دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

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Lecture 5_1: Dynamics: Newton-Euler

Advanced Robotics

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Outlines

- Introduction
- Acceleration of a Rigid Body
- Mass Distribution
- Newton's Equation, Euler's Equation
- Iterative Newton-Euler Dynamic Formulation
- The Structure of a Manipulator’s Dynamic Equations
- Dynamic Equations in Cartesian Space
- Inclusion of Non-rigid Body Effects

Introduction

- **Kinematics:** Study of positions, static forces, and velocities.

- **Dynamics:** Study of forces required to cause motion.

- Motion of the manipulator arises from:
  - **Torques** applied by the actuators
  - **External forces** applied to the manipulator

- Dynamics of mechanisms is an extensive field.
  - Here, we consider **certain formulations** of the dynamics problem seem particularly well suited to application to **serial manipulators**.

- **Two problems** related to the dynamics:
  - Given a trajectory point, Θ, $\dot{\Theta}$ and $\ddot{\Theta}$, find the required vector of joint torques, τ (Control).
  - Given a torque vector, τ, calculate the resulting motion of the manipulator, Θ, $\dot{\Theta}$ and $\ddot{\Theta}$ (Simulation).

Acceleration of a Rigid Body

- The linear and angular velocity vectors have derivatives that are called the linear and angular accelerations.

$$^{B}\dot{V}_Q = \frac{d}{dt}\,^{B}V_Q = \lim_{\Delta t \to 0} \frac{^{B}V_Q(t+\Delta t) - \,^{B}V_Q(t)}{\Delta t}$$

$$^{A}\dot{\Omega}_B = \frac{d}{dt}\,^{A}\Omega_B = \lim_{\Delta t \to 0} \frac{^{A}\Omega_B(t+\Delta t) - \,^{A}\Omega_B(t)}{\Delta t}$$

- When the reference frame of the differentiation is universal reference frame, {U}:

$$\dot{v}_A = \,^{U}\dot{V}_{AORG}$$

$$\dot{\omega}_A = \,^{U}\dot{\Omega}_A$$

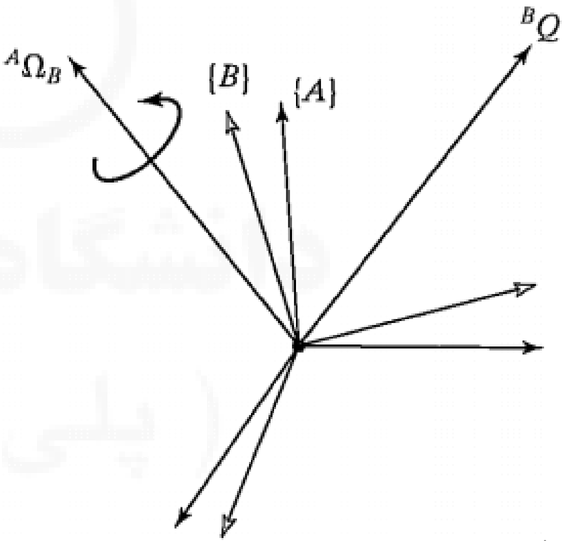
Acceleration of a Rigid Body

❑ Linear Acceleration

- **Remark:** The velocity of a vector ${}^{B}Q$ as seen from frame $\{A\}$ when the origins are coincident:

$$ {}^{A}Q = {}^{A}R_B\, {}^{B}Q $$

$$ {}^{A}V_Q = {}^{A}R_B\, {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}Q $$



Acceleration of a Rigid Body

- **Linear Acceleration**
- **Remark:**

$$^{A}Q = {}^{A}R_{B}\,{}^{B}Q$$

$$^{A}V_{Q} = {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q$$

(1)

- The LHS describes how ^{A}Q is changing in time, so

$$^{A}V_{Q} = \frac{d}{dt}({}^{A}Q) = \frac{d}{dt}({}^{A}R_{B}\,{}^{B}Q) = {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q$$

(2)

- Differentiating Eq. (1):

$$^{A}\dot{V}_{Q} = \frac{d}{dt}\left({}^{A}R_{B}\,{}^{B}V_{Q}\right) + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q + {}^{A}\Omega_{B} \times \frac{d}{dt}({}^{A}R_{B}\,{}^{B}Q)$$

(3)

- Applying Eq. (2) twice (to the first & last term), the RHS of Eq. (3) becomes

$$^{A}\dot{V}_{Q} = \left\{{}^{A}R_{B}\,{}^{B}\dot{V}_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}V_{Q}\right\} + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q + \left\{{}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q\right\}$$

(4)

Acceleration of a Rigid Body

Linear Acceleration

- Combining two similar terms in Eq. (4),

$$
{}^{A}\dot{V}_{Q} = {}^{A}R_{B}\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q
+ {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q) \tag{5}
$$

- Generalize to the case in which the origins are not coincident:

$$
{}^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BORG} + {}^{A}R_{B}\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q
+ {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q) \tag{6}
$$

- For a particular case that ${}^{B}Q$ is constant:

$$
{}^{B}V_{Q} = {}^{B}\dot{V}_{Q} = 0
$$

$$
{}^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BORG} + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q) \tag{7}
$$

- For a prismatic joint: Eq. (6)
- For a revolute joint: Eq. (7)

Acceleration of a Rigid Body

Angular Acceleration

- Assume $\{B\}$ is rotating relative to $\{A\}$ with ${}^{A}\Omega_{B}$ and $\{C\}$ is rotating relative to $\{B\}$ with ${}^{B}\Omega_{C}$.

- Therefore

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}R_{B}\,{}^{B}\Omega_{C} \tag{8} $$

- By differentiating, we obtain

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}\left({}^{A}R_{B}\,{}^{B}\Omega_{C}\right) \tag{9} $$

- Applying Eq. (2) to the last term of Eq. (9):

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + {}^{A}R_{B}\,{}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}\Omega_{C} \tag{10} $$

Acceleration of a Rigid Body

Angular Acceleration

$$ {}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}R_B\, {}^{B}\dot{\Omega}_C + {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}\Omega_C \qquad (10) $$

- For a particular case that ${}^{B}\Omega_C = {}^{B}\dot{\Omega}_C = 0$:

$$ {}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B \qquad (11) $$

- Use these results to calculate the angular acceleration of the links of a manipulator.

- For a revolute joint: Eq. (10)
- For a prismatic joint: Eq. (11)

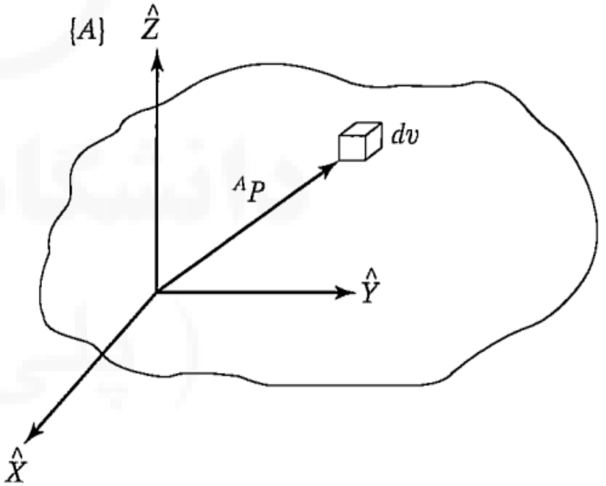
Mass Distribution

- For a single DOF system:
  - Mass of a rigid body is considered for linear motions.
  - Moment of inertia is considered for rotational motion.

