Mathematical Foundations of Computer Science

Project 13

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Warmups

7 Is (5.34) true also when $k < 0$?

Solution. It is also true that

$$r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}}, \text{when } k < 0$$

The proof is as follows.

$$\begin{aligned}
r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} &= \frac{r}{r^{\underline{-k+1}}}\frac{r-\frac{1}{2}}{\left(r-\frac{1}{2}\right)^{\underline{-k+1}}} \\
&= \frac{1}{(r+\frac{1}{2})(r+1)(r+\frac{3}{2})(r+2)\cdots(r-\frac{1}{2}-k)(r-k)} \\
&= \frac{2^{-2k}}{(2r+1)(2r+2)\cdots(2r-1-2k)(2r-2k)} \\
\frac{(2r)^{\underline{2k}}}{2^{2k}} &= \frac{2^{-2k}2r}{(2r)^{\underline{-2k+1}}} \\
&= \frac{2^{-2k}}{(2r+1)(2r+2)\cdots(2r-2k)}
\end{aligned}$$

And they are the same. □

8 Evaluate

$$\sum_k \binom{n}{k}(-1)^k\left(1-\frac{k}{n}\right)^n$$

What is the approximate value of this sum, when n is very large? Hint: The sum is $\Delta^n f(0)$ for some function f.

Solution. According to (5.40),

$$\begin{aligned}
\Delta^n f(x) &= \sum_k \binom{n}{k}(-1)^{n-k} f(x+k) \\
&= \sum_k \binom{n}{n-k}(-1)^{n-k} f(x+k) \\
&= \sum_k \binom{n}{k}(-1)^{k} f(x+n-k)
\end{aligned}$$

In other word,

$$\Delta^n f(0) = \sum_k \binom{n}{k}(-1)^k f(n-k)$$

Compare to the formula to solve, the function f is

$$\begin{aligned}
f(n-k) &= \left(1-\frac{k}{n}\right)^n = \left(\frac{n-k}{n}\right)^n \\
f(x) &= \left(\frac{x}{n}\right)^n
\end{aligned}$$

As a result,

$$\begin{aligned}
\sum_k \binom{n}{k}(-1)^k \left(1-\frac{k}{n}\right)^n &= \Delta^n f(0) \\
&= \left(\Delta^n \left(\frac{x}{n}\right)^n\right)(0) \\
&= \frac{(n-1)!}{n^n} \to 0, n \to +\infty
\end{aligned}$$

□

9 Show that the generalized exponentials of (5.58) obey the law

$$\mathcal{E}_t(z) = \mathcal{E}(tz)^{1/t}, \text{if } t \neq 0,$$

where $\mathcal{E}(z)$ is an abbreviation for $\mathcal{E}_1(z)$.

Proof. By (5.60),

$$\mathcal{E}_t(z)^r = \sum_{k\geq 0} r\frac{(tk+r)^{k-1}}{k!}z^k$$

By assigning $r = t$, and notice that $t \neq 0$,

$$\begin{aligned}
\mathcal{E}_t(z)^t &= \sum_{k\geq 0} t\frac{(tk+t)^{k-1}}{k!}z^k \\
&= \sum_{k\geq 0}(k+1)^{k-1}\frac{(tz)^k}{k!} \\
&= \mathcal{E}(tz)
\end{aligned}$$

which is the same as the original formula. □

Basics

14 Prove identity (5.25) by negating the upper index in Vandermonde's convolution (5.22). Then show that another negation yields (5.26).

Proof. For (5.25),

$$\begin{aligned}
\sum_{k\le l}\binom{l-k}{m}\binom{s}{k-n}(-1)^k &= \sum_{k\le l}\binom{l-k}{l-k-m}\binom{s}{k-n}(-1)^k && \text{(symmetry)}\\
&= \sum_{k\le l}(-1)^{l-k-m}\binom{-m-1}{l-k-m}\binom{s}{k-n}(-1)^k && \text{(negation)}\\
&= (-1)^{l-m}\sum_{k}\binom{s}{-n+k}\binom{-m-1}{l-m-k} && (m\ge 0)\\
&= (-1)^{l-m}\binom{s-m-1}{l-m-n} && \text{(by 5.22)}\\
&= (-1)^{l+m}\binom{s-m-1}{l-m-n} && ((-1)^{2m}=1)
\end{aligned}$$

