Mathematical Foundations of Computer Science

Project 8

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Warmups

6 Can something interesting be said about $\lfloor f(x) \rfloor$ when $f(x)$ is a continuous, monotonically *decreasing* function that takes integer values only when x is an integer?

7 Solve the recurrence

$$\begin{aligned} X_n &= n, && \text{for } 0 \le n < m \\ X_n &= X_{n-m} + 1, && \text{for } n \ge m. \end{aligned}$$

8 Prove the *Dirichlet box* principle: If n objects are put into m boxes, some box must contain $\ge \lceil n/m \rceil$ objects, and some box must contain $\le \lfloor n/m \rfloor$.

9 Egyptian mathematicians in 1800 B.C. represented rational numbers between 0 and 1 as sums of unit fractions $1/x_1 + \cdots + 1/x_k$, where the x's were distinct positive integers. For example, they wrote $1/3 + 1/15$ instead of $2/5$. Prove that it is always possible to do this in a systematic way: If $0 < m/n < 1$, then

$$\frac{m}{n} = \frac{1}{q} + \left\{ \text{representation of } \frac{m}{n} - \frac{1}{q} \right\}, \quad q = \left\lceil \frac{n}{m} \right\rceil$$

(This is *Fibonacci's algorithm*, due to Leonardo Fibonacci, A.D. 1202.)

Basics

10 Show that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

11 Give details of the proof alluded to in the text, that the open interval$(\alpha..\alpha)$contains exactly $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$ integers when $\alpha < \beta$. Why doesthe case $\alpha = \beta$ have to be excluded in order to make the proof correct?

12 Prove that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

for all integers n and all positive integers m. [This identity gives usanother way to convert ceilings to floors and vice versa, instead of using the reflective law (3.4).]

13 Let α and β be positive real numbers. Prove that $\text{Spec}(\alpha)$ and $\text{Spec}(\beta)$ partition the positive integers if and only if α and β are irrational and $1/\alpha + 1/\beta = 1$.