دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

بسم الله الرحمن الرحیم

دانشکده مهندسی مکانیک
دانشگاه صنعتی امیرکبیر

Lecture 4_1: Velocities

Advanced Robotics

Hamed Ghafarirad

Outlines

- Time Varying Position and Orientation
- Linear Velocity of Rigid Bodies
- More on Linear Velocity Due to the Rotational Motion
- Angular Velocity of Rigid Bodies
- Motion of the Links of a Robot
- Velocity “Propagation”

Time Varying Position and Orientation

The Linear Velocity Vector

- **The velocity of a position vector:**

  The linear velocity of the point in space represented by the position vector:

$$^{B}V_{Q} = \frac{d}{dt}\,^{B}Q = \lim_{\Delta t \to 0} \frac{^{B}Q(t+\Delta t) - {^{B}Q(t)}}{\Delta t}$$

- It is the derivative of Q relative to frame $\{B\}$.

- If Q is not changing in time relative to $\{B\}$, then the velocity calculated is zero (even if there is some other frame in which Q is varying).

- A velocity vector can be described in terms of any frame:

$$^{A}\left(^{B}V_{Q}\right) = \frac{^{A}d}{dt}\,^{B}Q$$

- $^{A}\left(^{B}V_{Q}\right)$ is the calculated velocity vector when expressed in terms of frame $\{A\}$.

Time Varying Position and Orientation

The Linear Velocity Vector

- The velocity numerical values depend on **two frames**:
  - With respect to which the differentiation was done (**Differentiation**).
  - In which the resulting velocity vector is expressed (**Expression**).

- When both superscripts are the same, do not indicate the outer one.

$$^{B}\left(^{B}V_{Q}\right) = {}^{B}V_{Q}$$

- The outer leading superscript can always be removed, by explicitly including the **rotation matrix.**

$$^{A}\left(^{B}V_{Q}\right) = {}^{A}R_{B}\,{}^{B}V_{Q}$$

- Consider the velocity of the origin of a frame relative to some understood universe reference frame.

- For this special case, a shorthand notation is used:

$$v_{C} = {}^{U}V_{CORG}$$

Time Varying Position and Orientation

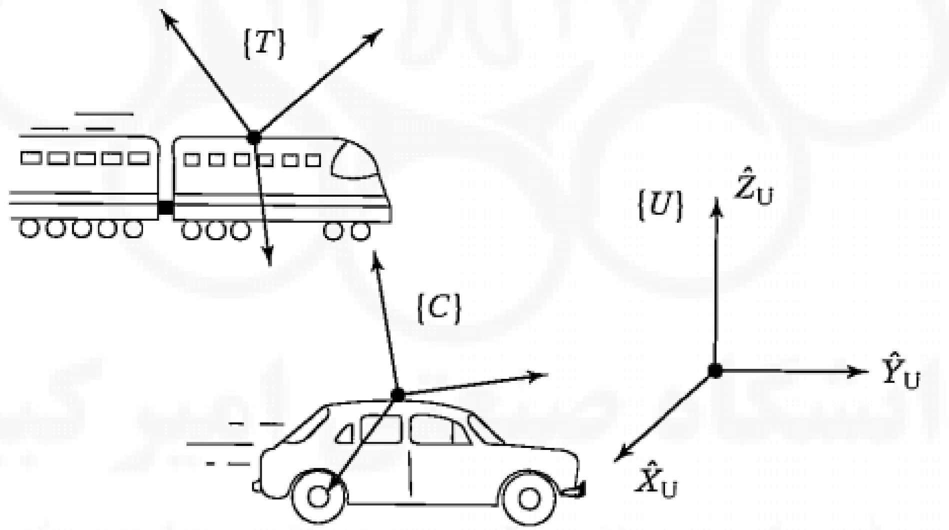
❑ The Linear Velocity Vector

$$v_c = {}^{U}V_{CORG}$$

- ${}^{A}v_{C}$ is the velocity of the origin of $\{C\}$ expressed in $\{A\}$, although differentiation is done relative to $\{U\}$.

Time Varying Position and Orientation

- **The Linear Velocity Vector**
- **Example:**
- Assume: A fixed universe frame, $\{U\}$,
- A frame attached to a train , $\{T\}$, traveling at 100 mph in the $\hat{Y}_U$ direction
- A frame attached to a car, $\{C\}$, traveling at 30 mph in the $\hat{Y}_U$ direction
- The rotation matrices, ${}^{U}R_{T}$ and ${}^{U}R_{C}$ are known and constant.
- Calculate ${}^{U}\frac{d}{dt}\,{}^{U}P_{CORG}$, ${}^{C}({}^{U}V_{TORG})$, ${}^{C}({}^{T}V_{CORG})$?

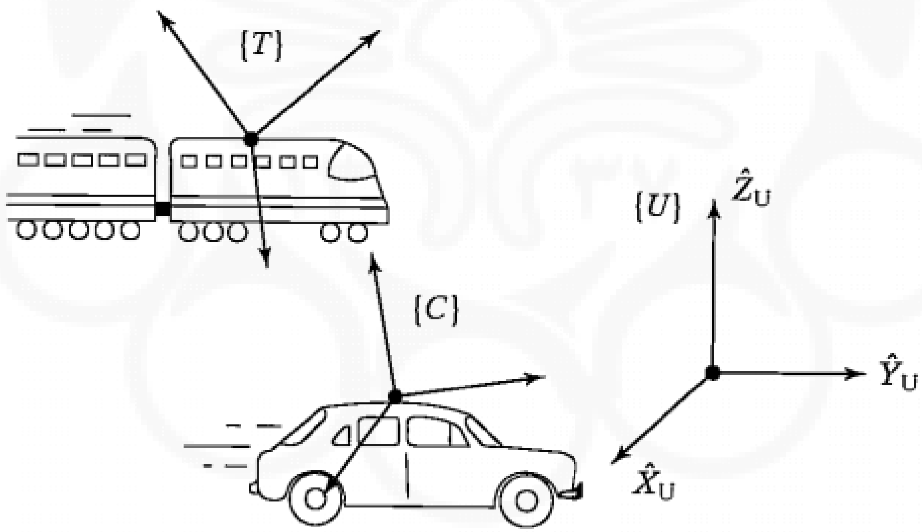


Time Varying Position and Orientation

The Linear Velocity Vector

Example:

- Train traveling at 100 mph, car traveling at 30 mph in $\hat{Y}_U$ direction
- Calculate ${}^{U}\frac{d}{dt}\,{}^{U}P_{CORG}$, ${}^{C}({}^{U}V_{TORG})$, ${}^{C}({}^{T}V_{CORG})$



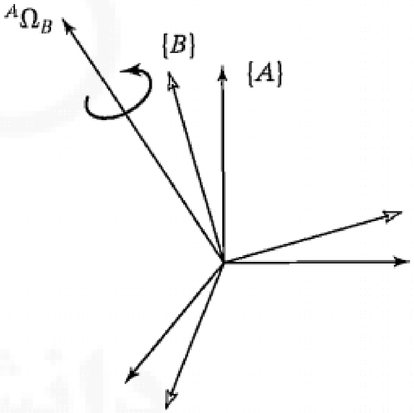
- ${}^{U}\frac{d}{dt}\,{}^{U}P_{CORG} = {}^{U}V_{CORG} = v_c = 30\,\hat{Y}$
- ${}^{C}({}^{U}V_{TORG}) = {}^{C}v_T = {}^{C}R_U v_T = {}^{C}R_U(100\,\hat{Y}) = {}^{U}R_C{}^{-1}(100\,\hat{Y})$
- ${}^{C}({}^{T}V_{CORG}) = {}^{C}R_T\,{}^{T}V_{CORG} = {}^{U}R_C{}^{-1}(-70\,\hat{Y}) = {}^{U}R_C{}^{-1}\,\cancel{{}^{U}R_T}(-70\,\hat{Y})$

Time Varying Position and Orientation

- **The Angular Velocity Vector**
- Linear velocity (V) describes an attribute of a point .
- **Angular velocity** (Ω) describes an attribute of a **body.**

- **Frames** are always attached to the bodies, so angular velocity is described as rotational motion of a frame.

