Project 7

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Warmups

1 When we analyzed the Josephus problem in Chapter 1, we represented an arbitrary positive integer n in the form $n = 2^m + l$, where $0 \le l < 2^m$. Give explicit formulas for l and m as functions of n, using floor and/or ceiling brackets.

Solution.

$$m = \lfloor \log_2 n \rfloor$$
$$l = n - 2^{\lfloor \log_2 n \rfloor}$$

□

2 What is a formula for the nearest integer to a given real number x? In case of ties, when x is exactly halfway between two integers, give an expression that rounds (a) up – that is, to $\lceil x \rceil$; (b) down – that is, to $\lfloor x \rfloor$.

Solution. Let $\boxed{x}$ be the nearest integer for real number x. It is either $\lfloor x \rfloor$ or $\lceil x \rceil$, which is decided by comparing $x - \lfloor x \rfloor$ and $\lceil x \rceil - x$.

Case (a): rounds up Because

$$\frac{\lfloor x \rfloor + \lceil x \rceil}{2} = \frac{2\lceil x \rceil - 1}{2} = \lceil x \rceil - \frac{1}{2}$$

$$\boxed{x} = \begin{Bmatrix} \lceil x \rceil, & x \ge \dfrac{\lfloor x \rfloor + \lceil x \rceil}{2}, \\ \lfloor x \rfloor, & x < \dfrac{\lfloor x \rfloor + \lceil x \rceil}{2} \end{Bmatrix} = \begin{Bmatrix} \lceil x \rceil, & x + \dfrac{1}{2} \ge \lceil x \rceil, \\ \lfloor x \rfloor, & x + \dfrac{1}{2} < \lceil x \rceil \end{Bmatrix}$$
$$= \begin{Bmatrix} \lceil x \rceil, & \lceil x \rceil + 1 > x + \dfrac{1}{2} \ge \lceil x \rceil, \\ \lfloor x \rfloor, & \lfloor x \rfloor \le x + \dfrac{1}{2} < \lceil x \rceil \end{Bmatrix} = \left\lfloor x + \frac{1}{2} \right\rfloor$$

Case (b): rounds down Because

$$\frac{\lfloor x\rfloor + \lceil x\rceil}{2} = \frac{2\lfloor x\rfloor + 1}{2} = \lfloor x\rfloor + \frac{1}{2}$$

$$\boxed{x} = \begin{Bmatrix} \lceil x\rceil, & x > \dfrac{\lfloor x\rfloor + \lceil x\rceil}{2}, \\ \lfloor x\rfloor, & x \le \dfrac{\lfloor x\rfloor + \lceil x\rceil}{2} \end{Bmatrix} = \begin{Bmatrix} \lceil x\rceil, & x - \dfrac{1}{2} > \lfloor x\rfloor, \\ \lfloor x\rfloor, & x - \dfrac{1}{2} \le \lfloor x\rfloor \end{Bmatrix}$$

$$= \begin{Bmatrix} \lceil x\rceil, & \lceil x\rceil \ge x - \dfrac{1}{2} > \lfloor x\rfloor, \\ \lfloor x\rfloor, & \lfloor x\rfloor - 1 < x - \dfrac{1}{2} \le \lfloor x\rfloor \end{Bmatrix} = \left\lceil x - \frac{1}{2}\right\rceil$$

□

3 Evaluate $\lfloor\lfloor m\alpha\rfloor n/\alpha\rfloor$, when m and n are positive integers and α is an irrational number greater than n.

Solution.

$$\left\lfloor \frac{\lfloor m\alpha\rfloor n}{\alpha}\right\rfloor < \left\lfloor \frac{m\alpha n}{\alpha}\right\rfloor = \lfloor mn\rfloor = mn$$

In fact,

$$\left\lfloor \frac{\lfloor m\alpha\rfloor n}{\alpha}\right\rfloor = \left\lfloor \frac{(m\alpha - \{m\alpha\})n}{\alpha}\right\rfloor = \left\lfloor mn - \frac{\{m\alpha\}n}{\alpha}\right\rfloor = mn - 1$$

because

$$\{m\alpha\}\frac{n}{\alpha} < 1 \times 1 = 1$$

□

4 The text describes problems at levels 1 through 5. What is a level 0 problem? (This, by the way, is not a level 0 problem.)

Answer. Something doesn't need any proof, just a guess or an a conjecture.

5 Find a necessary and sufficient condition that $\lfloor nx\rfloor = n\lfloor x\rfloor$, when n is a positive integer. (Your condition should involve $\{x\}$.)

Solution.

$$x = \lfloor x\rfloor + \{x\}$$

Then,

$$\lfloor nx\rfloor = \lfloor n\lfloor x\rfloor + n\{x\}\rfloor = n\lfloor x\rfloor + \lfloor n\{x\}\rfloor$$

Thus, $\lfloor nx\rfloor = n\lfloor x\rfloor$ holds when

$$n\{x\} < 1 \Leftrightarrow \{x\} < \frac{1}{n}$$

where $n \in \mathbb{N}_+$.

□