Project 2

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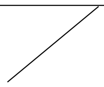
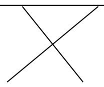
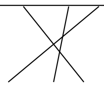
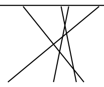
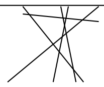
Warmups

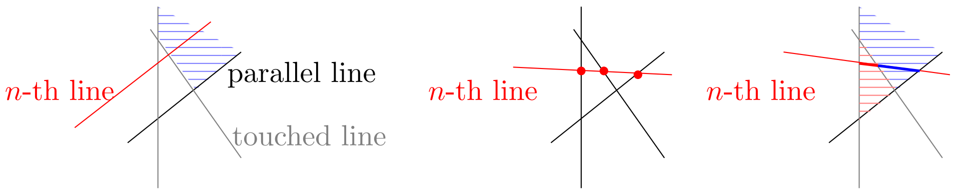
6 Some of the regions defined by n lines in the plane are infinite, while others are bounded. What's the maximum possible number of bounded regions?

Solution. First, enumerate the maximum possible number S_n of bounded regions:

n	0	1	2	3	4	5	$\cdots$
Case							$\cdots$
I_n	0	0	1	3	6	10	$\cdots$
ΔI_n		0	1	2	3	4	$\cdots$
S_n	0	0	0	1	3	6	$\cdots$
ΔS_n		0	0	1	2	3	$\cdots$

I_n represents the number of intersections. Three intersections are required to form a bounded region, so no bounded regions while $n \le 2$.

Similar to the observation for finding the maximum possible number L_n of regions, we could know that to get maximum of bounded regions, **the n-th line must hits the previous lines in $n-1$ different places.** Otherwise, there must be a parallel line to the n-th line, and the number of bounded regions will reduce, for it doesn't contribute to form the bounded region for the infinite regoin formed by parrell line and the lines it touched, shown in (a).



(a) Bad case with a new parrell line. (b) Intersections. (c) Segments split the bounded region and the infinte region.

So, it is the same case for L_n and S_n. As the new n-th line will add $\Delta I_n = n-1$ intersections, it will only increase $\Delta S_n = (n-1)-1 = n-2$ bounded regions **generated by the middle $(n-1)-1 = n-2$ segments**. Because the two end of the new line will not have any intersections to form another bounded regions and point to infinity, shown in (b).

Every Segment introduces $\Delta S_n = 1$. Segements could split bounded regions and infinite regions. When it splits the bounded one (shown in red), it will split the region into two because there is no overlapping on intersections, which contributes $\Delta S_n = 1$. While it splits (shown in blue), it will split th region into a bounded one and an infinite one. Because the two intersections touch two lines seperately and the two lines must intersect and not parallel, then there must introduce a closed triangle or polygon, which also contributes $\Delta S_n = 1$. The above is shown in (c).

Thus, when $n \geq 3$,

$$S_n = \sum_{i=1}^{n-2} i = \frac{(n-2)(n-1)}{2}$$

The closed formula is

$$S_n = \begin{cases} 0, & n = 0, 1, 2; \\ \dfrac{(n-2)(n-1)}{2}, & n \geq 3. \end{cases}$$

□

7 Let $H(n) = J(n+1) - J(n)$. Equation (1.8) tells us that $H(2n) = 2$, and $H(2n+1) = J(2n+2) - J(2n+1) = (2J(n+1) - 1) - (2J(n) + 1) = 2H(n) - 2$, for all $n > 1$. Therefore it seems possible to prove that $H(n) = 2$ for all n, by induction on n. What's wrong here?

Solution. It ignores the case of $n = 1$. $H(1) = J(2) - J(1) = 1 - 1 = 0$, the basic step is wrong for the conclusion of "$H(n) = 2$ for all n".

Now the recurrence is

$$\begin{aligned} H(1) &= 0, \\ H(2n) &= 2, && \forall n > 1; \\ H(2n+1) &= 2H(n) - 2, && \forall n > 1. \end{aligned}$$

As $H(n) = 2$ always holds for the even number n, it is odd number that needs consideration. List the result of $H(n)$ into a table, where n is an odd number.

n	1	3	5	7	9	11	13	15	17	19	$\cdots$
$H(n)$	0	-2	2	-6	2	2	2	-14	2	2	$\cdots$

And it is discovered that for $n = (\underbrace{1 \cdots 1}_{k \text{ of 1's}})_2 = \sum_{i=0}^{k} 1 = 2^{k+1} - 1$, $H(n) \neq 2$. For other odd numbers, $H(n) = 2$. And the proof is as follows:

Case 1: $n = 2^{k+1} - 1 (k \in \mathbb{N})$ In this case, the recurrence goes:

$$\begin{aligned} H(2^1 - 1) &= 0 \\ H(2^{k+1} - 1) &= 2H(2^k - 1) - 2 && k > 0 \end{aligned}$$

The last equation could be deduced by:

$$\begin{aligned} H(2^{k+1}-1)-2 &= 2\left[H(2^k-1)-2\right] \\ H(2^{k+1}-1)-2 &= 2^k\cdot(-2) \\ H(2^{k+1}-1) &= -2^{k+1}+2 \end{aligned}$$

Case 2: $n \neq 2^{k+1}-1 (k \in \mathbb{N})$ From the closed formula of $J(n)$:

$$J(2^m+l) = 2l+1 \qquad\qquad \text{for } m \geq 0 \text{ and } 0 \leq l < 2^m$$

Because $n+1 \neq 2^{k+1}$, or $n+1 = 2^m + l, 0 < l < 2^m$, so n and $n+1$ are in the same domain of $2^m + l$ where m is a specific number. It could be calculate that

$$H(n) = J(n+1) - J(n) = 2l+1-[2(l-1)+1] = 2$$

To sum up, the closed formula of $H(n)$ is:

$$H(n) = \begin{cases} -2^{k+1}+2, & n = 2^{k+1}-1, \quad \forall k \in \mathbb{N}; \\ 2, & \text{otherwise.} \end{cases}$$

□

Homework

8 Solve the recurrence
$Q_0 = \alpha; Q_1 = \beta;$
$Q_n = (1+Q_{n-1})/Q_{n-2};$ for $n > 1$.
Assume that $Q_n \neq 0$ for all $n \geq 0$. *Hint:* $Q_4 = (1+\alpha)/\beta$.

Solution. Due to $Q_n \neq 0$ for all $n \geq 0$, it could be assumed that all the coefficents there are reducible (otherwise a zero on the denominator could introduce a zero on one Q_n). And calculate the result based on the recurrence as the following table:

n	0	1	2	3	4	5	6	$\cdots$
Q_n	α	β	$\frac{1+\beta}{\alpha}$	$\frac{1+\alpha+\beta}{\alpha\beta}$	$\frac{1+\alpha}{\beta}$	α	β	$\cdots$

And it is discovered that $Q_5 = Q_1 = \alpha, Q_6 = Q_2 = \beta$, thus the loop will continue, and the closed formula of the recurrence is:

$$Q_n = \begin{cases} \alpha, & n \mod 5 = 0; \\ \beta, & n \mod 5 = 1; \\ \dfrac{1+\beta}{\alpha}, & n \mod 5 = 2; \\ \dfrac{1+\alpha+\beta}{\alpha\beta}, & n \mod 5 = 3; \\ \dfrac{1+\alpha}{\beta}, & n \mod 5 = 4. \end{cases}$$

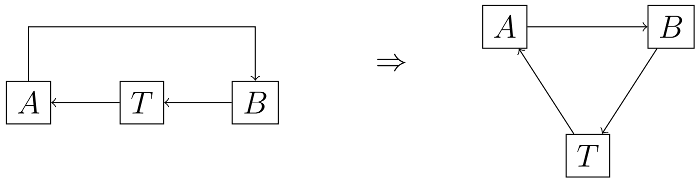
□

10 Let Q_n be the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be clockwise —that is, from A to B, or from B to the other peg, or from the other peg to A. Also let R_n be the minimum number of moves needed to go from B back to A under this restriction. Prove that

$$Q_n = \begin{cases} 0, & \text{if } n = 0; \\ 2R_{n-1} + 1, & \text{if } n > 0; \end{cases} \quad R_n = \begin{cases} 0, & \text{if } n = 0; \\ Q_n + Q_{n-1} + 1, & \text{if } n > 0 \end{cases}$$

(You need not solve these recurrences; we'll see how to do that in Chapter 7.)