- ${}^{A}\Omega_{B}$ describes the time varying rotation of frame {B} relative to {A}.



- The angular velocity vector is given by
 (One of the all representations is):

- ${}^{A}\Omega_{B} = \hat{k} \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \hat{k}\dot{\theta}$

- **Direction** of ${}^{A}\Omega_{B}$: the instantaneous axis of rotation of $\{B\}$ relative to $\{A\}$.
- **Magnitude** of ${}^{A}\Omega_{B}$: the speed of rotation.

Time Varying Position and Orientation

The Angular Velocity Vector

- Angular velocity vector may be expressed in any coordinate system.
- ${}^{C}({}^{A}\Omega_{B})$ is the angular velocity of frame $\{B\}$ relative to $\{A\}$ expressed in terms of frame $\{C\}$.
- For the case in which there is an understood reference frame, it need not be mentioned in the notation.

$$\omega_c = {}^{U}\Omega_C$$

- ω_C is the angular velocity of frame $\{C\}$ relative to some understood reference frame, i.e. $\{U\}$.
- ${}^{A}\omega_C$ is the angular velocity of frame $\{C\}$ expressed in terms of $\{A\}$ (though the angular velocity is with respect to $\{U\}$).

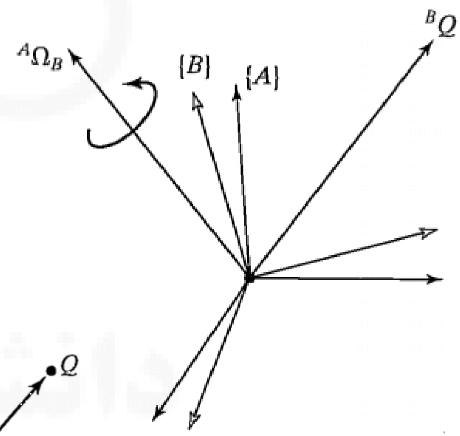
Time Varying Position and Orientation

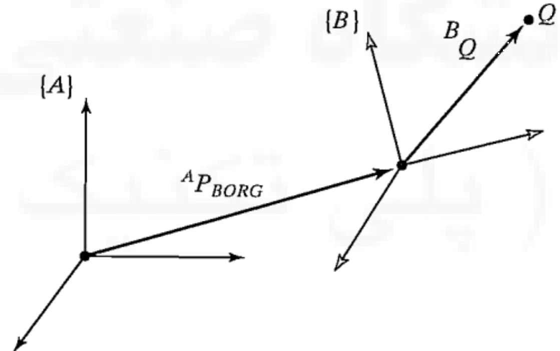
- **Note:**
- The velocity vector is the differentiation of position vector but it seems that the angular velocity vector does not have such a description.

- Differentiation of orientation ?!!

Linear Velocity of Rigid Bodies

- Investigate the description of motion of a rigid body, as far as **velocity.**
- Extend the notions of translations and orientations to the time-varying case.
- Frames are attached to rigid bodies.
- Motion of rigid bodies can be equivalently studied as the motion of frames relative to one another.
- Linear Velocity could be caused by:
  - Linear motion
  - Rotational motion
  - Simultaneous linear and rotational motions

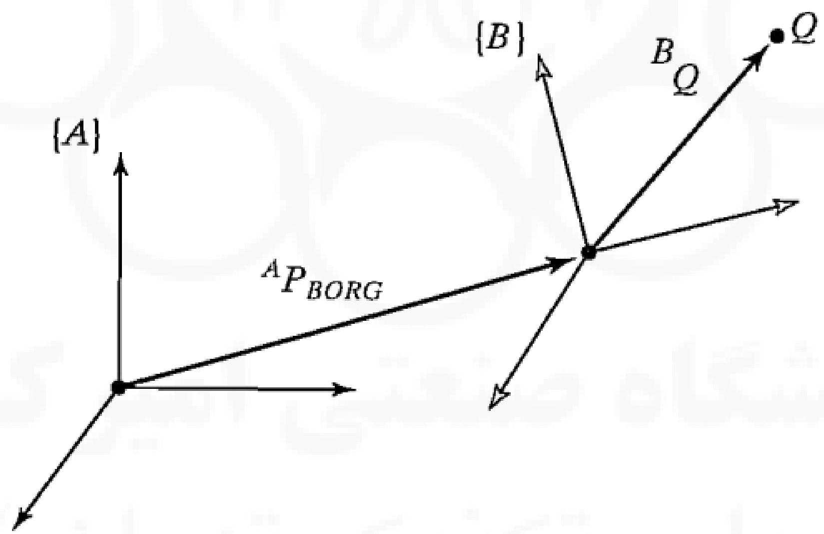




Linear Velocity of Rigid Bodies

❑ Linear Motion

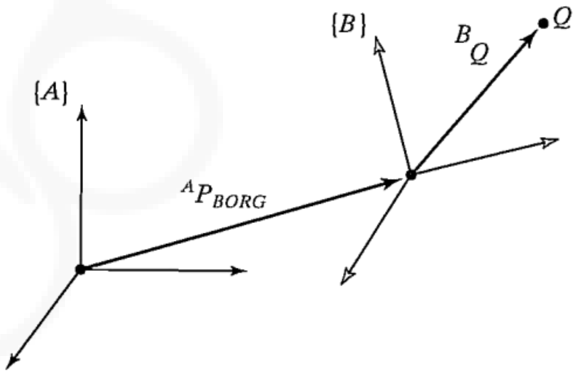
- Consider a frame {B} attached to a rigid body.
- Frame {B} is located relative to {A}, as described by ${}^{A}R_{B}$ & ${}^{A}P_{BORG}$.
- {B} is linearly moving relative to {A}.
- The motion of Q relative to frame {B} (${}^{B}V_{Q}$) is known.
- Describe the motion of Q relative to frame {A} (${}^{A}V_{Q} = ?$).



Linear Velocity of Rigid Bodies

Linear Motion

- The motion of Q relative to frame {B} (${}^{B}V_{Q}$) is known. (${}^{A}V_{Q}= ?$).
- **Assumptions:**
  - {A} is fixed.
  - ${}^{A}R_{B}$ is not changing with time.
  - {B} is linearly moving relative to {A}, i.e. $\frac{d}{dt}\,{}^{A}P_{BORG} = {}^{A}V_{BORG}$.

