Mathematical Foundations of Computer Science

Project 8

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Warmups

6 Can something interesting be said about $\lfloor f(x) \rfloor$ when $f(x)$ is a continuous, monotonically *decreasing* function that takes integer values only when x is an integer?

Solution.

$$\begin{aligned} \lfloor f(x) \rfloor &= \lfloor f(\lceil x \rceil) \rfloor \\ \lceil f(x) \rceil &= \lceil f(\lfloor x \rfloor) \rceil \end{aligned} \tag{1}$$

where $f(x)$, $f(\lfloor x \rfloor)$, $f(\lceil x \rceil)$ are defined. Because $f(x)$ is a continuous, monotonically decreasing function, $-f(x)$ is a continuous, monotonically increasing function. Then plug $-f$ to the increasing properties:

$$\begin{aligned} \lfloor -f(x) \rfloor &= \lfloor -f(\lfloor x \rfloor) \rfloor \\ \lceil -f(x) \rceil &= \lceil -f(\lceil x \rceil) \rceil \end{aligned}$$

By applying $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$,

$$\begin{aligned} -\lceil f(x) \rceil &= -\lceil f(\lfloor x \rfloor) \rceil \\ -\lfloor f(x) \rfloor &= -\lfloor f(\lceil x \rceil) \rfloor \end{aligned}$$

which is the same as Equations (1). □

7 Solve the recurrence

$$\begin{aligned} X_n &= n, && \text{for } 0 \leq n < m \\ X_n &= X_{n-m} + 1, && \text{for } n \geq m. \end{aligned}$$

Solution. Then closed formula is

$$X_n = (n \mod m) + \left\lfloor \frac{n}{m} \right\rfloor, \quad \text{for } n \in \mathbb{N} \tag{2}$$

Prove by discussing different scenarios.

Case 1: $n < m$**.** $X_n = n = n + \lfloor 0 \rfloor$ is true.

Case 2: $n = m$**.** $X_n = X_0 + 1 = 1$ is still hold for $X_n = 0 + \lfloor 1 \rfloor = 1$.

Case 3: $n > m$. $X_n = X_{n-m} + 1 = X_{n-2m} + 2 = \cdots = X_{n-lm} + l = n - lm + l$ when $n - lm < m \Rightarrow \frac{n}{m} \geq l > \frac{n}{m} - 1$. i.e., $l = \lfloor \frac{n}{m} \rfloor$. Thus,

$$X_n = n - \left\lfloor \frac{n}{m} \right\rfloor m + \left\lfloor \frac{n}{m} \right\rfloor = (n \mod m) + \left\lfloor \frac{n}{m} \right\rfloor$$

where $n - \left\lfloor \frac{n}{m} \right\rfloor m = (n \mod m)$ is a definition.

Thus, Equation (2) is true for all $n \in \mathbb{N}$. □

8 Prove the *Dirichlet box* principle: If n objects are put into m boxes, some box must contain $\geq \lceil n/m \rceil$ objects, and some box must contain $\leq \lfloor n/m \rfloor$.

Proof. Prove by contradition. If all boxes contain $< \lceil n/m \rceil$, then the total number of objects

$$n \leq m \left(\left\lceil \frac{n}{m} \right\rceil - 1 \right) \Leftrightarrow \frac{n}{m} + 1 \leq \left\lceil \frac{n}{m} \right\rceil$$

which is impossible because $\lceil x \rceil < x + 1$. If all boxes contain $> \lfloor n/m \rfloor$, then

$$n \geq m \left(\left\lfloor \frac{n}{m} \right\rfloor + 1 \right) \Leftrightarrow \frac{n}{m} - 1 \geq \left\lfloor \frac{n}{m} \right\rfloor$$

which is also impossible because $\lfloor x \rfloor > x - 1$. □

9 Egyptian mathematicians in 1800 B.C. represented rational numbers between 0 and 1 as sums of unit fractions $1/x_1 + \cdots + 1/x_k$, where the x's were distinct positive integers. For example, they wrote $1/3 + 1/15$ instead of $2/5$. Prove that it is always possible to do this in a systematic way: If $0 < m/n < 1$, then

$$\frac{m}{n} = \frac{1}{q} + \left\{ \text{representation of } \frac{m}{n} - \frac{1}{q} \right\}, \quad q = \left\lceil \frac{n}{m} \right\rceil$$

(This is *Fibonacci's algorithm*, due to Leonardo Fibonacci, A.D. 1202.)

