Mathematical Foundations of Computer Science

Project 13

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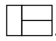
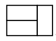
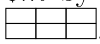
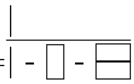
Warmups

1 An eccentric collector of $2 \times n$ domino tilings pays \$4 for each vertical domino and \$1 for each horizontal domino. How many tilings are worth exactly \$$m$ by this criterion? For example, when $m = 6$ there are three solutions: , , and .

Solution. Like what has been done in (7.5), (7.4) could be transferred into

$T =$  $= \frac{1}{1-z^4-z^2}$

whose sequence is

$$\frac{1}{1 - z^4 - z^2} = \frac{1}{1 - z^2 - (z^2)^2} \leftrightarrow \langle 0, F_2, 0, F_3, 0, F_4, \cdots \rangle$$

Thus, the number of solutions

$$N = \begin{cases} 0, & m \text{ is odd,} \\ F_{\frac{m}{2}+1}, & m \text{ is even} \end{cases}$$

where F is the Fibonacci number. $\square$

2 Give the generating function and the exponential generating function for the sequence $\langle 2, 5, 13, 35, \cdots \rangle = \langle 2^n + 3^n \rangle$ in closed form.

Solution. The generating function for $\langle c^n \rangle$ is $\frac{1}{1-cz}$, according to linearity,

$$\langle 2^n + 3^n \rangle \leftrightarrow \frac{1}{1 - 2z} + \frac{1}{1 - 3z}$$

is its generating function.

The exponential generating function for $\langle c^n \rangle$ is

$$\hat{G}(z) = \sum_{n \geq 0} c^n \frac{z^n}{n!} = \sum_{n \geq 0} \frac{(cz)^n}{n!} = e^{cz}$$

Thus, the exponential generating function is

$$\langle 2^n + 3^n \rangle \hat{\leftrightarrow} e^{2z} + e^{3z}$$

$\square$

3 What is $\sum_{n\geq 0} H_n/10^n$?

Solution. By (7.57),

$$\langle H_n \rangle \leftrightarrow \frac{1}{1-z} \ln \frac{1}{1-z}$$

The convergence radius R of $\langle H_n \rangle$ could be represented as

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \leq \sum_{k=1}^{n} 1 = n \Rightarrow R = \frac{1}{\limsup_{n\geq 0} \sqrt[n]{H_n}} = \frac{1}{1} = 1$$

(In fact, by recurrence we could get $1 = H_1 = H_0 + 1 \Rightarrow H_0 = 0$, which is satisfiable.)
Since $\frac{1}{10} < 1$ is within the radius,

$$\sum_{n\geq 0} \frac{H_n}{10^n} = \frac{1}{1-\frac{1}{10}} \ln \frac{1}{1-\frac{1}{10}} = \frac{10}{9} \ln \frac{10}{9}$$

□

4 The general expansion theorem for rational functions $P(z)/Q(z)$ is not completely general, because it restricts the degree of P to be less than the degree of Q. What happens if P has a larger degree than this?

Solution. The problem could be reduced to the scenario where $\deg P < \deg Q$, by applying polynomial division:

$$\frac{P(z)}{Q(z)} = S(z) + \frac{R(z)}{Q(z)}$$

where $\deg R < \deg Q$. Since $S(z)$ only influcence finite amount of items (based on its degree), Rational Expansion Theorem can be applied on the second term. □

Basics

6 Show that the recurrence (7.32) can be solved by the repertoire method, without using generating functions.

待定系数法.

Solution.

迷

$$g_0 = g_1 = 1;$$
$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n, \qquad \text{for } n \geq 2$$

$g_0 = g_1 = \alpha$

$\begin{cases} 2^n \\ (-1)^n \\ n(-1)^n \end{cases}$

不适用.

Consider a general form of

$$g_0 = \alpha;$$
3个初值 $$g_1 = \beta;$$
$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n\gamma, \qquad \text{for } n \geq 2$$

$(-1)^n$

has the closed form of

$$g_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

2^n

不解这一套方法求系数

只要有解，就说明方程

的解是存在的

Case 1: $g_n = 2^n$

$$
\begin{aligned}
\alpha &= 1 \\
\beta &= 2 \\
2^n &= 2^{n-1} + 2 \times 2^{n-2} + (-1)^n\gamma \Rightarrow \gamma = 0 \\
g_n &= A(n) + 2B(n) = 2^n \qquad (1)
\end{aligned}
$$

Case 2: $g_n = (-1)^n$

$$
\begin{aligned}
\alpha &= 1 \\
\beta &= -1 \\
(-1)^n &= (-1)^{n-1} + 2(-1)^{n-2} + (-1)^n\gamma \Rightarrow \gamma = 0 \\
g_n &= A(n) - B(n) = (-1)^n \qquad (2)
\end{aligned}
$$

Case 3: $g_n = n(-1)^n$

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= -1 \\
n(-1)^n &= (n-1)(-1)^{n-1} + 2(n-2)(-1)^{n-2} + (-1)^n\gamma \\
&\Rightarrow n = -n + 1 + 2(n-2) + \gamma \\
\gamma &= 3 \\
g_n &= -B(n) + 3C(n) = n(-1)^n \qquad (3)
\end{aligned}
$$

