

Project 6

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Warmups

7 Let $\nabla f(x) = f(x) - f(x-1)$. What is $\nabla(x^{\overline{m}})$?

Solution.

$$\begin{aligned}\nabla(x^{\overline{m}}) &= \nabla(x(x+1) \cdots (x+m-1)) \\ &= x(x+1) \cdots (x+m-1) - (x-1)x \cdots (x+m-2) \\ &= x(x+1) \cdots (x+m-2) [(x+m-1) - (x-1)] \\ &= mx^{\overline{m-1}}\end{aligned}$$

□

8 What is the value of $0^{\overline{m}}$, when m is a given integer?

Solution. Case 1: $m > 0$.

$$0^{\overline{m}} = 0 \cdot (-1)^{\overline{m-1}} = 0$$

Case 2: $m = 0$.

$$0^{\overline{0}} = 1$$

which is defined as the empty product.

Case 3: $m < 0$.

$$\begin{aligned}1 = 0^{\overline{0}} &= 0^{\overline{m}}(-m)^{\overline{-m}} = 0^{\overline{m}}|m|! \\ 0^{\overline{m}} &= \frac{1}{|m|!}\end{aligned}$$

□

9 What is the law of exponents for rising factorial powers, analogous to (2.52)? Use this to define $x^{\overline{-n}}$.

Solution.

$$x^{\overline{m+n}} = x(x+1) \cdots (x+m-1) \cdots (x+m) \cdots (x+m+n-1) = x^{\overline{m}}(x+m)^{\overline{n}}$$

$$\begin{aligned} 1 &= x^{\overline{0}} = x^{\overline{(-n)+n}} = x^{\overline{-n}}(x-n)^{\overline{n}} \\ x^{\overline{-n}} &= \frac{1}{(x-n)^{\overline{n}}} = \frac{1}{(x-n)(x-n+1) \cdots (x-1)} \end{aligned}$$

□

10 The text derives the following formula for the difference of a product:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

How can this formula be correct, when the left-hand side is symmetric with respect to u and v but the right-hand side is not?

Solution. Because the following formula could also be correct with tht respect of u :

$$\begin{aligned} \Delta(uv) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x+1)v(x) + u(x+1)v(x) - u(x)v(x) \\ &= u(x+1)\Delta v + v\Delta u \\ &= Eu\Delta v + v\Delta u \end{aligned}$$

It is an inner layer of symmetric.

$$Eu\Delta v + v\Delta u = u\Delta v + Ev\Delta u$$

□

Basics

14 Evaluate $\sum_{k=1}^n 2^k$ by rewriting it as the multiple sum $\sum_{1 \leq j \leq k \leq n} 2^k$.

$$\begin{aligned} \sum_{1 \leq j \leq k \leq n} 2^k &= \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} 2^k \\ &= \sum_{1 \leq j \leq n} [(2^1 + 2^2 + \cdots + 2^n) - (2^1 + 2^2 + \cdots + 2^{j-1})] \\ &= \sum_{1 \leq j \leq n} [(2^{n+1} - 2) - (2^j - 2)] \\ &= n2^{n+1} - (2^{n+1} - 2) \end{aligned}$$