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Research on Kinematic Modeling and Analysis Methods of UR Robot

Qiang Liu, Daoguo Yang, Weidong Hao, Yao Wei
School of Mechanical and Electrical Engineering, Guilin University Of Electronic Technology, Guilin, China

liuqiangrobotics@foxmail.com, daoguo_yang@163.com, 477927257@qq.com, 1020352311@qq.com

***Abstract*—Kinematic modeling and analysis is the foundation and focus of robotics research. In this paper, taking a 6 degrees of freedom (DOF) UR10 robot as the research object. Based on the homogenous transformation and screw theory, the kinematic modeling of the robot was performed with Denavit-Hartenberg (D-H) parameterization method and product-of-exponentials (POE) method respectively. In addition, the forward and inverse kinematics analysis are carried out based on the established kinematics models. Finally, the D-H parameterization method and POE method are compared in kinematic modeling and analysis, the advantages and disadvantages of the two methods are pointed out as well, which provide guidance for the modeling and analysis of a robot with specific configuration.**

Keywords—forward kinematics; inverse kinematics; UR10 robot; D-H method; screw theory; POE method

I. INTRODUCTION

The 3C industry is a labor-intensive industry. It has the characteristics of large output, fast product updates, and short cycle times. The characteristics of the 3C industry require that manufacturers have the ability to quickly adjust production plans according to market demand and flexibly configure production lines to achieve rapid production. Therefore, factories are required to import automatic production lines represented by robots. The robotic manipulators in the production lines of enterprises are usually 6 or 7 DOF, which have the ability to greatly improve the level of factory automation and intelligence. The kinematic modeling and analysis of the robot are the basis and focus of the robotics research, and they are the precondition for the follow-up research on robotic dynamics and trajectory planning.

Methods of robotic kinematis modeling mainly include D-H parameterization method and POE method. The D-H method is the basic method for kinematic modeling, and it is also a very mature and widely used method. The disadvantages of this method are that it needs to establish n+1 coordinate systems for a specific configuration (n is the number of degrees of freedom), and the inverse kinematics solver of a specific robot is not universal. The advantages of this method are that a kinematics model requires fewer parameters (Each link only needs 4 parameters to describe) and less requirements for mathematical knowledge. The advantages of POE method are that it provides a forward kinematics solution with simple form and has nothing to do with the joint type. POE method is a relatively new method, and the modeling process is easier than that of the D-H parameterization method. By decomposing the POE formula, the whole inverse kinematics problem is decomposed into several solvable subproblems. However, the disadvantages of this method are that more parameters are needed, the complete description of each joint requires 6 parameters, and not all of the robotic POE formula can be decomposed into several solvable subproblems.

In this paper, Taking UR10 robot as the research object, which is a modular 6 DOF robot. At present, many researches on UR robots are limited to D-H parameterization method and improved D-H parameterization method. There are few studies based on POE method, especially on the inverse kinematics analysis of UR robots. The structure is organized as follows: Section 2 kinematics modeling and analysis with D-H method, which includes forward and inverse kinematics analysis. Section 3 kinematics modeling and analysis with POE method, which includes forward and inverse kinematics analysis. Section 4 compare the above two modeling and kinematic analysis methods, and point out their respective advantages and disadvantages.

II. KINEMATIC MODELING AND ANALYSIS WITH D-H METHOD

A. Kinematic Modeling

D-H parameterization method is a basic method in kinematic modeling. The principle of this basic method is to establish a coordinate system at each joint of the robot, then determine the kinematics parameters between two adjacent coordinates, and finally get the homogeneous transformation matrix between two adjacent coordinate systems based on the determined kinematic parameters. The homogeneous transformation matrix can be determined with only four kinematic parameters. Therefore, as long as these parameters are determined, the coordinate transformation between the adjacent joints of the robot can be completely described. Here, we suppose arbitrary two adjacent coordinate systems are $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$ and $o_i x_i y_i z_i$.