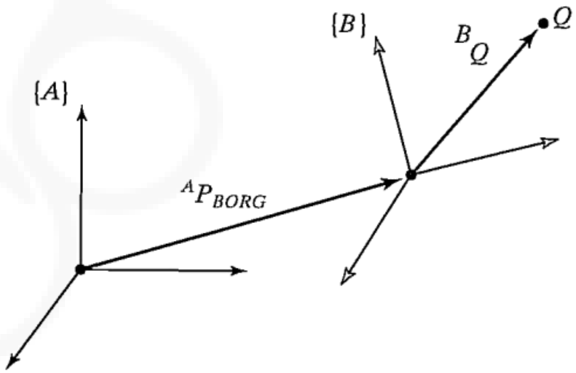


- **1st Case:** Vector ${}^{B}Q$ locates a point fixed in {B}, i.e. ${}^{B}V_{Q} = 0$.

$${}^{A}V_{Q} = ???$$

Linear Velocity of Rigid Bodies

- **Linear Motion**
- The motion of Q relative to frame {B} (${}^{B}V_Q$) is known. (${}^{A}V_Q$= ?).
- **Assumptions:**
  - {A} is fixed.
  - ${}^{A}R_B$ is not changing with time.
  - {B} is linearly moving relative to {A}, i.e. $\frac{d}{dt}\,{}^{A}P_{BORG} = {}^{A}V_{BORG}$.



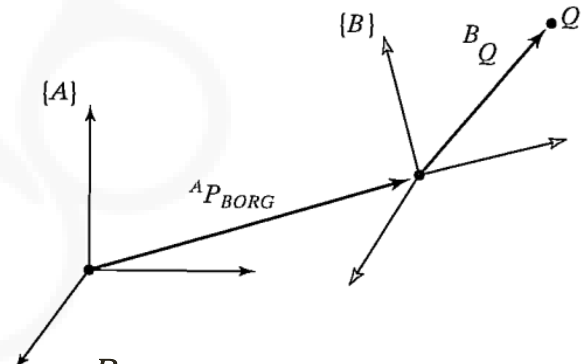
- **1[st] Case:** Vector ${}^{B}Q$ locates a point fixed in {B}, i.e. ${}^{B}V_Q = 0$.
- The motion of point Q relative to {A} is due to ${}^{A}P_{BORG}$ changing in time.

$$ {}^{A}V_Q = {}^{A}V_{BORG} $$

Linear Velocity of Rigid Bodies

Linear Motion

- The motion of Q relative to frame {B} (${}^{B}V_{Q}$) is known. (${}^{A}V_{Q}$= ?).
- **Assumptions:**
  - {A} is fixed.
  - ${}^{A}R_{B}$ is not changing with time.
  - {B} is linearly moving relative to {A}, i.e. $\frac{d}{dt}\,{}^{A}P_{BORG} = {}^{A}V_{BORG}$.



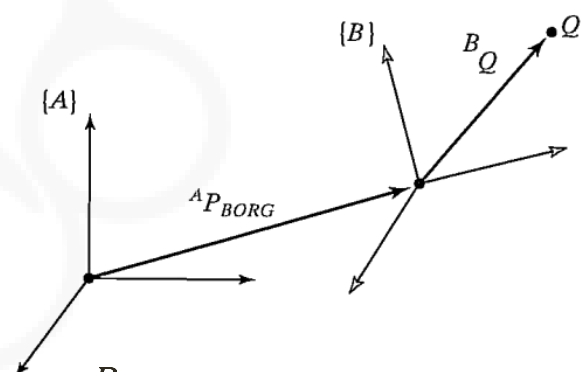
- **2nd Case:** Vector ${}^{B}Q$ is a moving point in {B}, i.e. ${}^{B}V_{Q} \neq 0$.

$${}^{A}V_{Q} = ???$$

Linear Velocity of Rigid Bodies

❑ Linear Motion

- The motion of Q relative to frame {B} $({}^{B}V_Q)$ is known. $({}^{A}V_Q = ?)$.
- **Assumptions:**
  - {A} is fixed.
  - ${}^{A}R_B$ is not changing with time.
  - {B} is linearly moving relative to {A}, i.e. $\frac{d}{dt}\,{}^{A}P_{BORG} = {}^{A}V_{BORG}$.



- **2^{nd} Case:** Vector ${}^{B}Q$ is a moving point in {B}, i.e. ${}^{B}V_Q \neq 0$.
- The motion of point Q relative to {A} is due to ${}^{A}P_{BORG}$ & ${}^{B}Q$ changing in time.

$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}R_B\,{}^{B}V_Q$$

Linear Velocity of Rigid Bodies

❑ Rotational Motion

- Two frames with coincident origins and with zero linear relative velocity.

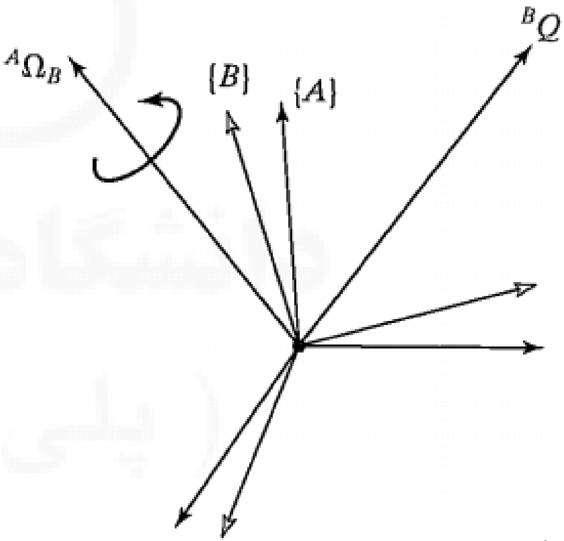
- Their origins will remain coincident for all time.

- The orientation of frame {B} with respect to frame {A} is changing in time.
- Rotational velocity of {B} relative to {A} is ${}^{A}\Omega_{B}$.

- How does a vector change with time as viewed from {A} when it is fixed in {B}?

- **Two Solution** Methods:
  - **Geometrical**
  - **Mathematical**

- Now, follow the **Geometrical** solution.



Linear Velocity of Rigid Bodies

❑ Rotational Motion

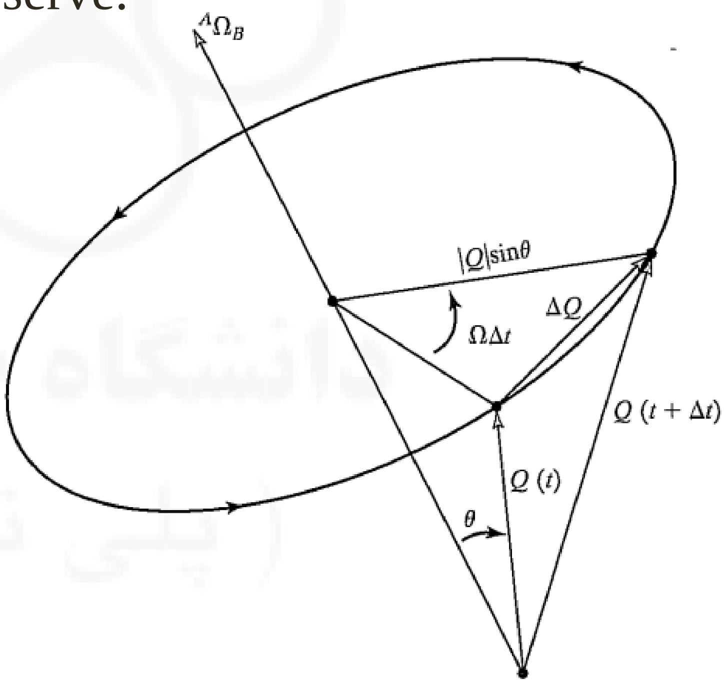
- **1st Case:** Vector ${}^{B}Q$ locates a point fixed in {B}, i.e. ${}^{B}V_Q = 0$.