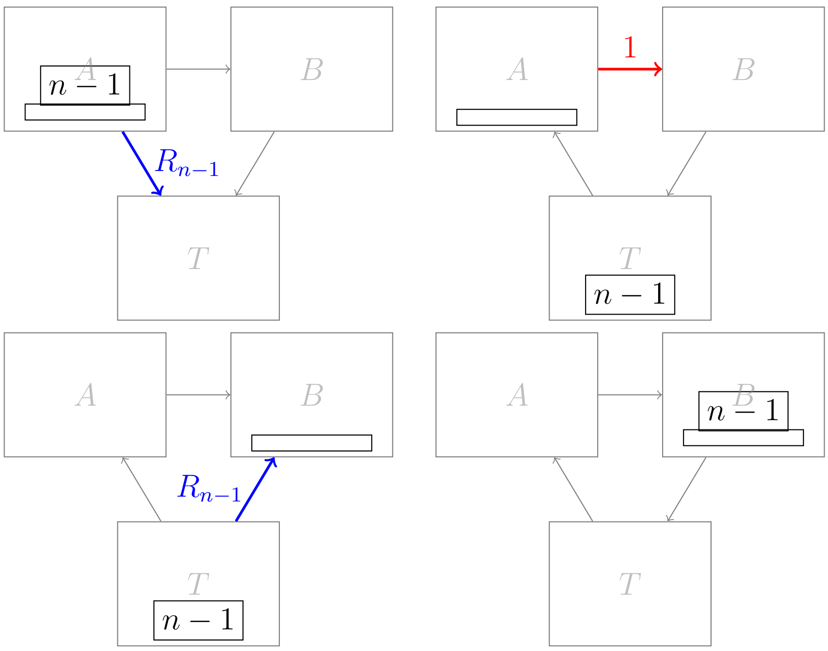
Solution. The restriction can be summarized as the left side:



However, the three nodes on the loop, in fact, are **in the equivalent positions**. Thus, the right side draws them on a equilateral triangle in order to show the equivalence. Thus, any operation follows the clockwise direction is equals Q, otherwise equals R.

$Q_0 = 0$ and $R_0 = 0$ are obvious.

Try to solve Q_n. As the restriction goes, the minimum number of moves are performed in the following way:



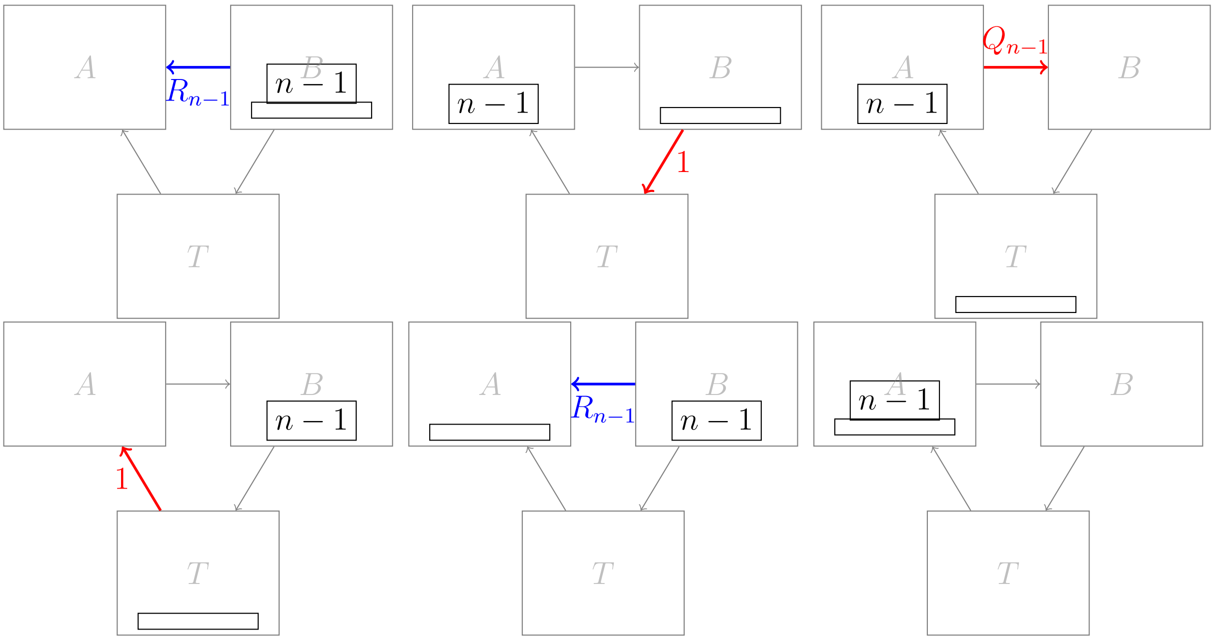
As a result, when $n > 0$,

$$Q_n = R_{n-1} + 1 + R_{n-1} \tag{1}$$

So,

$$Q_n = \begin{cases} 0, & \text{if } n = 0; \\ 2R_{n-1} + 1, & \text{if } n > 0; \end{cases}$$

Then, solve R_n.



As a result, when $n > 0$,

$$R_n = R_{n-1} + 1 + Q_{n-1} + 1 + R_{n-1} = Q_{n-1} + 2R_{n-1} + 1$$

Combine Equation (1), we could get that

$$R_n = \begin{cases} 0, & \text{if } n = 0; \\ Q_n + Q_{n-1} + 1, & \text{if } n > 0. \end{cases}$$

$\square$