Mathematical Foundations of Computer Science

Project 15

Log Creative

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Warmups

5 Find a generating function $S(z)$ such that

$$[z^n]S(z) = \sum_k \binom{r}{k}\binom{r}{n-2k}$$

Solution. Let $F(z) = (1+z^2)^r$ and $G(z) = (1+z)^r$, then

$$F(z)G(z) = \sum_n \sum_k \binom{r}{k} z^{2k} \binom{r}{n-2k} z^{n-2k} = \sum_n \sum_k \binom{r}{k}\binom{r}{n-2k} z^n$$

So that

$$S(z) = F(z)G(z) = (1+z^2)^r(1+z)^r = (1+z+z^2+z^3)^r$$

$\square$

Basics

8 What is $[z^n](\ln(1-z))^2/(1-z)^{m+1}$?

Solution. Let $F(z) = (\ln(1-z))^2$ and $G(z) = 1/(1-z)^{m+1}$. Since

$$-\ln(1-z) = \ln\frac{1}{1-z} = \sum_{n\geq 1}\frac{z^n}{n}$$

Then,

$$\begin{aligned}
[z^n](\ln(1-z))^2 &= [z^n](-\ln(1-z))(-\ln(1-z)) \\
&= \sum_{k=1}^{n-1}\frac{1}{k}\frac{1}{n-k} = \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right) \\
&= \frac{2}{n}H_{n-1}
\end{aligned}$$

and

$$[z^n]\frac{1}{(1-z)^{m+1}} = \binom{m+n}{m}$$

Then

$$[z^n](\ln(1-z))^2/(1-z)^{m+1} = [z^n]F(z)G(z) = \sum_{l=1}^{n} \frac{2}{l}\binom{m+n-l}{m} H_{l-1}$$

□

9 Use the result of the previous exercise to evaluate $\sum_{k=0}^{n} H_k H_{n-k}$.

Solution. The generating function $H(z)$ is

$$H(z) = \frac{1}{1-z}\ln\frac{1}{1-z} = \sum_{n\geq 0} H_n z^n$$

Then the problem is the coefficent of z^n for function $H^2(z)$

$$\sum_{k=0}^{n} H_k H_{n-k} = [z^n]H^2(z) = [z^n]\frac{(\ln(1-z))^2}{(1-z)^2}$$

which is the result of the previous exercise when $m = 1$.

□

10 Set $r = s = -1/2$ in identity (7.62) and then remove all occurrences of $1/2$ by using tricks like (5.36). What amazing identity do you deduce?

$$\sum_k \binom{r+k}{k}\binom{s+n-k}{n-k}(H_{r+k} - H_r) = \binom{r+s+n+1}{n}(H_{r+s+n+1} - H_{r+s+1})$$

Solution. Set $r = s = -\frac{1}{2}$,

$$\sum_k \binom{k-1/2}{k}\binom{n-k-1/2}{n-k}(H_{k-1/2} - H_{-1/2}) = \binom{n}{n}(H_n - H_0) = H_n$$

Then, with the help of (5.36),

$$\binom{k-1/2}{k}\binom{n-k-1/2}{n-k} = \frac{\binom{2k}{k}}{2^{2k}}\frac{\binom{2(n-k)}{n-k}}{2^{2(n-k)}} = \frac{\binom{2k}{k}\binom{2(n-k)}{n-k}}{2^{2n}}$$

Thus,

$$\sum_k \binom{2k}{k}\binom{2n-2k}{n-k}(H_{k-1/2} - H_{-1/2}) = 2^{2n} H_n$$

And since

$$\begin{aligned}
H_{k-1/2} - H_{-1/2} &= \frac{1}{k-\frac{1}{2}} + \frac{1}{k-\frac{1}{2}-1} + \cdots + \frac{1}{\frac{1}{2}} \\
&= \frac{2}{2k-1} + \frac{2}{2k-3} + \cdots + \frac{2}{1} \\
&= \frac{2}{2k} + \frac{2}{2k-1} + \frac{2}{2k-2} + \frac{2}{2k-3} + \cdots + \frac{2}{2} + \frac{2}{1} - \left(\frac{2}{2k} + \frac{2}{2k-2} + \cdots + \frac{2}{2}\right) \\
&= 2H_{2k} - H_k
\end{aligned}$$

$$\sum_k \binom{2k}{k}\binom{2n-2k}{n-k}(2H_{2k} - H_k) = 2^{2n} H_n$$

□