Firstly, establish a series of joint coordinate systems as shown in Fig.1, and define D-H parameters as shown in Table I. Next, use the following four steps to achieve a homogeneous transformation between adjacent coordinate systems[1]:

(1) Move from z_{i-1} to z_i along axis x_{i-1}, and the distance is a_{i-1}, we use $Trans_{x_{i-1}}(a_{i-1})$ to represent it;

(2) Rotate from z_{i-1} to z_i about axis x_{i-1}, and the angle is α_{i-1}, we use $Rot_{x_{i-1}}(\alpha_{i-1})$ to represent it;

(3) Move from x_{i-1} to x_i along axis z_i, and the distance is d_i, we use $Trans_{z_i}(d_i)$ to represent it;

(4) Rotate from x_{i-1} to x_i about axis z_i, and the angle is θ_i, we use $Rot_{z_i}(\theta_i)$ to represent it.

$$\mathrm{Trans}_{x_{i-1}}(a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

$$\mathrm{Rot}_{x_{i-1}}(\alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha_{i-1} & -sin\alpha_{i-1} & 0 \\ 0 & sin\alpha_{i-1} & cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

$$\mathrm{Trans}_{z_i}(d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

$$\mathrm{Rot}_{z_i}(\theta_i) = \begin{bmatrix} cos\theta_i & -sin\theta_i & 0 & 0 \\ sin\theta_i & cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

After multiplying formulas (1), (2), (3) and (4), we can get general form homogeneous transformation matrix ${}^{i-1}_{i}T$, with i = 1, 2, 3, 4, 5, and 6.

$$ {}^{i-1}_{i}T = \mathrm{Trans}_{x_{i-1}}(a_{i-1})\mathrm{Rot}_{x_{i-1}}(\alpha_{i-1})\mathrm{Trans}_{z_i}(d_i)\mathrm{Rot}_{z_i}(\theta_i)$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -d_i s\alpha_{i-1} \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & d_i c\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

TABLE I. DEFINITION OF D-H PARAMETERS

Parameter	Description
a_{i-1}	Distance, move from z_{i-1} to z_i along axis x_{i-1}
α_{i-1}	Angle, rotate from z_{i-1} to z_i about axis x_{i-1}
d_i	Distance, move from x_{i-1} to x_i along axis z_i
θ_i	Angle, rotate from x_{i-1} to x_i about axis z_i

TABLE II. LIST OF VALUES OF D-H PARAMETERS

Joint(i)	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	d_1	θ_1
2	0	$\alpha_1 = \pi/2$	0	θ_2
3	a_2	0	0	θ_3
4	a_3	0	d_4	θ_4
5	0	$\alpha_4 = \pi/2$	d_5	θ_5
6	0	$\alpha_5 -\pi/2$	d_6	θ_6

B. Forward Kinematics Analysis

Take each set of D-H parameters in TABLE II into formula (5), let $s_i = sin\theta_i$, and $c_i = cos\theta_i$, we can get the following homogeneous transformation matrix:

$${}^{0}_{1}T(\theta_1) = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$${}^{1}_{2}T(\theta_2) = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

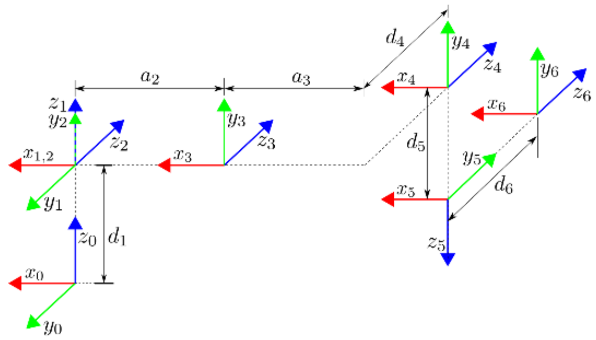

Fig. 1. 6-DOF UR10 robot

$${}^{2}_{3}T(\theta_3) = \begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

$${}^{3}_{4}T(\theta_4) = \begin{bmatrix} c_4 & -s_4 & 0 & a_3 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

$${}^{4}_{5}T(\theta_5) = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & -1 & -d_5 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (10)$$

$${}^{5}_{6}T(\theta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ -s_6 & -c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(11)

$${}^{0}_{6}T = {}^{0}_{1}T(\theta_1){}^{1}_{2}T(\theta_2){}^{2}_{3}T(\theta_3){}^{3}_{4}T(\theta_4){}^{4}_{5}T(\theta_5){}^{5}_{6}T(\theta_6)$$

$$= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

Put formulas (6), (7), (8), (9), (10) and (11) into formula (12), and simplify the calculated results, we can finally get the forward kinematics solver:

$n_x = c_6(c_1c_5c_{234} + s_1s_5) - s_6c_1s_{234}$
$o_x = -c_1c_6s_{234} - s_6(s_1s_5 + c_1c_5c_{234})$
$a_x = c_5s_1 - s_5c_1c_{234}$
$p_x = d_6(c_5s_1 - s_5c_1c_{234}) + d_5c_1s_{234}$
$+c_1(a_2c_2 + a_3c_{23}) + d_4s_1$
$n_y = c_6(s_1c_5c_{234} - c_1s_5) - s_1s_6s_{234}$
$o_y = s_6(c_1s_5 - s_1c_5c_{234}) - c_6s_1s_{234}$
$a_y = -c_1s_5 - s_1s_5c_{234}$
$p_y = -d_6(c_1c_5 + s_1s_5c_{234}) + d_5s_1s_{234}$
$+s_1(a_3c_{23} + a_2c_2) - d_4c_1$
$n_z = s_6c_{234} + c_5c_6s_{234}$
$o_z = c_6c_{234} - c_5s_6s_{234}$
$a_z = -s_5s_{234}$
$p_z = -d_6s_5s_{234} - d_5c_{234} + a_3s_{23} + a_2s_2 + d_1$

In the equations above-mentioned, $s_{234} = sin(\theta_2 + \theta_3 + \theta_4)$, and $c_{234} = cos(\theta_2 + \theta_3 + \theta_4)$.

C. Inverse Kinematics Analysis

The inverse kinematics problem of a robot refers to the pose of the tool coordinate system of a given robot relative to the base coordinate system in Cartesian space, and calculate all of the joint angles that is able to reach the given pose of the end-effector(i.e., homogeneous transformation matrix ${}_6^0T$ is given, and we need to calculate the value of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\ and\ \theta_6$).

In order to solve the first joint angle θ_1 of the UR10 robot, a vertical view is shown in Figure 2.

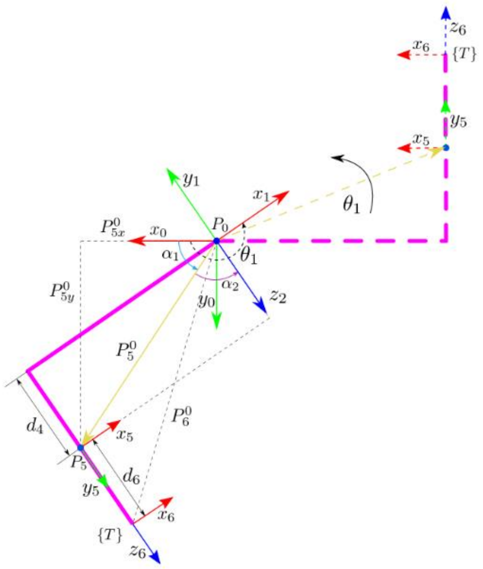

Fig. 2. Solve θ_1

Figure 2 shows that the robot rotates about z_1 from the initial position of the dashed line to solid line position, and rotates angle θ_1 counterclockwise. At this time, the projection components of the vector P_{5xy}^0 on axis x_0 and axis y_0 are P_{5x}^0 and P_{5y}^0 respectively[2].

$$\theta_1 = \alpha_1 + \alpha_2 + \frac{\pi}{2}, \alpha_1 = atan2(y,x) = atan2\left(P_{5y}^0, P_{5x}^0\right),$$

$$cos(\alpha_2) = \frac{d_4}{P_{5xy}^0}, \alpha_2 = \pm arccos\left(\frac{d_4}{\sqrt{{P_{5x}^0}^2 + {P_{5y}^0}^2}}\right).$$

$$\theta_1 = atan2\left(P_{5y}^0, P_{5x}^0\right) \pm arccos\left(\frac{d_4}{\sqrt{{P_{5x}^0}^2 + {P_{5y}^0}^2}}\right) + \frac{\pi}{2} \quad (13)$$

To finally solve the value of θ_1, we need to further solve the specific values of P_{5x}^0 and P_{5y}^0.

$$P_5^0 = P_6^0 - d_6 \cdot \hat{z}_6$$

Let ${}_6^0T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $P_6^0 = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$, $\hat{z}_6 = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$, and we can get the following form:

$$\begin{bmatrix} P_5^0 \\ 1 \end{bmatrix} = {}_6^0T \cdot \begin{bmatrix} 0 \\ 0 \\ d_6 \\ 1 \end{bmatrix} = \begin{bmatrix} p_x - d_6 \cdot a_x \\ p_y - d_6 \cdot a_y \\ p_z - d_6 \cdot a_z \\ 1 \end{bmatrix} = \begin{bmatrix} P_{5x}^0 \\ P_{5y}^0 \\ P_{5z}^0 \\ 1 \end{bmatrix}$$

Therefore, $P_{5x}^0 = p_x - d_6 \cdot a_x$, $P_{5y}^0 = p_y - d_6 \cdot a_y$.

