

Project 7

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Warmups

- 1 When we analyzed the Josephus problem in Chapter 1, we represented an arbitrary positive integer n in the form $n = 2^m + l$, where $0 \leq l < 2^m$. Give explicit formulas for l and m as functions of n , using floor and/or ceiling brackets.

Solution.

$$\begin{aligned} m &= \lfloor \log_2 n \rfloor \\ l &= n - 2^{\lfloor \log_2 n \rfloor} \end{aligned}$$

□

- 2 What is a formula for the nearest integer to a given real number x ? In case of ties, when x is exactly halfway between two integers, give an expression that rounds (a) up – that is, to $\lceil x \rceil$; (b) down – that is, to $\lfloor x \rfloor$.

Solution. Let \boxed{x} be the nearest integer for real number x . It is either $\lfloor x \rfloor$ or $\lceil x \rceil$, which is decided by comparing $x - \lfloor x \rfloor$ and $\lceil x \rceil - x$.

Case (a): rounds up Because

$$\begin{aligned} \frac{\lfloor x \rfloor + \lceil x \rceil}{2} &= \frac{2\lceil x \rceil - 1}{2} = \lceil x \rceil - \frac{1}{2} \\ \boxed{x} &= \begin{cases} \lceil x \rceil, & x \geq \frac{\lfloor x \rfloor + \lceil x \rceil}{2} \\ \lfloor x \rfloor, & x < \frac{\lfloor x \rfloor + \lceil x \rceil}{2} \end{cases} = \begin{cases} \lceil x \rceil, & x + \frac{1}{2} \geq \lceil x \rceil \\ \lfloor x \rfloor, & x + \frac{1}{2} < \lceil x \rceil \end{cases} \\ &= \begin{cases} \lceil x \rceil, & \lceil x \rceil + 1 > x + \frac{1}{2} \geq \lceil x \rceil \\ \lfloor x \rfloor, & \lfloor x \rfloor \leq x + \frac{1}{2} < \lceil x \rceil \end{cases} = \left\lfloor x + \frac{1}{2} \right\rfloor \end{aligned}$$

Case (b): rounds down Because

$$\frac{\lfloor x \rfloor + \lceil x \rceil}{2} = \frac{2\lfloor x \rfloor + 1}{2} = \lfloor x \rfloor + \frac{1}{2}$$

$$\begin{aligned} \boxed{x} &= \left\{ \begin{array}{ll} \lceil x \rceil, & x > \frac{\lfloor x \rfloor + \lceil x \rceil}{2}, \\ \lfloor x \rfloor, & x \leq \frac{\lfloor x \rfloor + \lceil x \rceil}{2} \end{array} \right\} = \left\{ \begin{array}{ll} \lceil x \rceil, & x - \frac{1}{2} > \lfloor x \rfloor, \\ \lfloor x \rfloor, & x - \frac{1}{2} \leq \lfloor x \rfloor \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \lceil x \rceil, & \lceil x \rceil \geq x - \frac{1}{2} > \lfloor x \rfloor, \\ \lfloor x \rfloor, & \lfloor x \rfloor - 1 < x - \frac{1}{2} \leq \lfloor x \rfloor \end{array} \right\} = \left\lceil x - \frac{1}{2} \right\rceil \end{aligned}$$

□

- 3** Evaluate $\lfloor \lfloor m\alpha \rfloor n / \alpha \rfloor$, when m and n are positive integers and α is an irrational number greater than n .

Solution.

$$\left\lfloor \frac{\lfloor m\alpha \rfloor n}{\alpha} \right\rfloor < \left\lfloor \frac{m\alpha n}{\alpha} \right\rfloor = \lfloor mn \rfloor = mn$$

In fact,

$$\left\lfloor \frac{\lfloor m\alpha \rfloor n}{\alpha} \right\rfloor = \left\lfloor \frac{(m\alpha - \{m\alpha\})n}{\alpha} \right\rfloor = \left\lfloor mn - \frac{\{m\alpha\}n}{\alpha} \right\rfloor = mn - 1$$

because

$$\{m\alpha\} \frac{n}{\alpha} < 1 \times 1 = 1$$

□

- 4** The text describes problems at levels 1 through 5. What is a level 0 problem? (This, by the way, is not a level 0 problem.)

Answer. Something doesn't need any proof, just a guess or an a conjecture.

- 5** Find a necessary and sufficient condition that $\lfloor nx \rfloor = n\lfloor x \rfloor$, when n is a positive integer. (Your condition should involve $\{x\}$.)

Solution.

$$x = \lfloor x \rfloor + \{x\}$$

Then,

$$\lfloor nx \rfloor = \lfloor n\lfloor x \rfloor + n\{x\} \rfloor = n\lfloor x \rfloor + \lfloor n\{x\} \rfloor$$

Thus, $\lfloor nx \rfloor = n\lfloor x \rfloor$ holds when

$$n\{x\} < 1 \Leftrightarrow \{x\} < \frac{1}{n}$$

where $n \in \mathbb{N}_+$.

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