- For a free rigid body in 3D space, there are *infinitely* many possible rotation axes.

- In the case of rotation about an arbitrary axis, it is necessary to characterize the mass distribution of a rigid body, i.e. the **inertia tensor.**

- A rigid body with an attached frame is considered.



Mass Distribution

- Inertia tensors can be defined relative to any frame, but always consider a frame attached to the rigid body (*Why?*)

- The inertia tensor relative to frame $\{A\}$ is expressed in the matrix form:

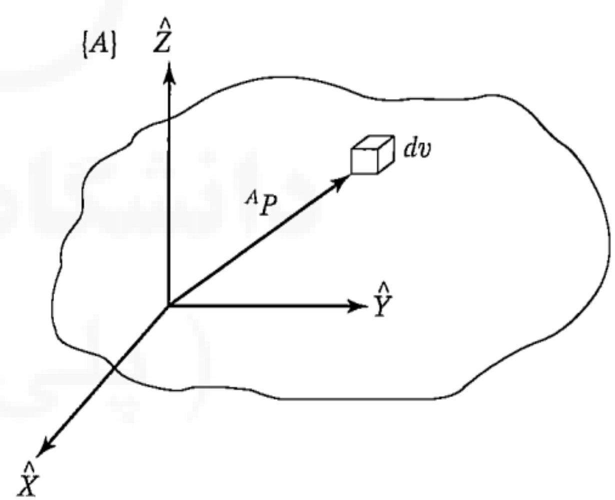
$$
{}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

- where

$$
I_{xx} = \iiint_V (y^2 + z^2)\rho dv, \qquad I_{xy} = \iiint_V xy\rho dv,
$$

$$
I_{yy} = \iiint_V (x^2 + z^2)\rho dv, \qquad I_{xz} = \iiint_V xz\rho dv,
$$

$$
I_{zz} = \iiint_V (x^2 + y^2)\rho dv, \qquad I_{yz} = \iiint_V yz\rho dv,
$$



Mass Distribution

- The elements I_{xx}, I_{yy} and I_{zz} are called the **mass moments of inertia.**

- **Note:** Integrating the mass elements, ρdv, times the squares of the perpendicular distances from the corresponding axis.

- The elements with mixed indices are called the **mass products of inertia.**

- This set of six independent quantities depend on the position and orientation of the frame in which they are defined.

- If it is free to choose the orientation of the reference frame, So for any specified position, it is possible to find an orientation, which cause the products of inertia to be zero.

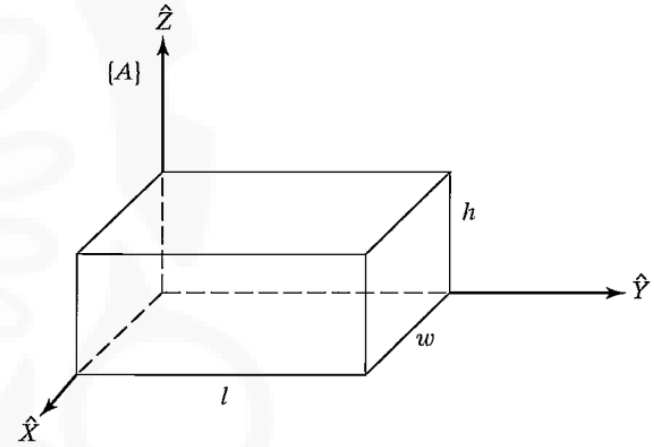
- The axes of the mentioned reference frame are called the **principal axes.**

- The corresponding mass moments are the **principal moments of inertia.**

Mass Distribution

❖ **Example 1:**

- Find the inertia tensor for the rectangular body of uniform density ρ with respect to the coordinate system?



- Moments of inertia

$$
\begin{aligned}
I_{xx} &= \int_0^h \int_0^l \int_0^\omega (y^2 + z^2)\rho \, dx \, dy \, dz \\
&= \int_0^h \int_0^l (y^2 + z^2)\omega\rho dy \, dz \\
&= \int_0^h \left(\frac{l^3}{3} + z^2 l\right) \omega\rho dz \\
&= \left(\frac{hl^3\omega}{3} + \frac{h^3 l\omega}{3}\right)\rho \\
&= \frac{m}{3}(l^2 + h^2),
\end{aligned}
$$

$$I_{yy} = \frac{m}{3}(\omega^2 + h^2)$$

$$I_{zz} = \frac{m}{3}(l^2 + \omega^2).$$

Mass Distribution

❖ **Example 1:**

- Products of inertia

$$I_{xy} = \int_0^h \int_0^l \int_0^\omega xy\rho\, dx\, dy\, dz$$
$$= \int_0^h \int_0^l \frac{\omega^2}{2} y\rho\, dy\, dz$$
$$= \int_0^h \frac{\omega^2 l^2}{4} \rho dz$$
$$= \frac{m}{4}\omega l.$$

$$I_{xz} = \frac{m}{4} h\omega$$

$$I_{yz} = \frac{m}{4} hl.$$

- Hence, the inertia tensor for this object is

$${}^A I = \begin{bmatrix} \frac{m}{3}(l^2 + h^2) & -\frac{m}{4}\omega l & -\frac{m}{4}h\omega \\ -\frac{m}{4}\omega l & \frac{m}{3}(\omega^2 + h^2) & -\frac{m}{4}hl \\ -\frac{m}{4}h\omega & -\frac{m}{4}hl & \frac{m}{3}(l^2 + \omega^2) \end{bmatrix}$$

Mass Distribution

Parallel-axis theorem

- The inertia tensor is a function of the **position** and **orientation** of the reference frame.

- **Parallel-axis theorem** describes how the inertia tensor changes under ***translations*** of the reference coordinate system.

- It relates the inertia tensor in a frame with origin at the center of mass to the inertia tensor with respect to another reference frame:

$$
\begin{aligned}
{}^{A}I_{zz} &= {}^{C}I_{zz} + m(x_c^2 + y_c^2), \\
{}^{A}I_{xy} &= {}^{C}I_{xy} - m x_c y_c,
\end{aligned}
$$

- where $\{C\}$ is located at the center of mass of the body, and $\{A\}$ is an arbitrarily translated frame.

