Project 13

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Warmups

1 An eccentric collector of $2 \times n$ domino tilings pays \$4 for each vertical domino and \$1 for each horizontal domino. How many tilings are worth exactly \$$m$ by this criterion? For example, when $m = 6$ there are three solutions: ⊟, ⊟, and ▭.

Solution. Like what has been done in (7.5), (7.4) could be transferred into

$$T = | \text{-} \square \text{-} \boxminus = \frac{1}{1 - z^4 - z^2}$$

whose sequence is

$$\frac{1}{1 - z^4 - z^2} = \frac{1}{1 - z^2 - (z^2)^2} \leftrightarrow \langle 0, F_2, 0, F_3, 0, F_4, \cdots \rangle$$

Thus, the number of solutions

$$N = \begin{cases} 0, & m \text{ is odd}, \\ F_{\frac{m}{2}+1}, & m \text{ is even} \end{cases}$$

where F is the Fibonacci number. □

2 Give the generating function and the exponential generating function for the sequence $\langle 2, 5, 13, 35, \cdots \rangle = \langle 2^n + 3^n \rangle$ in closed form.

Solution. The generating function for $\langle c^n \rangle$ is $\frac{1}{1-cz}$, according to linearity,

$$\langle 2^n + 3^n \rangle \leftrightarrow \frac{1}{1 - 2z} + \frac{1}{1 - 3z}$$

is its generating function.

The exponential generating function for $\langle c^n \rangle$ is

$$\hat{G}(z) = \sum_{n \geq 0} c^n \frac{z^n}{n!} = \sum_{n \geq 0} \frac{(cz)^n}{n!} = e^{cz}$$

Thus, the exponential generating function is

$$\langle 2^n + 3^n \rangle \leftrightarrow e^{2z} + e^{3z}$$

□

3   What is $\sum_{n\geq 0} H_n/10^n$?

Solution. By (7.57),

$$\langle H_n \rangle \leftrightarrow \frac{1}{1-z} \ln \frac{1}{1-z}$$

The convergence radius R of $\langle H_n \rangle$ could be representated as

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \leq \sum_{k=1}^{n} 1 = n \Rightarrow R = \frac{1}{\limsup_{n\geq 0} \sqrt[n]{H_n}} = \frac{1}{1} = 1$$

(In fact, by recurrence we could get $1 = H_1 = H_0 + 1 \Rightarrow H_0 = 0$, which is satisfiable.)
Since $\frac{1}{10} < 1$ is within the radius,

$$\sum_{n\geq 0} \frac{H_n}{10^n} = \frac{1}{1-\frac{1}{10}} \ln \frac{1}{1-\frac{1}{10}} = \frac{10}{9} \ln \frac{10}{9}$$

$\square$

4   The general expansion theorem for rational functions $P(z)/Q(z)$ is not completely general, because it restricts the degree of P to be less than the degree of Q. What happens if P has a larger degree than this?

Solution. The problem could be reduced to the scenario where $\deg P < \deg Q$, by applying polynomial division:

$$\frac{P(z)}{Q(z)} = S(z) + \frac{R(z)}{Q(z)}$$

where $\deg R < \deg Q$. Since $S(z)$ only influcence finite amount of items (based on its degree), Rational Expansion Theorem can be applied on the second term. $\square$

Basics

6   Show that the recurrence (7.32) can be solved by the repertoire method, without using generating functions.

Solution.

$$g_0 = g_1 = 1;$$
$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n, \qquad\qquad \text{for } n \geq 2$$

Consider a general form of

$$g_0 = \alpha;$$
$$g_1 = \beta;$$
$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n \gamma, \qquad\qquad \text{for } n \geq 2$$

has the closed form of

$$g_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

Case 1: $g_n = 2^n$

$$\alpha = 1$$
$$\beta = 2$$
$$2^n = 2^{n-1} + 2 \times 2^{n-2} + (-1)^n \gamma \Rightarrow \gamma = 0$$
$$g_n = A(n) + 2B(n) = 2^n \tag{1}$$

Case 2: $g_n = (-1)^n$

$$\alpha = 1$$
$$\beta = -1$$
$$(-1)^n = (-1)^{n-1} + 2(-1)^{n-2} + (-1)^n \gamma \Rightarrow \gamma = 0$$
$$g_n = A(n) - B(n) = (-1)^n \tag{2}$$

Case 3: $g_n = n(-1)^n$

$$\alpha = 0$$
$$\beta = -1$$
$$n(-1)^n = (n-1)(-1)^{n-1} + 2(n-2)(-1)^{n-2} + (-1)^n \gamma$$
$$\Rightarrow n = -n + 1 + 2(n-2) + \gamma$$
$$\gamma = 3$$
$$g_n = -B(n) + 3C(n) = n(-1)^n \tag{3}$$

Combining Eq. (1) – (3), the solution is

$$A(n) = \frac{2^n + 2(-1)^n}{3}$$
$$B(n) = \frac{2^n - (-1)^n}{3}$$
$$C(n) = \frac{n(-1)^n}{3} + \frac{2^n - (-1)^n}{9}$$

So plug in $\alpha = 1, \beta = 1, \gamma = 1$, the closed form is found:

$$g_n = \frac{7}{9}2^n + \left(\frac{1}{3}n + \frac{2}{9}\right)(-1)^n$$

$\square$

7   Solve the recurrence

$$g_0 = 1$$
$$g_n = g_{n-1} + 2g_{n-2} + \cdots + ng_0, \quad \text{for } n > 0$$

Solution. The recurrence can be representated by the single equation

$$g_n = \sum_{i=0}^{n-1}(n-i)g_i + [n = 0]$$

Write down $G(z) = \sum_n g_n z^n$,

$$G(z) = \sum_n g_n z^n = \sum_n \sum_{i=0}^{n-1} (n-i) g_i z^n + \sum_n [n=0] z^n$$

$$= 1 + \sum_{i=1}^{n} i z^i G(z)$$

$$= 1 + \frac{z}{(1-z)^2} G(z)$$

$$G(z) = \frac{1 - 2z + z^2}{1 - 3z + z^2}$$

$$= 1 + \frac{z}{1 - 3z + z^2}$$

$$= 1 + \frac{z}{(1 - \beta_1 z)(1 - \beta_2 z)}$$

Theorem gives that

$$g_n = a_1 \beta_1^n + a_2 \beta_2^n$$

where

$$a_1 = \frac{-\beta_1 \frac{1}{\beta_1}}{1 - (\beta_1 + \beta_2) + 2\beta_1 \beta_2 \frac{1}{\beta_1}} = -\frac{1}{2\beta_2 - 2}$$

and

$$a_2 = \frac{-\beta_2 \frac{1}{\beta_2}}{1 - (\beta_1 + \beta_2) + 2\beta_1 \beta_2 \frac{1}{\beta_2}} = -\frac{1}{2\beta_1 - 2}$$

Thus,

$$g_n = -\frac{1}{2}\left(\frac{\beta_1^n}{\beta_2 - 1} + \frac{\beta_2^n}{\beta_1 - 1} \right)$$

$$= \frac{1}{2}\left(\beta_1^n (\beta_1 - 1) + \beta_2^n (\beta_2 - 1) \right)$$

where $\beta_1 = \frac{3 + \sqrt{5}}{2}, \beta_2 = \frac{3 - \sqrt{5}}{2}$. $\qquad\square$