Integer Programming

The Knapsack Problem.

The simplest IP - a playground for developing methods

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Outline

1. Formulation and Variants
2. Complexity
3. LP Solution
4. IP Solution Heuristic
5. Correlation between profits and weights
6. Specialized Branch-and-Bound Approach
7. Dynamic Programming Approaches
8. Extended formulation
9. Variants

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Example: Portfolio optimization

Given a portfolio of possible investments, select a subset that maximizes expected return subject to a budget constraint:

$$\begin{aligned}
\textbf{maximize} \quad & \text{profit}_1\, x_1 + \text{profit}_2\, x_2 + \text{profit}_3\, x_3 \\
\textbf{st} \quad & \text{invest}_1\, x_1 + \text{invest}_2\, x_2 + \text{invest}_3\, x_3 \leq \text{budget} \\
& x_i = 0 \text{ or } 1 \quad \text{for } i = 1, \ldots, 3
\end{aligned}$$

Applications

- Hiker
- Finance
- Packing: f.i. airplane freit
- Cutting: f.i. window frame raw material
- IP subproblem: any single constraint extracted from an IP

Formulation : 0-1 or Pure integer program

$$\max \sum_{i=1}^{n} p_i \, x_i$$

s.t.

$$\sum_{i=1}^{n} w_i \, x_i \leq W$$

$$x_i \in \{0, 1\} \text{ or } \in \mathbb{N} \quad \forall i$$

With bounds:

$$x_i \leq u_i \leq \left\lfloor \frac{W}{w_i} \right\rfloor$$

Assumptions

i $w_i \leq W \quad \forall i$

ii $\sum_i w_i > W$

iii $p_i > 0$ and $w_i > 0 \quad \forall i$

Indeed,

- If $p_i > 0$ and $w_i \leq 0$, setting $x_i = 1$ (or $x_i = u_i$ in the IP case) is optimum;
- If $p_i \leq 0$ and $w_i \geq 0$, setting $x_i = 0$ is optimum;
- If $p_i < 0$ and $w_i < 0$, setting $y_i = 1 - x_i$ in remplacement if x_i (0-1 case) or $y_i = u_i - x_i$ (IP case).

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NP-Hardness

Proposition

The binary knapsack problem is NP-Hard

Proof: polynomial reduction from PARTITION

PARTITION:

$$\exists\ ?\ x \in \{0,1\}^n : \sum_{i=1}^{n} w_i x_i = W = \frac{1}{2}\sum_{i=1}^{n} w_i$$

KNAPSACK:

$$\max\{\sum_{i=1}^{n} p_i x_i : \sum_{i=1}^{n} w_i x_i \leq W,\ x \in \{0,1\}^n\}$$

PART $\propto$ *SAD* Algo for PARTITION:

Set $p_i = w_i$.

Call Knapsack algo

If x^* yields a profit $= W$, then return x^*.

Else, retrun NO.

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Preview on solution approaches: a numerical example

Sort items by non-increasing profit ratio

$$\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \dots \geq \frac{p_n}{w_n}$$

$n = 5,\ W = 6$

i	1	2	3	4	5
p_i	30	28	27	34	7
w_i	2	2	3	4	1
$\frac{p_i}{w_i}$	15	14	9	8.5	7

$$\text{Lp sol} : x^*_{LP} = (1, 1, \frac{2}{3}, 0, 0) \rightarrow UB = 76$$

$$\text{Ip Heuristic} : x^a_{IP} = (1, 1, 0, 0, 1) \rightarrow LB = 65$$

$$65 \leq c\, x^*_{IP} \leq 76$$

$$\text{Gap} : \alpha = \frac{76 - 65}{65} = 0.169$$

Proposition

A greedy algorithm solves the LP in $o(n \log n)$

GREEDY ALGO Sort items by non-increasing profit ratio; let c such that

$$\sum_{i=1}^{c-1} w_i \leq W \text{ but } \sum_{i=1}^{c} w_i > W$$

Set

$$\begin{aligned} x_i &= 1 \text{ pour } i = 1, \ldots, c-1 \\ x_c &= \frac{(W - \sum_{i=1}^{c-1} w_i)}{w_c} \\ x_i &= 0 \text{ pour } i > c+1 \end{aligned}$$

