دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

دانشکده مهندسی مکانیک
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Lecture 4_3:
Jacobians:
Singularity & Inverse Velocity

Advanced Robotics

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Outlines

- Singularities
  - Velocity Domain
  - Decoupled Manipulator
  - Force Domain
- Inverse Velocity & Acceleration

Singularities

- **Velocity Domain**

- We have a linear transformation relating joint velocity to Cartesian velocity by means of Jacobians.

$$v = J(\Theta)\dot{\Theta}$$

- **Objective:** Calculate the necessary joint rates to move the hand of the robot with a certain velocity vector in Cartesian space.

$$\dot{\Theta} = J^{-1}(\Theta)\, v$$

- Is this matrix invertible (nonsingular)?
  - If the matrix is nonsingular, then invert it to calculate joint rates from given Cartesian velocities.

- Manipulators have values of Θ where the Jacobian becomes singular (Non-invertible).

- Those are called singularities of the mechanism or **singularities** for short.

Singularities

- **Velocity Domain**

- The $6 \times n$ Jacobian $J(\Theta)$ defines a mapping:
$$\nu = J(\Theta)\dot{\Theta}$$

- All possible end-effector velocities are linear combinations of the columns of the Jacobian matrix.
$$\nu = J_1\dot{\theta}_1 + J_2\dot{\theta}_2 + \cdots + J_n\dot{\theta}_n$$

- It is necessary that $J(\Theta)$ have six linearly independent columns for the end-effector to be able to achieve any arbitrary velocity.

- The **rank** of a matrix
  = the number of linearly independent columns (or rows) in the matrix.

- When $rank\ J = 6$, the end-effector can execute any arbitrary velocity.

- For a matrix $J \in R^{6\times n}$, it is always the case that $rank\ J \ \leq\ \min(6, n)$.

Singularities

- **Velocity Domain**
- The rank of the manipulator Jacobian matrix will depend on the configuration Θ.

- Configurations for which the $rank\ J(\Theta)$ is less than its maximum value (**Rank Deficiency**) are called **singularities** or **singular configurations.**

- A **square matrix** is singular when its determinant is equal to zero.

$$det\, J(\Theta) = 0$$

- **Note:**
- A configuration Θ is singular iff (if & only if):
  - $J^{-1}(\Theta)$ does not exist.
  - $rank\ (J)$ is less than its maximum value.
  - $det\, J(\Theta) = 0$.

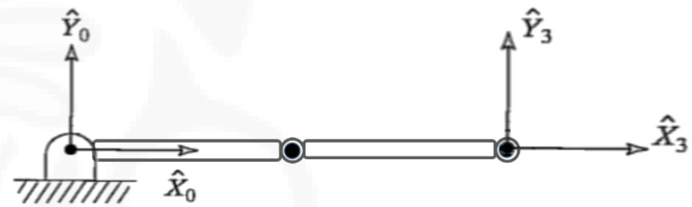
Singularities

❑ Velocity Domain

- Classify singularities into two categories:

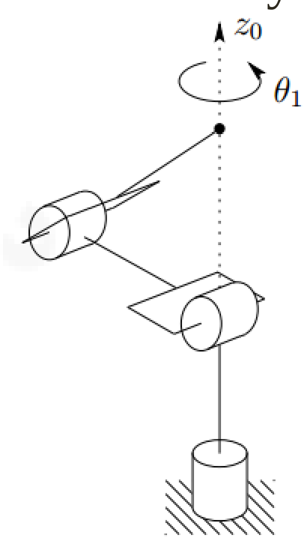
➢ **Workspace-boundary singularities**

When that the end-effector is at or very near the boundary of the workspace (<u>fully stretched out</u> or <u>folded back</u> on itself).



➢ **Workspace-interior singularities**

Away from the workspace boundary; they generally are caused by a <u>lining up</u> of two or more joint axes.



Singularities

- **Velocity Domain**

- All manipulators have singularities at the boundary of their workspace.

- Most have loci of singularities inside their workspace.

- In a singular configuration, it has lost one or more degrees of freedom (as viewed from Cartesian space).

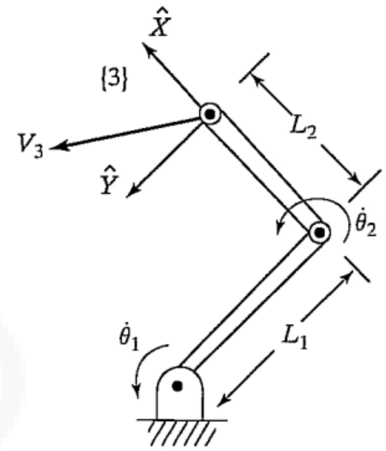
- This means that there is some direction (or subspace) in Cartesian space along which it is impossible to move the end-effector (no matter what joint rates are selected).

- Obviously, it happens at the workspace boundary of robots.

Singularities

- **Velocity Domain**
- **Example 1:** Two-link RR Manipulator
- To find the singularities, examine the determinant of its Jacobian.
- Remember:

$$^{3}J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$$



- Where the determinant is equal to zero, the Jacobian has lost full rank and is singular.

$$det(^{3}J(\Theta)) = det\left(\begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}\right) = l_2 l_1 s_2$$

- A singularity of the mechanism exists when θ_2 is 0 or 180 degrees.
- When $\theta_2 = 0$, the arm is stretched straight out.
- In this configuration, motion of the end-effector is possible along **only** one Cartesian direction (the one perpendicular to the arm). It has lost one degree of freedom.

