

## Project 12

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### Warmups

- 1 What is  $11^4$ ? Why is this number easy to compute, for a person who knows binomial coefficients?

**Solution.**

$$11^4 = (10 + 1)^4 = \sum_{k=0}^4 \binom{4}{k} 10^k \cdot 1^{4-k} = 14641$$

□

- 2 For which value(s) of  $k$  is  $\binom{n}{k}$  a maximum, when  $n$  is a given positive integer? Prove your answer.

**Proof.**  $k = \lfloor \frac{n}{2} \rfloor$  or  $\lceil \frac{n}{2} \rceil$ .

$k$  satisfies:

$$\begin{cases} \binom{n}{k-1} \leq \binom{n}{k} \\ \binom{n}{k+1} \leq \binom{n}{k} \end{cases} \Rightarrow \begin{cases} \frac{n!}{(k-1)!(n-k+1)!} \leq \frac{n!}{k!(n-k)!} \\ \frac{n!}{(k+1)!(n-k-1)!} \leq \frac{n!}{k!(n-k)!} \end{cases} \Rightarrow \begin{cases} k \leq n-k+1 \\ n-k \leq k+1 \end{cases}$$

In other word,

$$\frac{n-1}{2} \leq k \leq \frac{n+1}{2}$$

If  $n = 2l$  ( $l \in \mathbb{N}$ ), then  $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = \frac{n}{2} \in [\frac{n-1}{2}, \frac{n+1}{2}]$ ; if  $n = 2l + 1$  ( $l \in \mathbb{N}$ ), then  $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$  and  $\frac{n+1}{2} = \lceil \frac{n}{2} \rceil$ . □

- 3 Prove the hexagon property,

$$\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n-1}{k} \binom{n+1}{k+1} \binom{n}{k-1}$$

**Proof.**

$$\begin{aligned}
\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} &= \binom{n-1}{k} \binom{n+1}{k+1} \binom{n}{k-1} \\
&\Leftarrow \frac{(n-1)!n!(n+1)!}{(k-1)!(n-k)!(k+1)!(n-k-1)!k!(n+1-k)!} \\
&= \frac{(n-1)!n!(n+1)!}{k!(n-1-k)!(k+1)!(n-k)!(k-1)!(n-k+1)!}
\end{aligned}$$

Two sides are same. □

- 4 Evaluate  $\binom{-1}{k}$  by negating (actually un-negating) its upper index.

**Solution.**

$$\binom{-1}{k} = \frac{(-1)^{\bar{k}}}{k!} = \frac{(-1)^k 1^{\bar{k}}}{k!} = \frac{(-1)^k k^{\bar{k}}}{k!} = (-1)^k \binom{k}{k} = (-1)^k [k \geq 0]$$

or by upper negation,

$$\binom{-1}{k} = (-1)^k \binom{k - (-1) - 1}{k} = (-1)^k \binom{k}{k} = (-1)^k [k \geq 0]$$

□

- 5 Let  $p$  be prime. Show that  $\binom{p}{k} \bmod p = 0$  for  $0 < k < p$ . What does this imply about the binomial coefficients  $\binom{p-1}{k}$ ?

**Solution.**  $\binom{p}{k} = \frac{p!}{k!(p-k)!} = p \cdot \frac{(p-1)!}{k!(p-k)!}$  because  $p$  is prime and every term in  $k!(p-k)!$  is smaller than  $p$  so that no one can cancel  $p$  where  $0 < k < p$ . Thus,  $\binom{p}{k}, p \in \mathbb{N} \Rightarrow \frac{(p-1)!}{k!(p-k)!} \in \mathbb{N} \Rightarrow p \mid \binom{p}{k}$ .

Since  $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$ , then

$$\binom{p-1}{k} \equiv -\binom{p-1}{k-1} \pmod{p}$$

and assume  $A_k = \binom{p-1}{k}$ , then

$$A_k \equiv -A_{k-1} \pmod{p}$$

and  $A_0 = \binom{p-1}{0} = 1 \pmod{p}$ , so

$$\binom{p-1}{k} = A_k \equiv (-1)^k \pmod{p}$$

□

- 6 Fix up the text's derivation in Problem 6, Section 5.2, by correctly applying symmetry.

**Solution.**

$$\frac{1}{n+1} \sum_k \binom{n+k}{k} \binom{n+1}{k+1} (-1)^k = \frac{1}{n+1} \sum_k \binom{n+k}{n} \binom{n+1}{k+1} (-1)^k$$

is wrong. The correct one should be

$$\frac{1}{n+1} \sum_k \binom{n+k}{k} \binom{n+1}{k+1} (-1)^k = \frac{1}{n+1} \sum_{k \geq 0} \binom{n+k}{n} \binom{n+1}{k+1} (-1)^k$$

because

$$\binom{n-1}{-1} = 0 \quad \binom{n-1}{n} = \frac{(n-1)^n}{n!} = [n=0]$$

which is cancelled and the conversion is not correct. And the remaining result should be

$$\begin{aligned} \frac{1}{n+1} \sum_k \binom{n+k}{n} \binom{n+1}{k+1} (-1)^k - \frac{1}{n+1} \binom{n-1}{n} \binom{n+1}{0} (-1)^{-1} \\ = \frac{1}{n+1} \sum_k \binom{n+k}{n} \binom{n+1}{k+1} (-1)^k + [n=0] \\ = [n=0] \end{aligned}$$

as the result desired. □

## Basics

- 13** Find relations between the superfactorial function  $P_n = \prod_{k=1}^n k!$  of exercise 4.55, the hyperfactorial function  $Q_n = \prod_{k=1}^n k^k$ , and the product  $R_n = \prod_{k=0}^n \binom{n}{k}$ .

**Solution.**

$$R_n = \prod_{k=0}^n \binom{n}{k} = \prod_{k=0}^n \frac{n!}{k!(n-k)!} = \frac{n!^{n+1}}{P_n^2} = \frac{Q_n n(n-1)^2 \cdots 1^n}{P_n^2} = \frac{Q_n P_n}{P_n^2} = \frac{Q_n}{P_n}$$

□