

Project 13

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Warmups

7 Is (5.34) true also when $k < 0$?

Solution. It is also true that

$$r^{\underline{k}} \left(r - \frac{1}{2}\right)^{\underline{k}} = \frac{(2r)^{2k}}{2^{2k}}, \text{ when } k < 0$$

The proof is as follows.

$$\begin{aligned} r^{\underline{k}} \left(r - \frac{1}{2}\right)^{\underline{k}} &= \frac{r}{r^{-k+1}} \frac{r - \frac{1}{2}}{\left(r - \frac{1}{2}\right)^{-k+1}} \\ &= \frac{1}{(r + \frac{1}{2})(r + 1)(r + \frac{3}{2})(r + 2) \cdots (r - \frac{1}{2} - k)(r - k)} \\ &= \frac{2^{-2k}}{(2r + 1)(2r + 2) \cdots (2r - 1 - 2k)(2r - 2k)} \\ \frac{(2r)^{2k}}{2^{2k}} &= \frac{2^{-2k} 2r}{(2r)^{-2k+1}} \\ &= \frac{2^{-2k}}{(2r + 1)(2r + 2) \cdots (2r - 2k)} \end{aligned}$$

And they are the same. □

8 Evaluate

$$\sum_k \binom{n}{k} (-1)^k \left(1 - \frac{k}{n}\right)^n$$

What is the approximate value of this sum, when n is very large? Hint: The sum is $\Delta^n f(0)$ for some function f .

Solution. According to (5.40),

$$\begin{aligned}\Delta^n f(x) &= \sum_k \binom{n}{k} (-1)^{n-k} f(x+k) \\ &= \sum_k \binom{n}{n-k} (-1)^{n-k} f(x+k) \\ &= \sum_k \binom{n}{k} (-1)^k f(x+n-k)\end{aligned}$$

In other word,

$$\Delta^n f(0) = \sum_k \binom{n}{k} (-1)^k f(n-k)$$

Compare to the formula to solve, the function f is

$$\begin{aligned}f(n-k) &= \left(1 - \frac{k}{n}\right)^n = \left(\frac{n-k}{n}\right)^n \\ f(x) &= \left(\frac{x}{n}\right)^n\end{aligned}$$

As a result,

$$\begin{aligned}\sum_k \binom{n}{k} (-1)^k \left(1 - \frac{k}{n}\right)^n &= \Delta^n f(0) \\ &= \left(\Delta^n \left(\frac{x}{n}\right)^n\right)(0) \\ &= \frac{(n-1)!}{n^n} \rightarrow 0, n \rightarrow +\infty\end{aligned}$$

□

- 9 Show that the generalized exponentials of (5.58) obey the law

$$\mathcal{E}_t(z) = \mathcal{E}(tz)^{1/t}, \text{ if } t \neq 0,$$

where $\mathcal{E}(z)$ is an abbreviation for $\mathcal{E}_1(z)$.

Proof. By (5.60),

$$\mathcal{E}_t(z)^r = \sum_{k \geq 0} r \frac{(tk+r)^{k-1}}{k!} z^k$$

By assigning $r = t$, and notice that $t \neq 0$,

$$\begin{aligned}\mathcal{E}_t(z)^t &= \sum_{k \geq 0} t \frac{(tk+t)^{k-1}}{k!} z^k \\ &= \sum_{k \geq 0} (k+1)^{k-1} \frac{(tz)^k}{k!} \\ &= \mathcal{E}(tz)\end{aligned}$$

which is the same as the original formula.

□

Basics

- 14 Prove identity (5.25) by negating the upper index in Vandermonde's convolution (5.22). Then show that another negation yields (5.26).

Proof. For (5.25),

$$\begin{aligned}
\sum_{k \leq l} \binom{l-k}{m} \binom{s}{k-n} (-1)^k &= \sum_{k \leq l} \binom{l-k}{l-k-m} \binom{s}{k-n} (-1)^k && \text{(symmetry)} \\
&= \sum_{k \leq l} (-1)^{l-k-m} \binom{-m-1}{l-k-m} \binom{s}{k-n} (-1)^k && \text{(negation)} \\
&= (-1)^{l-m} \sum_k \binom{s}{-n+k} \binom{-m-1}{l-m-k} && (m \geq 0) \\
&= (-1)^{l-m} \binom{s-m-1}{l-m-n} && \text{(by 5.22)} \\
&= (-1)^{l+m} \binom{s-m-1}{l-m-n} && ((-1)^{2m} = 1)
\end{aligned}$$

The step on $(m \geq 0)$ holds for the reason when $k > l$:

$$l - k - m \leq -1 - m < 0 \Rightarrow \binom{-m-1}{l-k-m} = 0, \quad (m \geq 0) \quad (1)$$