Singularities

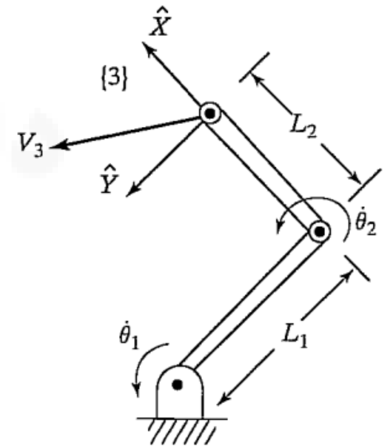
- **Velocity Domain**
- **Example 1:** Two-link RR Manipulator

- When $\theta_2 = 180$, the arm is folded completely back on itself (motion is possible only in one Cartesian direction instead of two).

- Those are workspace-boundary singularities.

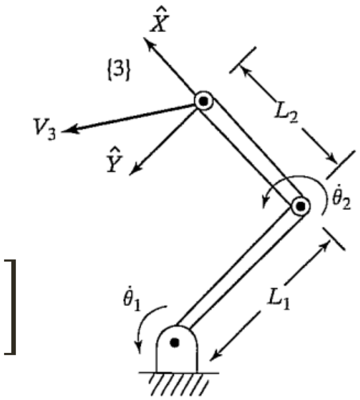
- ***Q:*** The Jacobian was written with respect to frame {3}, what about if it is expressed in other frames, e.g. {0}?

$$^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$



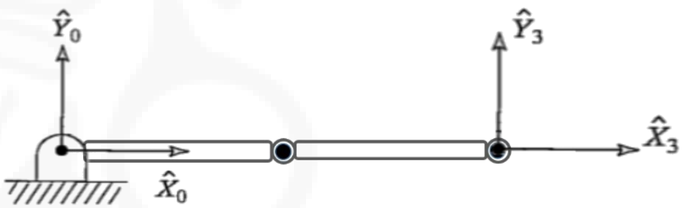
Singularities

- **Velocity Domain**
- **Example 1:** Two-link RR Manipulator
- **Singularity Analysis:**



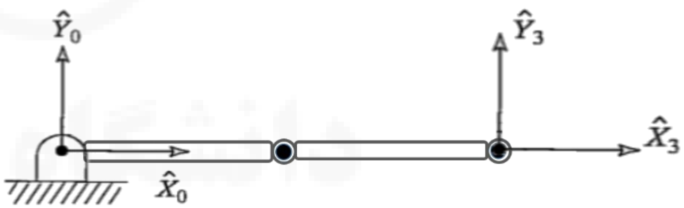
$$^3J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \quad , \quad ^0J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

- At singularity (e.g. $s_2 = 0$)



$$^3J(\Theta) = \begin{bmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{bmatrix}$$

- wrt. $v = J(\Theta)\dot{\Theta}$, it means that the EE has no motion in the direction of $^3\hat{X}$ (One Dof has been missed)



$$^0J(\Theta) = \begin{bmatrix} -(l_1 + l_2)s_1 & -l_2 s_1 \\ (l_1 + l_2)c_1 & l_2 c_1 \end{bmatrix}$$

- wrt. $v = J(\Theta)\dot{\Theta}$, it means that the EE velocity Caused by $\dot{\theta}_1$ and $\dot{\theta}_2$ has the same direction (One Dof has been missed)

Singularities

- **Velocity Domain**
- **Example 2:** Two-link RR Manipulator
- The two-link robot is moving its end-effector along the X axis at 1.0 m/s.
- Calculate the required joint rates ($\dot{\Theta}$).
- Remember

$$^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

- Calculate the inverse of the Jacobian written in (0). (*Why?*)

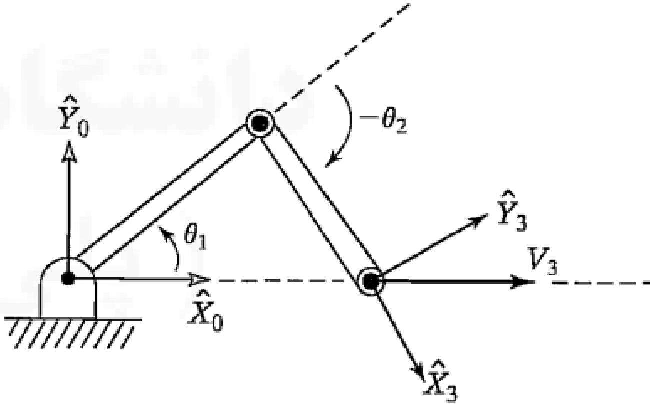
$$^{0}J^{-1}(\Theta) = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{bmatrix}$$

- Calculate joint rates as a function of manipulator configuration.

$$\dot{\Theta} = J^{-1}(\Theta)\, v \quad , \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dot{\theta}_1 = \frac{c_{12}}{l_1\, s_2}$$

$$\dot{\theta}_2 = -\frac{c_1}{l_2\, s_2} - \frac{c_{12}}{l_1\, s_2}$$



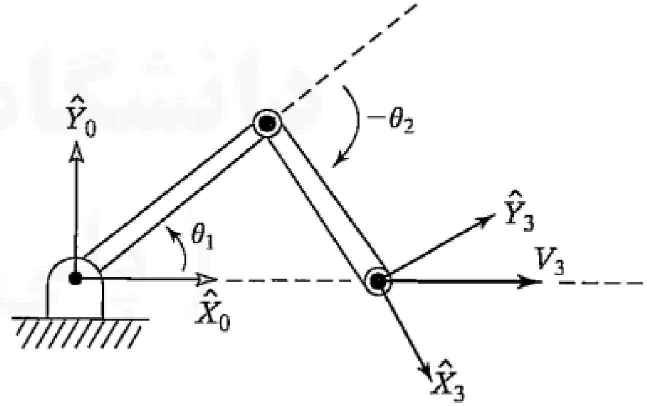
Singularities

- **Velocity Domain**
- **Example 2:** Two-link RR Manipulator

$$\dot{\theta}_1 = \frac{c_{12}}{l_1\, s_2}$$

$$\dot{\theta}_2 = -\frac{c_1}{l_2\, s_2} - \frac{c_{12}}{l_1\, s_2}$$

- Joint rates are reasonable when far from a singularity.
- As the arm stretches out toward, i.e. $\theta_2 = 0$ (Singularity), both joint rates go to infinity.
- A dangerous case for a robot *control system* !!!



