Algorithms: Efficiency, Analysis, and Order

Foundations of Algorithms

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01 Algorithms:
Efficiency, Analysis, and Order

Order

Rigorous Introduction to Order (Big O)

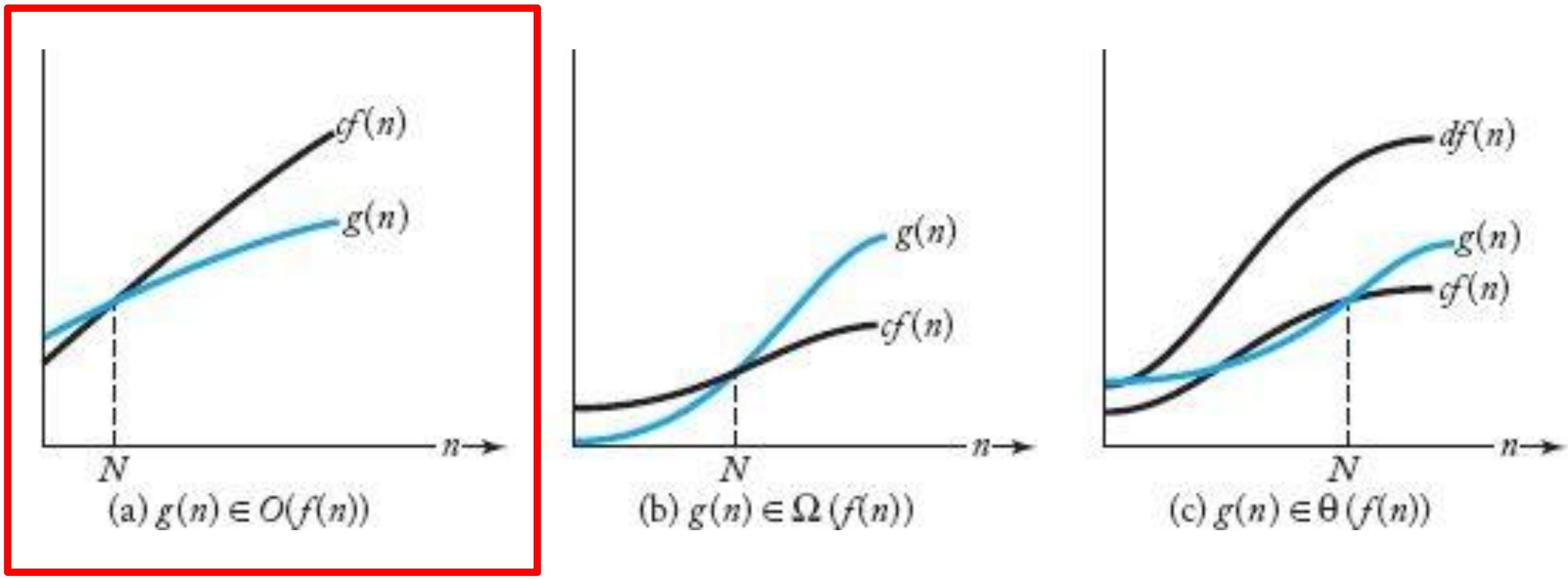
- Definition of 'Big O' Notation

Definition
For a given complexity function $f(n)$, $O(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constant c and some nonnegative integer N such that for all $n \geq N$,

$$g(n) \leq c \times f(n).$$

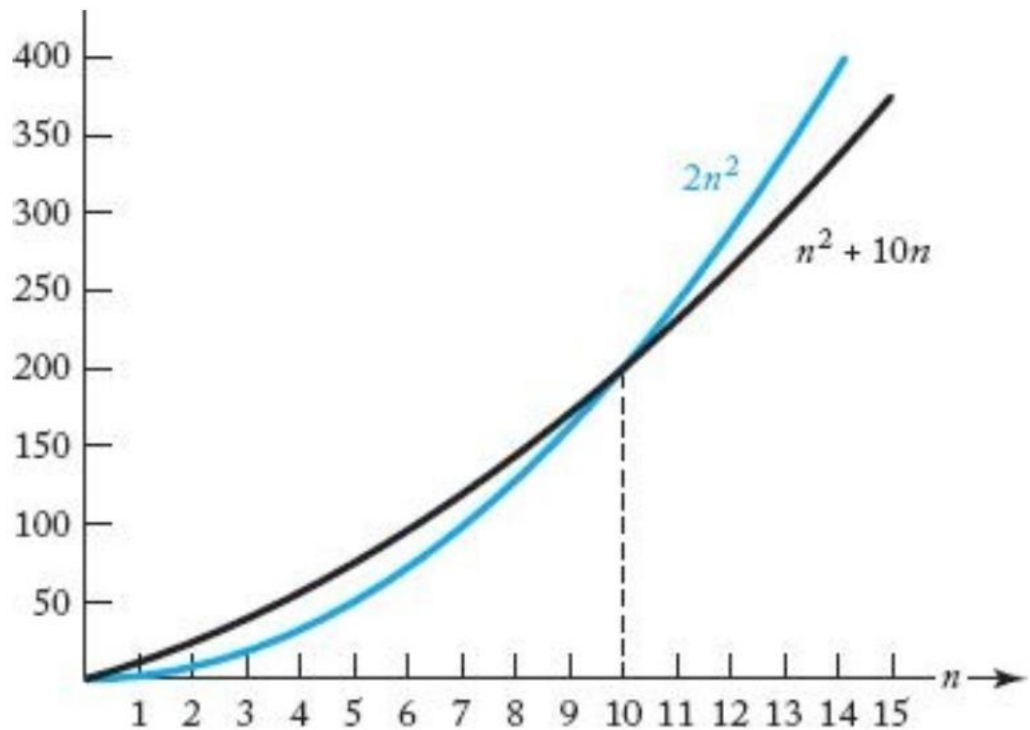
Rigorous Introduction to Order (Big O)