The step on $(m \ge 0)$ holds for the reason when $k > l$:

$$l-k-m \le -1-m < 0 \Rightarrow \binom{-m-1}{l-k-m} = 0, \quad (m\ge 0) \tag{1}$$

For (5.26),

$$\begin{aligned}
\sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{n} &= \sum_{0\le k\le l}\binom{l-k}{l-k-m}\binom{q+k}{q+k-n} && \text{(symmetry)}\\
&= \sum_{0\le k\le l}\binom{-m-1}{l-k-m}\binom{-n-1}{q+k-n}(-1)^{l-k-m+q+k-n} && \text{(nagation)}\\
&= (-1)^{l-m+q-n}\sum_{k}\binom{-m-1}{l-k-m}\binom{-n-1}{q+k-n} && (m\ge 0, n\ge q)\\
&= (-1)^{l-m+q-n}\binom{-m-n-2}{l-m+q-n} && \text{(by 5.22)}\\
&= \binom{l+q+1}{l-m+q-n} && \text{(negation)}\\
&= \binom{l+q+1}{m+n-1} && \text{(symmetry)}
\end{aligned}$$

and the step on $(m \ge 0, n \ge q)$ holds for

$$k<0 \Rightarrow q+k-n \le n+k-n = k < 0 \Rightarrow \binom{-n-1}{q+k-n} = 0$$

the first term holds for the same reason as Eq. (1). □

15 What is $\sum_k \binom{n}{k}^3(-1)^k$? Hint: See (5.29).

Solution. By (5.29),

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!b!c!}, \quad a,b,c \in \mathbb{N}.$$

when $n = 2m(m \in \mathbb{N})$, the original formula could be deduced to

$$\begin{aligned}
\sum_k \binom{n}{k}^3 (-1)^k &= \sum_k \binom{n}{n-k}^3 (-1)^k && \text{(symmetry)} \\
&= \sum_k \binom{n}{n+k}^3 (-1)^{-k} && (k \to -k) \\
&= \sum_k \binom{n}{n+k}\binom{n}{n+k}\binom{n}{n+k}(-1)^k(-1)^{-2k} && ((-1)^{-2k} = 1) \\
&= \frac{(m+m+m)!}{m!m!m!} && \text{(by 5.29)} \\
&= \frac{(3m)!}{m!^3}
\end{aligned}$$

when $n = 2m+1(m \in \mathbb{N})$,

$$\sum_k \binom{n}{k}^3 (-1)^k = \sum_k \binom{n}{n-k}^3 (-1)^k$$

expand both parts,

$$\begin{aligned}
&\binom{n}{0}^3 + \binom{n}{1}^3 \times (-1) + \cdots + \binom{n}{n-1}^3 + \binom{n}{n}^3 \times (-1) \\
&= \binom{n}{n}^3 + \binom{n}{n-1}^3 \times (-1) + \cdots + \binom{n}{1}^3 + \binom{n}{0}^3 \times (-1) \\
&= -\left[\binom{n}{0}^3 + \binom{n}{1}^3 \times (-1) + \cdots + \binom{n}{n-1}^3 + \binom{n}{n}^3 \times (-1)\right]
\end{aligned}$$

then,

$$2\sum_k \binom{n}{k}^3 (-1)^k = 0 \Rightarrow \sum_k \binom{n}{k}^3 (-1)^k = 0$$

when $n < 0$,

$$\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}$$

and the original formula comes to

$$\sum_k \binom{n}{k}^3 (-1)^k = \sum_{k \geq 0} \binom{-n+k-1}{k}^3$$

and $(-n+k-1)-k = -n-1 \geq 0$, the symmetry property dispears. Every term is close to 1 when n grows, so this is $+\infty$ when $n \to +\infty$. In conclusion,

$$\sum_k \binom{n}{k}^3 (-1)^k = \begin{cases} \frac{(3m)!}{m!^3}, & \text{if } n = 2m(m \in \mathbb{N}), \\ 0, & \text{if } n = 2m+1(m \in \mathbb{N}), \\ +\infty, & \text{if } n < 0. \end{cases}$$

□

16 Evaluate the sum

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k$$

when a, b, c are nonnegative integers.

