Robotic Manipulators: Lecture 8

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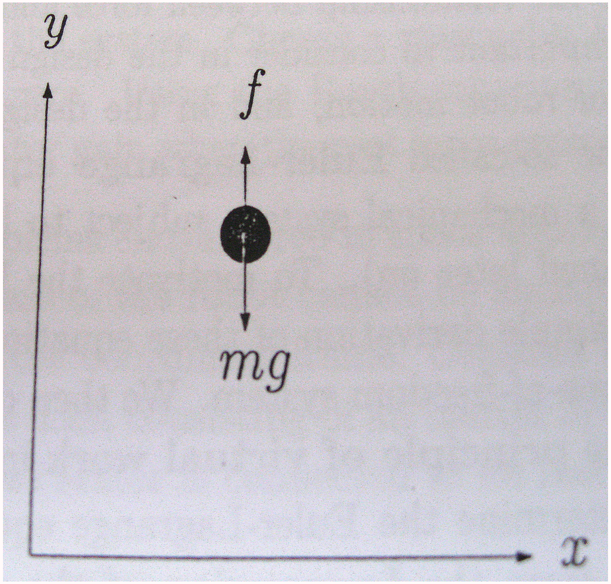
Outline

- Dynamic Equations of Motion
  - Euler-Lagrange Equation
  - Lagrangian
  - Kinetic Energy
  - Potential Energy
  - Properties of Dynamic Equations
  - ~~Nonrigid Body Effects~~

Manipulator Dynamics

- **Euler-Lagrange Equations:** Dynamic equations for mechanical systems satisfying the principle of virtual work
- **Motivation:**
  - Derive the Euler-lagrange equations for a 1 DOF system:



  - Equations of motion:

$$m\ddot{y} = f - mg$$

Lagrangian

$$m\ddot{y} \;=\; \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$$

- where $K = \left(\frac{1}{2}m\dot{y}^2\right)$ is the **kinetic energy**.
- Similarly, the gravitational force can be expressed as

$$mg \;=\; \frac{\partial}{\partial y}(mgy) = \frac{\partial P}{\partial y}$$

- where $P = mgy$ is the **potential energy** due to the gravity.
- Define:

$$L \;=\; K - P = \frac{1}{2}m\dot{y}^2 - mgy \cancel{=}$$

- Note that:

$$\begin{aligned} \frac{\partial L}{\partial \dot{y}} &= \frac{\partial K}{\partial \dot{y}} \\ \frac{\partial L}{\partial y} &= -\frac{\partial P}{\partial y} \end{aligned}$$

Lagrangian

- Hence, the equation of motion can be expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = f \tag{1}$$

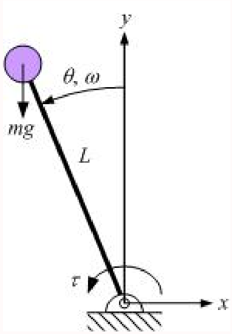
- The difference between the kinetic and potential energy (L) is called the **Lagrangian** and (1) is called the **Euler-Lagrange** Equation.
- General procedure: write the kinetic and potential energy in terms of generalized coordinates ($q_1, \cdots, q_n$), where n is the number of DOF .
- Compute the equations of motion of n-DOF system:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k, \quad k = 1, \cdots, n$$

- where τ_k is the (generalized) force associated with q_k
- Results in a set of coupled ordinary differential equations

Lagrangian

- **Example An Inverted Pendulum:**



- A pendulum coupled through a gear to a DC motor.
- Let θ_l and θ_m denote the angle of the pendulum and motor shaft, respectively.
- $\theta_l = r\theta_m$, where $1 : r$ is the gear ratio.
- System has only 1 DOF $\Longrightarrow$ generalized coordinate could be θ_l or θ_m.
- Kinetic energy in terms of θ_l:

$$K = \frac{1}{2}J_m\,\dot{\theta}_m^2 + \frac{1}{2}J_l\,\dot{\theta}_l^2 + \frac{1}{2}Ml^2\dot{\theta}_l^2 = \frac{1}{2}(J_m/r^2 + J_l + Ml^2)\,\dot{\theta}_l^2$$