- Q will have a velocity as seen from {A} due to the rotational velocity ${}^{A}\Omega_B$.

- Consider two instants of time as vector Q rotates around ${}^{A}\Omega_B$.

- This is what an observer in {A} would observe.

- $|\Delta Q| = (|{}^{A}Q| \sin\theta)(|{}^{A}\Omega_B|\Delta t)$



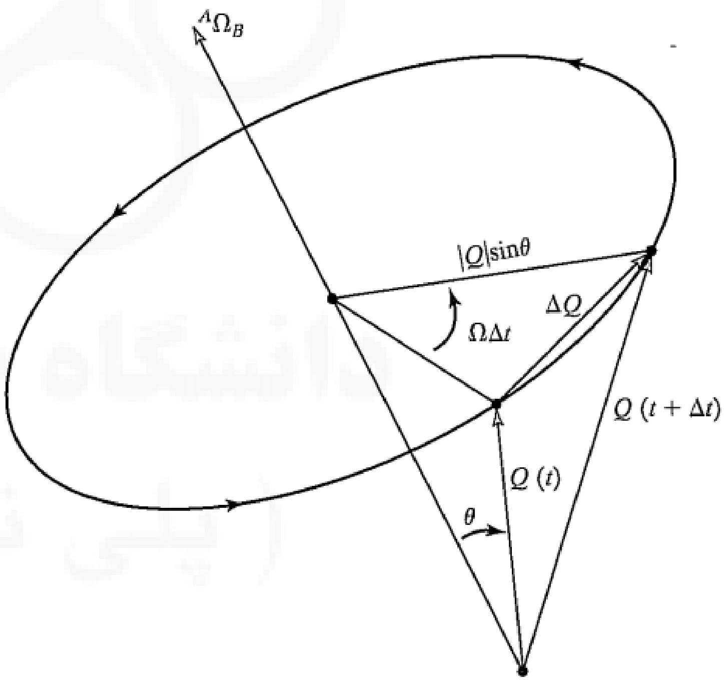
Linear Velocity of Rigid Bodies

- **Rotational Motion**
- **1st Case:** Vector ^{B}Q locates a point fixed in {B}, i.e. $^{B}V_{Q}= 0$.

$$|\Delta Q| = (|^{A}Q|\sin\theta)(|^{A}\Omega_{B}|\Delta t)$$

- These conditions on magnitude and direction immediately suggest the vector cross product.

$$^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}Q$$



Linear Velocity of Rigid Bodies

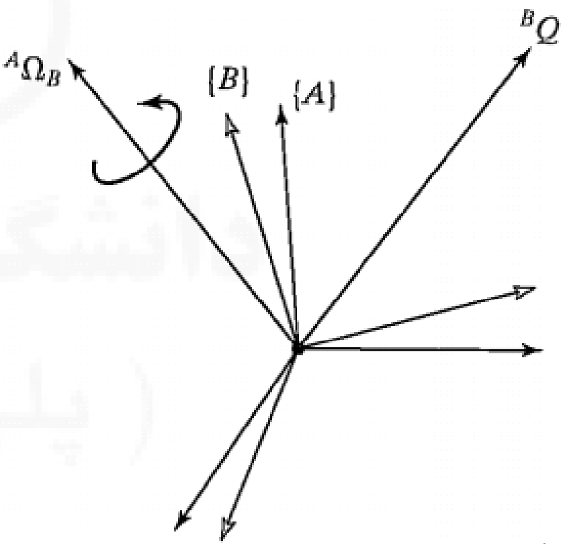
Rotational Motion

- **2nd Case:** Vector ${}^{B}Q$ is a moving point in {B}, i.e. ${}^{B}V_Q \neq 0$.

$$ {}^{A}V_Q = {}^{A}\left({}^{B}V_Q\right) + {}^{A}\Omega_B \times {}^{A}Q $$

- Using a rotation matrix to remove the dual-superscript, and noting that the description of ${}^{A}Q$ at any instant is ${}^{A}R_B\,{}^{B}Q$.

$$ {}^{A}V_Q = {}^{A}R_B\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}R_B\,{}^{B}Q $$



Linear Velocity of Rigid Bodies

Simultaneous Linear and Rotational Motions

- {B} is rotating relative to {A} by ${}^{A}\Omega_B$ & ${}^{B}Q$ locates a point fixed in {B}.

$$ {}^{A}V_Q = {}^{A}\Omega_B \times {}^{A}Q = {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}Q $$

- The vector Q could also be changing with respect to frame {B}.

$$ {}^{A}V_Q = {}^{A}R_B\, {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}Q $$

- The case where origins are not coincident is the final result for the derivative of a vector in a moving frame as seen from a stationary frame.

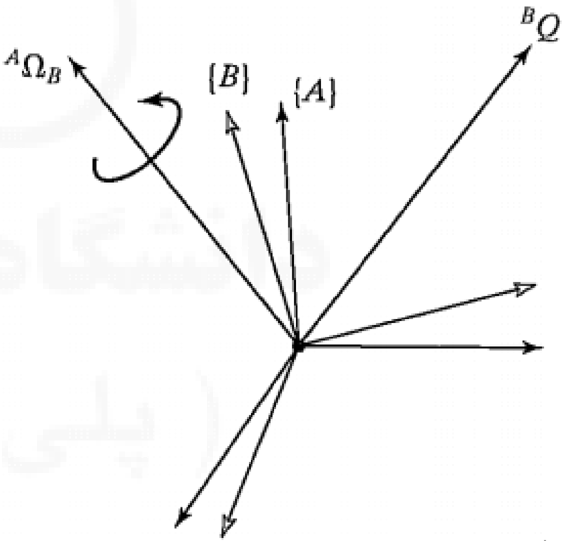
$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}R_B\, {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}Q $$

← Outline

More on Linear Velocity Due to the Rotational Motion

Rotational Motion

- Consider the same problem:
  - Two frames with coincident origins and with zero linear relative velocity.
  - Their origins will remain coincident for all time.
  - The orientation of frame {B} with respect to frame {A} is changing in time.
  - Rotational velocity of {B} relative to {A} is ${}^{A}\Omega_{B}$.
  - How does a vector change with time as viewed from {A} when it is fixed in {B}?



- Now, follow the **Mathematical** solution.

More on Linear Velocity Due to the Rotational Motion

❑ A Property of the Derivative of an Orthonormal Matrix

- For any $n \times n$ orthonormal matrix, R,

$$R\,R^T = I_n$$

- where I_n is the $n \times n$ identity matrix (*Why?*)
- Our interest, is in the case where $n = 3$ and R is the rotation matrix.
- Differentiating:

$$\dot{R}\,R^T + R\,\dot{R}^T = 0_n \qquad \text{or} \qquad \dot{R}\,R^T + \left(\dot{R}\,R^T\right)^T = 0_n$$

- Defining:

$$S = \dot{R}\,R^T$$
$$S + S^T = 0_n$$

- S is a skew-symmetric matrix.
- A property relating the derivative of orthonormal matrices with skew-symmetric matrices:

$$S = \dot{R}\,R^{-1} \qquad \text{or} \qquad \dot{R} = S\,R$$

More on Linear Velocity Due to the Rotational Motion

Velocity of a Point Due to Rotating Reference Frame

- Consider ${}^{B}P$ is a fixed vector in frame $\{B\}$, its description in another frame $\{A\}$ with the same origin is given as:

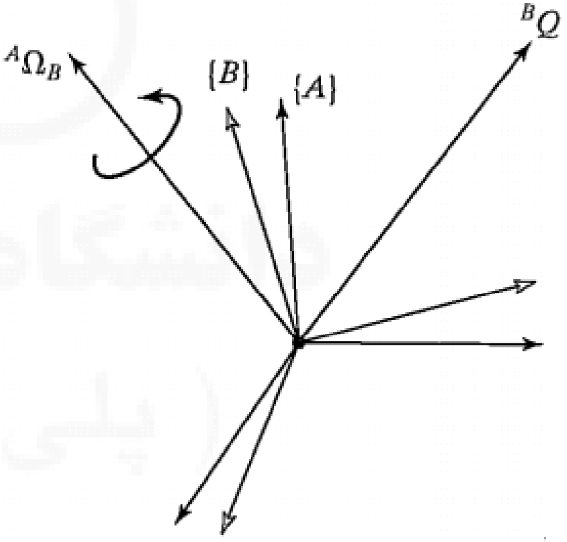
$$ {}^{A}P = {}^{A}R_{B}\,{}^{B}P $$

- If frame $\{B\}$ is rotating (i.e., the derivative is nonzero),

$$ {}^{A}\dot{P} = {}^{A}\dot{R}_{B}\,{}^{B}P \qquad \text{or} \qquad {}^{A}V_{P} = {}^{A}\dot{R}_{B}\,{}^{B}P $$

- Substituting for ${}^{B}P$:

$$ {}^{A}V_{P} = {}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1}\,{}^{A}P $$



More on Linear Velocity Due to the Rotational Motion

Velocity of a Point Due to Rotating Reference Frame

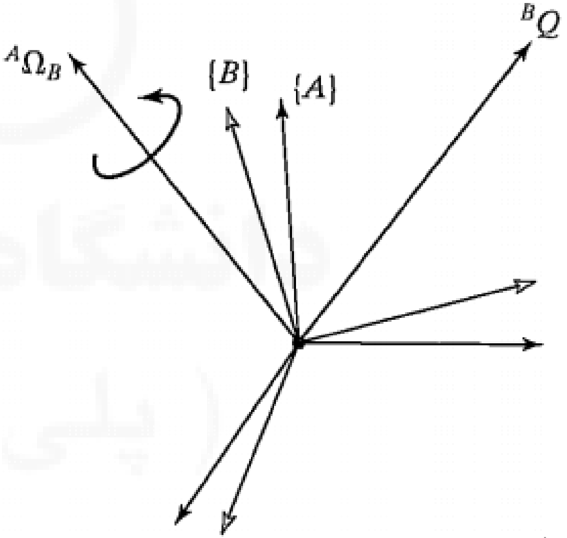
$$^{A}V_{P} = {}^{A}\dot{R}_{B}\ {}^{A}R_{B}^{-1}\,{}^{A}P$$

- Using the result for orthonormal matrices:

$$^{A}V_{P} = {}^{A}S_{B}\,{}^{A}P$$

- $^{A}S_{B}$:
  The skew-symmetric matrix associated with the rotation matrix $^{A}R_{B}$.

- $^{A}S_{B}$ is called the **angular-velocity matrix.**



More on Linear Velocity Due to the Rotational Motion

❑ Skew-Symmetric Matrices and the Vector Cross-Product

- Assign the elements in a skew-symmetric matrix S as

$$S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

- Define the 3×1 column vector: $$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$

- It is easily verified that:

$$S\,P = \Omega \times P$$

  P is any vector, and $\times$ is the vector cross-product.

- The 3×1 vector which corresponds to the 3×3 angular-velocity matrix, is called the **angular-velocity vector.**

- Hence

$${}^{A}V_P = {}^{A}S_B\,{}^{A}P = {}^{A}\Omega_B \times {}^{A}P$$

- The same notation as in previous section.

More on Linear Velocity Due to the Rotational Motion

Gaining Physical Insight Concerning the Angular-Velocity Vector

- Having concluded that there exists some vector Ω such that:

$$ {}^{A}V_{P} = {}^{A}S_{B}\,{}^{A}P = {}^{A}\Omega_{B} \times {}^{A}P $$

Now, explore its physical meaning.

- Derive Ω by direct differentiation of a rotation matrix.

$$ \dot{R} = \lim_{\Delta t \to 0} \frac{R(t + \Delta t) - R(t)}{\Delta t} $$

- Write $R(t + \Delta t)$ as the composition of two matrices

$$ R(t + \Delta t) = R_{K}(\Delta\theta)\, R(t) $$

over the interval Δt, a small rotation of $\Delta\theta$ has occurred about axis $\hat{K}$.

- So,

$$ \dot{R} = \left(\lim_{\Delta t \to 0} \frac{R_{K}(\Delta\theta) - I_{3}}{\Delta t} \right) R(t) $$

More on Linear Velocity Due to the Rotational Motion

❑ Gaining Physical Insight Concerning the Angular-Velocity Vector

$$\dot{R} = \left(\lim_{\Delta t \to 0} \frac{R_K(\Delta\theta) - I_3}{\Delta t} \right) R(t)$$

- Remember

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z\, s\theta & k_x k_z v\theta + k_y\, s\theta \\ k_x k_y v\theta + k_z\, s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x\, s\theta \\ k_x k_z v\theta - k_y\, s\theta & k_y k_z v\theta + k_x\, s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

- From small angle ($\Delta\theta$) substitution:

$$R_K(\Delta\theta) = \begin{bmatrix} 1 & -k_z\,\Delta\theta & k_y\,\Delta\theta \\ k_z\,\Delta\theta & 1 & -k_x\,\Delta\theta \\ -k_y\,\Delta\theta & k_x\,\Delta\theta & 1 \end{bmatrix}$$

More on Linear Velocity Due to the Rotational Motion

- **Gaining Physical Insight Concerning the Angular-Velocity Vector**

$$\dot{R} = \left(\lim_{\Delta t \to 0} \frac{R_K(\Delta\theta) - I_3}{\Delta t} \right) R(t)$$

$$R_K(\Delta\theta) = \begin{bmatrix} 1 & -k_z\,\Delta\theta & k_y\,\Delta\theta \\ k_z\,\Delta\theta & 1 & -k_x\,\Delta\theta \\ -k_y\,\Delta\theta & k_x\,\Delta\theta & 1 \end{bmatrix}$$

- so

$$\dot{R} = \left(\lim_{\Delta t \to 0} \frac{\begin{bmatrix} 0 & -k_z\,\Delta\theta & k_y\,\Delta\theta \\ k_z\,\Delta\theta & 0 & -k_x\,\Delta\theta \\ -k_y\,\Delta\theta & k_x\,\Delta\theta & 0 \end{bmatrix}}{\Delta t} \right) R(t)$$

- Finally, dividing the matrix through by Δt and then taking the limit,

$$\dot{R} = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix} R(t)$$

More on Linear Velocity Due to the Rotational Motion

❑ Gaining Physical Insight Concerning the Angular-Velocity Vector

$$\dot{R} = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix} R(t)$$

- Hence

$$\dot{R}R^{-1} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

- where

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} k_x\dot{\theta} \\ k_y\dot{\theta} \\ k_z\dot{\theta} \end{bmatrix} = \dot{\theta}\hat{K}$$

- **Physical meaning** of the angular-velocity vector:
  - ➢ At any instant, the change in orientation of a rotating frame can be viewed as a rotation about some axis $\hat{K}$.
  - ➢ Angular-velocity vector is the instantaneous axis of rotation scaled by the speed of rotation ($\dot{\theta}$).