Mass Distribution

Parallel-axis theorem

$$^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2),$$
$$^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c,$$

- Assume $P_c = [x_c, y_c, z_c]^T$ locates the center of mass $\{C\}$ relative to $\{A\}$.
- It may be stated in vector-matrix form as:

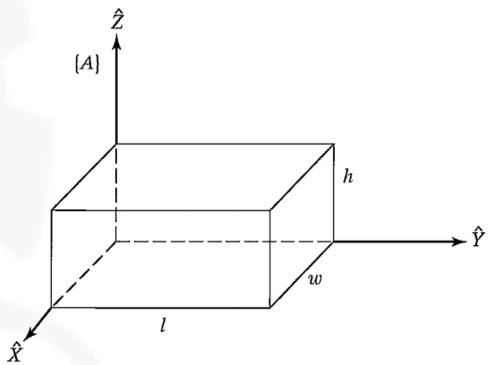
$$^{A}I = {}^{C}I + m[P_c^T P_c I_3 - P_c P_c^T]$$

- where I_3 is the 3×3 identity matrix.

Mass Distribution

❖ **Example 2:**

- Find the inertia tensor described in a coordinate system with origin at the body's center of mass?



- Applying the parallel-axis theorem,

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \omega \\ l \\ h \end{bmatrix}$$

$$^{C}I_{zz} = \frac{m}{12}(\omega^2 + l^2)$$

$$^{C}I_{xy} = 0.$$

- The resulting inertia tensor written in the frame at the center of mass is:

$$^{C}I = \begin{bmatrix} \frac{m}{12}(h^2 + l^2) & 0 & 0 \\ 0 & \frac{m}{12}(\omega^2 + h^2) & 0 \\ 0 & 0 & \frac{m}{12}(l^2 + \omega^2) \end{bmatrix}$$

- The result is diagonal, so frame $\{C\}$ must represent the **principal axes** of this body.

Mass Distribution

Properties of inertia tensors

1) If two axes of the reference frame form a plane of symmetry for the mass distribution of the body, the products of inertia having as an index that is normal to the plane of symmetry will be zero.

2) Moments of inertia must always be positive. Products of inertia may have either sign.

3) The sum of the three moments of inertia is invariant under orientation changes in the reference frame.

4) The eigenvalues of an inertia tensor are the principal moments for the body. The associated eigenvectors are the principal axes.

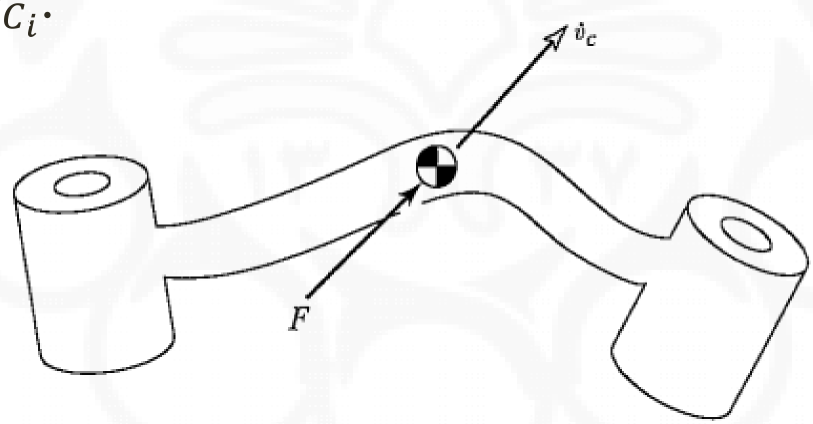
- **Note:**
- Most manipulators have links with complicated geometry and composition. So, calculating the inertia tensor is difficult in practice.
- An alternative approach is using a measuring device (e.g., an *inertia pendulum*).

Newton's Equation, Euler's Equation

- **Newton's equation**, along with its rotational analog, **Euler's equation**, describes how forces, inertias, and accelerations relate.

Newton's Equation

- Assume a rigid body whose center of mass is accelerating with acceleration $\dot{v}_{C_i}$.



- The force, F_i, acting at the center of mass and causing this acceleration is as follows:

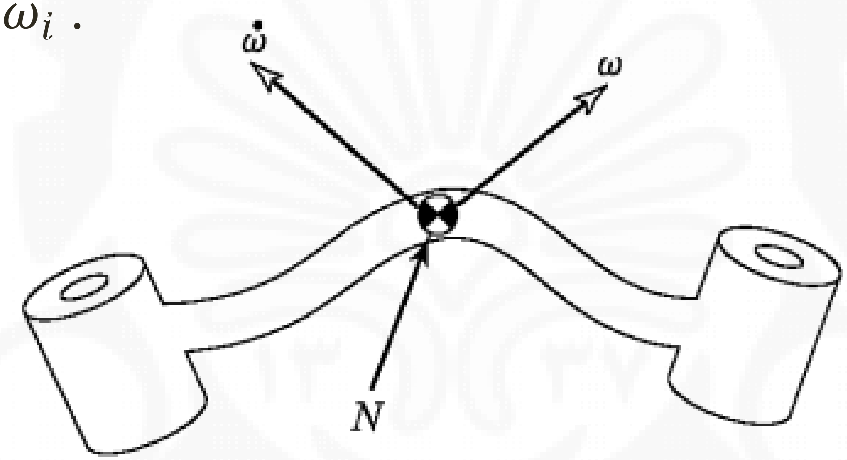
$$F_i = m_i \, \dot{v}_{C_i}$$

- m_i is the total mass of the body.

Newton's Equation, Euler's Equation

Euler's Equation

- Assume a rigid body rotating with angular velocity ${}^{i}\omega_i$ and with angular acceleration ${}^{i}\dot{\omega}_i$.



- The moment N_i, which must be acting on the body to cause this motion, is as follows:

$$N_i = {}^{C_i}I_i \, {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \, {}^{i}\omega_i$$

- ${}^{C_i}I_i$ is the inertia tensor of the body written in a frame, $\{C\}$, whose origin is located at the center of mass.

Iterative Newton-Euler Dynamic Formulation

- **Problem:** Computing the torques that correspond to a given trajectory of a manipulator.

  - **Inputs:**
  - Known position, velocity, and acceleration of the joints, $(\theta, \dot{\theta}, \ddot{\theta})$
  - Knowledge of the kinematics
  - Mass-distribution information

  - **Output:**
  - The joint torques required to cause this motion

- The method includes **three steps**:

1) Outward iterations to compute velocities and accelerations.
2) Compute the inertial forces and torques acting on the links.
3) Inward iterations to compute forces and torques.

Iterative Newton-Euler Dynamic Formulation

❑ Outward Iterations to Compute Velocities and Accelerations

- **Newton's equation**

$$F_i = m_i \, \dot{v}_{C_i}$$

- **Euler's equation**

$$N_i = {}^{C_i}I_i \, {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \, {}^{i}\omega_i$$

- To compute inertial forces, it is necessary to compute the rotational velocity (ω) and linear and rotational acceleration ($\dot{v}_C$, $\dot{\omega}$) of the center of mass of each link.

- It will be done in an iterative way, starting with link 1 and moving successively, outward to link n .