Proof. The process could be described as follows if denote x as $\frac{m}{n}$:

$$\begin{aligned}
x &= \frac{1}{\lceil 1/x \rceil} + y_1 && \Rightarrow x > y_1 \\
y_1 &= \frac{1}{\lceil 1/y_1 \rceil} + y_2 && \Rightarrow y_1 > y_2 \\
&\vdots \\
y_i &= \frac{1}{\lceil 1/y_i \rceil} + y_{i+1} && \Rightarrow y_i > y_{i+1} \\
&\vdots
\end{aligned}$$

where right column holds because $0 < x < 1$ and

$$y_{i+1} = y_i - \frac{1}{\lceil 1/y_i \rceil} \geq y_i - \frac{1}{1/y_i} = 0$$

With all numbers are positive, it is indicated that

$$1 > x > y_1 > y_2 > \cdots > y_i > y_{i+1} > \cdots \geq 0$$

They are in strictly decreasing order, which are distinct *real numbers*. However, this process must be terminated in some point, because

$$y_1 = x - \frac{1}{\lceil 1/x \rceil} = \frac{x\lceil 1/x \rceil - 1}{\lceil 1/x \rceil} < \frac{x(1/x+1) - 1}{\lceil 1/x \rceil} = \frac{x}{\lceil 1/x \rceil} \tag{3}$$

Plug $x = \frac{m}{n}$ into Equation (3)

$$0 \leq y_1 = \frac{m\lceil n/m \rceil - n}{n\lceil n/m \rceil} < \frac{m}{n\lceil n/m \rceil}$$

Denote $y_1 = \frac{m_1}{n_1}$, then

$$0 \leq y_1 = \frac{m_1}{n_1} < \frac{m}{n_1} \Rightarrow m_1 < m$$

By the same process by applying $y_i = \frac{m_i}{n_i}$ and $y_{i+1} = \frac{m_{i+1}}{n_{i+1}}$, it is similar that

$$y_{i+1} < \frac{y_i}{\lceil 1/y_i \rceil} \Rightarrow m_{i+1} < m_i$$

then all *integer* numerators for y_i will be in an decreasing order.

$$0 \leq \cdots < m_{i+1} < m_i < \cdots < m_2 < m_1 < m \tag{4}$$

There are only finite *integers* between $[0, m)$, then **the process must terminate at some point**.

$\square$

Basics

10 Show that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

Proof.

$$\begin{aligned}
\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor &= \left\lceil \frac{2x+1}{2} \right\rceil - \left(\left\lceil \frac{2x+1}{4} \right\rceil - \left\lfloor \frac{2x+1}{4} \right\rfloor \right) \\
&= \left\lceil \frac{2x+1}{2} \right\rceil - \left[\frac{2x+1}{4} \notin \mathbb{Z} \right] \\
&= \begin{cases} \left\lceil \frac{2x+1}{2} \right\rceil - 1, & \frac{2x+1}{4} \notin \mathbb{Z} \\ \left\lceil \frac{2x+1}{2} \right\rceil, & \frac{2x+1}{4} \in \mathbb{Z} \end{cases}
\end{aligned} \tag{5}$$

Then, the result is divided into two cases:

Case 1: $\frac{2x+1}{4}$ is not an integer. The result could also be divided into two cases:

Case 1a: $\frac{2x+1}{2}$ is an integer. Then assume the integer is k,[1]

$$\left\lceil \frac{2x+1}{2} \right\rceil - 1 = \frac{2x+1}{2} - 1 = k - 1 \tag{6}$$

while

$$x = \frac{2k-1}{2} = k - \frac{1}{2} \Rightarrow \lfloor x \rfloor = k - 1 \tag{7}$$

By combining Equation (6) and Equation (7),

$$\left\lceil \frac{2x+1}{2} \right\rceil - 1 = \lfloor x \rfloor, \quad \frac{2x+1}{2} \in \mathbb{Z} \tag{8}$$