Combining Eq. (1) – (3), the solution is

$$
\begin{aligned}
A(n) &= \frac{2^n + 2(-1)^n}{3} \\
B(n) &= \frac{2^n - (-1)^n}{3} \\
C(n) &= \frac{n(-1)^n}{3} + \frac{2^n - (-1)^n}{9}
\end{aligned}
$$

So plug in $\alpha = 1, \beta = 1, \gamma = 1$, the closed form is found:

$$
g_n = \frac{7}{9}2^n + \left(\frac{1}{3}n + \frac{2}{9}\right)(-1)^n
$$

□

7 Solve the recurrence

$$
\begin{aligned}
g_0 &= 1 \\
g_n &= g_{n-1} + 2g_{n-2} + \cdots + ng_0, \quad \text{for } n > 0
\end{aligned}
$$

Solution. The recurrence can be representated by the single equation

$$
g_n = \sum_{i=0}^{n-1}(n-i)g_i + [n = 0]
$$

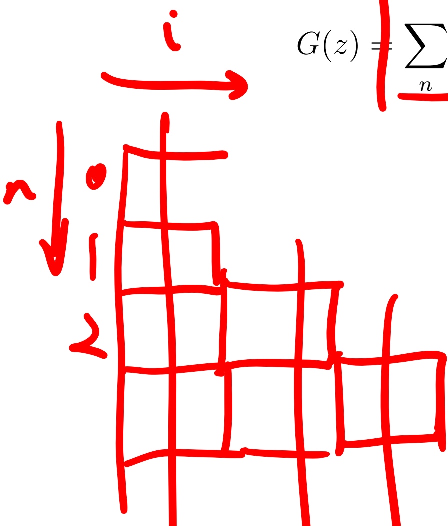
$$\sum_{i=0}^{n-1}\sum_n (n-i) g_i z^n$$

Write down $G(z) = \sum_n g_n z^n$,

$$\sum_{i=1}^{n}\sum_n i\, g_{n-i} z^n$$

$$G(z) = \boxed{\sum_n g_n z^n} = \sum_n \sum_{i=0}^{n-1} (n-i) g_i z^n + \sum_n [n=0] z^n$$

$$\sum_{i=1}\sum_n i\, g_n z^{n+i}$$



$$= 1 + \sum_{i=1}^{n \to \infty} i z^i G(z)$$

$$\sum_{i=1}^{n\to\infty} i \boxed{\sum_n g_n z^{n+i}}$$

$$= 1 + \frac{z}{(1-z)^2} G(z)$$

$$G(z) = \frac{1-2z+z^2}{1-3z+z^2}$$

$$\sum_{i=1}^{n\to\infty} i z^i \left(\sum_n g_n z^n\right)$$

$$= 1 + \frac{z}{1-3z+z^2}$$

$$= 1 + \frac{z}{(1-\beta_1 z)(1-\beta_2 z)}$$

$$\sum_{i=1}^{n\to\infty} i z^i G(z)$$

Theorem gives that

$$g_n = a_1 \beta_1^n + a_2 \beta_2^n$$

$$\frac{1}{(1-z)^2} \leftrightarrow \langle 1, 2, 3, \dots \rangle$$

where

$$a_1 = \frac{-\beta_1 \frac{1}{\beta_1}}{1 - (\beta_1 + \beta_2) + 2\beta_1\beta_2 \frac{1}{\beta_1}} = -\frac{1}{2\beta_2 - 2}$$

$$\frac{z}{(1-z)^2} \leftrightarrow \langle 0, 1, 2, \dots \rangle$$

and

$$a_2 = \frac{-\beta_2 \frac{1}{\beta_2}}{1 - (\beta_1 + \beta_2) + 2\beta_1\beta_2 \frac{1}{\beta_2}} = -\frac{1}{2\beta_1 - 2}$$

Thus,

$$= 1 + \frac{z}{(1-\Phi^2 z)(1-\hat{\Phi}^2 z)}$$

$$g_n = -\frac{1}{2}\left(\frac{\beta_1^n}{\beta_2 - 1} + \frac{\beta_2^n}{\beta_1 - 1}\right)$$
$$= \frac{1}{2}\left(\beta_1^n(\beta_1 - 1) + \beta_2^n(\beta_2 - 1)\right)$$

$$F(z) = \frac{z}{(1-\Phi z)(1-\hat{\Phi} z)}$$

where $\beta_1 = \frac{3+\sqrt{5}}{2}$, $\beta_2 = \frac{3-\sqrt{5}}{2}$. □

$$\frac{3+\sqrt5}{2} = \left(\frac{1+\sqrt5}{2}\right)^2 = \frac{1+5+2\sqrt5}{4} = \frac{3+\sqrt5}{2}$$

$$\frac{3-\sqrt5}{2} = \left(\frac{\sqrt5-1}{2}\right)^2 = \frac{6-2\sqrt5}{4} = \frac{3-\sqrt5}{2}$$

$$\frac{\Phi^n - \hat{\Phi}^n}{\sqrt5}$$

$$\frac{1}{2}\left(\Phi^n\right.$$