Iterative Newton-Euler Dynamic Formulation

- **Outward Iterations to Compute Velocities and Accelerations**
- **Rotational Velocity**
- The "propagation" of rotational velocity:
- **Revolute joint**

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}R_i \, {}^{i}\omega_i + \dot{\theta}_{i+1} \, {}^{i+1}\hat{Z}_{i+1} $$

- **Prismatic joint (!)**

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}R_i \, {}^{i}\omega_i $$

Iterative Newton-Euler Dynamic Formulation

- **Outward Iterations to Compute Velocities and Accelerations**
- **Angular Acceleration**
- The "propagation" of angular acceleration:
- **Revolute joint**

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{\omega}_i + {}^{i+1}R_i\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

- How?
- 1) Direct Differentiation of rotational velocity:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}R_i\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

Or

- 2) Remember:

$$ {}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}R_B\,{}^{B}\dot{\Omega}_C + {}^{A}\Omega_B \times {}^{A}R_B\,{}^{B}\Omega_C $$

- Assume:

$$ C = i + 1 $$
$$ B = i $$
$$ A = 0 $$

- **Prismatic joint (!)**

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{\omega}_i $$

Iterative Newton-Euler Dynamic Formulation

- **Outward Iterations to Compute Velocities and Accelerations**
- **Linear Acceleration**
- The "propagation" of linear acceleration of each link-frame origin:
- **Revolute joint (!)**

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right) + {}^{i}\dot{v}_i \right) $$

- How?
- Remember:

$$ {}^{A}\dot{V}_Q = {}^{A}\dot{V}_{BORG} + {}^{A}\dot{\Omega}_B \times {}^{A}R_B {}^{B}Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}R_B {}^{B}Q) $$

- Assume:

$$ Q = i + 1 $$
$$ B = i $$
$$ A = 0 $$

- **Prismatic joint**

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right) + {}^{i}\dot{v}_i \right) + 2\, {}^{i+1}\omega_{i+1} \times \dot{d}_{i+1} {}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1} {}^{i+1}\hat{Z}_{i+1} $$

Iterative Newton-Euler Dynamic Formulation

- **Outward Iterations to Compute Velocities and Accelerations**
- **Linear Acceleration**

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right) + {}^{i}\dot{v}_i \right) $$

- **Note:** a frame, $\{C_i\}$, attached to each link, having its origin located at the center of mass of the link and having the same orientation as the link frame, $\{i\}$**.**

- The "propagation" of linear acceleration of the center of mass of each link (!)

$$ {}^{C_i}\dot{v}_{C_i} = {}^{C_i}R_i \left({}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{C_i} \right) + {}^{i}\dot{v}_i \right) $$

$$ {}^{C_i}\dot{v}_{C_i} = {}^{C_i}R_i {}^{i}\dot{v}_{C_i} = {}^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{C_i} \right) + {}^{i}\dot{v}_i \qquad (*) $$

- Eq. $(*)$ doesn't involve joint motion and is valid for joint i, regardless of whether it is revolute or prismatic. (*Why?*)

- Apply the equations to link 1 by ${}^{0}\omega_0 = {}^{0}\dot{\omega}_0 = 0$.

Iterative Newton-Euler Dynamic Formulation

❑ The Force and Torque Acting on a Link

- Apply the Newton—Euler equations to compute the inertial force and torque acting at the center of mass of each link.

$$F_i = m_i \, \dot{v}_{C_i}$$

$$N_i = {}^{C_i}I_i \, {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \, {}^{i}\omega_i$$

Iterative Newton-Euler Dynamic Formulation

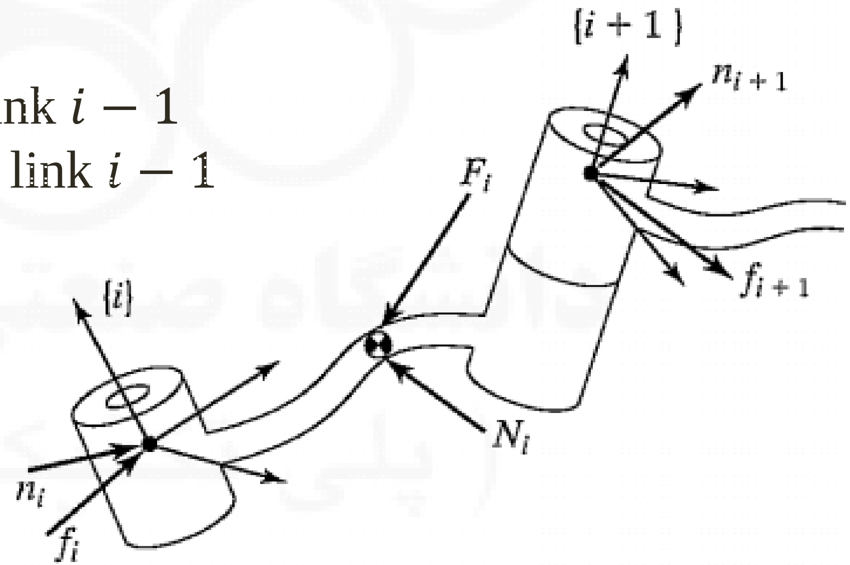
Inward Iterations to Compute Forces and Torques

- Calculate the joint torques that will result in these net forces and torques being applied to each link.

- Writing a force-balance and moment-balance equation based on a free-body diagram of a typical link.

- Each link has forces and torques exerted on it by
  1) Its neighbors
  2) Inertial force and torque

- f_i = force exerted on link i by link $i-1$
- n_i = torque exerted on link i by link $i-1$



Iterative Newton-Euler Dynamic Formulation

❑ Inward Iterations to Compute Forces and Torques

- Force-balance relationship

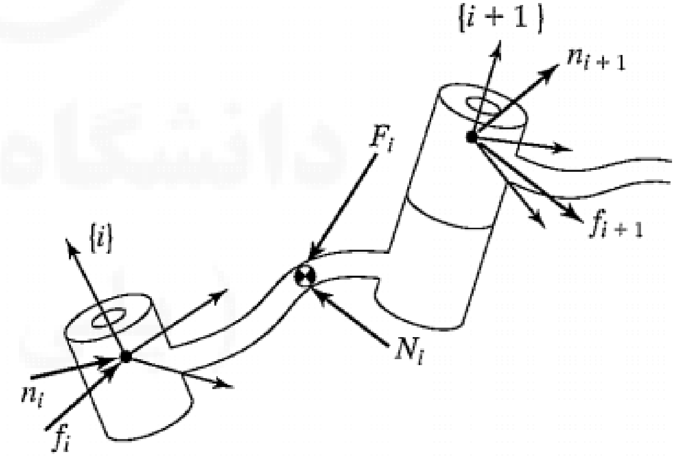
$$^{i}F_i = {}^{i}f_i - {}^{i}R_{i+1}\,{}^{i+1}f_{i+1}$$

- Torque-balance equation (about the center of mass)

$$^{i}N_i = {}^{i}n_i - {}^{i}n_{i+1} + \left(-{}^{i}P_{C_i}\right) \times {}^{i}f_i - \left({}^{i}P_{i+1} - {}^{i}P_{C_i}\right) \times {}^{i}f_{i+1}$$

- Using the result from the force-balance relation and adding a few rotation matrices

$$^{i}N_i = {}^{i}n_i - {}^{i}R_{i+1}\,{}^{i+1}n_{i+1} + \left(-{}^{i}P_{C_i}\right) \times {}^{i}F_i - {}^{i}P_{i+1} \times {}^{i}R_{i+1}\,{}^{i+1}f_{i+1}$$



Iterative Newton-Euler Dynamic Formulation

❑ Inward Iterations to Compute Forces and Torques

- Rearrange the equations to be appeared as iterative relationships (higher to lower numbered neighbor).

$$ {}^{i}f_i = {}^{i}R_{i+1}\,{}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}N_i + {}^{i}R_{i+1}\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}R_{i+1}\,{}^{i+1}f_{i+1} $$

- These equations are evaluated link by link, starting from link n going inward toward the base of the robot.