LP Dual Bound

Proposition

A greedy algorithm solves the LP in $o(n \log n)$

Proof

$$\lambda = \frac{p_c}{w_c},\ \nu_i = w_i \left(\frac{p_i}{w_i} - \frac{p_c}{w_c}\right) \text{ pour } i = 1, \ldots, c-1,$$

while $\nu_i = 0$ for $i = c, \ldots, n$ provides a feasible dual solution that achieve the same value:

$$\begin{array}{rclc|rcl}
\max \sum_i p_i x_i & & & & \min W\lambda + \sum_i \nu_i & & \\
\sum_i w_i x_i & \leq & W & & w_i\lambda + \nu_i & \geq & p_i \ \forall i \\
x_i & \leq & 1 \ \forall i & & \nu_i & \geq & 0 \ \forall i \\
x_i & \geq & 0 & & \lambda & \geq & 0
\end{array}$$

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Greedy IP heuristic

for $k = 1, \ldots, n$ do
    if $(w_k \leq W)$, then $x_k = 1$ and $W = W - w_k$;
        else $x_k = 0$.

Example $n = 5$, $W = 6$,

i	1	2	3	4	5
p_i	30	28	27	34	7
w_i	2	2	3	4	1
$\frac{p_i}{w_i}$	15	14	9	8.5	7

Solution: $x = (1, 1, 0, 0, 1) \rightarrow LB = 65$

Proposition

$$Z^G \leq Z^* \leq Z^{LP} \leq \sum_{i=1}^{c} p_i \leq Z^G + p_c \leq Z^G + p_{\max}$$

Greedy Worse Case Performance is ∞

Example: $n = 2$, $W = k$

i	1	2
p_i	1	k
w_i	1	k
$\frac{p_i}{w_i}$	1	1

$$\text{greedy} : x = (1, 0) \rightarrow Z^G = 1$$

$$\text{optimum} : x = (0, 1) \rightarrow Z^* = k$$

$$Performance\ ratio = \frac{Z^G}{Z^*} \rightarrow 0$$

Alternative Greedy with Worse Case Analysis

noindent Sort items per non decrasing profitp_i:

$$p_1 \geq p_2 \geq \ldots \geq p_n$$

and apply greedy...
Example: n, $W = n - 1$

i	1	2	$\ldots$	n
p_i	2	1	$\ldots$	1
w_i	n-1	1	$\ldots$	1

$$\textbf{Performance ratio} = \frac{Z^{G'}}{Z^*} = \frac{2}{n-1} \to 0$$

Combined Greedy: G''

Take the best solution from both procedures

Proposition

The Combined Greedy has

Performance Ratio $= \max \frac{\max\{Z^G, Z^{G'}\}}{Z^*} = \frac{1}{2}$.

<u>PROOF:</u> $\max\{Z^G, Z^{G'}\} \geq \frac{Z^G + Z^{G'}}{2} \geq \frac{Z^* - p_{\max} + p_{\max}}{2} \geq \frac{Z^*}{2}$

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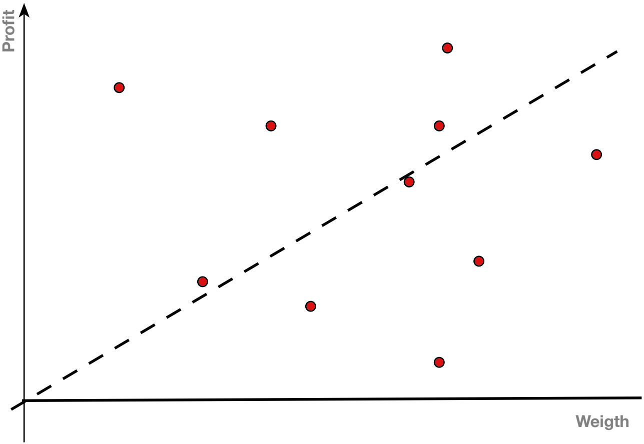
Correlation is bad for Greedy Heuristics



Variable Fixing

Let $U_i^0 = UB(x_i = 0)$ and $U_i^1 = UB(x_i = 1)$.
Let INC be the best known LB. Then,

$$\begin{aligned} U_i^0 \leq INC & \quad \Rightarrow x_i^* = 1 \\ U_i^1 \leq INC & \quad \Rightarrow x_i^* = 0 \end{aligned}$$

The Core of a Knapsack problem

Index items by non-decreasing profit ratio

$$\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \geq \frac{p_n}{w_n}$$

The core is the smallest interval $[a, b]$ such that $1 \leq a \leq b \leq n$ t.q.

$$Z^* = \sum_{i<a} p_i + \max\{\sum_{i=a}^{b} p_j\, x_j : \sum_{i=a}^{b} w_i\, x_i \leq W - \sum_{i<a} w_i,$$

$$x_i \in \{0, 1\}\ i = a, \ldots, b\}$$

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Depth-First-Search B-a-B (Horowitz et Sahni - 1974)

Step 0: **Sort** items s.t. $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \geq \frac{p_n}{w_n}$

Step 1: **Initialize:** $i = 1$, $c = W$, $z = 0$,
$INC = 0$, $UB = \infty$, $x_k^* = x_k = 0$ pour $k = 1, \ldots, n$

Step 2: **Compute UB:** $UB = z + PL(i, n, c)$
Si $UB \leq INC$, goto Step 6.