More on Linear Velocity Due to the Rotational Motion

- **Remark:**
- **Simultaneous Linear and Rotational Velocity (*Geometrical Sol.*)**

- {B} is rotating relative to {A} by ${}^{A}\Omega_B$ & ${}^{B}Q$ locates a point fixed in {B}

$$ {}^{A}V_Q = {}^{A}\Omega_B \times {}^{A}Q = {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}Q $$

- The vector Q could also be changing with respect to frame {B}

$$ {}^{A}V_Q = {}^{A}R_B\, {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}Q $$

- The case where origins are not coincident is the final result for the derivative of a vector in a moving frame as seen from a stationary frame

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}R_B\, {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}R_B\, {}^{B}Q $$

More on Linear Velocity Due to the Rotational Motion

- **Simultaneous Linear and Rotational Velocity (*Mathematical Sol*)**
- In the general case:

$$ {}^{A}Q = {}^{A}P_{BORG} + {}^{A}R_{B}\,{}^{B}Q $$

- By differentiation of two sides:

$$ \frac{d}{dt}{}^{A}Q = \frac{d}{dt}[{}^{A}P_{BORG} + {}^{A}R_{B}\,{}^{B}Q] $$

- So:

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\dot{R}_{B}\,{}^{B}Q $$

- It was shown:

$$ {}^{A}\dot{R}_{B} = {}^{A}\Omega_{B} \times {}^{A}R_{B} $$

- Therefore:

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q $$

Angular Velocity of Rigid Bodies

Angular Velocity

- Assume
  - $\{B\}$ is rotating relative to $\{A\}$ with ${}^{A}\Omega_{B}$
  - $\{C\}$ is rotating relative to $\{B\}$ with ${}^{B}\Omega_{C}$

- Therefore

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}R_{B}\, {}^{B}\Omega_{C} $$

Angular Velocity of Rigid Bodies

Other Representations of Angular Velocity

- Assume the orientation of the rotating frame relative to the base frame is described by the set of Z-Y-Z Euler angles (α, β, γ) (one of the 24 angle sets).

$$R = f(\Theta) \qquad \Theta_{Z'Y'Z'} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

- **Objective:** Express the angular velocity (Ω) of a rotating frame as rates of the set of Z-Y-Z Euler angles $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$.

$$\Omega = f\left(\Theta, \dot{\Theta}\right) \qquad \dot{\Theta}_{Z'Y'Z'} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

Angular Velocity of Rigid Bodies

Other Representations of Angular Velocity

$$\Omega = f\left(\Theta, \dot{\Theta}\right) \qquad \Theta_{Z'Y'Z'} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \qquad \dot{\Theta}_{Z'Y'Z'} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

- We have:

$$\dot{R}R^T = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

- From this matrix equation, one can extract three independent equations.

$$\begin{aligned} \Omega_x &= \dot{r}_{31} r_{21} + \dot{r}_{32} r_{22} + \dot{r}_{33} r_{23} \\ \Omega_y &= \dot{r}_{11} r_{31} + \dot{r}_{12} r_{32} + \dot{r}_{13} r_{33} \\ \Omega_z &= \dot{r}_{21} r_{11} + \dot{r}_{22} r_{12} + \dot{r}_{23} r_{13} \end{aligned}$$

- From the symbolic description of R in terms of an angle set (α, β, γ), derive the expressions that relate the equivalent angular-velocity vector (Ω) to the angle-set velocities $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$.

Angular Velocity of Rigid Bodies

Other Representations of Angular Velocity

$$\Omega_x = \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23}$$
$$\Omega_y = \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33}$$
$$\Omega_y = \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13}$$

- It can be expressed in matrix form:

$$\Omega = E_{Z'Y'Z'}(\Theta_{Z'Y'Z'})\,\dot{\Theta}_{Z'Y'Z'}$$

- $E(.)$ is a Jacobian relating an angle-set velocity vector (Ω) to the angular-velocity vector $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ and is a function of the instantaneous values of the angle set (α, β, γ) .

- Hint

Angular Velocity of Rigid Bodies

Other Representations of Angular Velocity

❖ Example:

- Construct the E matrix that relates Z-Y-Z Euler angles to the angular-velocity vector.

$$ {}^{A}R_{B\,Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} $$

$$ \dot{R}R^T = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \quad \text{or} \quad \begin{aligned} \Omega_x &= \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23} \\ \Omega_y &= \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33} \\ \Omega_y &= \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13} \end{aligned} $$

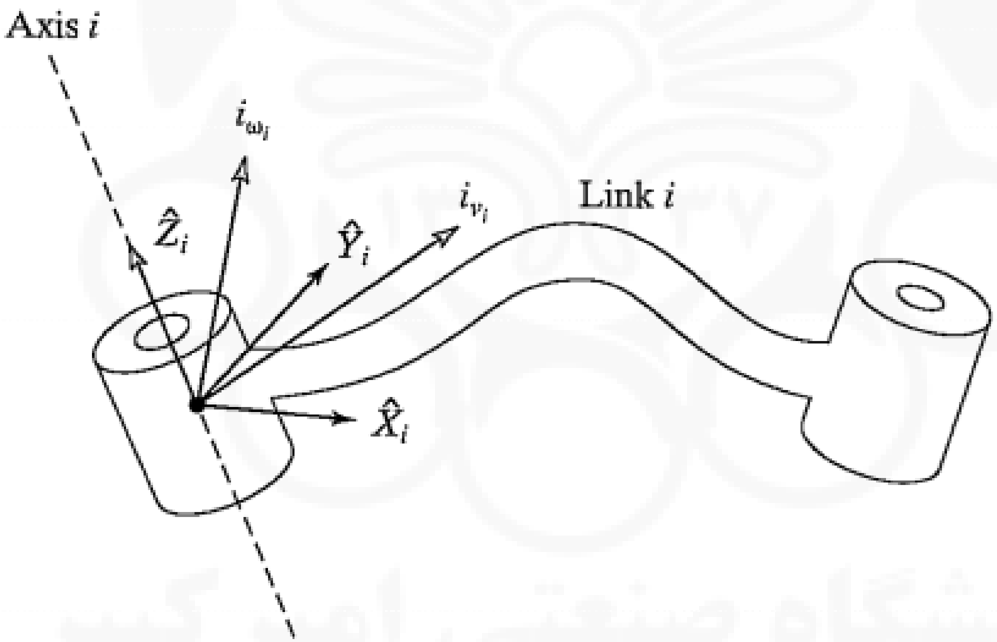
$$ \Omega = E_{Z'Y'Z'}(\Theta_{Z'Y'Z'})\,\dot{\Theta}_{Z'Y'Z'} $$

$$ E_{Z'Y'Z'} = \begin{bmatrix} 0 & -s\alpha & c\alpha\, s\beta \\ 0 & c\alpha & s\alpha\, s\beta \\ 1 & 0 & c\beta \end{bmatrix} $$

Motion of the Links of a Robot

- We will always use link frame {0} as the reference frame.

- v_i is the linear velocity of the origin of link frame $\{i\}$.
- ω_i is the angular velocity of link frame $\{i\}$.



- It is indicated that they are expressed in frame $\{i\}$.