- The method is analogous to the static force iterations, except that inertial forces and torques are now considered at each link.

- The required joint torques are found by taking the Z_i component of the torque (force) applied i.e. ${}^{i}n_i$ $\left({}^{i}f_i\right)$ (As like as the static case).

Iterative Newton-Euler Dynamic Formulation

❑ Inward Iterations to Compute Forces and Torques

- Revolute joint

$$\tau_i = {}^{i}n_i^T \hat{Z}_i$$

- Prismatic joint

$$\tau_i = {}^{i}f_i^T \hat{Z}_i$$

- The symbol τ_i is used for a linear actuator force as like as rotary actuator torque.

- For a robot moving in free space, ${}^{N+1}f_{N+1}$ and ${}^{N+1}n_{N+1}$ are set equal to zero.

- If the robot is in contact with the environment, the forces and torques due to this contact can be included in the force balance.

Iterative Newton-Euler Dynamic Formulation

❑ The Iterative Newton—Euler Dynamics Algorithm

- **Summary:**

1) Link velocities and accelerations are iteratively computed from link 1 out to link n.

2) The Newton-Euler equations are applied to each link.

3) Forces and torques of interaction and joint actuator torques are computed recursively from link n back to link 1.

Iterative Newton-Euler Dynamic Formulation

- **The Iterative Newton—Euler Dynamics Algorithm**
- **Summary: (Revolute joints)**
- **Outward iterations:** $i\ :\ 0 \rightarrow 5$
  - $^{i+1}\omega_{i+1} = {}^{i+1}R_i\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$
  - $^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{\omega}_i + {}^{i+1}R_i\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$
  - $^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right)$
  - $^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1}$
  - $^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$
  - $^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}$

- **Inward iterations:** $i\ :\ 6 \rightarrow 1$
  - $^{i}f_i = {}^{i}R_{i+1}\,{}^{i+1}f_{i+1} + {}^{i}F_i$
  - $^{i}n_i = {}^{i}N_i + {}^{i}R_{i+1}\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}R_{i+1}\,{}^{i+1}f_{i+1}$
  - $\tau_i = {}^{i}n_i^T\,\hat{Z}_i$

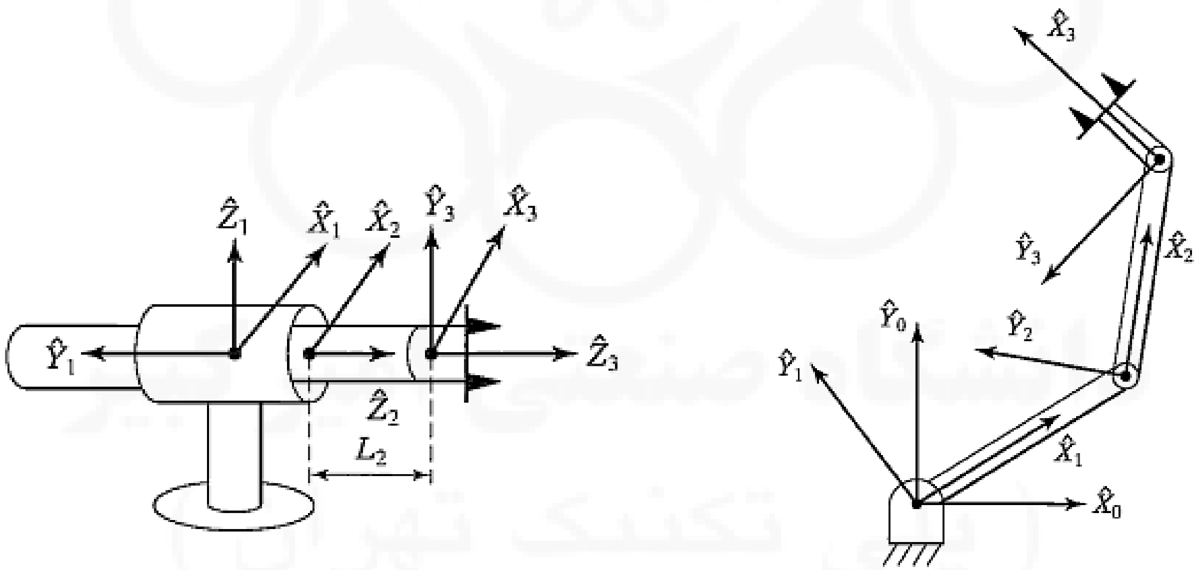
Iterative Newton-Euler Dynamic Formulation

❑ Inclusion of Gravity in the Dynamics Algorithm

- The effect of gravity can be included by setting ${}^{0}\dot{v}_{0} = G$.

- G has the magnitude of the gravity vector but points in the opposite direction.

- It is equivalent to saying that the base of the robot is accelerating upward with 1 g acceleration.



Iterative Newton-Euler Dynamic Formulation

❑ Iterative Vs. Closed Form

- This algorithm gives a computational scheme (given (θ, $\dot{\theta}$, $\ddot{\theta}$), compute the required joint torques), as like as development of equations to compute the Jacobian.

- It can be used in two ways:
  - ➢ **As a Numerical Algorithm**
    - Once ${}^{C}I_i$, m_i , ${}^{i}P_{C_i}$ and ${}^{i+1}R_i$ are specified, the equations can be applied directly to compute the joint torques corresponding to any motion.

  - ➢ **As an Analytical Algorithm**
    - To analyze the structure of the equations (e.g. the gravity effects, the inertial effects & …), it is useful to apply the recursive Newton-Euler equations **symbolically** to develop **closed form equations** (analogous to derive the symbolic form of the Jacobian).