Finally, we can get θ_1:

$$\theta_1 = atan2\left(p_y - d_6 \cdot a_y, p_x - d_6 \cdot a_x\right) \pm arccos\left(\frac{d_4}{\sqrt{(p_x - d_6 \cdot a_x)^2 + (p_y - d_6 \cdot a_y)^2}}\right) + \frac{\pi}{2} \quad (14)$$

In order to solve the joint angle θ_5 of the robot, a vertical view is shown in Figure 3.

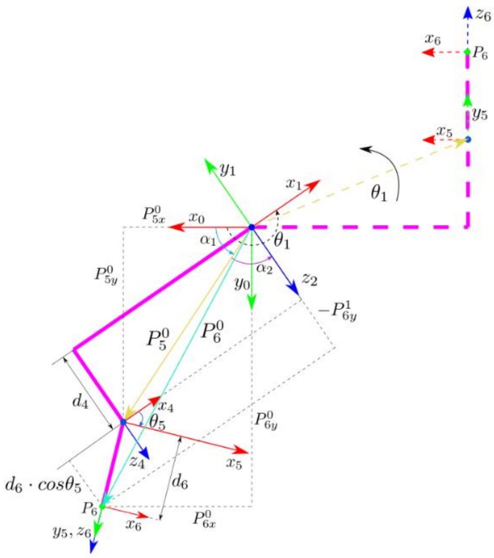

Fig. 3. Solve θ_5

From Figure 3, we can get:

$$-P_{6y}^1 = d_4 + d_6 \cdot cos\theta_5 \quad (15)$$

As d_4 and d_6 are known, P_{6y}^1 can be solved by algebraic method.

Because $P_6^0 = R_1^0 \cdot P_6^1$, there is $P_6^1 = {R_1^0}^{-1} \cdot P_6^0$. Based on the property of orthogonal rotation matrix that ${R_1^0}^{-1} = {R_1^0}^T$, we can get :

$$\begin{bmatrix} P_{6x}^1 \\ P_{6y}^1 \\ P_{6z}^1 \end{bmatrix} = \begin{bmatrix} cos\theta_1 & sin\theta_1 & 0 \\ -sin\theta_1 & cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{6x}^0 \\ P_{6y}^0 \\ P_{6z}^0 \end{bmatrix} \quad (16)$$

According to formula (16), we know that $P_{6y}^1 = -P_{6y}^0 \cdot sin\theta_1 + P_{6y}^0 \cdot cos\theta_1$, and put it into formula (15), we can solve θ_5:

$$\theta_5 = \pm arccos\left(\frac{P_{6x}^0 \cdot sin\theta_1 - P_{6y}^0 \cdot cos\theta_1 - d_4}{d_6}\right) \quad (17)$$

As axes of 2th, 3th and 4th are parallel to each other, any rotation of 2th, 3th and 4th axes causes axis y_1 parallel to z_2, z_3

and z_4. Use the end-effector coordinate system to represent the pose of unit magnitude vector $\hat{y}_1$ [4] , as is shown in Figure 4.

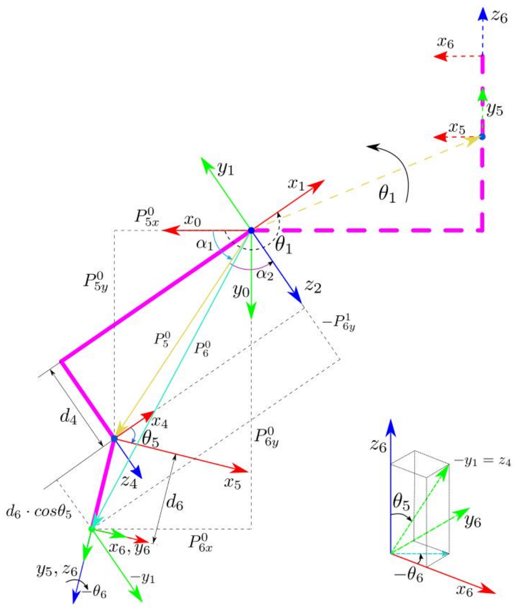

Fig. 4. Solve θ_6

The inverse unit magnitude vector of y_1 on the coordinate system $ox_6y_6z_6$ can be expressed as $-\hat{y}_1^6$:

$$-\hat{y}_1^6 = \begin{bmatrix} sin\theta_5 cos(-\theta_6) \\ sin\theta_5 sin(-\theta_6) \\ cos\theta_5 \end{bmatrix} \text{(i.e. } \hat{y}_1^6 = \begin{bmatrix} -sin\theta_5 cos\theta_6 \\ sin\theta_5 sin\theta_6 \\ -cos\theta_5 \end{bmatrix}\text{).}$$

