Mathematical Foundations of Computer Science

Project 13

Log Creative

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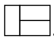
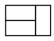
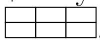
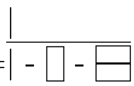
Warmups

1 An eccentric collector of $2 \times n$ domino tilings pays \$4 for each vertical domino and \$1 for each horizontal domino. How many tilings are worth exactly $\$m$ by this criterion? For example, when $m = 6$ there are three solutions: , , and .

Solution. Like what has been done in (7.5), (7.4) could be transferred into

$T =$  $= \frac{1}{1-z^4-z^2}$

whose sequence is

$$\frac{1}{1-z^4-z^2} = \frac{1}{1-z^2-(z^2)^2} \leftrightarrow \langle 0, F_2, 0, F_3, 0, F_4, \cdots \rangle$$

Thus, the number of solutions

$$N = \begin{cases} 0, & m \text{ is odd,} \\ F_{\frac{m}{2}+1}, & m \text{ is even} \end{cases}$$

where F is the Fibonacci number. □

2 Give the generating function and the exponential generating function for the sequence $\langle 2, 5, 13, 35, \cdots \rangle = \langle 2^n + 3^n \rangle$ in closed form.

Solution. The generating function for $\langle c^n \rangle$ is $\frac{1}{1-cz}$, according to linearity,

$$\langle 2^n + 3^n \rangle \leftrightarrow \frac{1}{1-2z} + \frac{1}{1-3z}$$

is its generating function.

The exponential generating function for $\langle c^n \rangle$ is

$$\hat{G}(z) = \sum_{n \geq 0} c^n \frac{z^n}{n!} = \sum_{n \geq 0} \frac{(cz)^n}{n!} = e^{cz}$$

Thus, the exponential generating function is

$$\langle 2^n + 3^n \rangle \hat{\leftrightarrow} e^{2z} + e^{3z}$$

□

3 What is $\sum_{n\geq 0} H_n/10^n$?

Solution. By (7.57),

$$\langle H_n\rangle \leftrightarrow \frac{1}{1-z}\ln\frac{1}{1-z}$$

The convergence radius R of $\langle H_n\rangle$ could be representated as

$$H_n = \sum_{k=1}^{n}\frac{1}{k} \leq \sum_{k=1}^{n} 1 = n \Rightarrow R = \frac{1}{\limsup_{n\geq 0}\sqrt[n]{H_n}} = \frac{1}{1} = 1$$

(In fact, by recurrence we could get $1 = H_1 = H_0 + 1 \Rightarrow H_0 = 0$, which is satisfiable.)
Since $\frac{1}{10} < 1$ is within the radius,

$$\sum_{n\geq 0}\frac{H_n}{10^n} = \frac{1}{1-\frac{1}{10}}\ln\frac{1}{1-\frac{1}{10}} = \frac{10}{9}\ln\frac{10}{9}$$

□

4 The general expansion theorem for rational functions $P(z)/Q(z)$ is not completely general, because it restricts the degree of P to be less than the degree of Q. What happens if P has a larger degree than this?

Solution. The problem could be reduced to the scenario where $\deg P < \deg Q$, by applying polynomial division:

$$\frac{P(z)}{Q(z)} = S(z) + \frac{R(z)}{Q(z)}$$

where $\deg R < \deg Q$. Since $S(z)$ only influcence finite amount of items (based on its degree), Rational Expansion Theorem can be applied on the second term. □

Basics

6 Show that the recurrence (7.32) can be solved by the repertoire method, without using generating functions.

Solution.

$$\begin{aligned} g_0 &= g_1 = 1; \\ g_n &= g_{n-1} + 2g_{n-2} + (-1)^n, && \text{for } n \geq 2 \end{aligned}$$

Consider a general form of

$$\begin{aligned} g_0 &= \alpha; \\ g_1 &= \beta; \\ g_n &= g_{n-1} + 2g_{n-2} + (-1)^n\gamma, && \text{for } n \geq 2 \end{aligned}$$

has the closed form of

$$g_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

Case 1: $g_n = 2^n$

$$\begin{aligned}\alpha &= 1\\ \beta &= 2\\ 2^n &= 2^{n-1} + 2 \times 2^{n-2} + (-1)^n\gamma \Rightarrow \gamma = 0\\ g_n &= A(n) + 2B(n) = 2^n \qquad (1)\end{aligned}$$