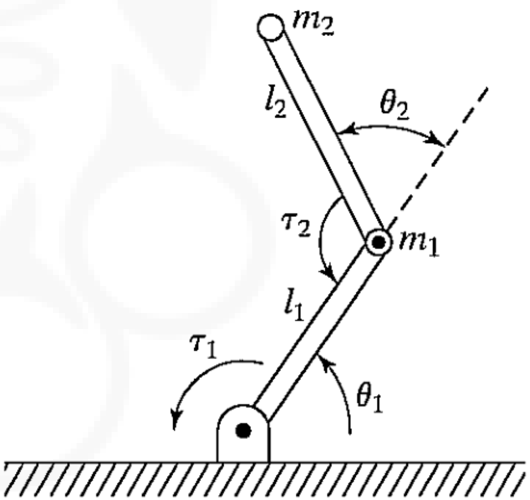
Iterative Newton-Euler Dynamic Formulation

❖ **Example:**

- Compute the closed-form dynamic equations for the two-link RR planar manipulator
- **Assumption**: Point masses at the end of each link (m_1 and m_2)

- Center of mass vectors:

$$ {}^{1}P_{C_1} = l_1 \hat{X}_1, $$
$$ {}^{2}P_{C_2} = l_2 \hat{X}_2. $$



- Due to the point-mass assumption:

$$ {}^{C_1}I_1 = 0, $$
$$ {}^{C_2}I_2 = 0. $$

- No forces acting on the end-effector:

$$ f_3 = 0, $$
$$ n_3 = 0. $$

Iterative Newton-Euler Dynamic Formulation

❖ **Example:**

- The base of the robot is not rotating, so:

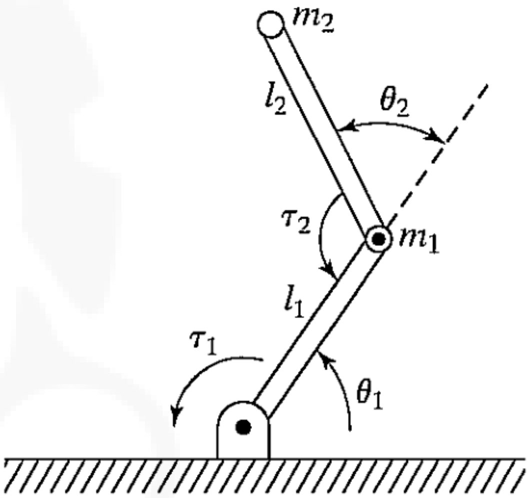
$$\omega_0 = 0,$$

$$\dot{\omega}_0 = 0.$$

- To include gravity forces:

$${}^0\dot{v}_0 = g\hat{Y}_0.$$

- The rotation between successive link frames:



$${}^{i}_{i+1}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0.0 \\ s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$${}^{i+1}_{i}R = \begin{bmatrix} c_{i+1} & s_{i+1} & 0.0 \\ -s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

Iterative Newton-Euler Dynamic Formulation

❖ **Example:**

- The outward iterations for link 1:

$$^1\omega_1 = \dot\theta_1\,{}^1\hat Z_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix},$$

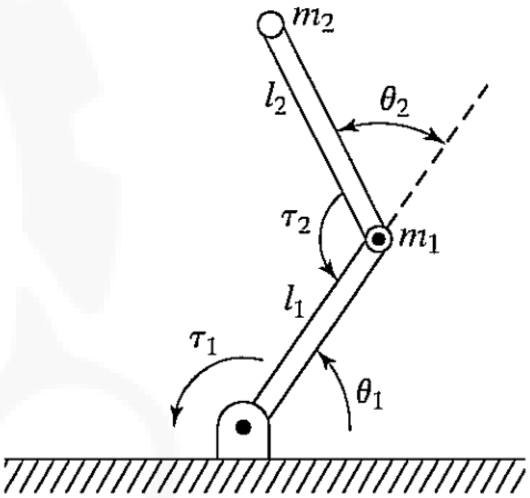
$$^1\dot\omega_1 = \ddot\theta_1\,{}^1\hat Z_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 \end{bmatrix},$$

$$^1\dot v_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix},$$

$$^1\dot v_{C_1} = \begin{bmatrix} 0 \\ l_1\ddot\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot\theta_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot\theta_1^2 + gs_1 \\ l_1\ddot\theta_1 + gc_1 \\ 0 \end{bmatrix},$$

$$^1F_1 = \begin{bmatrix} -m_1l_1\dot\theta_1^2 + m_1gs_1 \\ m_1l_1\ddot\theta_1 + m_1gc_1 \\ 0 \end{bmatrix},$$

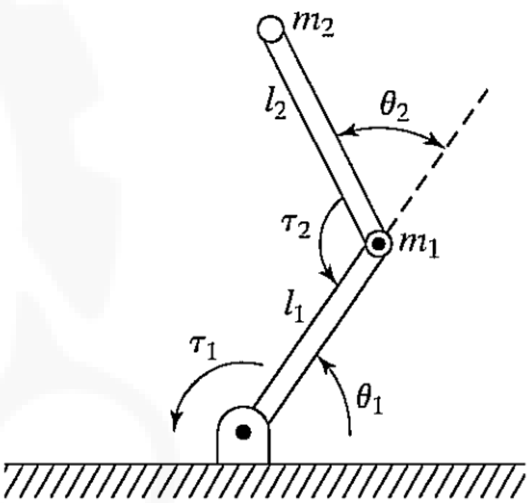
$$^1N_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$



Iterative Newton-Euler Dynamic Formulation

❖ **Example:**

- The outward iterations for link 2:



$$^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},$$

$$^{2}\dot{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix},$$

$$^{2}\dot{v}_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + gs_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix},$$

$$^{2}\dot{v}_{C_2} = \begin{bmatrix} 0 \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + gs_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix},$$

$$^{2}F_2 = \begin{bmatrix} m_2 l_1\ddot{\theta}_1 s_2 - m_2 l_1\dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1\ddot{\theta}_1 c_2 + m_2 l_1\dot{\theta}_1^2 s_2 + m_2 g c_{12} + m_2 l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix},$$

$$^{2}N_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Iterative Newton-Euler Dynamic Formulation

❖ **Example:**

- The inward iterations for link 2:

$$^2f_2 = {}^2F_2,$$

$$^2n_2 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

Iterative Newton-Euler Dynamic Formulation

❖ **Example:**

- The inward iterations for link 1:

$$
{}^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}
+ \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix},
$$

$$
{}^1n_1 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}
+ \begin{bmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g c_1 \end{bmatrix}
+ \begin{bmatrix} 0 \\ 0 \\ m_2 l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 g s_2 s_{12} \\ + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}.
$$

Iterative Newton-Euler Dynamic Formulation

- **Example:**
- Extracting the $\hat{Z}_i$ components of the ${}^{i}n_i$ to find the joint torques:

$$\tau_1 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2$$
$$-2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1,$$
$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2).$$

The Structure of a Manipulator's Dynamic Equations

- As shown, dynamic equations relates the (**acceleration of manipulator joints)** to (**torques** acting at it).