$$R_1^6 = ((R_1^0)^T \cdot R_6^0)^T \tag{18}$$

$$R_6^0 = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix} \tag{19}$$

$$R_1^0 = \begin{bmatrix} cos\theta_1 & -sin\theta_1 & 0 \\ sin\theta_1 & cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$R_1^6 = \begin{bmatrix} n_x c\theta_1 + n_y s\theta_1 & -n_x s\theta_1 + n_y c\theta_1 & n_z \\ o_x c\theta_1 + o_y s\theta_1 & -o_x s\theta_1 + o_y c\theta_1 & o_z \\ a_x c\theta_1 + a_y s\theta_1 & -a_x s\theta_1 + a_y c\theta_1 & a_z \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} -n_x sin\theta_1 + n_y cos\theta_1 \\ -o_x sin\theta_1 + o_y cos\theta_1 \\ -a_x sin\theta_1 + a_y cos\theta_1 \end{bmatrix} = \begin{bmatrix} -sin\theta_5 cos\theta_6 \\ sin\theta_5 sin\theta_6 \\ -cos\theta_5 \end{bmatrix} \tag{22}$$

According to the first raw and the second raw of of formula (22), θ_6 can be calculated:

$$\theta_6 = atan2\left(\frac{-o_x s\theta_1 + o_y c\theta_1}{s\theta_5}, \frac{n_x s\theta_1 - n_y c\theta_1}{s\theta_5}\right) \tag{23}$$

The UR10 robot's 2th, 3th, and 4th joints form a RRR-type plane arm. Multiplying homogeneous transformation matrixs , we can get ${}^1_4T$:

$$^1_4T = {}^1_2T {}^2_3T {}^3_4T = \begin{bmatrix} c_{234} & -s_{234} & 0 & a_2c_2 + a_3c_{23} \\ 0 & 0 & -1 & -d_4 \\ s_{234} & c_{234} & 0 & a_2s_2 + a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{24}$$

$$p^1_{4x} = a_2c_2 + a_3c_{23}$$
$$p^1_{4z} = a_2s_2 + a_3s_{23}$$
$$n^1_{4x} = c_{234}$$
$$n^1_{4z} = s_{234}$$

According to p^1_{4x} and p^1_{4z}, we can get θ_3:

$$\theta_3 = \pm arccos\left(\frac{(p^1_{4x})^2 + (p^1_{4z})^2 - a_2^2 - a_3^2}{2a_2a_3}\right) \tag{25}$$

Put θ_3 into p^1_{4x} and p^1_{4z}, we can get:

$$\theta_2 = atan2((a_3c_3 + a_2)p^1_{4z} - a_3s_3p^1_{4x}, (a_3c_3 + a_2)p^1_{4x} + a_3s_3p^1_{4z}) \tag{26}$$

Finally, we can get θ_4 based on θ_3 and θ_2:

$$\theta_4 = atan2(n^1_{4z}, n^1_{4x}) - \theta_2 - \theta_3 \tag{27}$$

As θ_1,θ_5 and θ_6 have been solved, in order to calculate ${}^1_4T$, we need to calculate the inverse of ${}^0_1T$, ${}^5_6T$ and ${}^4_5T$ separately[3].

$$^1_4T = ({}^0_1T)^{-1}({}^0_6T)({}^5_6T)^{-1}({}^4_5T)^{-1} \tag{28}$$

$$({}^0_1T)^{-1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$$({}^5_6T)^{-1} = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_6 & -c_6 & 0 & -d_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

$$({}^4_5T)^{-1} = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_5 & c_5 & 0 & -d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

Finally, we can get P^1_{4x}, P^1_{4z}, n^1_{4x} and n^1_{4z}:

$$P^1_{4x} = -d_6(a_xc_1 + a_ys_1) + d_5(o_xc_1c_6 + o_ys_1c_6 + n_xc_1s_6 + n_ys_1s_6) + p_xc_1 + p_ys_1 \tag{32}$$

$$P^1_{4z} = -d_6a_z + d_5o_zc_6 + d_5n_zs_6 + p_z - d_1 \tag{33}$$

$$n^1_{4x} = -s_5(a_xc_1 + a_ys_1) + c_5(n_xc_1c_6 + n_ys_1c_6 - o_xc_1s_6 - o_ys_1s_6) \tag{34}$$

$$n^1_{4z} = n_zc_5c_6 - o_zc_5s_6 - a_zs_5 \tag{35}$$

After solving P^1_{4x}, P^1_{4z}, n^1_{4x} and n^1_{4z}, values of θ_3, θ_2 and θ_4 are easy to get.