For (5.26),

$$\begin{aligned}
\sum_{0 \leq k \leq l} \binom{l-k}{m} \binom{q+k}{n} &= \sum_{0 \leq k \leq l} \binom{l-k}{l-k-m} \binom{q+k}{q+k-n} && \text{(symmetry)} \\
&= \sum_{0 \leq k \leq l} \binom{-m-1}{l-k-m} \binom{-n-1}{q+k-n} (-1)^{l-k-m+q+k-n} && \text{(negation)} \\
&= (-1)^{l-m+q-n} \sum_k \binom{-m-1}{l-k-m} \binom{-n-1}{q+k-n} && (m \geq 0, n \geq q) \\
&= (-1)^{l-m+q-n} \binom{-m-n-2}{l-m+q-n} && \text{(by 5.22)} \\
&= \binom{l+q+1}{l-m+q-n} && \text{(negation)} \\
&= \binom{l+q+1}{m+n-1} && \text{(symmetry)}
\end{aligned}$$

and the step on $(m \geq 0, n \geq q)$ holds for

$$k < 0 \Rightarrow q + k - n \leq n + k - n = k < 0 \Rightarrow \binom{-n-1}{q+k-n} = 0$$

the first term holds for the same reason as Eq. (1). □

- 15 What is $\sum_k \binom{n}{k}^3 (-1)^k$? Hint: See (5.29).

Solution. By (5.29),

$$\sum_k \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} (-1)^k = \frac{(a+b+c)!}{a!b!c!}, \quad a, b, c \in \mathbb{N}.$$

when $n = 2m (m \in \mathbb{N})$, the original formula could be deduced to

$$\begin{aligned} \sum_k \binom{n}{k}^3 (-1)^k &= \sum_k \binom{n}{n-k}^3 (-1)^k && \text{(symmetry)} \\ &= \sum_k \binom{n}{n+k}^3 (-1)^{-k} && (k \rightarrow -k) \\ &= \sum_k \binom{n}{n+k} \binom{n}{n+k} \binom{n}{n+k} (-1)^k (-1)^{-2k} && ((-1)^{-2k} = 1) \\ &= \frac{(m+m+m)!}{m!m!m!} && \text{(by 5.29)} \\ &= \frac{(3m)!}{m!^3} \end{aligned}$$

when $n = 2m + 1 (m \in \mathbb{N})$,

$$\sum_k \binom{n}{k}^3 (-1)^k = \sum_k \binom{n}{n-k}^3 (-1)^k$$

expand both parts,

$$\begin{aligned} &\binom{n}{0}^3 + \binom{n}{1}^3 \times (-1) + \cdots + \binom{n}{n-1}^3 + \binom{n}{n}^3 \times (-1) \\ &= \binom{n}{n}^3 + \binom{n}{n-1}^3 \times (-1) + \cdots + \binom{n}{1}^3 + \binom{n}{0}^3 \times (-1) \\ &= - \left[\binom{n}{0}^3 + \binom{n}{1}^3 \times (-1) + \cdots + \binom{n}{n-1}^3 + \binom{n}{n}^3 \times (-1) \right] \end{aligned}$$

then,

$$2 \sum_k \binom{n}{k}^3 (-1)^k = 0 \Rightarrow \sum_k \binom{n}{k}^3 (-1)^k = 0$$

when $n < 0$,

$$\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}$$

and the original formula comes to

$$\sum_k \binom{n}{k}^3 (-1)^k = \sum_{k \geq 0} \binom{-n+k-1}{k}^3$$

and $(-n+k-1) - k = -n-1 \geq 0$, the symmetry property disappears. Every term is close to 1 when n grows, so this is $+\infty$ when $n \rightarrow +\infty$. In conclusion,

$$\sum_k \binom{n}{k}^3 (-1)^k = \begin{cases} \frac{(3m)!}{m!^3}, & \text{if } n = 2m (m \in \mathbb{N}), \\ 0, & \text{if } n = 2m + 1 (m \in \mathbb{N}), \\ +\infty, & \text{if } n < 0. \end{cases}$$

□

16 Evaluate the sum

$$\sum_k \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} (-1)^k$$

when a, b, c are nonnegative integers.