- The structure of dynamic equations can be expressed as:
  - The State-Space Equations
  - The Configuration-Space Equations

The Structure of a Manipulator’s Dynamic Equations

The State-Space Equation

- A dynamic equation can be written in the form:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

- $M(\Theta)$ is the $n \times n$ mass matrix of the manipulator.
- $V(\Theta,\dot{\Theta})$ is an $n \times 1$ vector of centrifugal and Coriolis terms.
- $G(\Theta)$ is an $n \times 1$ vector of gravity terms.

- It is called the state-space equation because the term $V(\Theta,\dot{\Theta})$ has both position and velocity dependence.

- Each element of $M(\Theta)$ and $G(\Theta)$ is a complex functions of Θ (the position of all joints).

- Each element of $V(\Theta,\dot{\Theta})$ is a complex function of both Θ and $\dot{\Theta}$.

The Structure of a Manipulator's Dynamic Equations

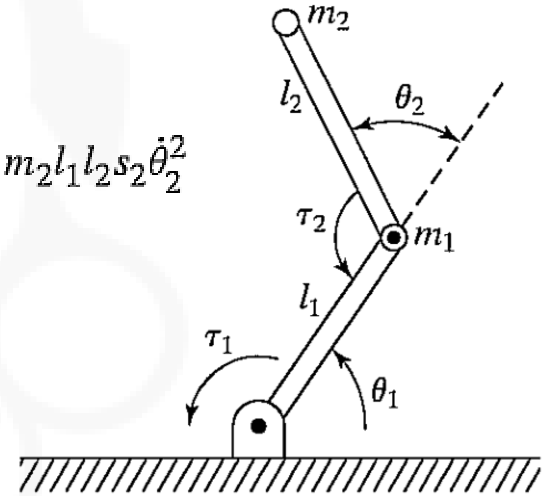
The State-Space Equation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

- **Notes:**
- $M(\Theta)$ is composed of all those terms which multiply $\ddot{\Theta}$, i.e. inertia forces.
- Any manipulator mass matrix, $M(\Theta)$, is symmetric and positive definite, therefore, **always invertible** (*Why?*).
- The velocity term, $V(\Theta, \dot{\Theta})$, contains all those terms that have any dependence on joint velocity, i.e. centrifugal and Coriolis forces.
- The gravity term, $G(\Theta)$, contains all those terms in which the gravitational constant, g, appears i.e. potential forces.

The Structure of a Manipulator's Dynamic Equations

- **The State-Space Equation**
- **Example 1:**
- Calculate $M(\Theta)$, $V(\Theta, \dot{\Theta})$, and $G(\Theta)$ for two-link RR planar manipulator?



- Remember:

$$\tau_1 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2$$
$$-2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1,$$
$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2).$$

- The manipulator mass matrix:

$$M(\Theta) = \begin{bmatrix} m_2 l_2^2 + 2m_2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix}$$

- $M(\Theta)$ is a function of Θ.

The State-Space Equation

Example 1:

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

- A term like $-m_2 l_1 l_2 s_2 \dot{\theta}_2^2$ is caused by a centrifugal force, because it depends on the square of a joint velocity.

- A term such as $-2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2$ is caused by a Coriolis force, because it always contain the product of two different joint velocities.

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

- The gravity term depends only on Θ and not on its derivatives.

The Structure of a Manipulator's Dynamic Equations

The Configuration-Space Equation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

- Writing the velocity-dependent term, $V(\Theta, \dot{\Theta})$, in a different form:

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2] + G(\Theta)$$

- $B(\Theta)$: $n \times n(n - 1)/2$ matrix of Coriolis coefficients.

- $[\dot{\Theta}\dot{\Theta}]$: $n(n - 1)/2 \times 1$ vector of joint velocity products given by:

$$[\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad \dots \quad \dot{\theta}_{n-1}\dot{\theta}_n]^T$$

- $C(\Theta)$: $n \times n$ matrix of centrifugal coefficients.

- $[\dot{\Theta}^2]$: $n \times 1$ vector given by

$$[\dot{\Theta}^2] = [\dot{\theta}_1^2 \quad \dot{\theta}_2^2 \quad \dots \quad \dot{\theta}_n^2]^T$$

The Structure of a Manipulator's Dynamic Equations

The Configuration-Space Equation

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2] + G(\Theta)$$

- It is called the configuration-space equation, because the matrices are functions only of manipulator position.

- It is useful in applications in which the dynamic equations must be updated as the manipulator moves. (e.g. control of a manipulator)

The Structure of a Manipulator's Dynamic Equations

- **The Configuration-Space Equation**
- **Example 2:**
- Calculate $B(\Theta)$ and $C(\Theta)$ for two-link RR planar manipulator
- Remember

$$V(\Theta,\dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

- For the two-link manipulator

$$[\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1 \dot{\theta}_2]$$

$$[\dot{\Theta}^2] = [\dot{\theta}_1^2 \quad \dot{\theta}_2^2]^T$$

- So,

$$B(\Theta) = \begin{bmatrix} -2m_2 l_1 l_2 s_2 \\ 0 \end{bmatrix}$$

$$C(\Theta) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 \end{bmatrix}$$

Dynamic Equations in Cartesian Space

- Remember

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

- It has been developed in joint space because the serial-link nature of the mechanism was utilized in deriving the equations.

- It relates the (**acceleration of manipulator joints)** to (**torques** acting at it).

- Now, find the formulation that relate **(acceleration of the end-effector** expressed in **Cartesian space)** to **(joints torques)** ***or*** (**Cartesian forces and moments acting at the end-effector).**

- It can be expressed as:
  1) The Cartesian State-Space Torque Equations
  2) The Cartesian State-Space Force Equations
  3) The Cartesian Configuration-Space Torque Equations
  4) The Cartesian Configuration-Space Force Equations

Dynamic Equations in Cartesian Space

1) The Cartesian State-Space Torque Equations

- The dynamics of a manipulator with respect to Cartesian variables:

$$\tau = M_X(\Theta)\ddot{X} + V_X\big(\Theta,\dot{\Theta}\big) + G_X(\Theta)$$

- X is an appropriate Cartesian vector representing position and orientation of the end-effector.

$$\dot{X} = \begin{bmatrix} \dot{d}(\Theta) \\ \dot{\Theta}(\Theta) \end{bmatrix} \quad or \quad \dot{X} = \begin{bmatrix} \dot{d}(\Theta) \\ \omega(\Theta) \end{bmatrix}$$

- $M_X(\Theta)$ is the Cartesian mass matrix.
- $V_X\big(\Theta,\dot{\Theta}\big)$ is a vector of velocity terms in Cartesian space.
- $G_X(\Theta)$ is a vector of gravity terms in Cartesian space.