Singularities

❑ Decoupled Manipulator

- Recall that for manipulators having 3 intersecting axes, e.g., spherical wrists, the Kinematic problem is decoupled (**Decoupled Manipulators**).

- For decoupled manipulators, decouple the determination of singular configurations, into two simpler problems.

  - Determine arm singularities, singularities resulting from motion of the arm, which consists of the first three or more links.

  - Determine wrist singularities resulting from motion of the spherical wrist.

- Suppose that $n = 6$, the manipulator consists of a 3-DOF arm with a 3-DOF spherical wrist.

- The Jacobian is a 6×6 matrix and a configuration Θ is singular iff (if & only if)

$$det\, J(\Theta) = 0$$

Singularities

- **Decoupled Manipulator**
- Partition the Jacobian J into 3×3 blocks as:

$$J = [J_P \,|\, J_O] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

- Since the final three joints are always revolute:

$$J_O = \begin{bmatrix} z_4 \times (o_e - o_4) & z_5 \times (o_e - o_5) & z_6 \times (o_e - o_6) \\ z_4 & z_5 & z_6 \end{bmatrix}$$

- Since the wrist axes intersect at a common point o, if we choose the coordinate frames so that $o_4 = o_5 = o_6 = o_e$, then J_O becomes:

$$J_O = \begin{bmatrix} 0 & 0 & 0 \\ z_4 & z_5 & z_6 \end{bmatrix}$$

- Therefore

$$J = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

Singularities

Decoupled Manipulator

$$J = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

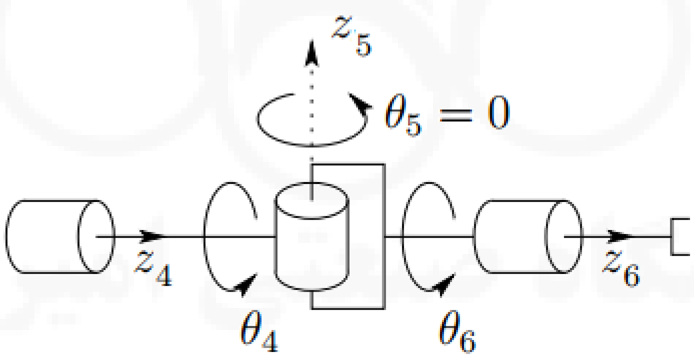
- Determinant

$$det(J) = det(J_{11})\, det(J_{22})$$

- The manipulator singular configurations =
- Arm configurations satisfying $det\, J_{11} = 0$ + Wrist configurations satisfying $det\, J_{22} = 0$

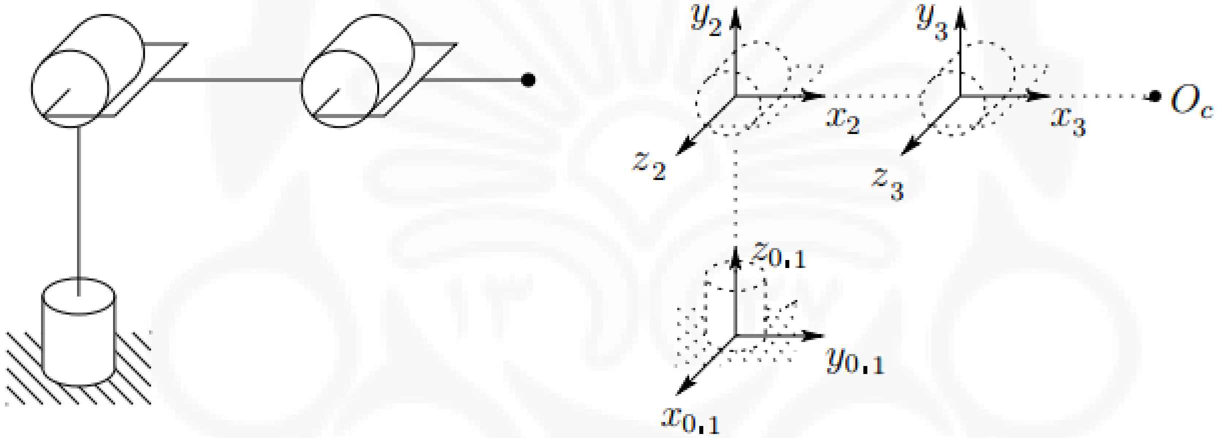
Singularities

- **Decoupled Manipulator**
- **Wrist Singularities**
  - A spherical wrist is in a singular configuration whenever the vectors z_4, z_5 and z_6 are linearly dependent.
  - It happens when the joint axes z_4 and z_6 are collinear (Lining Up).
  - This is the only singularity of the spherical wrist, and is unavoidable without imposing *mechanical limits* on the wrist design to restrict its motion.



