

## Project 5

Zilong Li

Student ID: 518070910095

March 22, 2021

### Warmups

- 6 What is the value of  $\sum_k [1 \leq j \leq k \leq n]$ , as a function of  $j$  and  $n$ ?

**Solution.**

$$\sum_k [1 \leq j \leq k \leq n](j, n) = \begin{cases} 0, & j > n \text{ or } j < 1, \\ n - j + 1, & \text{else.} \end{cases}$$

□

### Basics

- 11 The general rule (2.56) for summation by parts is equivalent to

$$\sum_{0 \leq k < n} (a_{k+1} - a_k)b_k = a_nb_n - a_0b_0 - \sum_{0 \leq k < n} a_{k+1}(b_{k+1} - b_k), \quad \text{for } n \geq 0$$

Prove this formula directly by using the distributive, associative, and commutative laws.

**Proof.**

$$\begin{aligned} \sum_{0 \leq k < n} (a_{k+1} - a_k)b_k &= \sum_{0 \leq k < n} a_{k+1}b_k - \sum_{0 \leq k < n} a_kb_k && \text{(associative law)} \\ &= \sum_{0 \leq k < n} a_{k+1}b_k - \sum_{-1 \leq k < n-1} a_{k+1}b_{k+1} && \text{(commutative law)} \\ &= \sum_{0 \leq k < n} a_{k+1}b_k - \left( \sum_{0 \leq k < n} a_{k+1}b_{k+1} + a_0b_0 - a_nb_n \right) && \text{(commutative law)} \\ &= a_nb_n - a_0b_0 + \sum_{0 \leq k < n} a_{k+1}b_k - \sum_{0 \leq k < n} a_{k+1}b_{k+1} \\ &= a_nb_n - a_0b_0 + \sum_{0 \leq k < n} a_{k+1}(b_{k+1} - b_k) && \text{(distributive law)} \end{aligned}$$

□

- 12 Show that the function  $p(k) = k + (-1)^kc$  is a permutation of the set of all integers, whenever  $c$  is an integer.

**Proof.** Assume  $p(k) = n$ , then

$$n + c = k + ((-1)^k + 1) c$$

By beging the index of  $-1$ ,

$$(-1)^{n+c} = (-1)^{k+((-1)^k+1)c}$$

because  $((-1)^k + 1)$  is always even, thus

$$(-1)^{n+c} = (-1)^k$$

So, plug  $k = n - (-1)^k c = n - (-1)^{n+c} c$  back to  $p(k)$

$$p(k) = n - (-1)^k c + (-1)^k c = n$$

So a given  $n$ , there is always  $k = (-1)^{n+c} c$  such that  $p(k) = n$ . Because this is a surjective function,  $k$ 's are unique (otherwise it is not a function). It is available for all integers, thus  $k$ 's will expand to all integers and  $p(k)$  is a bijective. Thus,  $p(k)$  is a permutation of the set of all integers.  $\square$

- 13** Use the repertoire method to find a closed form for  $\sum_{k=0}^n (-1)^k k^2$ .

**Solution.** Consider the recurrence of

$$\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows a closed form of

$$R(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$$

**Case 1:**  $R(n) = 1$

$$\begin{aligned} 1 &= \alpha \\ 1 &= 1 + (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows  $\beta = \gamma = \delta = 0$ .

$$A(n) = 1$$

**Case 2:**  $R(n) = (-1)^n$

$$\begin{aligned} 1 &= \alpha \\ (-1)^n &= (-1)^{n-1} + (-1)^n (\beta + n\gamma + n^2\delta) \\ 2 \times (-1)^n &= (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows  $\beta = 2, \gamma = \delta = 0$

$$\begin{aligned} (-1)^n &= 1 + 2B(n) \\ B(n) &= \frac{(-1)^n - 1}{2} \end{aligned}$$

**Case 3:**  $R(n) = (-1)^n n$

$$0 = \alpha$$

$$\begin{aligned} (-1)^n n &= (-1)^{n-1} (n-1) + (-1)^n (\beta + n\gamma + n^2\delta) \\ (-1)^n (2n-1) &= (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows  $\beta = -1, \gamma = 2, \delta = 0$

$$\begin{aligned} (-1)^n n &= -B(n) + 2C(n) \\ C(n) &= \frac{2(-1)^n n + (-1)^n - 1}{4} \end{aligned}$$

**Case 4:**  $R(n) = (-1)^n n^2$

$$0 = \alpha$$

$$\begin{aligned} (-1)^n n^2 &= (-1)^{n-1} (n-1)^2 + (-1)^n (\beta + n\gamma + n^2\delta) \\ (-1)^n (2n^2 - 2n + 1) &= (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows  $\beta = 1, \gamma = -2, \delta = 2$

$$\begin{aligned} (-1)^n n^2 &= B(n) - 2C(n) + 2D(n) \\ (-1)^n n^2 &= \frac{(-1)^n - 1}{2} - 2 \frac{2(-1)^n n + (-1)^n - 1}{4} + 2D(n) \\ D(n) &= \frac{(-1)^n}{2} (n^2 + n) \end{aligned}$$

In this problem,  $\alpha = 0, \beta = 0, \gamma = 0, \delta = 1$ , thus

$$\begin{aligned} \sum_{k=0}^n (-1)^k k^2 &= D(n) \\ &= \frac{(-1)^n}{2} (n^2 + n) \end{aligned}$$

□