- Illustration of 'Big O' Notation



Rigorous Introduction to Order (Big O)

- Big O Function Example 1
  - $n^2 + 10n \in O(n^2)$
    - $n^2 + 10n \le 2n^2$ (N=10, c=2)



Rigorous Introduction to Order (Big O)

- Big O Function Example 2

We show that $5n^2 \in O(n^2)$. Because, for $n \geq 0$,

$$5n^2 \leq 5n^2,$$

we can take $c = 5$ and $N = 0$ to obtain our desired result.

Rigorous Introduction to Order (Big O)

- Big O Function Example 3

Recall that the time complexity of Algorithm 1.3 (Exchange Sort) is given by

$$T(n) = \frac{n(n-1)}{2}.$$

Because, for $n \geq 0$,

$$\frac{n(n-1)}{2} \leq \frac{n(n)}{2} = \frac{1}{2}n^2,$$

we can take $c = 1/2$ and $N = 0$ to conclude that $T(n) \in O(n^2)$.

Rigorous Introduction to Order (Big O)

- Big O Function Example 4

We show that $n^2 + 10n \in O(n^2)$. Because, for $n \geq 1$,

$$n^2 + 10n \leq n^2 + 10n^2 = 11n^2,$$

we can take $c = 11$ and $N = 1$ to obtain our result.

Rigorous Introduction to Order (Omega Ω)

