p_j as weighting factor and ϵ as an error limit (usually about 0.1 μm).

The solution described above can be used for any geometric element and corresponding algorithms and even procedures are available for a large number of two- and three-dimensional geometric elements. As can be seen from Eqs. (16) and (17) there exist some similarities to the Gaussian approximation by means of the Householder algorithms for the least-squares method. Thus there is a possibility to link both solutions to a more general one for best-fit evaluation according to Gauss and Tsebyshev at once.

In order to get an impression of the Tsebyshev approximation by means of linear programming in respect to one parameter a, a two-dimensional graph as shown in Fig. 4 can be sketched.

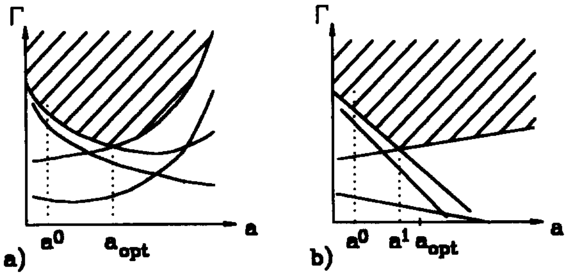

Fig. 4. Non-linear and linearised constraints.

Four constraints according to Eqs. (8) and (9) are drawn depending on the parameter a. The four lines are limiting the marked solution field. The lowest point of this field is the searched point of the objective function Γ with parameter a_{opt}. Linearisation according to eqs. (14) and (15) in $a = \mathring{a}$ leads to four straight lines instead of curves. These lines describe an approaching solution field with lowest intersection point a^1 as solution. In the case of a sufficient starting solution a the remaining difference $a^1 - a_{\text{opt}}$ becomes smaller from step to step and the solution is reached as the intersection point of the actual curves similar to Newton's method for solution of non-linear equation systems. A comparison with the Gaussian objective function is explained in Ref. [14].

4. Random error suppression

In order to compare the properties and accuracies of the Gaussian and Tsebyshev approximation of geometric elements, a large number of measured and simulated circles have been evaluated. In distinction to known results the random errors of the measuring points have been taken into accout. Thus the measured as well as the simulated profiles are always taken as a super-position of a systematic (real) form deviation and random measuring errors. There are many methods for eliminating the random errors as for example by means of Fourier transformation, digital filters or moving average. In order to desensitize Tsebyshev routines in respect to random errors, two different digital filters are discussed and tested:

- outlyer detection and elimination by means of well-known method;
- digital low-pass filter like in surface and roundness measurement.

As an example Fig. 5 shows the principle of random error elimination by a very simple filtering process (mean value of every four points). This is unusual until now and some words about digital filters in co-ordinate measurement seem to be in place.

Digital filters are widely used in signal processing based on sampled data. One of the prerequisites is the equidistant sampling of the signal or consequently of the measured profile. Thus only controlled CMM with scanning equipment can be used for picking up a large number of measuring points.

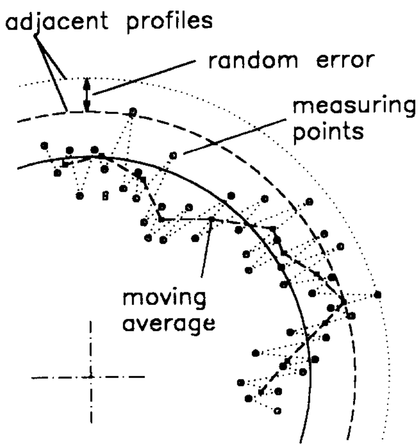

Fig. 5. Principle of low-pass filtering of measuring points; the average of four points is used for demonstration.

Furthermore only the distances f_i of the measuring points from the best-fit profile can be processed. Consequently the following procedure has been carried out step by step:

- measurement of workpieces by a large number of measuring points;
- calculation of the Gaussian best-fit profile and of the of distances f_i;
- smoothing the values f_i by means of a digital low-pass filter according to the following equation [21]:

$$f_i^* = \frac{1}{C} \sum_{k=i-r}^{i+r} p_k f_k,$$

$$C = \sum_{k=i-r}^{i+r} p_k; \qquad (21)$$

- recalculate the processed measuring points by means of corresponding normal direction $\boldsymbol{n}_i$

$$\boldsymbol{x}_i^* = \boldsymbol{x}_i - (f_i - f_i^*)\boldsymbol{n}_i;$$

- calculation of adjacent profiles by means of Tsebyshev algorithms.

5. Results and conclusions

As final results of application experimental examinations of cylindrical workpieces have been carried out by means of Gaussian low-pass filters according to Ref. [21] with a cut-off $\lambda = 0.8$; 2.5; 8; 12 mm (the last value according to 15 waves per revolution as used in roundness measurement). Some results are shown in Table 1.

As final results of application and testing of the Tsebyshev approximation of geometric elements and digital low-pass filters were found:

- The adjacent profiles according to ISO 1101 based on the minimum-zone principle (called the Tsebyshev approximation) can be calculated by means of optimisation algorithms.
- The non-linear optimisation problem can be solved iteratively by means of linearisation of objective function and constraints and solving

Table 1
Decreasing of form departure uncertainty by means of digital low-pass filter (mean values of 50 tests)

Test	Cut-off (mm)	Form departure (μm)	Uncertainty of form departure (%)	
		$N=250$	$N=250$	N = 630
1	–	13.4	5.7	5.3
2	0.8	13.1	5.9	3.3
3	2.5	12.1	4.6	2.6
4	8	10.9	3.7	2.1
5	12	10.3	3.2	1.9

the linear programming problem by means of the Simplex method. Solutions and algorithms are easy and fast and they are available for all important geometric elements in co-ordinate measurement.

- In order to eliminate or to lower the influence of random errors as well as of outlyers, a special averaging pre-processing of measuring data must be introduced. Whereas the best-fit approximation according to Tsebyshev is only a mathematical problem, the filtering process must be optimised experimentally.
- For testing and approving Tsebyshev algorithms test data and reference algorithms must be agreed upon internationally.
- Averaging or filtering algorithms can be used for lowering random errors of measuring points and adjacent profiles. Such algorithms should be developed in accordance with form and roughness measurement as standardised and they must be standardised too.

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