- where J_m and J_l are the inertias of motor and pendulum, respectively

Lagrangian

- The potential energy:

$$P \;=\; Mgl\cos\theta_l$$

- Sometimes, the potential energy is expressed s.t. it has a zero minimum value. This corresponds to defining the potential energy relative to an arbitrary zero reference height:

$$P \;=\; Mgl(1+\cos\theta_l)$$

- where M is the total mass of the link and l is the distance from the joint axis to the link center of mass:
- Let $I = J_l + J_m/r^2 + Ml^2$, then the Lagrangian L is given by:

$$L \;=\; \frac{1}{2}I\,\dot{\theta}_l^2 - Mgl(1+\cos\theta_l)$$

- Equation of motion:

$$I\,\ddot{\theta}_l - Mgl\sin\theta_l = \tau_l$$

Kinetic and Potential Energy

- For this example, τ_l is input motor torque $u = \tau_m / r$ reflected to the link and damping torques $B_m \dot{\theta}_m$ and $B_l \dot{\theta}_l$:

$$\begin{aligned} \tau_l &= u - B\dot{\theta}_l \\ B &= B_m / r^2 + B_l \end{aligned}$$

- Dynamic Equations:

$$I\,\ddot{\theta}_l + B\dot{\theta}_l - Mgl\sin\theta_l = u$$

- **Kinetic and Potential Energy**
  - Heart of Lagrangian formulation: compute kinetic and potential energy
  - The generalized coordinates for manipulators with rigid links are the joint variables
  - The kinetic energy of a rigid object consists of two terms, translational kinetic energy concentrating the entire mass at COM and the rotational kinetic energy about COM
  - Consider the frame attached at the body's center of mass

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T I \omega$$

Kinetic Energy

- where I is the inertia tensor expressed in the frame attached to COM
- Consider a manipulator with n link
- Linear and angular velocities of any point on any link can be expressed in terms of the Jacobian matrix and the derivatives the joint variables:

$$
\begin{aligned}
v_i &= J_{\boxed{v_i}}(\theta)\dot{\theta}_{\cancel{i}} \\
\omega_i &= J_{\omega_i}(\theta)\dot{\theta}_{\cancel{i}}
\end{aligned}
$$

- Let the mass of link i be m_i and the inertia tensor about COM be I_i
- The overall kinetic energy:

$$
\begin{aligned}
& \frac{1}{2}\dot{\theta}^T \left[\sum_{i=1}^{n} \left\{ m_i J_{v_i}(\theta)^T J_{v_i}(\theta) + J_{\omega_i}(\theta)^T \boxed{R_i(\theta) I_i R_i(\theta)^T} J_{\omega_i}(\theta) \right\} \right] \dot{\theta} \\
= & \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} \qquad \text{where} \\
M(\theta) = & \left[\sum_{i=1}^{n} \left\{ m_i J_{v_i}(\theta)^T J_{v_i}(\theta) + J_{\omega_i}(\theta)^T R_i(\theta) I_i R_i(\theta)^T J_{\omega_i}(\theta) \right\} \right]
\end{aligned}
$$

is $n \times n$ inertia (mass) matrix

Potential Energy

- The inertia matrix is a symmetric positive definite matrix. Symmetry is clear from the above and the positive-definiteness is inferred from the fact that the kinetic energy is positive for nonzero velocities of the joints.
- **Potential Energy:**
  - For a rigid body, the only source of potential energy is gravity
  - Considering the mass of the entire object is concentrated at COM:

$$P_i = -m_i g^T r_{ci} + P_{ref}$$

    where P_{ref} is selected s.t. P has zero minimum value.
  - where g is the vector giving the direction of the gravity in the inertial frame and r_{ci} gives the coordinates of the COM of link i.
  - The total potential energy:

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} -m_i g^T r_{ci}$$

  - For the case of elastic joints or flexible links the potential energy contains the terms stored in elastic terms
  - It is a function of generalized coordinates only.