Singularities

- **Decoupled Manipulator**
- **Arm Singularities**
  - To investigate arm singularities, compute J_{11}:



$$J_{11} = \begin{bmatrix} -a_2 s_1 c_2 - a_3 s_1 c_{23} & -a_2 s_2 c_1 - a_3 s_{23} c_1 & -a_3 c_1 s_{23} \\ a_2 c_1 c_2 + a_3 c_1 c_{23} & -a_2 s_1 s_2 - a_3 s_1 s_{23} & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{bmatrix}$$

  - So:

$$\det J_{11} = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

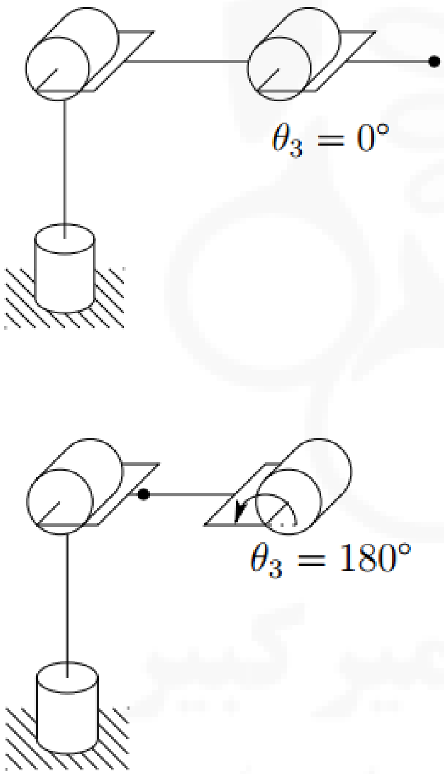
  - Therefore:

$$\begin{aligned} s_3 &= 0, \quad \textit{that is, } \theta_3 = 0 \textit{ or } \pi \\ a_2 c_2 + a_3 c_{23} &= 0 \end{aligned}$$

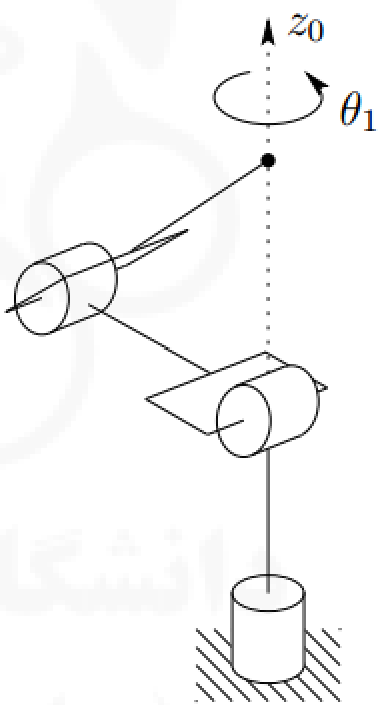
Singularities

- **Decoupled Manipulator**
  - **Arm Singularities**

$$s_3 \quad = \quad 0, \quad \textit{that is, } \theta_3 = 0 \textit{ or } \pi$$



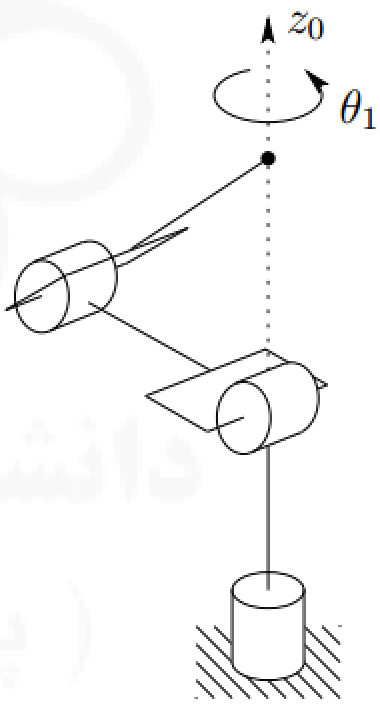
$$a_2c_2 + a_3c_{23} \quad = \quad 0$$



Singularities

- **Decoupled Manipulator**
- **Arm Singularities**

- **Note 1: Singular points Characteristic**
- There are infinitely many singular configurations and infinitely many solutions to the *inverse position kinematics* when the wrist center is along this axis.



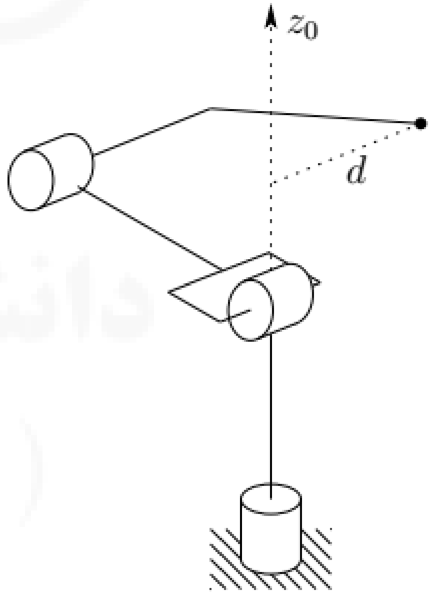
Singularities

- **Decoupled Manipulator**
- **Arm Singularities**

- **Note 2: Singular points Characteristic**
- Points which are reachable at singular configurations **may** not be reachable under arbitrarily small perturbations of the manipulator parameters.

- It may be also used for **Singularity avoidance** by changing the design.

- In this case:
  An offset in either the elbow or the shoulder.



Singularities

- **Force Domain**
- Recall

$$\tau = J^T \digamma$$

- When the Jacobian loses full rank (i.e. Singular), there are certain directions in which the end-effector cannot exert desired static forces.

- If the Jacobian is singular:
  $\digamma$ could be increased or decreased in certain directions (the *null-space of the Jacobian*) without effect on the value calculated for τ.

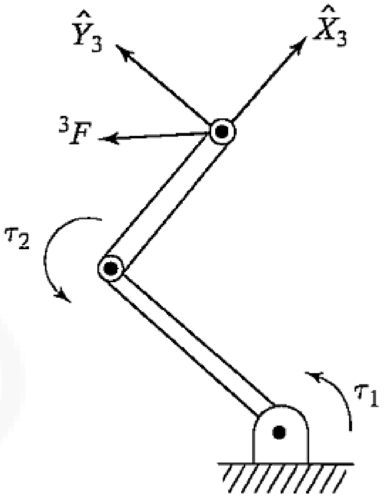
- Near singular configurations, with small joint torques, large forces could be generated at the end-effector.

