27 Compute $\Delta(c^{\underline{x}})$, and use it to deduce the value of $\sum_{k=1}^{n}(-2)^{\underline{k}}/k$.

28 At what point does the following derivation go astray?

Exam problems

29 Evaluate the sum $\sum_{k=1}^{n}(-1)^k k/(4k^2-1)$.

27 Prove that infinitely many of the numbers $D_n^{(3)}$ defined by (3.20) are even, and that infinitely many are odd.

28 Solve the recurrence

$$\begin{aligned} a_0 &= 1\,; \\ a_n &= a_{n-1} + \lfloor\sqrt{a_{n-1}}\rfloor, \qquad \text{for } n > 0. \end{aligned}$$

31 Prove or disprove: $\lfloor x\rfloor + \lfloor y\rfloor + \lfloor x+y\rfloor \leqslant \lfloor 2x\rfloor + \lfloor 2y\rfloor$.

33 Show that if $f(m)$ and $g(m)$ are multiplicative functions, then so is $h(m) = \sum_{d\backslash m} f(d)\,g(m/d)$.

factor of n.

47 Show that if $n^{m-1} \equiv 1 \pmod{m}$ and if $n^{(m-1)/p} \not\equiv 1 \pmod{m}$ for all primes such that $p \backslash (m-1)$, then m is prime. *Hint:* Show that if this condition holds, the numbers $n^k \bmod m$ are distinct, for $1 \leqslant k < m$.

37 Show that an analog of the binomial theorem holds for factorial powers. That is, prove the identities

$$\begin{aligned}(x+y)^{\underline{n}} &= \sum_k \binom{n}{k} x^{\underline{k}}\, y^{\underline{n-k}}\,,\\ (x+y)^{\overline{n}} &= \sum_k \binom{n}{k} x^{\overline{k}}\, y^{\overline{n-k}}\,,\end{aligned}$$

for all nonnegative integers n.

47 The sum

$$\sum_k \binom{rk+s}{k}\binom{rn-rk-s}{n-k}$$

is a polynomial in r and s. Show that it doesn't depend on s.

11 This problem, whose three parts are independent, gives practice in the manipulation of generating functions. We assume that $A(z) = \sum_n a_n z^n$, $B(z) = \sum_n b_n z^n$, $C(z) = \sum_n c_n z^n$, and that the coefficients are zero for negative n.

Superman.

a If $c_n = \sum_{j+2k \leqslant n} a_j b_k$, express C in terms of A and B.

b If $nb_n = \sum_{k=0}^{n} 2^k a_k/(n-k)!$, express A in terms of B.

c If r is a real number and if $a_n = \sum_{k=0}^{n} \binom{r+k}{k} b_{n-k}$, express A in terms of B; then use your formula to find coefficients $f_k(r)$ such that $b_n = \sum_{k=0}^{n} f_k(r) a_{n-k}$.

26 The second-order Fibonacci numbers $\langle \mathfrak{F}_n \rangle$ are defined by the recurrence

$$\begin{aligned} \mathfrak{F}_0 &= 0; \qquad \mathfrak{F}_1 = 1; \\ \mathfrak{F}_n &= \mathfrak{F}_{n-1} + \mathfrak{F}_{n-2} + F_n, \qquad \text{for } n > 1. \end{aligned}$$

Express $\mathfrak{F}_n$ in terms of the usual Fibonacci numbers F_n and F_{n+1}.

27 A $2 \times n$ domino tiling can also be regarded as a way to draw n disjoint