Dynamic Equations in Cartesian Space

1) The Cartesian State-Space Torque Equations

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

- Develop a relationship between joint space and Cartesian acceleration,

$$\dot{X} = J(\Theta)\dot{\Theta}$$
$$\ddot{X} = J(\Theta)\ddot{\Theta} + \dot{J}(\Theta)\dot{\Theta}$$
$$\ddot{\Theta} = J^{-1}(\Theta)\ddot{X} - J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta} \qquad (*)$$

- Substituting

$$\tau = M(\Theta)\left(J^{-1}\ddot{X} - J^{-1}\dot{J}\dot{\Theta}\right) + V(\Theta,\dot{\Theta}) + G(\Theta)$$

- Therefore

$$\tau = M_X(\Theta)\ddot{X} + V_X(\Theta,\dot{\Theta}) + G_X(\Theta)$$

$$M_X(\Theta) = M(\Theta)J^{-1}(\Theta)$$
$$V_X(\Theta,\dot{\Theta}) = V(\Theta,\dot{\Theta}) - M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}$$
$$G_X(\Theta) = G(\Theta)$$

- **Note:** the Jacobian, $J(\Theta)$, is written in the same frame as $\ddot{X}$. The choice of this frame is **arbitrary** but is usually the tool frame, $\{T\}$.

Dynamic Equations in Cartesian Space

2) The Cartesian State-Space Force Equations

- The dynamics of a manipulator with respect to Cartesian variables

$$\mathcal{F} = M_X(\Theta)\ddot{X} + V_X\left(\Theta, \dot{\Theta}\right) + G_X(\Theta)$$

- $\mathcal{F}$ is a force-torque vector acting on the end-effector

- **Remark:** The acting end-effector forces, $\mathcal{F}$, could in fact be applied by the actuators at the joints:

$$\tau = J^T(\Theta)\,\mathcal{F}$$

- It means that $\mathcal{F}$ is the Cartesian expression of joints torques

- **Note:** the Jacobian, $J(\Theta)$, is written in the same frame as $\mathcal{F}$ and $\ddot{X}$. The choice of this frame is **arbitrary** but is usually the tool frame, $\{T\}$

❖ ***Q:*** What does this dynamic represent?

Dynamic Equations in Cartesian Space

2) The Cartesian State-Space Force Equations

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

- Premultiplying by the inverse of the Jacobian transpose

$$J^{-T}\tau = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}\,V(\Theta,\dot{\Theta}) + J^{-T}G(\Theta)$$

- or

$$\mathcal{F} = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}\,V(\Theta,\dot{\Theta}) + J^{-T}G(\Theta)$$

- Using $\ddot{\Theta}$ as Equ. $(*)$

$$\mathcal{F} = J^{-T}M(\Theta)\,J^{-1}\ddot{X} - J^{-T}M(\Theta)\,J^{-1}\,\dot{J}\,\dot{\Theta} + J^{-T}V(\Theta,\dot{\Theta}) + J^{-T}G(\Theta)$$

- Therefore

$$\mathcal{F} = M_X(\Theta)\ddot{X} + V_X(\Theta,\dot{\Theta}) + G_X(\Theta)$$

$$M_X(\Theta) = J^{-T}(\Theta)M(\Theta)\,J^{-1}(\Theta)$$
$$V_X(\Theta,\dot{\Theta}) = J^{-T}\left(V(\Theta,\dot{\Theta}) - M(\Theta)\,J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$$
$$G_X(\Theta) = J^{-T}(\Theta)\,G(\Theta)$$

Dynamic Equations in Cartesian Space

2) The Cartesian State-Space Force Equations

❖ **Example:**

- Derive the Cartesian-space dynamic equations for the two-link RR planar arm in terms of a frame attached to the end of the second link

- the Jacobian

$$J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$$

- the inverse Jacobian

$$J^{-1}(\Theta) = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 & 0 \\ -l_1 c_2 - l_2 & l_1 s_2 \end{bmatrix}$$

- the time derivative of the Jacobian:

$$\dot{J}(\Theta) = \begin{bmatrix} l_1 c_2 \dot{\theta}_2 & 0 \\ -l_1 s_2 \dot{\theta}_2 & 0 \end{bmatrix}$$

Dynamic Equations in Cartesian Space

2) The Cartesian State-Space Force Equations

❖ **Example:**

- Therefore

$$\mathcal{F} = M_X(\Theta)\ddot{X} + V_X(\Theta,\dot{\Theta}) + G_X(\Theta)$$

$$M_x(\Theta) = \begin{bmatrix} m_2 + \dfrac{m_1}{s_2^2} & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V_x(\Theta,\dot{\Theta}) = \begin{bmatrix} -(m_2 l_1 c_2 + m_2 l_2)\dot{\theta}_1^2 - m_2 l_2 \dot{\theta}_2^2 - \left(2m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \dfrac{c_2}{s_2^2}\right)\dot{\theta}_1\dot{\theta}_2 \\ m_2 l_1 s_2 \dot{\theta}_1^2 + m_2 l_1 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

$$G_x(\Theta) = \begin{bmatrix} m_1 g \dfrac{c_1}{s_2} + m_2 g\, s_{12} \\ m_2 g\, c_{12} \end{bmatrix}$$

*

Dynamic Equations in Cartesian Space

3) The Cartesian Configuration-Space Torque Equations

$$\tau = M_X(\Theta)\ddot{X} + B_X(\Theta)[\dot{\Theta}\dot{\Theta}] + C_X(\Theta)[\dot{\Theta}^2] + G_X(\Theta)$$

- $B_X(\Theta) \neq B(\Theta)$
- $C_X(\Theta) \neq C(\Theta)$
- $G_X(\Theta) = G(\Theta)$

4) The Cartesian Configuration-Space Force Equations

$$\mathcal{F} = M_X(\Theta)\ddot{X} + B_X(\Theta)[\dot{\Theta}\dot{\Theta}] + C_X(\Theta)[\dot{\Theta}^2] + G_X(\Theta)$$

- $B_X(\Theta) \neq B(\Theta)$
- $C_X(\Theta) \neq C(\Theta)$
- $G_X(\Theta) \neq G(\Theta)$

Inclusion of Non-rigid Body Effects

- Derived dynamic equations do not encompass all the effects acting on a manipulator.

- The most important source of forces that are not included is **friction.**

- **Viscous friction:** the torque due to friction is proportional to the velocity of joint motion.

$$\tau_{friction} = v\dot{\theta}$$

- v is a viscous-friction constant.

- **Coulomb friction:** the torque due to friction is constant except for a sign dependence on the joint velocity.

$$\tau_{friction} = c\ sgn(\dot{\theta})$$

- c is a Coulomb-friction constant.

- c is often taken at one value when $\dot{\theta} = 0$, i.e. the **static coefficient**, but at a lower value, when $\dot{\theta} \neq 0$, i.e. the **dynamic coefficient.**

Inclusion of Non-rigid Body Effects

- To include both

$$\tau_{friction} = c\ sgn(\dot{\theta}) + v\dot{\theta}$$

- In many joints, friction displays a dependence on the joint position.

- A major cause might be not perfectly round gears (Also eccentricity).

- A fairly complex friction model

$$\tau_{friction} = F(\theta, \dot{\theta})$$

- Adding to the dynamic terms derived from the rigid-body model

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})$$

- In addition **bending effects** (which give rise to resonances) are neglected (Flexible links and joints).

The END

- **References:**

1) …