Equations of Motion

- **The kinetic energy is a quadratic function of $\dot{\theta}$**

$$K = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} \cancel{=} \sum_{i,j} m_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j$$

  where m_{ij} is the entries of symmetric positive definite inertia matrix $M(\theta)$

- **Potential energy is independent of $\dot{\theta}$**
- Using the above definition, write the Lagrangian as:

$$L = K - P = \frac{1}{2}\sum_{i,j} m_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j - P(\theta)$$

- Partial derivative wrt kth joint velocity:

$$\frac{\partial L}{\partial \dot{\theta}_k} = \sum_j m_{kj}\dot{\theta}_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_k} = \sum_j m_{kj}\ddot{\theta}_j + \sum_j \frac{d}{dt} m_{kj}\dot{\theta}_j \cancel{=} m_{kj}\ddot{\theta}_j + \sum_{i,j}\frac{\partial m_{kj}}{\partial \theta_i}\dot{\theta}_i\dot{\theta}_j$$

Equations of Motion

- Partial derivative wrt kth joint position:

$$\frac{\partial L}{\partial \theta_k} = \frac{1}{2}\sum_{i,j}\frac{\partial m_{ij}}{\partial \theta_k}\dot{\theta}_i\dot{\theta}_j - \frac{\partial P}{\partial \theta_k}$$

- The Euler-Lagrange equations:

$$\sum_j m_{kj}\ddot{\theta}_j + \sum_{i,j}\left\{\frac{\partial m_{kj}}{\partial \theta_i} - \frac{1}{2}\frac{\partial m_{\cancel{k}j}}{\partial \theta_k}\right\}\dot{\theta}_i\dot{\theta}_j + \frac{\partial P}{\partial \theta_k} = \tau_k$$

- It can be shown that:

$$\sum_{i,j}\left\{\frac{\partial m_{kj}}{\partial \theta_i}\right\}\dot{\theta}_i\dot{\theta}_j \;=\; \frac{1}{2}\sum_{i,j}\left\{\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j}\right\}\dot{\theta}_i\dot{\theta}_j$$

- Hence:

$$\begin{aligned}\sum_{i,j}\left\{\frac{\partial m_{kj}}{\partial \theta_i} - \frac{1}{2}\frac{\partial m_{\cancel{k}j}}{\partial \theta_k}\right\}\dot{\theta}_i\dot{\theta}_j &= \sum_{i,j}\frac{1}{2}\left\{\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right\}\dot{\theta}_i\dot{\theta}_j \\ &= \sum_{i,j} c_{ijk}\dot{\theta}_i\dot{\theta}_j\end{aligned}$$

Equations of motion

where

$$c_{ijk} := \frac{1}{2}\left\{\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right\}$$

- Note that for a fixed k, $c_{ijk} = c_{jik}$. Define:

$$g_k = \frac{\partial P}{\partial \theta_k}$$

- Then, the euler-Lagrange equations can be written as:

$$\sum_j m_{kj}\ddot{\theta}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(\theta)\dot{\theta}_i\dot{\theta}_j + g_k(\theta) \;=\; \tau_k \quad k = 1, \cdots, n$$

- As can be seen, similar terms as Newton-Euler equations presents: inertia terms, Coriolis and centrifugal, and potential energy (gravity)
- State space form:

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) \;=\; \tau \tag{2}$$

Example: Two-link planar manipulator

- where

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(\theta)\dot{\theta}_i = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right\}\dot{\theta}_i$$

and the gravity vector is given by

$$g(q) = \begin{bmatrix} g_1(\theta) & \cdots & g_n(\theta) \end{bmatrix}^T$$

- **Example: A planar two-link manipulator**
  - The Jacobian (in base frame):

$$v_{c1} = J_{v_{c1}}\dot{\theta}$$

where

$$J_{v_{c1}} = \begin{bmatrix} -l_{c1}s_1 & 0 \\ l_{c1}c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: Two-link planar manipulator