III. KINEMATIC MODELING AND ANALYSIS WITH POE METHOD

The POE method is a modeling method based on screw theory, which is able to effectively overcome the limitations of the D-H parameterization method(e.g. Regardless of whether it is a rotating joint or a moving joint, the POE method provides a uniform forward kinematics expression. Moreover, rigid body motion can be described by screw axes from a global perspective, thus avoiding the singularity caused by using the local coordinate system to describe the rigid body motion. In addition, because of the pose of the robot is described by screw axes and the initial pose, the local coordinate system can be arbitrarily selected).

A. Kinematic Modeling

To establish the forward kinematics formula of the UR10 robot, we need to establish a schematic diagram in its home position(Figure 5). We choose frame $x_0y_0z_0$ as the base frame {S}, and the end-effector frame {T} attached to the last link. Place the robot in its zero position by setting all joint values to zero, with the direction of positive displacement.

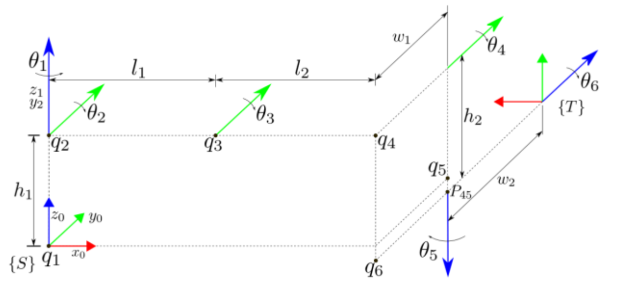

Fig. 5. POE forward kinematics for UR10 robot

Now, suppose that $\xi_n = (w_n; v_n)$ is the screw axis of joint n. As joint n is revolute then $w_n \in R^3$ is a unit magnitude vector in the positive direction of joint axis n. $v_n = q_n \times w_n$, with q_n any arbitrary point on joint axis n as written in coordinates in the fixed base frame[5-6].

It can be seen from Figure 5 that w_1, w_2, w_3, w_4, w_5 and w_6 are as follows:

$$w_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}, w_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, w_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}, w_4 = \begin{bmatrix}0\\1\\0\end{bmatrix}, w_5 = \begin{bmatrix}0\\0\\-1\end{bmatrix}, w_6 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

For the convenience of calculation, we choose the position of q_1, q_2, q_3, q_4, q_5 and q_6 as follows:

$$q_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}, q_2 = \begin{bmatrix}0\\0\\h_1\end{bmatrix}, q_3 = \begin{bmatrix}l_1\\0\\h_1\end{bmatrix}, q_4 = \begin{bmatrix}l_1+l_2\\0\\h_1\end{bmatrix}, q_5 = \begin{bmatrix}l_1+l_2\\w_1\\0\end{bmatrix},$$

$$q_6 = \begin{bmatrix}l_1+l_2\\0\\h_1-h_2\end{bmatrix}$$

$\xi_n = (w_n; v_n) = (w_n; q_n \times w_n)$, with n=1, 2, 3, 4, 5, and 6.

$$\xi = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6] = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & -1 & 0\\ 0 & -h_1 & -h_1 & -h_1 & -w_1 & h_2-h_1\\ 0 & 0 & 0 & 0 & l_1+l_2 & 0\\ 0 & 0 & l_1 & l_1+l_2 & 0 & l_1+l_2\end{bmatrix} \tag{36}$$

$$T_n = e^{\theta_n\widehat{\xi_n}}T_n = \begin{bmatrix}e^{\theta_n\widehat{\omega_n}} & \left(I - e^{\theta_n\widehat{\omega_n}}\right)(\omega_n \times v_n) + \theta_n\omega_n\omega_n^T v_n\\ 0 & 1\end{bmatrix} \tag{37}$$

$$T_n = \begin{bmatrix}R_n & P_n\\ 0 & 1\end{bmatrix}, \text{and } \widehat{\xi_n} = \begin{bmatrix}\widehat{\omega_n} & v_n\\ 0 & 0\end{bmatrix}$$
$$\widehat{\omega_n} = \begin{bmatrix}0 & -\omega_{n_z} & \omega_{n_y}\\ \omega_{n_z} & 0 & -\omega_{n_x}\\ -\omega_{n_y} & \omega_{n_x} & 0\end{bmatrix} \tag{38}$$