- Singularities manifest themselves in the force domain as well as in the velocity domain.

Singularities

- **Force Domain**
- **Example:** Two-link RR Manipulator

- $\tau = {}^3J^T\, {}^3\mathrm{F}$
- ${}^3J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$
- So, …



Singularities

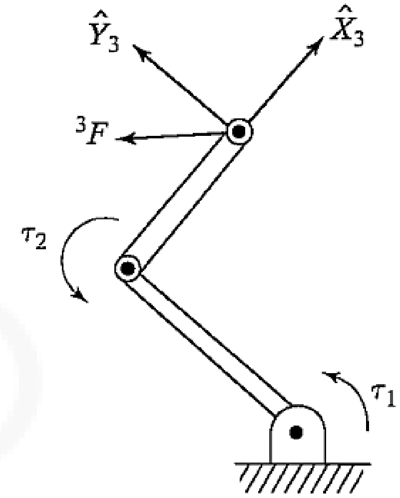
❑ Force Domain

❖ **Example:** Two-link RR Manipulator

- $\tau = {}^3J^T \, {}^3\mathrm{F}$

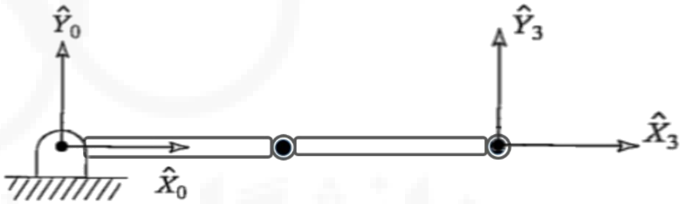
- ${}^3J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$

- $\tau = \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$



- At singularity (i.e. $s_2 = 0$):

$$\tau = \begin{bmatrix} 0 & l_1 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$



- **Note:** No joint torque is required to compensate any force applied in the direction of ${}^3\hat{X}$.

Singularities

- **At singularity:**

1) Certain directions of motion may be unattainable.

2) Bounded end-effector velocities may correspond to unbounded joint velocities.

3) Certain directions in which the end-effector cannot exert desired static forces.

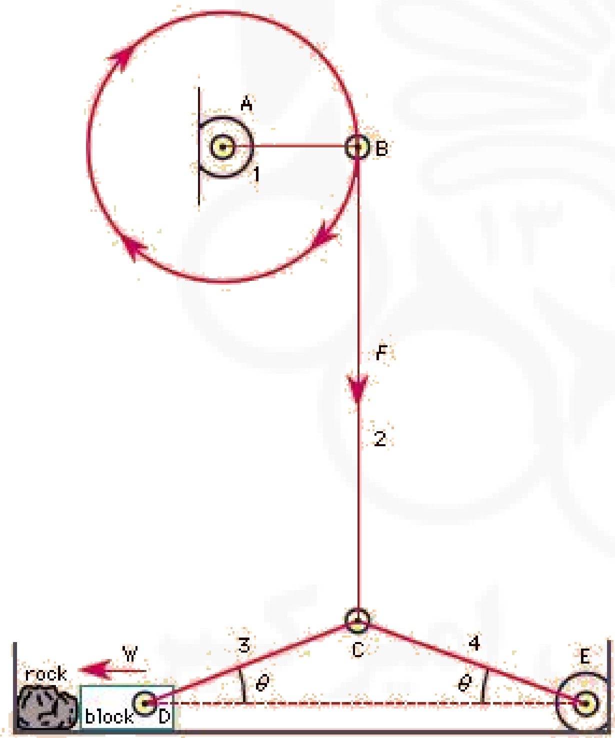
4) Bounded end-effector desired forces and torques may correspond to unbounded joint torques.

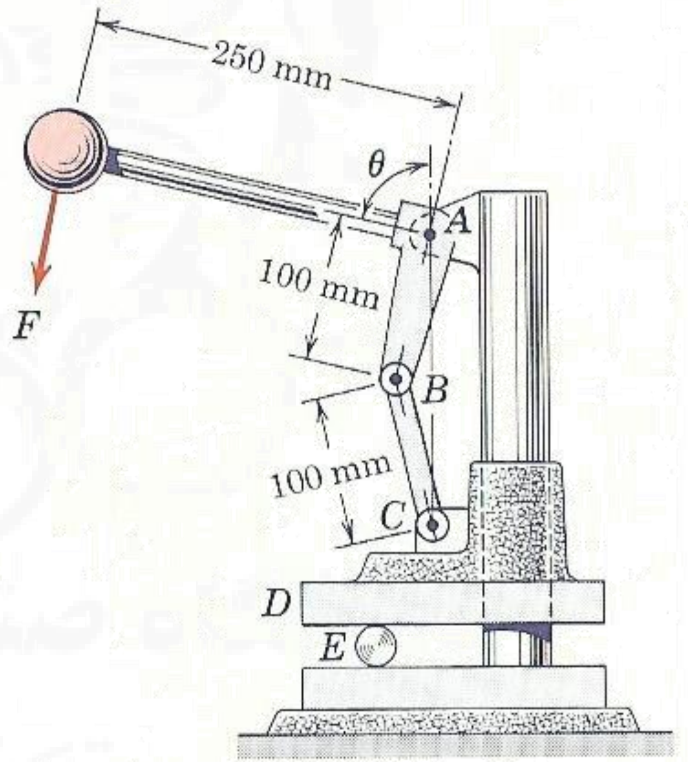
5) Bounded joint torques may correspond to unbounded end-effector forces and torques.

