Mathematical Foundations of Computer Science

Project 4

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March 15, 2021

Warmups

1 What does the notation

$$\sum_{i=4}^{0} q_i$$

mean?

Solution. In a programmer's perspective, this notation could be interpreted as the decreasing order on the index:

$$\sum_{i=4}^{0} q_i = q_4 + q_3 + q_2 + q_1 + q_0 = \sum_{i=0}^{4} q_k$$

But in mathematics, the formula also holds:

$$\sum_{i=4}^{0} q_i = \sum_{4 \le i \le 0} q_i = \sum_{\varnothing} q_i = 0$$

However, if $\sum_{k \le n} q_k$ and $\sum_{k<m} q_k$ are finite sums, the following interpretation also holds if the summation is regarded generally:

$$\sum_{i=4}^{0} q_i = \sum_{i \le 0} q_i - \sum_{i<4} q_i = -q_1 - q_2 - q_3$$

The interpretation is different when the definition differs, one should always use an increasing order on index to avoid misinterpretations. □

2 Simplify the expression $x \cdot ([x > 0] - [x < 0])$.

Solution. According to Iverson's convention,

$$x \cdot ([x > 0] - [x < 0]) = \begin{cases} x \cdot 1, & x > 0, \\ 0 \cdot 0, & x = 0, \\ x \cdot (-1), & x < 0 \end{cases} = |x|$$

□

3 Demonstrate your understanding of $\sum$-notation by writing out the sums

$$\sum_{0\le k\le 5} a_k \text{ and } \sum_{0\le k^2\le 5} a_{k^2}$$

in full. (Watch out the second sum is a bit tricky.)

Solution.

$$\sum_{0\le k\le 5} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$$

The second notation here impiles:

$$0 \le k^2 \le 5 \Rightarrow k = \{0, 1, 2, -1, -2\}$$

So, the summation follows:

$$\sum_{0\le k^2\le 5} a_{k^2} = a_0 + 2a_1 + 2a_2$$

□

4 Express the triple sum

$$\sum_{1\le i<j<k\le 4} a_{ijk}$$

as a three-fold summation (with three $\sum$'s),

a summing first on k, then j, then i;

b summing first on i, then j, then k.

Also write your triple sums out in full without the $\sum$-notation, using parentheses to show what is being added together first.

Solution.

$$\begin{aligned}
\sum_{1\le i<j<k\le 4} a_{ijk} &= \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} a_{ijk} \\
&= \sum_{i=1}^{4}\sum_{j=1}^{4}(a_{ij,1} + a_{ij,2} + a_{ij,3} + a_{ij,4}) \\
&= \sum_{i=1}^{4}[(a_{i,1,1} + a_{i,1,2} + a_{i,1,3} + a_{i,1,4}) + (a_{i,2,1} + a_{i,2,2} + a_{i,2,3} + a_{i,2,4}) \\
&+ (a_{i,3,1} + a_{i,3,2} + a_{i,3,3} + a_{i,3,4}) + (a_{i,4,1} + a_{i,4,2} + a_{i,4,3} + a_{i,4,4})] \\
&= [(a_{1,1,1} + a_{1,1,2} + a_{1,1,3} + a_{1,1,4}) + (a_{1,2,1} + a_{1,2,2} + a_{1,2,3} + a_{1,2,4}) \\
&+ (a_{1,3,1} + a_{1,3,2} + a_{1,3,3} + a_{1,3,4}) + (a_{1,4,1} + a_{1,4,2} + a_{1,4,3} + a_{1,4,4})] \\
&+ [(a_{2,1,1} + a_{2,1,2} + a_{2,1,3} + a_{2,1,4}) + (a_{2,2,1} + a_{2,2,2} + a_{2,2,3} + a_{2,2,4}) \\
&+ (a_{2,3,1} + a_{2,3,2} + a_{2,3,3} + a_{2,3,4}) + (a_{2,4,1} + a_{2,4,2} + a_{2,4,3} + a_{2,4,4})] \\
&+ [(a_{3,1,1} + a_{3,1,2} + a_{3,1,3} + a_{3,1,4}) + (a_{3,2,1} + a_{3,2,2} + a_{3,2,3} + a_{3,2,4}) \\
&+ (a_{3,3,1} + a_{3,3,2} + a_{3,3,3} + a_{3,3,4}) + (a_{3,4,1} + a_{3,4,2} + a_{3,4,3} + a_{3,4,4})] \\
&+ [(a_{4,1,1} + a_{4,1,2} + a_{4,1,3} + a_{4,1,4}) + (a_{4,2,1} + a_{4,2,2} + a_{4,2,3} + a_{4,2,4}) \\
&+ (a_{4,3,1} + a_{4,3,2} + a_{4,3,3} + a_{4,3,4}) + (a_{4,4,1} + a_{4,4,2} + a_{4,4,3} + a_{4,4,4})]
\end{aligned}$$