Expand $e^{\theta_n\widehat{\omega_n}}$ to the following form based on the Rodrigues' formula:

$$R_n = e^{\theta_n\widehat{\omega_n}} = I + \widehat{\omega_n}sin\theta_n + \widehat{\omega_n}^2(1-cos\theta)$$
$$= \begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix} \tag{39}$$
$$r_{11} = \omega_{n_x}^2(1-cos\theta) + cos\theta$$
$$r_{12} = \omega_{n_x}\omega_{n_y}(1-cos\theta) - \omega_{n_z}sin\theta$$
$$r_{13} = \omega_{n_x}\omega_{n_z}(1-cos\theta) + \omega_{n_y}sin\theta$$
$$r_{21} = \omega_{n_x}\omega_{n_y}(1-cos\theta) + \omega_{n_z}sin\theta$$
$$r_{22} = \omega_{n_y}^2(1-cos\theta) + cos\theta$$
$$r_{23} = \omega_{n_y}\omega_{n_z}(1-cos\theta) - \omega_{n_x}sin\theta$$
$$r_{31} = \omega_{n_x}\omega_{n_z}(1-cos\theta) - \omega_{n_y}sin\theta$$
$$r_{32} = \omega_{n_y}\omega_{n_z}(1-cos\theta) + \omega_{n_x}sin\theta$$
$$r_{33} = \omega_{n_z}^2(1-cos\theta) + cos\theta$$
$$T_n^{-1} = e^{-\theta_n\widehat{\xi_n}} = \begin{bmatrix}R_n^T & -R_n^T R_n\\ 0 & 1\end{bmatrix}$$

For UR10 robot, the home pose of the end-effector is $g_{st}(0)$.

$$g_{st}(0) = \begin{bmatrix}R_0 & P_0\\ 0 & 1\end{bmatrix}, \text{with } P_0 = \begin{bmatrix}l_1+l_2\\ \omega_1+\omega_2\\ h_1-h_2\end{bmatrix}$$

$$R_0 = \mathrm{Rot}\left(x, -\frac{\pi}{2}\right)\mathrm{Rot}\left(z, \frac{\pi}{2}\right)\mathrm{Rot}\left(z, \frac{\pi}{2}\right) = \begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix}-1 & 0 & 0 & l_1+l_2\\ 0 & 0 & 1 & \omega_1+\omega_2\\ 0 & 1 & 0 & h_1-h_2\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{40}$$

Finally, the forward kinematics formula is as follows:

$$g_{st}(\theta) = e^{\theta_1\widehat{\xi_1}}e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}e^{\theta_5\widehat{\xi_5}}e^{\theta_6\widehat{\xi_6}}g_{st}(0)$$

B. Forward Kinematics Analysis

As we can see from the previous section that the forward kinematics has the following form:

$$g_{st}(\theta) = e^{\theta_1\widehat{\xi_1}}e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}e^{\theta_5\widehat{\xi_5}}e^{\theta_6\widehat{\xi_6}}g_{st}(0) \tag{41}$$

Here, we suppose that $\theta_2 = \frac{-\pi}{2}$ and $\theta_5 = \frac{\pi}{2}$, with all other joint angles equal to zero.

$$T_2 = e^{\frac{-\pi}{2}\widehat{\xi_2}} = \begin{bmatrix}R_2 & P_2\\ 0 & 1\end{bmatrix} \tag{42}$$

$$T_5 = e^{\frac{\pi}{2}\widehat{\xi_5}} = \begin{bmatrix}R_5 & P_5\\ 0 & 1\end{bmatrix} \tag{43}$$

$$T_1 = T_3 = T_4 = T_6 = I \tag{44}$$

$$g_{st}(\theta) = T_2T_5g_{st}(0) \tag{45}$$

Just replace θ_2 with $\frac{-\pi}{2}$, and θ_5 with $\frac{\pi}{2}$ in the formula of section A, and finally we'll get the forward kinematics solution.

C. Inverse Kinematics Analysis

The configuration of UR robots are different from that of general industrial robots. Axes of the last three joints of most industrial robots intersect at one point, while UR robots do not have such a configuration. Therefore, many methods for solving inverse kinematics of industrial robots based on screw theory and product of exponential formula are not applicable to UR robots. As axes of UR robots' 2th, 3th and 4th joints are parallel to each other, according to Pieper criterion[1], UR robots have analytical solution (it can be assumed that the 2th, 3th and 4th joints axes intersect at infinity).