Singularities

- **At singularity:**

5) Bounded joint torques may correspond to unbounded end-effector forces and torques. (**Toggle Mechanism**)





Singularities

- **At singularity:**

1) Singularities correspond to points on the boundary of workspace, i.e., points of maximum reach of the manipulator.

2) Singularities correspond to points in the manipulator workspace that **may be** unreachable under small perturbations of the link parameters, such as length, offset, etc.

3) **Near interior singularities** there **may not exist** a unique solution to the inverse kinematics problem.

Singularities

❑ **Analytical Jacobians**

➢ The analytical Jacobian can be found by the kinematic problem.

$$\dot{X} = \begin{bmatrix} \dot{d}(\Theta) \\ \dot{\Theta}(\Theta) \end{bmatrix} = J_a(\Theta)\dot{\Theta}$$

➢ The analytical Jacobian, $J_a(\theta)$, may be computed from the geometric Jacobian as :

$$J_a(\Theta) = \begin{bmatrix} I & 0 \\ 0 & E^{-1}(\Theta) \end{bmatrix} J(\Theta) \qquad E_{Z'Y'Z'} = \begin{bmatrix} 0 & -s\alpha & c\alpha\, s\beta \\ 0 & c\alpha & s\alpha\, s\beta \\ 1 & 0 & c\beta \end{bmatrix}$$

provided $det\, E(\Theta) \neq 0$.

❑ **Singularity**

- Singularities of $E(\Theta)$ are called *representational singularities.*

- The singularities of the analytical Jacobian =
  The singularities of the geometric Jacobian, J + the representational singularities .

Inverse Velocity & Acceleration

❑ Inverse Velocity

$$\nu = J(\Theta)\dot{\Theta}$$

- *Assume $J(\Theta)$ is nonsingular:*
- The inverse velocity problem is finding the joint velocities $\dot{\Theta}$ that produce the desired end-effector velocity ν.

- When the Jacobian is square (i.e., $J \in R^{n\times n}$) and nonsingular, this problem can be solved by simply inverting the Jacobian matrix.

$$\dot{\Theta} = J^{-1}(\Theta)\,\nu$$

- For manipulators that do not have exactly six links, the Jacobian can not be inverted. *So what* !!!!

Inverse Velocity & Acceleration

Inverse Velocity

$$\nu = J(\Theta)\dot{\Theta}$$

- For manipulators that do not have exactly six links, the Jacobian can not be inverted.
- In this case, a solution exists if and only if ν lies in the **range space** of the Jacobian $J(\Theta)$.
- A vector ν belongs to the range of $J(\Theta)$ if and only if:

$$rank\ J(\Theta) = rank\ [\, J(\Theta) \mid \nu \,]$$

- Several algorithms exist, such as Gaussian elimination, for solving such systems of linear equations.
- …

Inverse Velocity & Acceleration

❑ Inverse Velocity

$v = J\dot{\theta} \quad \xrightarrow[J \in R^{6\times 6}]{} \quad \dot{\theta} = J^{-1} v$

اگر تعداد درجات آزادی ربات ۶ نباشد $\longrightarrow$ J^{-1} تعریف نشده

$J \in R^{m \times n}$, $m = 6$ — $m < n$ / $m > n$ $\longrightarrow \dot{\theta} = ?$

$A\cancel{x} = b \qquad A \in R^{m \times n} \qquad \longrightarrow \quad x = ?$

Inverse Velocity & Acceleration

❑ Inverse Velocity

$$Ax = b \qquad A \in R^{m \times n} \qquad \longrightarrow \qquad x = ?$$

a) $m > n$

تعداد درجات آزادی کمتر از b است ⟵ درحالت کلی جواب دقیق ندارد ⟵ هدف یافتن جوابی با حداقل خطا

$$e = Ax - b$$ بردار خطا

$$\min \quad \frac{1}{2} e^T e = C.F. \qquad \text{Cost Function}$$

$$C.F. = \frac{1}{2} e^T e = \frac{1}{2} (x^T A^T - b^T)(Ax - b) = \frac{1}{2} \left[x^T A^T A x - 2 x^T A^T b + b^T b \right]$$

Inverse Velocity & Acceleration

❑ Inverse Velocity

a) $m > n$

$$c.f. = \frac{1}{2} e^T e = \frac{1}{2}(x^T A^T - b^T)(Ax - b) = \frac{1}{2}\left[x^T A^T A x - 2 x^T A^T b + b^T b \right]$$

$$\boxed{\frac{\partial}{\partial x} c^T x = c \;,\quad \frac{\partial}{\partial x} x^T A x = 2Ax \;,\quad \frac{\partial x^T A}{\partial x} = A}$$

$$\frac{\partial c.f.}{\partial x} = A^T A x - A^T b = 0 \longrightarrow \boxed{x = (A^T A)^{-1} A^T b} = A_L^+ b$$

- Left Pseudo Inverse:

$$\boxed{A_L^+ = (A^T A)^{-1} A^T}$$

$$A_L^+ A = \left[(A^T A)^{-1} A^T\right] A = I$$

- **Note:**

if $m = n$ $\longrightarrow$ $(A^T A)^{-1} A^T = A^{-1}$

Inverse Velocity & Acceleration

Inverse Velocity

$Ax = b \qquad A \in R^{m \times n} \longrightarrow x = ?$

b) $m < n$

تعداد درجات آزادی بیشتر از b → بی‌نهایت جواب دارد → هدف پیدا کردن جواب با کمترین $\|x\|$

ضریب لاگرانژ

$$\begin{cases} \min & CF = x^T x \\ \text{Subject to} & Ax = b \end{cases} \Rightarrow \min_{x} CF = x^T x + \lambda^T (Ax - b)$$