$$
\begin{aligned}
\sum_{1\le i<j<k\le 4} a_{ijk} &= \sum_{k=1}^{4}\sum_{j=1}^{4}\sum_{i=1}^{4} a_{ijk} \\
&= \sum_{k=1}^{4}\sum_{j=1}^{4}(a_{1,jk} + a_{2,jk} + a_{3,jk} + a_{4,jk}) \\
&= \sum_{k=1}^{4}[(a_{1,1,k} + a_{2,1,k} + a_{3,1,k} + a_{4,1,k}) + (a_{1,2,k} + a_{2,2,k} + a_{3,2,k} + a_{4,2,k}) \\
&+ (a_{1,3,k} + a_{2,3,k} + a_{3,3,k} + a_{4,3,k}) + (a_{1,4,k} + a_{2,4,k} + a_{3,4,k} + a_{4,4,k})] \\
&= [(a_{1,1,1} + a_{2,1,1} + a_{3,1,1} + a_{4,1,1}) + (a_{1,2,1} + a_{2,2,1} + a_{3,2,1} + a_{4,2,1}) \\
&+ (a_{1,3,1} + a_{2,3,1} + a_{3,3,1} + a_{4,3,1}) + (a_{1,4,1} + a_{2,4,1} + a_{3,4,1} + a_{4,4,1})] \\
&+ [(a_{1,1,2} + a_{2,1,2} + a_{3,1,2} + a_{4,1,2}) + (a_{1,2,2} + a_{2,2,2} + a_{3,2,2} + a_{4,2,2}) \\
&+ (a_{1,3,2} + a_{2,3,2} + a_{3,3,2} + a_{4,3,2}) + (a_{1,4,2} + a_{2,4,2} + a_{3,4,2} + a_{4,4,2})] \\
&+ [(a_{1,1,3} + a_{2,1,3} + a_{3,1,3} + a_{4,1,3}) + (a_{1,2,3} + a_{2,2,3} + a_{3,2,3} + a_{4,2,3}) \\
&+ (a_{1,3,3} + a_{2,3,3} + a_{3,3,3} + a_{4,3,3}) + (a_{1,4,3} + a_{2,4,3} + a_{3,4,3} + a_{4,4,3})] \\
&+ [(a_{1,1,4} + a_{2,1,4} + a_{3,1,4} + a_{4,1,4}) + (a_{1,2,4} + a_{2,2,4} + a_{3,2,4} + a_{4,2,4}) \\
&+ (a_{1,3,4} + a_{2,3,4} + a_{3,3,4} + a_{4,3,4}) + (a_{1,4,4} + a_{2,4,4} + a_{3,4,4} + a_{4,4,4})]
\end{aligned}
$$

$\square$

5 What's wrong with the following derivation?

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k} = \sum_{k=1}^{n} n = n^2$$

Solution.

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} \neq \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k}$$

Because

$$
\begin{aligned}
\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} &= \sum_{j=1}^{n}\left(\frac{a_j}{a_1} + \cdots + \frac{a_j}{a_n}\right) \\
&= \left(\frac{a_1}{a_1} + \cdots + \frac{a_1}{a_n}\right) + \cdots + \left(\frac{a_n}{a_1} + \cdots + \frac{a_n}{a_n}\right)
\end{aligned}
$$

is not equal to

$$\sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k} = \sum_{k=1}^{n} n = n^2$$

$\square$