Step 3: **Plunge with fixation to 1:** while $i \leq n$ & $c \geq w_i$,
$x_i = 1$, $z = z + p_i$, $c = c - w_i$, $i = i + 1$

Step 4: **A fixation to 0:** If $(i < n)$,
set $x_i = 0$, $i = i + 1$ & goto Step 2.

Step 5: **Update LB:** If $(i = n)$,
si $z > LB$, $x^* = x$, $LB = z$.

Step 6: **Backtrack:** If $i = n$,
set $x_n = 0$, $z = z - p_n$, $c = c + w_n$, $i = i - 1$.
While $x_i = 0$, do $i = i - 1$. If $i = 0$, STOP.
Set $x_i = 0$, $z = z - p_i$, $c = c + w_i$, $i = i + 1$.
Goto Step 2.

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Dynamic Programming for the Knapsack Problem

$$KNP \equiv \max \sum_{i=1}^{n} p_i \; x_i$$

$$\sum_{i=1}^{n} w_i \; x_i \leq W$$

$$x_i \in \{0, 1\} \quad \forall i$$

- Stages $\equiv$ subsets of items $\{1, \ldots, k\}$.
- States $\equiv$ capacity used up to the current stage.

$$V^k(b) = \max\{\sum_{i=1}^{k} p_i x_i : \sum_{i=1}^{k} w_i x_i \leq b, x_i \in \{0, 1\} \; i = 1, \ldots k\}$$

defines the maximum profit possible using item $1, \ldots, k$ and capacity b.

Dynamic Programming for the Knapsack Problem

- $V^k(b) = \max\{\sum_{i=1}^k p_i\, x_i : \sum_{i=1}^k w_i\, x_i \leq b, x_i \in \{0,1\}\ i = 1, \ldots k\}$.
- **Algorithm:**

Step 1: Initialize

$$V^0(b) = 0 \qquad \text{for } b = 0, \ldots, W$$

Step 2: Recursively compute

$$V^k(b) = \max\{\underbrace{V^{k-1}(b)}_{x_k=0}, \underbrace{V^{k-1}(b - w_k) + p_k}_{x_k=1}\}$$

for $k = 1, \ldots, n$ and $b = 1, \ldots, W$.

Step 3: The optimum value is given by $\max_{b \leq W} V^n(b)$.

Backward Recursions for the 0-1 Knapsack : in practice

$$V^k(b) = \max\{\sum_{i=1}^{k} p_i x_i : \sum_{i=1}^{k} w_i x_i \leq b, x_i \in \{0,1\}\ i = 1,\ldots k\} \rightarrow O(nW)$$

b / k	0	1	2	3	4	5
0						
1						
⋮						
W						

Initialisation: For $b = 0, \ldots, W$, let $V[b] = 0$;
Recursion: For $k = 1, \ldots, n$, do
For $b = W, \ldots, w_k$, if $V[b - w_k] + p_k > V[b]$
then set $V[b] = V[b - w_k] + p_k$.
Output solution value : $V[W]$

Dynamic Programming for the Knapsack Problem

- **Outputing the Solution :**
  - either one **records** the associated **solutions** one can keep track of:

$$X^k(b) = \begin{cases} 1 & \text{si } V^{k-1}(b - w_k) + p_k > V^{k-1}(b) \\ 0 & \text{sinon} \end{cases}$$

  - or, one uses backtracking.
- **Complexity** : $O(n\,W) \ll O(2^n)$
  (pseudo-polynomial complexity)
- **Memory space** : $O(n\,W)$

$$B^k(p) = \min\{\sum_{i=1}^{k} w_i\, x_i \,:\, \sum_{i=1}^{k} p_i\, x_i \geq p, x_i \in \{0,1\}\ i = 1, \ldots k\} \rightarrow O(nP)$$

- **Algorithm:**

p / k	0	1	2	3	4	5
0						
1						
⋮						
P						

- **Complexity** : $O(n\,P) \ll O(2^n)$
  (pseudo-polynomial complexity)
- **Memory space** : $O(n\,P)$