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Solution.

$$\sum_k \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} (-1)^k = \sum_k \frac{(2a)!}{(a+k)!(a-k)!} \frac{(2b)!}{(b+k)!(b-k)!} \frac{(2c)!}{(c+k)!(c-k)!} (-1)^k$$

compared with (5.29),

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$$\sum_k \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} (-1)^k = \sum_k \frac{(a+b)!}{(a+k)!(b-k)!} \frac{(b+c)!}{(b+k)!(c-k)!} \frac{(c+a)!}{(c+k)!(a-k)!} (-1)^k$$

then,

$$\begin{aligned} \sum_k \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} (-1)^k &= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(c+a)!} \sum_k \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} (-1)^k \\ &= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(c+a)!} \frac{(a+b+c)!}{a!b!c!} \end{aligned}$$

□

17 Find a simple relation between $\binom{2n-1/2}{n}$ and $\binom{2n-1/2}{2n}$.

Solution. By (5.35),

$$\binom{r}{k} \binom{r-1/2}{k} = \binom{2r}{2k} \binom{2k}{k} / 2^{2k}$$

and assigning $r = 2n, k = n$,

$$\binom{2n-1/2}{n} = \frac{\binom{4n}{2n} \binom{2n}{n}}{2^{2n} \binom{2n}{n}} = \binom{4n}{2n} / 2^{2n}$$

assigning $r = 2n, k = 2n$,

$$\binom{2n-1/2}{2n} = \frac{\binom{4n}{4n} \binom{4n}{2n}}{2^{4n} \binom{2n}{2n}} = \binom{4n}{2n} / 2^{4n}$$

yields the simple relation of

$$\binom{2n-1/2}{n} = 2^{2n} \binom{2n-1/2}{2n}$$

□

18 Find an alternative form analogous to (5.35) for the product

$$\binom{r}{k} \binom{r-1/3}{k} \binom{r-2/3}{k}$$

Solution.

$$\begin{aligned}
 & r^{\underline{k}} \left(r - \frac{1}{3}\right)^{\underline{k}} \left(r - \frac{2}{3}\right)^{\underline{k}} \\
 &= r \left(r - \frac{1}{3}\right) \left(r - \frac{2}{3}\right) (r-1) \left(r - \frac{4}{3}\right) \left(r - \frac{5}{3}\right) \cdots (r-k+1) \left(r - k + \frac{2}{3}\right) \left(r - k + \frac{1}{3}\right) \\
 &= \frac{(3r)(3r-1)(3r-2)(3r-3)(3r-4)(3r-5) \cdots (3r-3k+3)(3r-3k+2)(3r-3k+1)}{3^{3k}}
 \end{aligned}$$

divide both sides by $k!^3$,

$$\begin{aligned}
 \frac{r^{\underline{k}}}{k!} \frac{\left(r - \frac{1}{3}\right)^{\underline{k}}}{k!} \frac{\left(r - \frac{2}{3}\right)^{\underline{k}}}{k!} &= \frac{1}{3^{3k}} \frac{(3r)^{3k}}{k!^3} \\
 \binom{r}{k} \binom{r - \frac{1}{3}}{k} \binom{r - \frac{2}{3}}{k} &= \frac{1}{3^{3k}} \frac{(3r)^{3k}}{(3k)!} \frac{(3k)!}{k!^3} \\
 \binom{r}{k} \binom{r - \frac{1}{3}}{k} \binom{r - \frac{2}{3}}{k} &= \frac{1}{3^{3k}} \binom{3r}{3k} \frac{(3k)! (2k)^{\underline{k}}}{(2k)! k!} \\
 \binom{r}{k} \binom{r - \frac{1}{3}}{k} \binom{r - \frac{2}{3}}{k} &= \frac{1}{3^{3k}} \binom{3r}{3k} \binom{3k}{2k} \binom{2k}{k}
 \end{aligned}$$

$$\prod \binom{j+1}{j}^k$$

$$\begin{aligned}
 &= \frac{((j+1)k)!}{(jk)! k!} \frac{(mk)!}{(k!)^m} \\
 &= \frac{(jk)!}{(j-1)k! k!} \square \\
 &= \frac{(j-1)k! k!}{(2k)!} \square
 \end{aligned}$$

$$\begin{aligned}
 \prod_{i=0}^{n-1} \binom{r - \frac{i}{m}}{k} &= \prod \frac{\left(r - \frac{i}{m}\right)^{\underline{k}}}{k!} \\
 &= \prod \frac{\left(r - \frac{i}{m}\right) \left(r - \frac{i}{m} - 1\right) \cdots \left(r - \frac{i}{m} - (k-1)\right)}{k!}
 \end{aligned}$$

$$= \prod \frac{(mr-i)(mr-i-m) \cdots (mr-i-(k-1)m)}{m^k k!}$$

$$\begin{aligned}
 &= \frac{(mr)^{\underline{mk}}}{m^{mk} (k!)^m} = \frac{1}{m^{mk}} \frac{(mr)^{\underline{mk}}}{(mk)!} \frac{(mk)!}{(k!)^m} \\
 &= \frac{1}{m^{mk}} \binom{mr}{mk} \frac{(mk)!}{(k!)^m}
 \end{aligned}$$

$$= \frac{1}{m^{mk}} \binom{mr}{mk} \prod_{j=1}^{m-1} \binom{j+1}{j}^k$$