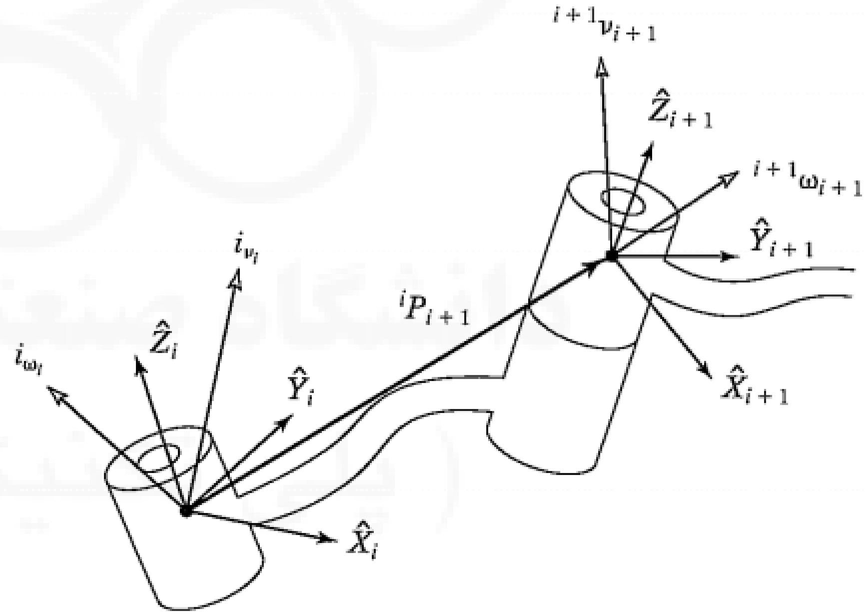
Velocity "Propagation"

- A manipulator is a chain of bodies, each one capable of motion relative to its neighbors.

- So, compute the velocity of each link in order, starting from the base.

- The velocity of link $i+1$ =
  (Velocity of link i) + (New velocity components added by joint $i+1$)

- Figure shows links i and $i+1$, along with their velocity vectors <u>expressed</u> in the link frames.



Velocity “Propagation”

- Assume Joint $i+1$ is **Revolute**

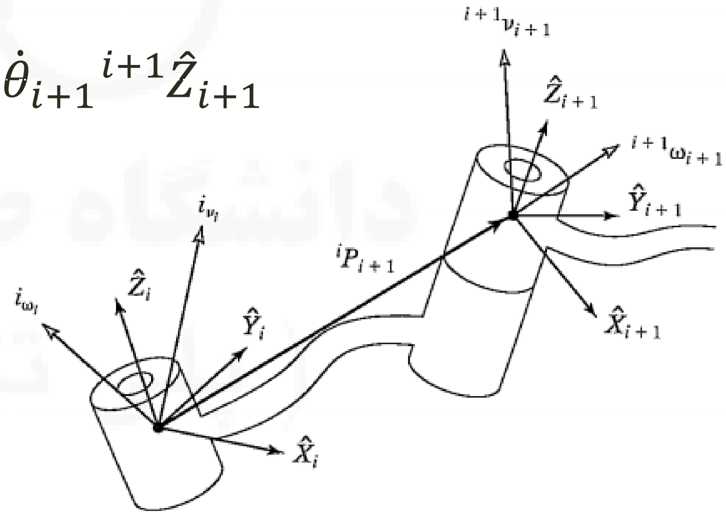
- Angular Velocity

➢ The angular velocity of link $i+1$ is the same as that of link i plus a new component caused by rotational velocity at joint $i+1$.

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}R_{i+1}\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \qquad , \qquad \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

➢ By premultiplying by ${}^{i+1}R_{i}$, the description of the angular velocity of link $i+1$ with respect to frame $\{i+1\}$ is as follow :

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}R_{i}\,{}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$



Velocity "Propagation"

- Assume Joint $i+1$ is **Revolute**

- Angular Velocity

$$^{i+1}\omega_{i+1} = {}^{i+1}R_i\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

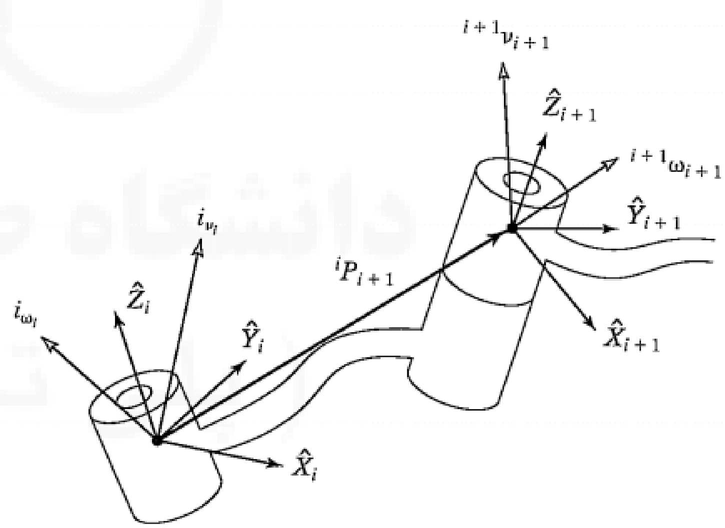
➢ **Alternative Method:**

- Remember:

$$^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}R_B\,{}^{B}\Omega_C$$

- Assume:

$$C = i+1$$
$$B = i$$
$$A = 0$$



Velocity "Propagation"

- Assume Joint $i+1$ is **Revolute**

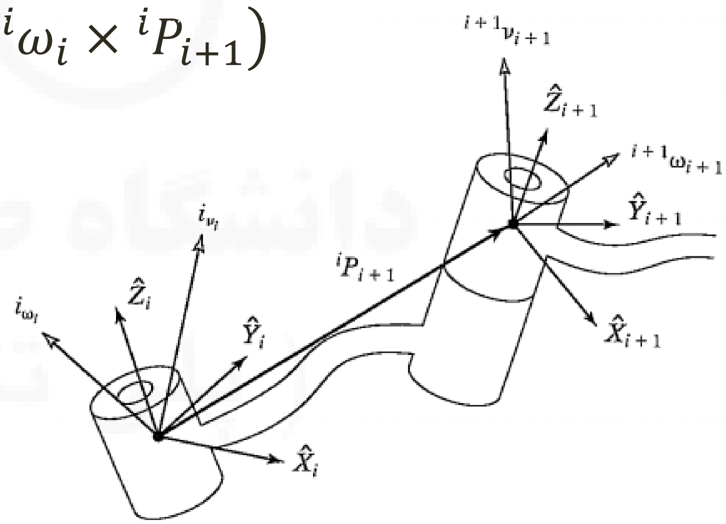
- Linear Velocity

➢ The linear velocity of the origin of frame $\{i+1\}$ is the same as that of the origin of frame $\{i\}$ plus a new component caused by rotational velocity of link i .

$${}^{i}v_{i+1} = {}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}$$

➢ Premultiplying both sides by ${}^{i+1}R_{i}$:

$${}^{i+1}v_{i+1} = {}^{i+1}R_{i}\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right)$$



Velocity "Propagation"

- Assume Joint $i+1$ is **Revolute**

- Linear Velocity

$$^{i+1}v_{i+1} = {}^{i+1}R_i\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right)$$

