Mathematical Foundations of Computer Science

Project 12

Log Creative

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Warmups

1 What is 11^4? Why is this number easy to compute, for a person who knows binomial coefficients?

Solution.

$$11^4 = (10+1)^4 = \sum_{k=0}^{4} \binom{4}{k} 10^k \cdot 1^{4-k} = 14641$$

$\square$

2 For which value(s) of k is $\binom{n}{k}$ a maximum, when n is a given positive integer? Prove your answer.

Proof. $k = \left\lfloor \frac{n}{2} \right\rfloor$ or $\left\lceil \frac{n}{2} \right\rceil$.

k satisfies:

$$\begin{cases} \binom{n}{k-1} \leq \binom{n}{k} \\ \binom{n}{k+1} \leq \binom{n}{k} \end{cases} \Rightarrow \begin{cases} \dfrac{n!}{(k-1)!(n-k+1)!} \leq \dfrac{n!}{k!(n-k)!} \\ \dfrac{n!}{(k+1)!(n-k-1)!} \leq \dfrac{n!}{k!(n-k)!} \end{cases} \Rightarrow \begin{cases} k \leq n-k+1 \\ n-k \leq k+1 \end{cases}$$

In other word,

$$\frac{n-1}{2} \leq k \leq \frac{n+1}{2}$$

If $n = 2l (l \in \mathbb{N})$, then $\left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil = \frac{n}{2} \in \left[\frac{n-1}{2}, \frac{n+1}{2}\right]$; if $n = 2l+1 (l \in \mathbb{N})$, then $\frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$ and $\frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$. $\square$

3 Prove the hexagon property,

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}$$

Proof.

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}$$
$$\Leftarrow \quad \frac{(n-1)!n!(n+1)!}{(k-1)!(n-k)!(k+1)!(n-k-1)!k!(n+1-k)!}$$
$$= \frac{(n-1)!n!(n+1)!}{k!(n-1-k)!(k+1)!(n-k)!(k-1)!(n-k+1)!}$$

Two sides are same. □

4 Evaluate $\binom{-1}{k}$ by negating (actually un-negating) its upper index.

Solution.

$$\binom{-1}{k} = \frac{(-1)^{\underline{k}}}{k!} = \frac{(-1)^k 1^{\overline{k}}}{k!} = \frac{(-1)^k k^{\underline{k}}}{k!} = (-1)^k\binom{k}{k} = (-1)^k[k \geq 0]$$

or by upper negation,

$$\binom{-1}{k} = (-1)^k\binom{k-(-1)-1}{k} = (-1)^k\binom{k}{k} = (-1)^k[k \geq 0]$$

□

5 Let p be prime. Show that $\binom{p}{k} \bmod p = 0$ for $0 < k < p$. What does this imply about the binomial coefficients $\binom{p-1}{k}$?

Solution. $\binom{p}{k} = \frac{p!}{k!(p-k)!} = p \cdot \frac{(p-1)!}{k!(p-k)!}$ because p is prime and every term in $k!(p-k)!$ is smaller than p so that no one can cancel p where $0 < k < p$. Thus, $\binom{p}{k}, p \in \mathbb{N} \Rightarrow \frac{(p-1)!}{k!(p-k)!} \in \mathbb{N} \Rightarrow p \,\Big|\, \binom{p}{k}$.

Since $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$, then

$$\binom{p-1}{k} \equiv -\binom{p-1}{k-1} \pmod{p}$$

and assume $A_k = \binom{p-1}{k}$, then

$$A_k \equiv -A_{k-1} \pmod{p}$$

and $A_0 = \binom{p-1}{0} = 1 \pmod{p}$, so

$$\binom{p-1}{k} = A_k \equiv (-1)^k \pmod{p}$$

□

6 Fix up the text's derivation in Problem 6, Section 5.2, by correctly applying symmetry.

Solution.

$$\frac{1}{n+1}\sum_k \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1}\sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k$$

is wrong. The correct one should be

$$\frac{1}{n+1}\sum_k \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1}\sum_{k\geq 0} \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k$$

because

$$\binom{n-1}{-1} = 0 \quad \binom{n-1}{n} = \frac{(n-1)^{\underline{n}}}{n!} = [n=0]$$

which is cancelled and the conversion is not correct. And the remaining result should be

$$\begin{aligned}\frac{1}{n+1}\sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k - \frac{1}{n+1}\binom{n-1}{n}\binom{n+1}{0}(-1)^{-1}\\ = \frac{1}{n+1}\sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k + [n=0]\\ = [n=0]\end{aligned}$$

as the result desired. □

Basics

13 Find relations between the superfactorial function $P_n = \prod_{k=1}^n k!$ of exercise 4.55, the hyperfactorial function $Q_n = \prod_{k=1}^n k^k$, and the product $R_n = \prod_{k=0}^n \binom{n}{k}$.

Solution.

$$R_n = \prod_{k=0}^n \binom{n}{k} = \prod_{k=0}^n \frac{n!}{k!(n-k)!} = \frac{n!^{n+1}}{P_n^2} = \frac{Q_n n(n-1)^2\cdots 1^n}{P_n^2} = \frac{Q_n P_n}{P_n^2} = \frac{Q_n}{P_n}$$

□