Inverse Velocity & Acceleration

❑ Inverse Velocity

b) $m < n$

ضریب لاگرانژ

$$\begin{cases} \min \quad CF = x^T x \\ \text{Subject to} \quad Ax = b \end{cases} \Rightarrow \min_x CF = x^T x + \lambda^T (Ax - b)$$

$$\frac{\partial CF}{\partial x} = 0 \Rightarrow 2x + A^T \lambda = 0 \quad \Rightarrow x = -\frac{1}{2} A^T \lambda$$

$$\frac{\partial CF}{\partial \lambda} = 0 \Rightarrow Ax - b = 0 \quad \Rightarrow Ax = b \Longrightarrow -\frac{1}{2} A A^T \lambda = b \quad \Rightarrow \lambda = -2 (AA^T)^{-1} b$$

$$\Rightarrow \boxed{x = A^T (AA^T)^{-1} b} = A_R^+ b$$

- Right Pseudo Inverse:

$$\boxed{A_R^+ = A^T (AA^T)^{-1}}$$

$$A A_R^+ = AA^T (AA^T)^{-1} = I$$

Inverse Velocity & Acceleration

❑ Inverse Velocity

$$Ax = b \qquad A \in R^{m \times n} \longrightarrow x = ?$$

b) $\quad m < n$

$$\Rightarrow \boxed{X = A^T (AA^T)^{-1} b} = A_R^+ b$$

$$\boxed{A_R^+ = A^T (AA^T)^{-1}}$$

- Final Response:

حل کلی : $\quad X = A_R^+ b + \overbrace{[I - A_R^+ A] Z}$

در null space ماتریس A قرار دارد

یعنی با تغییر Z ، X تغییر کند

ولی خروجی نخواهد داشت

بردار دلخواه

$$AX = AA_R^+ b + A[I - A_R^+ A] Z$$

$$= b + [A - A] Z = b$$

Inverse Velocity & Acceleration

Inverse Velocity

$$\nu = J(\Theta)\dot{\Theta} \qquad J(\Theta) \in R^{m\times n}$$

- **If $m > n$**

$$\dot{\Theta} = J_L^{+}\,\nu \qquad J_L^{+} \in R^{n\times m}$$

- where J_L^{+} is Left Pseudo Inverse.

$$J_L^{+} = (J^T J)^{-1} J^T$$

- Left Pseudo Inverse:

$$J_L^{+} J = \left((J^T J)^{-1} J^T\right) J = I_{n\times n}$$

Inverse Velocity & Acceleration

Inverse Velocity

$$\nu = J(\Theta)\dot{\Theta} \qquad J(\Theta) \in R^{m\times n}$$

- **If $m < n$ (Redundant Manipulators)**

$$\dot{\Theta} = J_R^{+}\,\nu + [I - J_R^{+}\,J]z \qquad J_R^{+} \in R^{n\times m}$$

- where J_R^{+} is Right Pseudo Inverse.

$$J_R^{+} = J^T(J\,J^T)^{-1}$$

- Right Pseudo Inverse :

$$J\,J_R^{+} = J\,(J^T(J\,J^T)^{-1}) = I_{m\times m}$$

- z is any arbitrary vector which $[I - J_R^{+}\,J]z$ is in the null space of $J(\Theta)$.
- It means $\dot{\Theta}$ can change with no effect on the end-effector velocity.
- It is special <u>advantage</u> of redundant manipulators for *obstacle avoidance* & … .

Inverse Velocity & Acceleration

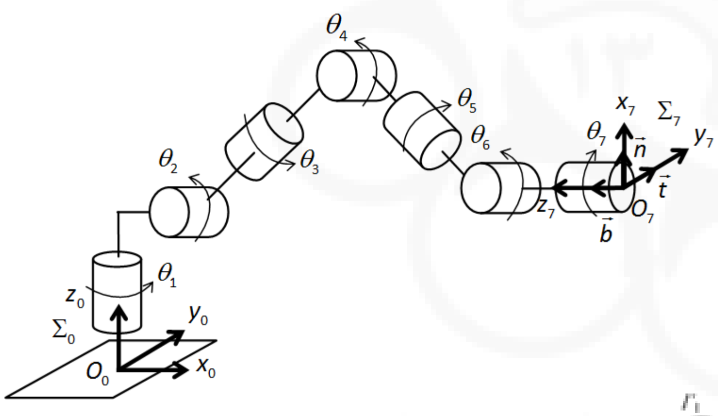
- **Inverse Velocity**

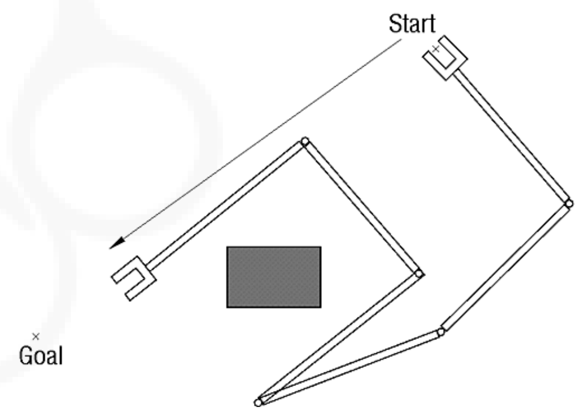
$$\nu = J(\Theta)\dot{\Theta} \qquad J(\Theta) \in R^{m\times n}$$