Case 1b: $\frac{2x+1}{2}$ is not an integer. Then, at the moment, $\{x\} \neq \frac{1}{2}$, otherwise assume $x = \lfloor x \rfloor + \frac{1}{2}$

$$\frac{2x+1}{2} = \frac{2\lfloor x \rfloor + 2}{2} = \lfloor x \rfloor + 1 \in \mathbb{Z}$$

which is contradictory.

$$\begin{aligned} \left\lceil \frac{2x+1}{2} \right\rceil - 1 &= \left\lfloor \frac{2x+1}{2} \right\rfloor \\ &= \left\lfloor x + \frac{1}{2} \right\rfloor \\ &= \left\lfloor \lfloor x \rfloor + \{x\} + \frac{1}{2} \right\rfloor \\ &= \lfloor x \rfloor + \left\lfloor \{x\} + \frac{1}{2} \right\rfloor \\ &= \begin{cases} \lfloor x \rfloor, & 0 \leq \{x\} < \dfrac{1}{2} \\ \lceil x \rceil, & \dfrac{1}{2} < \{x\} < 1 \end{cases} \end{aligned} \tag{9}$$

Case 2: $\frac{2x+1}{4}$ is an integer. Assume the integer is $n \in \mathbb{Z}$, thus

$$\left\lceil \frac{2x+1}{2} \right\rceil = \left\lceil 2 \times \frac{2x+1}{4} \right\rceil = 2n \tag{10}$$

On the other hand,

$$\begin{aligned} x &= \frac{4n-1}{2} \\ \lceil x \rceil &= \left\lceil 2n - \frac{1}{2} \right\rceil = 2n \end{aligned}$$

Plug it into Equation (10),

$$\left\lceil \frac{2x+1}{2} \right\rceil = \lceil x \rceil \tag{11}$$

[1] $\frac{2x+1}{4} = \frac{k}{2} \notin \mathbb{Z}$, k is an odd number.

As a matter of fact, $\frac{2x+1}{4}$ and $\frac{2x+1}{4}$ will lead to $\{x\} = \frac{1}{2}$. Thus,

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \begin{cases} \lfloor x \rfloor, & 0 \leq \{x\} < \dfrac{1}{2}, \\ \lceil x \rceil, & \dfrac{1}{2} < \{x\} < 1, \\ \lceil x \rceil - \left[\dfrac{2x+1}{4} \notin \mathbb{Z} \right], & \{x\} = \dfrac{1}{2}. \end{cases}$$

$\square$

11 Give details of the proof alluded to in the text, that the open interval $(\alpha..\beta)$ contains exactly $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$ integers when $\alpha < \beta$. Why does the case $\alpha = \beta$ have to be excluded in order to make the proof correct?

Proof. For integer $n \in (\alpha..\beta)$,

$$\lfloor \alpha \rfloor < n < \lceil \beta \rceil \tag{12}$$

has to be the full range for n. **Prove by contradiction.** Otherwise,

$$\begin{aligned} n &\leq \lfloor \alpha \rfloor \leq \alpha \\ n &\geq \lceil \beta \rceil \geq \beta \end{aligned} \tag{13}$$

which are contradictory with $\alpha < n < \beta$. Thus

$$\alpha < n < \beta \Leftrightarrow \lfloor \alpha \rfloor < n < \lceil \beta \rceil$$

And the number of integers in the interval of Equation (12) is

$$\#n = \lceil \beta \rceil - \lfloor \alpha \rfloor - 1$$

when $\alpha < \beta$. And when $\alpha = \beta$, Equation (13) won't lead to contradiction with $\alpha < n < \beta$, since $\alpha \leq n \leq \alpha$ has intersection with $\alpha < n < \alpha$, i.e.,

$$\alpha < n < \alpha \Rightarrow \alpha \leq n \leq \alpha$$

$\square$

12 Prove that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

for all integers n and all positive integers m. [This identity gives us another way to convert ceilings to floors and vice versa, instead of using the reflective law (3.4).]

Proof. Split it into two scenarios.

