Mathematical Foundations of Computer Science

Project 5

Log Creative

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Warmups

6 What is the value of $\sum_k [1 \le j \le k \le n]$, as a function of j and n?

Solution.

$$\sum_k [1 \le j \le k \le n](j,n) = \begin{cases} 0, & j > n \text{ or } j < 1, \\ n - j + 1, & \text{else.} \end{cases}$$

□

Basics

11 The general rule (2.56) for summation by parts is equivalent to

$$\sum_{0 \le k < n} (a_{k+1} - a_k) b_k = a_n b_n - a_0 b_0 - \sum_{0 \le k < n} a_{k+1}(b_{k+1} - b_k), \quad \text{for } n \ge 0$$

Prove this formula directly by using the distributive, associative, and commutative laws.

Proof.

$$\begin{aligned}
\sum_{0 \le k < n} (a_{k+1} - a_k) b_k &= \sum_{0 \le k < n} a_{k+1} b_k - \sum_{0 \le k < n} a_k b_k && \text{(associative law)} \\
&= \sum_{0 \le k < n} a_{k+1} b_k - \sum_{-1 \le k < n-1} a_{k+1} b_{k+1} && \text{(commutative law)} \\
&= \sum_{0 \le k < n} a_{k+1} b_k - \left(\sum_{0 \le k < n} a_{k+1} b_{k+1} + a_0 b_0 - a_n b_n \right) && \text{(commutative law)} \\
&= a_n b_n - a_0 b_0 + \sum_{0 \le k < n} a_{k+1} b_k - \sum_{0 \le k < n} a_{k+1} b_{k+1} \\
&= a_n b_n - a_0 b_0 + \sum_{0 \le k < n} a_{k+1}(b_{k+1} - b_k) && \text{(distributive law)}
\end{aligned}$$

□

12 Show that the function $p(k) = k + (-1)^k c$ is a permutation of the set of all integers, whenever c is an integer.

Proof. Assume $p(k) = n$, then

$$n + c = k + \left((-1)^k + 1\right) c$$

By beging the index of -1,

$$(-1)^{n+c} = (-1)^{k+\left((-1)^k+1\right)c}$$

because $\left((-1)^k + 1\right)$ is always even, thus

$$(-1)^{n+c} = (-1)^k$$

So, plug $k = n - (-1)^k c = n - (-1)^{n+c} c$ back to $p(k)$

$$p(k) = n - (-1)^k c + (-1)^k c = n$$

So a given n, there is always $k = (-1)^{n+c} c$ such that $p(k) = n$. Because this is a surjective function, k's are unique (otherwise it is not a funciton). It is avaliable for all integers, thus k's will expand to all integers and $p(k)$ is a bijective. Thus, $p(k)$ is a permutation of the set of all integers. □

13 Use the repertoire method to find a closed form for $\sum_{k=0}^{n} (-1)^k k^2$.

Solution. Consider the recurrence of

$$\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows a closed form of

$$R(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$$

Case 1: $R(n) = 1$

$$\begin{aligned} 1 &= \alpha \\ 1 &= 1 + (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows $\beta = \gamma = \delta = 0$.

$$A(n) = 1$$

Case 2: $R(n) = (-1)^n$

$$\begin{aligned} 1 &= \alpha \\ (-1)^n &= (-1)^{n-1} + (-1)^n (\beta + n\gamma + n^2\delta) \\ 2 \times (-1)^n &= (-1)^n (\beta + n\gamma + n^2\delta) \end{aligned}$$

follows $\beta = 2, \gamma = \delta = 0$

$$\begin{aligned} (-1)^n &= 1 + 2B(n) \\ B(n) &= \frac{(-1)^n - 1}{2} \end{aligned}$$

Case 3: $R(n) = (-1)^n n$

$$\begin{aligned}
0 &= \alpha \\
(-1)^n n &= (-1)^{n-1}(n-1) + (-1)^n(\beta + n\gamma + n^2\delta) \\
(-1)^n(2n-1) &= (-1)^n(\beta + n\gamma + n^2\delta)
\end{aligned}$$

follows $\beta = -1, \gamma = 2, \delta = 0$

$$\begin{aligned}
(-1)^n n &= -B(n) + 2C(n) \\
C(n) &= \frac{2(-1)^n n + (-1)^n - 1}{4}
\end{aligned}$$

Case 4: $R(n) = (-1)^n n^2$

$$\begin{aligned}
0 &= \alpha \\
(-1)^n n^2 &= (-1)^{n-1}(n-1)^2 + (-1)^n(\beta + n\gamma + n^2\delta) \\
(-1)^n(2n^2 - 2n + 1) &= (-1)^n(\beta + n\gamma + n^2\delta)
\end{aligned}$$

follows $\beta = 1, \gamma = -2, \delta = 2$

$$\begin{aligned}
(-1)^n n^2 &= B(n) - 2C(n) + 2D(n) \\
(-1)^n n^2 &= \frac{(-1)^n - 1}{2} - 2\frac{2(-1)^n n + (-1)^n - 1}{4} + 2D(n) \\
D(n) &= \frac{(-1)^n}{2}(n^2 + n)
\end{aligned}$$

In this problem, $\alpha = 0, \beta = 0, \gamma = 0, \delta = 1$,thus

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k k^2 &= D(n) \\
&= \frac{(-1)^n}{2}(n^2 + n)
\end{aligned}$$

$\square$