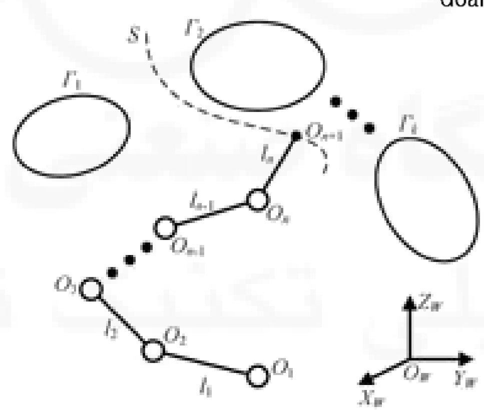
- **If $m < n$ (Redundant Manipulators)**

$$\dot{\Theta} = J_R^{+}\,\nu + [I - J_R^{+}\,J]z \qquad J_R^{+} \in R^{n\times m}$$

- ***Obstacle avoidance:***







← Outline

Inverse Velocity & Acceleration

Inverse Velocity

$$\nu = J(\Theta)\dot{\Theta} \qquad J(\Theta) \in R^{m\times n}$$

- If $m > n$

$$\dot{\Theta} = J_L^{+}\,\nu \qquad \& \qquad J_L^{+} = (J^T J)^{-1} J^T$$

- If $m < n$ (Redundant Manipulators)

$$\dot{\Theta} = J_R^{+}\,\nu + [I - J_R^{+}\,J]z \qquad \& \qquad J_R^{+} = J^T (J\,J^T)^{-1}$$

- **Remark:** A configuration Θ for the case $m > n$ is singular iff (if & only if), (Similarly, $m < n$)
  - $(J^T J)^{-1}$ does not exist (Similarly, $(J\,J^T)^{-1}$)
  - $rank\ (J^T J)$ is less than its maximum value (Similarly,$rank\ (J\,J^T)$)
  - $det(J^T J) = 0$ (Similarly, $det(J\,J^T)$)

- **Note:** for the square matrix J

$$det(J^T J) = det(J\,J^T) = det(J^T)\,det(J) = (det(J))^2 = 0$$

Inverse Velocity & Acceleration

Inverse Velocity

- **Singularity, Eigenvalue & Singular Value**
- Consider

$$v = J(\Theta)\dot{\Theta}$$

- Assume:

$$J(\Theta) \in R^{n\times n}$$

- λ_i and u_i are corresponding eigenvalue and eigenvector pairs for (J).

$$J\,u_i = \lambda_i\,u_i$$

- A configuration Θ is singular iff $det(J) = 0$, i.e. the eigenvalue of J become zero.

Inverse Velocity & Acceleration

❑ Inverse Velocity

- **Singularity, Eigenvalue & Singular Value**
- Consider

$$v = J(\Theta)\dot{\Theta}$$

- Assume:

$$J(\Theta) \in R^{m\times n} \quad \& \quad m < n$$

- λ_i' and u_i' are corresponding eigenvalue and eigenvector pairs for $(J\,J^T)$.

$$(J\,J^T)\,u_i' = \lambda_i'\,u_i'$$

- The ***singular values*** for the Jacobian matrix J are given by the square roots of the eigenvalues of $(J\,J^T)$,

$$\sigma_i = \sqrt{\lambda_i'}$$

- ***Generally***, a configuration Θ is singular iff $det(J\,J^T) = 0$, i.e. the singular value of J become zero.

Inverse Velocity & Acceleration

- **Inverse Velocity**
- **Singularity, Eigenvalue & Singular Value**
- **Note:**
- For **<u>the square matrix</u>**, the singular values of J & eigenvalues of J are **the same**,
- Eigenvalue

$$J\, u_i = \lambda_i\, u_i$$

- Singular Value

$$(J^T J)\, u_i' = \lambda_i'\, u_i'$$

$$\sigma_i = \sqrt{\lambda_i'}$$

- Therefore $\sigma_i = \lambda_i$. Prove it !!!
- Therefore, a configuration Θ is singular iff $det(J) = 0$, i.e. the eigenvalue become zero.

Inverse Velocity & Acceleration

❑ Inverse Acceleration

- Differentiating the velocity equation yields the acceleration equations.

$$a = \dot{v} = \frac{d}{dt}\left(J(\Theta)\dot{\Theta}\right) = J(\Theta)\ddot{\Theta} + \frac{d}{dt}\left(J(\Theta)\right)\dot{\Theta}$$

- The instantaneous joint acceleration vector $\ddot{\Theta}$ is given as:

$$\ddot{\Theta} = J^{-1}(\Theta)\left[a - \frac{d}{dt}\left(J(\Theta)\right)\dot{\Theta}\right]$$

- For 6-DOF manipulators the inverse velocity and acceleration equations are written as:

$$\dot{\Theta} = J^{-1}(\Theta)\, v$$

$$\ddot{\Theta} = J^{-1}(\Theta)\left[a - \frac{d}{dt}\left(J(\Theta)\right)\dot{\Theta}\right]$$

Inverse Velocity & Acceleration

- **Note:**
- We can apply a similar approach when the analytical Jacobian is used in place of the manipulator geometrical Jacobian.

- Recall

$$X = \begin{bmatrix} d(\Theta) \\ \Theta(\Theta) \end{bmatrix} \quad \rightarrow \quad \dot{X} = \begin{bmatrix} \dot{d}(\Theta) \\ \dot{\Theta}(\Theta) \end{bmatrix} = J_a(\Theta)\dot{\Theta} \qquad (*)$$

- Thus the inverse velocity problem is solving the system of linear equations (*) (i.e. all pseudo inverse & … will be defined based on J_a).

- For 6-DOF manipulators the inverse velocity and acceleration equations are written as

$$\dot{\Theta} = {J_a}^{-1}(\Theta)\,\dot{X}$$

$$\ddot{\Theta} = {J_a}^{-1}(\Theta)\left[\ddot{X} - \frac{d}{dt}(J_a(\Theta))\dot{\Theta}\right]$$

The END

- **References:**

1) …