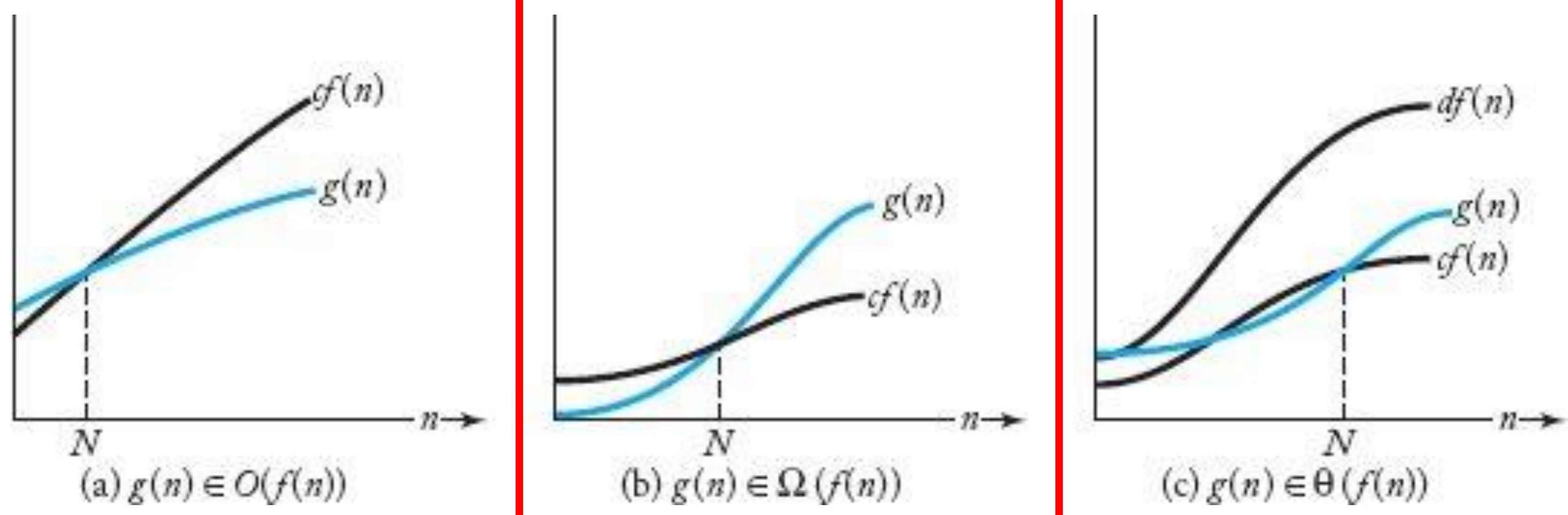
- Definition of 'Omega Ω' Notation

Definition
For a given complexity function $f(n)$, $\Omega(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constant c and some nonnegative integer N such that, for all $n \geq N$,

$$g(n) \geq c \times f(n).$$

Rigorous Introduction to Order (Omega Ω)

- Illustration of 'Omega Ω' Notation



Rigorous Introduction to Order (Omega Ω)

- Ω Function Example 1

We show that $5n^2 \in \Omega(n^2)$. Because, for $n \geq 0$,

$$5n^2 \geq 1 \times n^2,$$

we can take $c = 1$ and $N = 0$ to obtain our result.

Rigorous Introduction to Order (Omega Ω)

- Ω Function Example 2

We show that $n^2 + 10n \in \Omega(n^2)$. Because, for $n \geq 0$, $n^2 + 10n \geq n^2$,

$$n^2 + 10n \geq n^2,$$

we can take $c = 1$ and $N = 0$ to obtain our result.

Rigorous Introduction to Order (Omega Ω)

- Ω Function Example 3

Consider again the time complexity of Algorithm 1.3 (Exchange Sort). We show that

$$T(n) = \frac{n(n-1)}{2} \in \Omega\left(n^2\right).$$

For $n \geq 2$,

$$n - 1 \geq \frac{n}{2}.$$

Therefore, for $n \geq 2$,

$$\frac{n(n-1)}{2} \geq \frac{n}{2} \times \frac{n}{2} = \frac{1}{4}n^2,$$

which means we can take $c = 1/4$ and $N = 2$ to obtain our result.

Rigorous Introduction to Order (Omega Ω)

- Ω Function Example 4

We show that $n^3 \in \Omega(n^2)$. Because, if $n \geq 1$,

$$n^3 \geq 1 \times n^2,$$

we can take $c = 1$ and $N = 1$ to obtain our result.

Rigorous Introduction to Order (Theta θ)

- Definition of 'Theta θ' Notation

Definition
For a given complexity function $f(n)$,

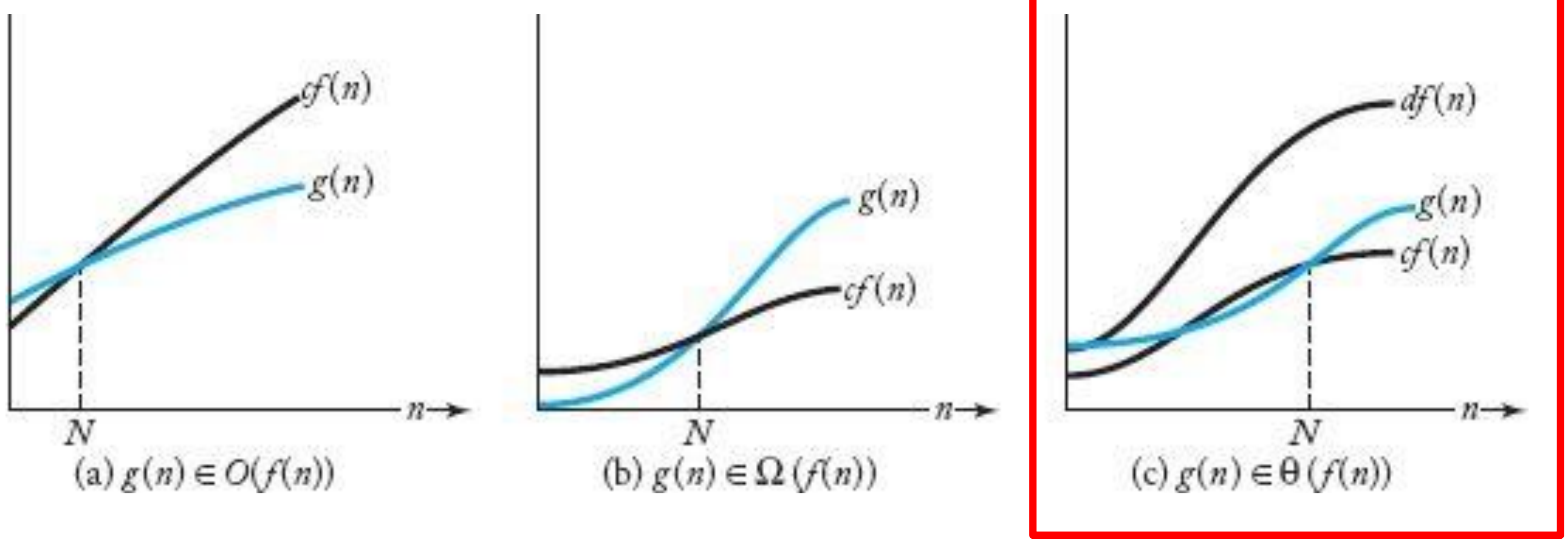
$$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n)).$$