Forward Recursion by list for the 0-1 Knapsack

- **State:** $= (w, p) =$ (current weight, current profit)
- **Dominance:** $\boxed{(w^1, p^1) \succ (w^2, p^2)}$,
  if $(w^1 < w^2$ and $p^1 \geq p^2)$ or $(w^1 = w^2$ and $p^1 > p^2)$
- **Forward DP:**

  Step 1: Let $list = \{(0, 0)\}$
  Step 2: For $k = 1, \ldots, n$, do
  {
  Let $list'$ be empty. For all $s = (w, p) \in list$, do
  { if $(w + w_k \leq W)$, **(feasibility test)**
  add $(w + w_k, p + p_k)$ to $list'$.
  }
  Merge $list'$ to $list$ while eliminating dominated states.
  }
  Step 3: The optimum value is given by

$$\max\{p : (w, p) \in list\}$$

- **Prune by bound:** if $p + LP(n - k, W - w) \leq INC$, cut (w, p).

Exploiting the Core

- Index items s.t.

$$\text{avec } \frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \geq \frac{p_n}{w_n}$$

- Let $V^{a,b}(d)$ be the maximum profit made using items $i \in [a, b]$ using capacity d, given that items $i < a$ are fixed to 1 and items $i > b$ are fixed to 0:

$$V^{a,b}(d) = \sum_{i<a} p_i + P^{ab}(d - \sum_{i<a} w_i)$$

where $P^{ab}(Cap) \equiv$

$$\max\{\sum_{i=a}^{b} p_i\, x_i : \sum_{i=a}^{b} w_i\, x_i \leq Cap, x_i \in \{0, 1\}\ i \in [a, b]\}$$

Exploiting the Core

- **Initialization**

$$V^{c,c-1}(d) = \sum_{i<c} p_i \quad \forall d > \sum_{i<c} w_i$$

- **Recursion** $\forall a, b$ **and** $d = 0, \ldots 2\,W$**:**

$$V^{a,b}(d) = \max\{\underbrace{V^{a+1,b}(d)}_{x_a=1},$$

$$\underbrace{V^{a+1,b}(d+w_a) - p_a}_{x_a=0},$$

$$\underbrace{V^{a,b-1}(d)}_{x_b=0},$$

$$\underbrace{V^{a,b-1}(d-w_b) + p_b}_{x_b=1}\}\ .$$

- **Sequence** $\forall d, \quad V^{c,c-1},\ V^{c,c},\ V^{c-1,c},\ V^{c-1,c+1}, \ldots,\ V^{1,n}$

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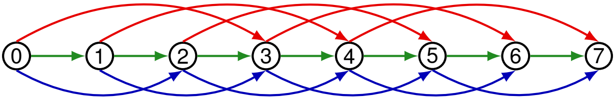
Unbounded Knapsack Problem: Network Flow Reformulation

a cutting pattern defines a path



$$\max \sum_{i=1}^{n} \sum_{b} p_i x_{ib} \tag{1}$$

$$\sum_{i} x_{i,b-w_i} - \sum_{i} x_{ib} = 0 \qquad b = 1, \cdots, W-1 \tag{2}$$

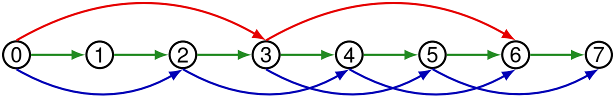
$$x_{ib} \geq 0 \qquad i, b \tag{3}$$

Arc flow:

$x_{ib} =$ nb of copies of item i cut from position b in a pattern.

Unbounded Knapsack Problem: Network Flow Reformulation

a cutting pattern defines a path



$$\max \sum_{i=1}^{n} \sum_{b} p_i x_{ib} \quad (1)$$

$$\sum_{i} x_{i,b-w_i} - \sum_{i} x_{ib} = 0 \qquad b = 1, \cdots, W-1 \quad (2)$$

$$x_{ib} \geq 0 \qquad i, b \quad (3)$$

Arc flow:

$x_{ib} =$ nb of copies of item i cut from position b in a pattern.

