Mathematical Foundations of Computer Science

Project 6

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Warmups

7 Let $\nabla f(x) = f(x) - f(x-1)$. What is $\nabla(x^{\overline{m}})$?

Solution.

$$\begin{aligned}\nabla(x^{\overline{m}}) &= \nabla(x(x+1)\cdots(x+m-1)) \\ &= x(x+1)\cdots(x+m-1) - (x-1)x\cdots(x+m-2) \\ &= x(x+1)\cdots(x+m-2)\left[(x+m-1)-(x-1)\right] \\ &= mx^{\overline{m-1}}\end{aligned}$$

$\square$

8 What is the value of $0^{\underline{m}}$, when m is a given integer?

Solution. Case 1: $m > 0$**.**

$$0^{\underline{m}} = 0 \cdot (-1)^{\underline{m-1}} = 0$$

Case 2: $m = 0$**.**

$$0^{\underline{0}} = 1$$

which is defined as the empty product.

Case 3: $m < 0$**.**

$$\begin{aligned}1 = 0^{\underline{0}} = 0^{\underline{m}}(-m)^{\underline{-m}} &= 0^{\underline{m}}|m|! \\ 0^{\underline{m}} &= \frac{1}{|m|!}\end{aligned}$$

$\square$

9 What is the law of exponents for rising factorial powers, analogous to (2.52)? Use this to define $x^{\overline{-n}}$.

Solution.

$$x^{\overline{m+n}} = x(x+1)\cdots(x+m-1)\cdots(x+m)\cdots(x+m+n-1) = x^{\overline{m}}(x+m)^{\overline{n}}$$

$$1 = x^{\overline{0}} = x^{\overline{(-n)+n}} = x^{\overline{-n}}(x-n)^{\overline{n}}$$

$$x^{\overline{-n}} = \frac{1}{(x-n)^{\overline{n}}} = \frac{1}{(x-n)(x-n+1)\cdots(x-1)}$$

□

10 The text derives the following formula for the difference of a product:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

How can this formula be correct, when the left-hand side is symmetric with respect to u and v but the right-hand side is not?

Solution. Because the following formula could also be correct with tht respect of u:

$$\begin{aligned}\Delta(uv) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x+1)v(x+1) - u(x+1)v(x) + u(x+1)v(x) - u(x)v(x)\\ &= u(x+1)\Delta v + v\Delta u\\ &= Eu\Delta v + v\Delta u\end{aligned}$$

It is an inner layer of symmetric.

$$Eu\Delta v + v\Delta u = u\Delta v + Ev\Delta u$$

□

Basics

$$\sum_{k=1}^{n}\sum_{j=1}^{k} 2^k.$$

14 Evaluate $\sum_{k=1}^{n} k2^k$ by rewriting it as the multiple sum $\sum_{1\le j\le k\le n} 2^k$.

$$\sum_{k=1}^{n} k2^k = \sum_{1\le j\le k\le n} 2^k$$

$$\begin{aligned}\sum_{1\le j\le k\le n} 2^k &= \sum_{1\le j\le n}\sum_{j\le k\le n} 2^k\\ &= \sum_{1\le j\le n}\left[(2^1+2^2+\cdots+2^n) - (2^1+2^2+\cdots+2^{j-1})\right]\\ &= \sum_{1\le j\le n}\left[(2^{n+1}-2) - (2^j-2)\right]\\ &= n2^{n+1} - (2^{n+1}-2)\end{aligned}$$

$2^1 + 2^2 + \cdots + 2^n.$