This means that $\Theta(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constants c and d and some nonnegative integer N such that, for all $n \geq N$,

$$c \times f(n) \leq g(n) \leq d \times f(n).$$

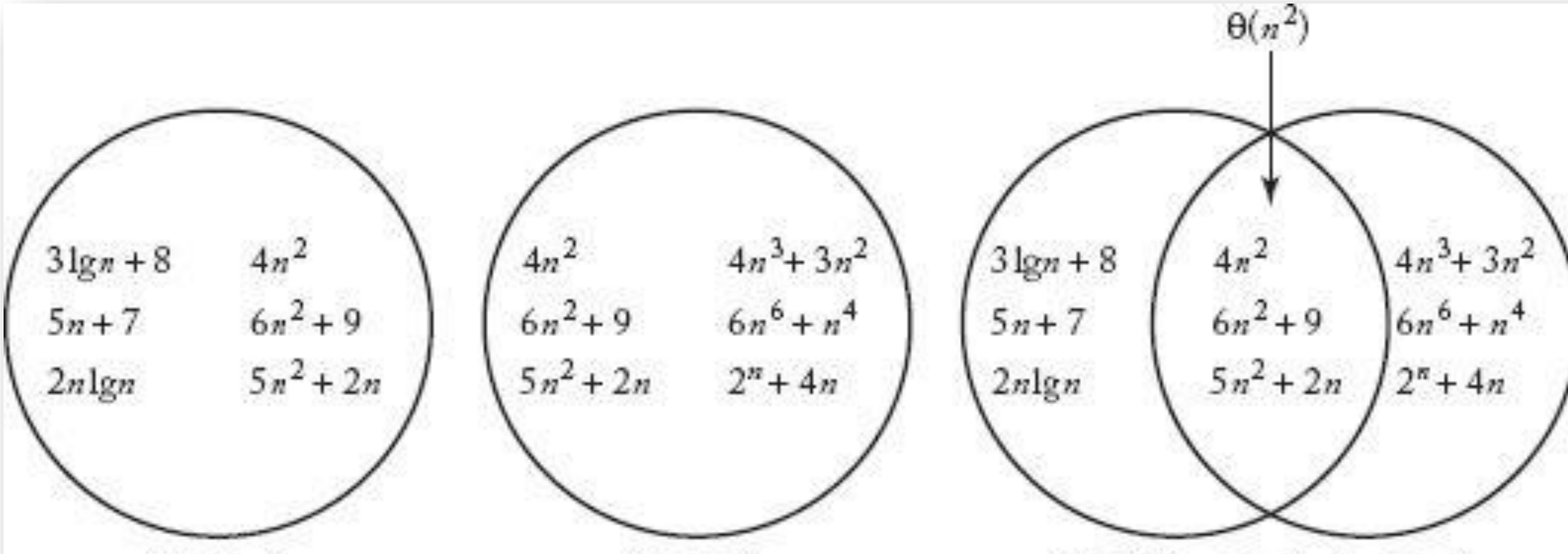
Rigorous Introduction to Order (Theta θ)

- Illustration of 'Theta θ' Notation



Rigorous Introduction to Order (Theta θ)

- O(n^2), $\Omega(n^2)$, $\theta(n^2)$



(a) $O(n^2)$ (b) $\Omega(n^2)$ (c) $\Theta(n^2) = O(n^2) \cap \Omega(n^2)$

Rigorous Introduction to Order (Theta θ)

- θ Function Example 1
  - $T(n) \in O(n^2) \cap \Omega(n^2) = \theta(n^2)$

$$T(n) = \frac{n(n-1)}{2} \quad \text{is in both} \quad O(n^2) \quad \text{and} \quad \Omega(n^2).$$

- Is $5n + 7$ in $\theta(n^2)$?
  - NO, since $5n + 7$ not in $\Omega(n^2)$
- Is $4n^3 + 3n^2$ in $\theta(n^2)$?
  - NO, since $4n^3 + 3n^2$ not in $O(n^2)$