Dominance rule to eliminate some symmetric solutions

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The Integer Knapsack problem: bounded case

$$\max \sum_{i=1}^{n} p_i \, x_i$$

$$\sum_{i=1}^{n} w_i \, x_i \leq W$$

$$u_i \geq x_i \in \mathbb{N} \quad \forall i$$

DP : $O(n\,W^2)$ (in case of a bounded problem)
Le $V^k(b)$ be the maximum profit that can be made with items 1 to k and capacity b.

$$V^0(b) = 0 \qquad\qquad \text{pour } b = 0, \ldots, W$$

$$V^k(b) = \max_{t=0,\ldots,u_i} \{\underbrace{V^{k-1}(b - t\,w_k) + t\,p_k}_{x_k = t}\}$$

for $k = 1, \ldots, n$ and $b = 1, \ldots, W$.
The optimum solution value is $\max_{b \leq W} V^n(b)$.

The Integer Knapsack problem: unbounded case

$$\max \sum_{i=1}^{n} p_i\, x_i$$

$$\sum_{i=1}^{n} w_i\, x_i \leq W$$

$$x_i \in \mathbb{N} \quad \forall i$$

DP: $O(n\,W)$ (in case of an unbounded problem)

$$\text{for } b = 0, \ldots, W, \quad V^k(b) = \max\{\underbrace{V^{k-1}(b)}_{x_k=0}, \underbrace{V^k(b - w_k) + p_k}_{x_k \geq 1}\}$$

where

$$V(b) = \max_{i:\, w_i \leq b} \{\underbrace{V(b - w_i) + p_i}_{x_i = x_i + 1}\}$$

Knapsack problem with SOS Constraints

Given the set of items $\{1, \ldots, I\}$ with weights $w_i \geq 0$, profits $p_i \geq 0$ and a set of knapsacks $\{1, \ldots, K\}$ with disjoint SOS Constraints over subsets S_k, solve the following problem:

$$
\begin{aligned}
\max \sum_{i=1}^{I} p_i x_i & & & \\
\text{s.t.} \sum_{i=1}^{I} w_i x_i & \leq & W & \\
\sum_{i \in S^k} x_i & = & 1 & \quad \forall k \in \{1, \ldots, K\} \\
x_i & \in & \{0, 1\} & \quad \forall i \in \{1, \ldots, I\}
\end{aligned}
$$

Knapsack problem with SOS Constraints: solution

Proposition

A greedy algorithm solves the LP in $O(n \log n)$

Proof

For each S_k sort the items $i \in S_k$ by $w_{k,j0} \leq w_{k,j1} \leq \ldots$.
Define $y_{kj} = x_{k,j} - x_{k,j-1}$ for $j = 1, \ldots, |S_k| - 1$

$$\max \sum_{k=1}^{K} \sum_{j=1}^{|S_k|-1} (p_{k,j} - p_{k,j-1}) y_{kj}$$

$$\begin{aligned}
\text{s.t.} \sum_{k=1}^{K} \sum_{j=1}^{|S_k|-1} (w_{k,j} - w_{k,j-1}) y_{kj} &\leq W - \sum_{k} w_{k,j0} \\
\sum_{j=1}^{|S_k|-1} y_{kj} &\leq 1 \quad \forall k \in \{1, \ldots, K\} \\
y_{kj} &\in \{0, 1\} \quad \forall kj
\end{aligned}$$

All-capacity knapsack problem

Given a set of knapsacks $\{1, \ldots, K\}$ with capacities W_k and usage costs c_k, solve the following problem:

$$
\begin{aligned}
\max \sum_{i=1}^{I} p_i x_i - \sum_{k=1}^{K} c_k z_k & & & \\
\text{s.t. } \sum_{i=1}^{I} w_i x_i & \leq & \sum_{k=1}^{K} W_k z_k & \\
\sum_{k=1}^{K} z_k & = & 1 & \forall k \in \{1, \ldots, K\} \\
x_i & \in & \{0, 1\} & \forall i \in \{1, \ldots, I\} \\
z_k & \in & \{0, 1\} & \forall k \in \{1, \ldots, K\}
\end{aligned}
$$

All-capacity knapsack problem: solution

Assume $0 = W_0 \leq W_1 \leq W_2 \leq \ldots \leq W_K$ and define $y_k = \sum_{\kappa=k}^{K} z_\kappa$

$$
\begin{aligned}
\max \sum_{i=1}^{I} p_i x_i - \sum_{k=1}^{K} (c_k - c_{k-1}) y_k & & & \\
\text{s.t.} \sum_{i=1}^{I} w_i x_i &\leq \sum_{k=1}^{K} (W_k - W_{k-1}) y_k & & \\
y_k &\geq y_{k+1} & \forall k \in \{1, \ldots, K-1\} & \\
x_i &\in \{0, 1\} & \forall i \in \{1, \ldots, I\} & \\
y_k &\in \{0, 1\} & \forall k \in \{1, \ldots, K\} &
\end{aligned}
$$