Solution.

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k = \sum_k \frac{(2a)!}{(a+k)!(a-k)!}\frac{(2b)!}{(b+k)!(b-k)!}\frac{(2c)!}{(c+k)!(c-k)!}(-1)^k$$

compared with (5.29),

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \sum_k \frac{(a+b)!}{(a+k)!(b-k)!}\frac{(b+c)!}{(b+k)!(c-k)!}\frac{(c+a)!}{(c+k)!(a-k)!}(-1)^k$$

then,

$$\begin{aligned}\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k &= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(c+a)!}\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k \\ &= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(c+a)!}\frac{(a+b+c)!}{a!b!c!}\end{aligned}$$

□

17 Find a simple relation between $\binom{2n-1/2}{n}$ and $\binom{2n-1/2}{2n}$.

Solution. By (5.35),

$$\binom{r}{k}\binom{r-\frac{1}{2}}{k} = \binom{2r}{2k}\binom{2k}{k}\Big/ 2^{2k}$$

and assigning $r = 2n, k = n$,

$$\binom{2n-\frac{1}{2}}{n} = \frac{\binom{4n}{2n}\binom{2n}{n}}{2^{2n}\binom{2n}{n}} = \binom{4n}{2n}\Big/ 2^{2n}$$

assigning $r = 2n, k = 2n$,

$$\binom{2n-\frac{1}{2}}{2n} = \frac{\binom{4n}{4n}\binom{4n}{2n}}{2^{2n}\binom{2n}{2n}} = \binom{4n}{2n}\Big/ 2^{4n}$$

yields the simple relation of

$$\binom{2n-\frac{1}{2}}{n} = 2^{2n}\binom{2n-\frac{1}{2}}{2n}$$

□

18 Find an alternative form analogous to (5.35) for the product

$$\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k}$$

Solution.

$$\begin{aligned}
&r^{\underline{k}}\left(r-\frac{1}{3}\right)^{\underline{k}}\left(r-\frac{2}{3}\right)^{\underline{k}}\\
&\quad=r\left(r-\frac{1}{3}\right)\left(r-\frac{2}{3}\right)(r-1)\left(r-\frac{4}{3}\right)\left(r-\frac{5}{3}\right)\cdots(r-k+1)\left(r-k+\frac{2}{3}\right)\left(r-k+\frac{1}{3}\right)\\
&\quad=\frac{(3r)(3r-1)(3r-2)(3r-3)(3r-4)(3r-5)\cdots(3r-3k+3)(3r-3k+2)(3r-3k+1)}{3^{3k}}
\end{aligned}$$

divide both sides by $k!^3$,

$$\begin{aligned}
\frac{r^{\underline{k}}}{k!}\frac{\left(r-\frac{1}{3}\right)^{\underline{k}}}{k!}\frac{\left(r-\frac{2}{3}\right)^{\underline{k}}}{k!}&=\frac{1}{3^{3k}}\frac{(3r)^{\underline{3k}}}{k!^3}\\
\binom{r}{k}\binom{r-\frac{1}{3}}{k}\binom{r-\frac{2}{3}}{k}&=\frac{1}{3^{3k}}\frac{(3r)^{\underline{3k}}}{(3k)!}\frac{(3k)!}{k!^3}\\
\binom{r}{k}\binom{r-\frac{1}{3}}{k}\binom{r-\frac{2}{3}}{k}&=\frac{1}{3^{3k}}\binom{3r}{3k}\frac{(3k)!}{2k!k!}\frac{(2k)^{\underline{k}}}{k!}\\
\binom{r}{k}\binom{r-\frac{1}{3}}{k}\binom{r-\frac{2}{3}}{k}&=\frac{1}{3^{3k}}\binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k}
\end{aligned}$$

$\square$