Case 1: $\frac{n}{m} \in \mathbb{Z}$. Then

$$\left\lfloor \frac{n+m-1}{m} \right\rfloor = \left\lfloor \frac{n}{m} + \frac{m-1}{m} \right\rfloor = \frac{n}{m} + \left\lfloor \frac{m-1}{m} \right\rfloor = \frac{n}{m} = \left\lceil \frac{n}{m} \right\rceil$$

Case 2: $\frac{n}{m} \notin \mathbb{Z}$**.** Assume

$$n = \left\lfloor \frac{n}{m} \right\rfloor m + r$$

where $1 \le r = n \mod m < m$. Then

$$\left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{r-1}{m} \right\rfloor = \left\lfloor \left\lfloor \frac{n}{m} \right\rfloor + \frac{r-1}{m} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{n}{m} \right\rfloor m + r - 1}{m} \right\rfloor = \left\lfloor \frac{n-1}{m} \right\rfloor$$

Thus,

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n}{m} \right\rfloor + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

□

13 Let α and β be positive real numbers. Prove that Spec(α) and Spec(β) partition the positive integers if and only if α and β are irrational and $1/\alpha + 1/\beta = 1$.

Proof. The number of elements in Spec(α) that are $\le n$, where $n \in \mathbb{Z}$:

$$\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [\lfloor k\alpha \rfloor \le n] \\
&= \sum_{k>0} [\lfloor k\alpha \rfloor < n+1] \\
&= \sum_{k>0} [k\alpha < n+1] \\
&= \sum_{k} [0 < k < \frac{n+1}{\alpha}] \\
&= \left\lceil \frac{n+1}{\alpha} \right\rceil - \lfloor 0 \rfloor - 1 \qquad \text{(By Problem 11)} \\
&= \left\lceil \frac{n+1}{\alpha} \right\rceil - 1
\end{aligned}$$

"$\Leftarrow$" for $\forall n > 0$:

$$\begin{aligned}
N(\alpha, n) + N(\beta, n) &= \left\lceil \frac{n+1}{\alpha} \right\rceil - 1 + \left\lceil \frac{n+1}{\beta} \right\rceil - 1 \\
&= \left\lfloor \frac{n+1}{\alpha} \right\rfloor + \left\lfloor \frac{n+1}{\beta} \right\rfloor \qquad (14) \\
&= \frac{n+1}{\alpha} - \left\{ \frac{n+1}{\alpha} \right\} + \frac{n+1}{\beta} - \left\{ \frac{n+1}{\beta} \right\} \\
&= (n+1)\left(\frac{1}{\alpha} + \frac{1}{\beta} \right) - \left(\left\{ \frac{n+1}{\alpha} \right\} + \left\{ \frac{n+1}{\beta} \right\} \right) \qquad (15) \\
&= n + 1 - 1 = n
\end{aligned}$$

Equation (14) holds because

$$\alpha, \beta \in \mathbb{R} \backslash \mathbb{Q} \Rightarrow \nexists n \in \mathbb{Z} : \alpha | n+1, \beta | n+1 \Rightarrow \frac{n+1}{\alpha}, \frac{n+1}{\beta} \notin \mathbb{Z} \qquad (16)$$

Equation (15) holds because of (16) and

$$(n+1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=n+1\in\mathbb{Z} \tag{17}$$

The sum of fractional parts has to be 1.

"$\Rightarrow$" **Prove by contradiction.**

Case 1: $\alpha, \beta \in \mathbb{Q}$. Suppose

$$\alpha=\frac{m_1}{n_1}, \beta=\frac{m_2}{n_2}$$

where $m_1, n_1, m_2, n_2 \in \mathbb{N}$ (because they are positive). Then the least common multiple $\operatorname{lcm}(m_1, m_2)$ of m_1 and m_2 will exist in both $\operatorname{Spec}(\alpha)$ and $\operatorname{Spec}(\beta)$, which will make it not a division of integers.

Case 2: $\frac{1}{\alpha}+\frac{1}{\beta}\neq 1$. The coefficient of n is $\frac{1}{\alpha}+\frac{1}{\beta}$ as is seen in (15) where samll constants are neglected when n is large whether or not Equation (14) holds. Then the coefficient will not match and $N(\alpha, n)+N(\beta, n)$ will not be n to be the partition of integers.

Case 3: $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{R}\backslash\mathbb{Q}$. Then

$$1\neq\frac{1}{\alpha}+\frac{1}{\beta}\in\mathbb{R}\backslash\mathbb{Q}$$

which is the same as the previous case.

$\square$