Case 2: $g_n = (-1)^n$

$$\begin{aligned}\alpha &= 1\\ \beta &= -1\\ (-1)^n &= (-1)^{n-1} + 2(-1)^{n-2} + (-1)^n\gamma \Rightarrow \gamma = 0\\ g_n &= A(n) - B(n) = (-1)^n \qquad (2)\end{aligned}$$

Case 3: $g_n = n(-1)^n$

$$\begin{aligned}\alpha &= 0\\ \beta &= -1\\ n(-1)^n &= (n-1)(-1)^{n-1} + 2(n-2)(-1)^{n-2} + (-1)^n\gamma\\ &\Rightarrow n = -n + 1 + 2(n-2) + \gamma\\ \gamma &= 3\\ g_n &= -B(n) + 3C(n) = n(-1)^n \qquad (3)\end{aligned}$$

Combining Eq. (1) – (3), the solution is

$$\begin{aligned}A(n) &= \frac{2^n + 2(-1)^n}{3}\\ B(n) &= \frac{2^n - (-1)^n}{3}\\ C(n) &= \frac{n(-1)^n}{3} + \frac{2^n - (-1)^n}{9}\end{aligned}$$

So plug in $\alpha = 1, \beta = 1, \gamma = 1$, the closed form is found:

$$g_n = \frac{7}{9}2^n + \left(\frac{1}{3}n + \frac{2}{9}\right)(-1)^n$$

□

7 Solve the recurrence

$$\begin{aligned}g_0 &= 1\\ g_n &= g_{n-1} + 2g_{n-2} + \cdots + ng_0, \quad \text{for } n > 0\end{aligned}$$

Solution. The recurrence can be representated by the single equation

$$g_n = \sum_{i=0}^{n-1}(n-i)g_i + [n=0]$$

Write down $G(z) = \sum_n g_n z^n$,

$$\begin{aligned}
G(z) = \sum_n g_n z^n &= \sum_n \sum_{i=0}^{n-1} (n-i) g_i z^n + \sum_n [n=0] z^n \\
&= 1 + \sum_n \sum_{i=1}^{n} i g_{n-i} z^n \\
&= 1 + \sum_{i=1}^{\infty} \sum_n i g_{n-i} z^n \\
&= 1 + \sum_{i=1}^{\infty} \sum_n i g_n z^{n+i} \\
&= 1 + \sum_{i=1}^{\infty} i z^i \sum_n g_n z^n \\
&= 1 + \sum_{i=1}^{\infty} i z^i G(z) \\
&= 1 + \frac{z}{(1-z)^2} G(z) \\
G(z) &= \frac{1-2z+z^2}{1-3z+z^2} \\
&= 1 + \frac{z}{1-3z+z^2} \\
&= 1 + \frac{z}{(1-\beta_1 z)(1-\beta_2 z)}
\end{aligned}$$

Theorem gives that

$$g_n = [n=0] + a_1 \beta_1^n + a_2 \beta_2^n$$

where

$$a_1 = \frac{-\beta_1 \frac{1}{\beta_1}}{-3 + 2\frac{1}{\beta_1}} = \frac{-\beta_1}{-3\beta_1 + 2} = -\frac{1}{\sqrt{5}}$$

and

$$a_2 = \frac{-\beta_2 \frac{1}{\beta_2}}{-3 + 2\frac{1}{\beta_2}} = \frac{-\beta_2}{-3\beta_2 + 2} = \frac{1}{\sqrt{5}}$$

Thus,

$$g_n = [n=0] + \frac{\beta_2^n - \beta_1^n}{\sqrt{5}}$$

where $\beta_1 = \frac{3+\sqrt{5}}{2}, \beta_2 = \frac{3-\sqrt{5}}{2}$. Notice that $\beta_1 = \hat{\Phi}^2, \beta_2 = \Phi^2$, thus,

$$\begin{aligned}
g_n &= [n=0] + \frac{\Phi^{2n} - \hat{\Phi}^{2n}}{\sqrt{5}} \\
&= [n=0] + F_{2n}
\end{aligned}$$

where $F_n = \frac{\Phi^n - \hat{\Phi}^n}{\sqrt{5}}$ is the Fibonacci number. $\square$