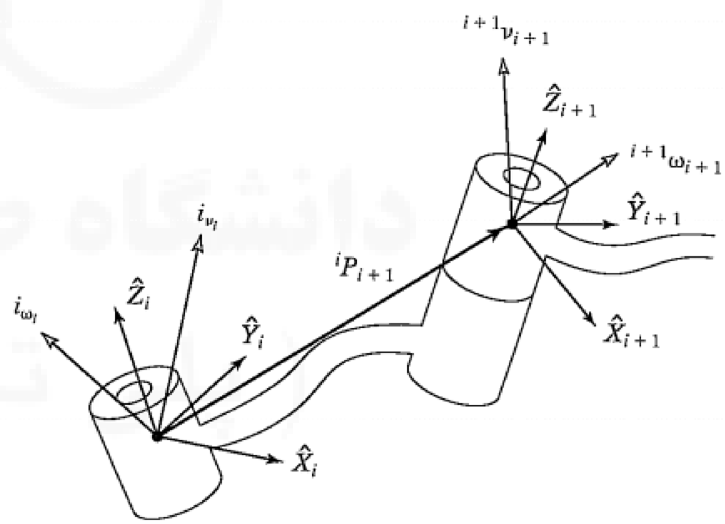
➢ **Alternative Method:**

- Remember:

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}R_B\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}R_B\,{}^{B}Q$$

- Assume:

$$Q = i+1$$
$$B = i$$
$$A = 0$$



Velocity "Propagation"

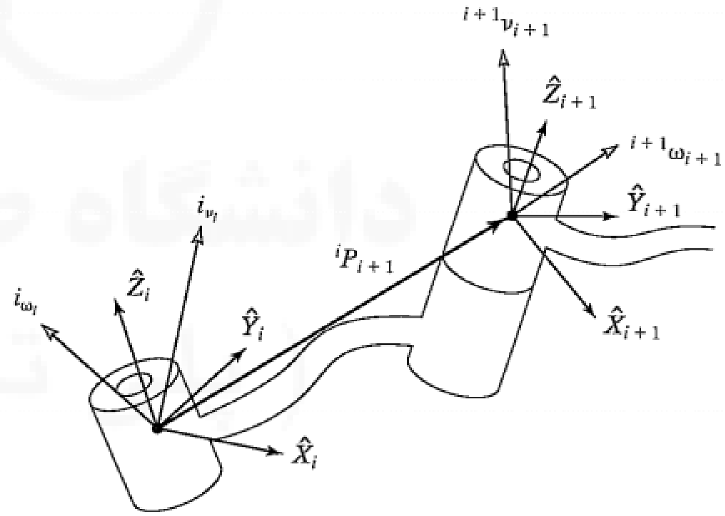
- Assume Joint $i+1$ is **Prismatic**

$$^{i+1}\omega_{i+1} = {}^{i+1}R_i\,{}^{i}\omega_i$$

$$^{i+1}v_{i+1} = {}^{i+1}R_i\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

- Using successively from link to link, we can compute ${}^{N}\omega_N$ and ${}^{N}v_N$ the rotational and linear velocities of the last link.

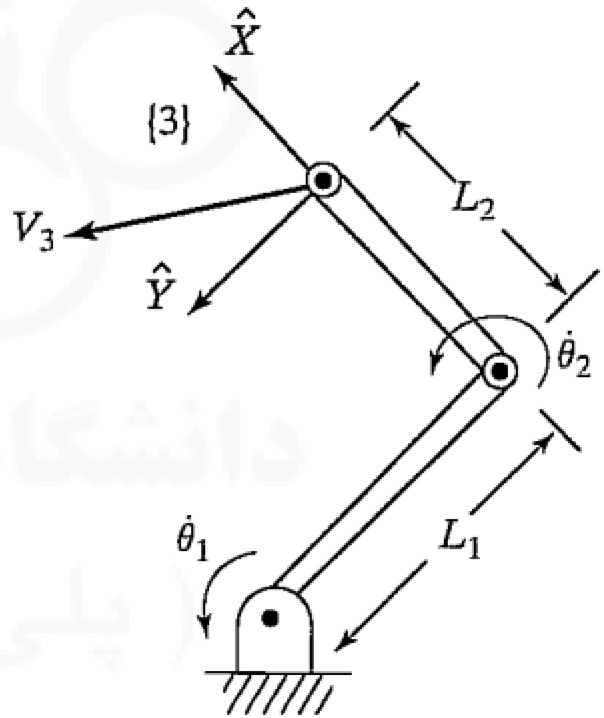
- They can be rotated into base coordinates by multiplication with ${}^{0}R_N$.



Velocity "Propagation"

❖ **Example:**

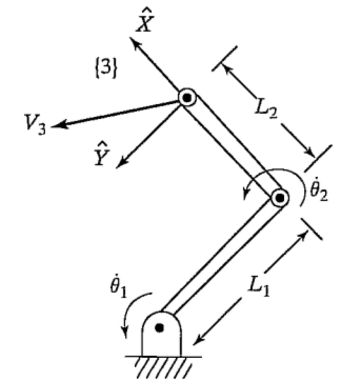
- A two-link manipulator with rotational joints (RR).
- Calculate the velocity of the tip of the arm as a function of joint rates.
- In terms of frame {3} and also in terms of frame {0}.
- ${}^{3}v_3$ & ${}^{3}\omega_3 = ?$
- ${}^{0}v_3$ & ${}^{0}\omega_3 = ?$



Velocity "Propagation"

❖ **Example:**

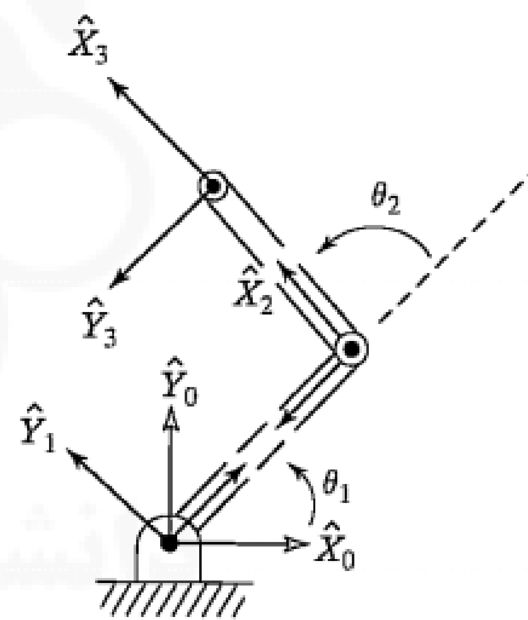
- ${}^{3}v_3$ & ${}^{3}\omega_3 = ?$
- ${}^{0}v_3$ & ${}^{0}\omega_3 = ?$



- Start by attaching frames to the links

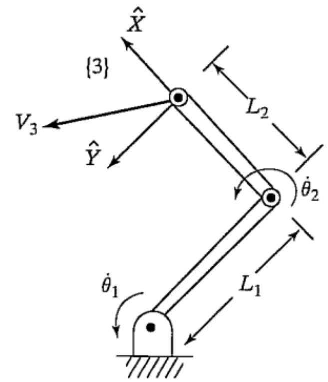
$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



- Compute the velocity of the origin of each frame, starting from the base frame {0}, which has zero velocity

Velocity "Propagation"



❖ **Example:**

- ${}^3v_3$ & ${}^3\omega_3 = ?$
- ${}^0v_3$ & ${}^0\omega_3 = ?$

$$ {}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \quad {}^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}, \quad {}^2v_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ {}^3\omega_3 = {}^2\omega_2, \quad {}^3v_3 = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

- To find these velocities with respect to the base frame, rotate them with the rotation matrix ${}^0R_3$.

$$ {}^0R_3 = {}^0R_1\, {}^1R_2\, {}^2R_3 = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^0v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

- What about ${}^0\omega_3 = ?$

The END

- **References:**

1) .