To solve the problem of inverse kinematics of serial robots in specific situations, it is necessary to solve common inverse kinematics problems first, and then try to decompose the whole problem of solving inverse kinematics into several sub-problems whose solutions are known. These sub-problems should have clear geometric meaning and numerical stability. The following three principles need to be followed in solving inverse kinematics problems[6]: (1) The principle of keeping the position unchanged; (2) The principle of keeping distance constant; (3) The principle of keeping posture uchanged.

For the principle of keeping the position unchanged, that is for a unit magnitude screw axis $\xi = (\omega; q \times \omega)$ with pure rotation, the position of any point P on the rotation axis remains unchanged, $e^{\theta\hat{\xi}}P = P$. For the principle of keeping distance constant, that is for a screw axis $\xi = (\omega; q \times \omega)$ with pure rotation, The distance from any point P outside the rotation axis to the fixed point q on the rotation axis remains unchanged, $\left\|e^{\theta\hat{\xi}}P - q\right\| = \|P - q\|$.

Next, we'll use the direct decomposition method(i.e. first principle) and the variable elimination method(i.e. second principle) to solve the inverse kinematics problem of UR10. Assume that the UR10 robot' fifth and sixth joint axis intersect at P_{56}.

$$e^{\theta_1\widehat{\xi_1}}e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}e^{\theta_5\widehat{\xi_5}}e^{\theta_6\widehat{\xi_6}} = g_{st}(\theta)g_{st}^{-1}(0) = g$$

$$e^{\theta_5\widehat{\xi_5}}e^{\theta_6\widehat{\xi_6}}P_{56} = P_{56} = \begin{bmatrix} l_1 + l_2 \\ \omega_1 \\ h_1 - h_2 \end{bmatrix}$$

$$e^{\theta_1\widehat{\xi_1}}e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}e^{\theta_5\widehat{\xi_5}}e^{\theta_6\widehat{\xi_6}}P_{56} = gP_{56}$$

$$e^{\theta_1\widehat{\xi_1}}e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}P_{56} = gP_{56}$$

We arbitrarily select points q_1, q_2, q_3 on axis of twist coordinate ξ_1, and we can get the following formulas:

$$e^{\theta_1\widehat{\xi_1}}q_1 = q_1,\ e^{\theta_1\widehat{\xi_1}}q_2 = q_2,\ e^{\theta_1\widehat{\xi_1}}q_3 = q_3$$

$$e^{\theta_1\widehat{\xi_1}}e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}P_{56} - q_1 = gP_{56} - q_1$$

$$e^{\theta_1\widehat{\xi_1}}\left(e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}P_{56} - q_1\right) = gP_{56} - q_1 \tag{46}$$

Applying the norm to each side of the above formula, we can get:

$$\left\|e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}P_{56} - q_1\right\| = \|gP_{56} - q_1\| \tag{47}$$

The formula above has only one independent equation. To solve θ_1, θ_2, θ_3, we need to establish three independent equations[7].

$$\left\|e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}P_{56} - q_2\right\| = \|gP_{56} - q_2\| \tag{48}$$

$$\left\|e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}P_{56} - q_3\right\| = \|gP_{56} - q_3\| \tag{49}$$

$$\left(e^{\theta_2\widehat{\xi_2}}e^{\theta_3\widehat{\xi_3}}e^{\theta_4\widehat{\xi_4}}P_{56} - P_{56}\right)\cdot\omega_2 = 0 \tag{50}$$

According to the above formulas, θ_1, θ_2 and θ_3 can be solved. After θ_1, θ_2 and θ_3 have been solved, we can get the following formula:

$$e^{\theta_1\widehat{\xi_1}}T_2T_3T_4P_{56} = gP_{56} \tag{51}$$

We can solve θ_1 based on Paden-Kahan's Subproblem 1, and solve θ_4 and θ_5 based on Paden-Kahan's Subproblem 2 [6].

IV. Conclusions

In this paper, UR10 robot's kinematic modeling and analysis are carried by D-H parameterization method and POE method respectively. The D-H method and POE method have their own advantages and disadvantages. For example, D-H method needs to establish n+1 coordinate systems, and the solver is only aimed at a specific configuration of the robot, and not universal, but D-H parameterization method needs fewer parameters. In some configurations, D-H parameterization method is easier to solve than POE method, and the computation load is relatively small. The POE method is clear in the process of kinematics modeling and analysis, and solvers of subproblems of many configurations are universal, but not all configurations are solvable by POE method, because some of their forward kinematics formulas can not be decomposed into solvable subproblems[8].

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