- Similarly

$$J_{v_{c2}} = \begin{bmatrix} -l_1 s_1 - l_{c2}s_{12} & -l_{c2}s_{12} \\ l_1 c_1 + l_{c2}c_{12} & l_{c2}c_{12} \\ 0 & 0 \end{bmatrix}$$

- **Translational part of kinetic energy:**

$$\frac{1}{2}m_1 v_{c1}^T v_{c1} + \frac{1}{2}m_1 v_{c2}^T v_{c2} = \frac{1}{2}\dot{\theta}\left\{m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}}\right\}\dot{\theta}$$

- The angular velocity terms:

$$\begin{aligned} \omega_1 &= \dot{\theta}_1 {}^1\hat{Z}_1 \\ \omega_{\cancel{1}} &= (\dot{\theta}_1 + \dot{\theta}_2)^2\hat{Z}_2 \end{aligned}$$

- **Rotational kinetic energy** is simply $I_i\omega_i^2$
- Rotational kinetic energy in matrix form:

$$\frac{1}{2}\dot{\theta}^T\left\{I_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}\dot{\theta}$$

Example: Two-link planar manipulator

- To get the inertia matrix $M(\theta)$ add the translational and rotational kinetic energy terms:

$$M(\theta) = m_1 J_{vc1}^T J_{vc1} + m_2 J_{vc2}^T J_{vc2} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

- Now, by using some algebraic manipulations:

$$\begin{aligned} m_{11} &= m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos\theta_2) + I_1 + I_2 \\ m_{12} &= m_{21} = m_2(l_{c2}^2 + l_1 l_{c2} \cos\theta_2) + I_2 \\ m_{22} &= m_2 l_{c2}^2 + I_2 \end{aligned}$$

- The velocity terms $C(\theta, \dot{\theta})$:

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_1} = 0 \\ c_{121} &= c_{211} = \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2} = -m_2 l_1 l_{c2} \sin\theta_2 =: h \\ c_{221} &= \frac{\partial m_{12}}{\theta_2} - \frac{1}{2}\frac{\partial m_{22}}{\partial \theta_1} = h, \quad c_{112} = \frac{\partial m_{21}}{\theta_1} - \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2} = -h \end{aligned}$$

Example: Two-link planar manipulator

$$c_{122} = c_{212} = \frac{1}{2}\frac{\partial m_{22}}{\partial \theta_1} = 0, \quad c_{222} = \frac{1}{2}\frac{\partial m_{22}}{\partial \theta_2} = 0$$

- **Potential energy of the manipulator:**

$$\begin{aligned} P_1 &= m_1 g l_{c1} \sin\theta_1, \\ P_2 &= m_2 g (l_1 \sin\theta_1 + l_{c2}\sin(\theta_1+\theta_2)) \\ P &= P_1 + P_2 = (m_1 l_{c1} + m_2 l_1) g \sin\theta_1 + m_2 l_{c2} g \sin(\theta_1+\theta_2) \end{aligned}$$

- The gravity terms:

$$\begin{aligned} g_1 &= \frac{\partial P}{\partial \theta_1} = (m_1 l_{c1} + m_2 l_1) g \cos\theta_1 + m_2 l_{c2} g \cos(\theta_1+\theta_2) \\ g_2 &= \frac{\partial P}{\partial \theta_2} = m_2 l_{c2} g \cos(\theta_1+\theta_2) \end{aligned}$$

- Dynamic Equations of motion:

$$\begin{aligned} m_{11}\ddot{\theta}_1 &+ m_{12}\ddot{\theta}_2 + c_{121}\dot{\theta}_1\dot{\theta}_2 + c_{211}\dot{\theta}_2\dot{\theta}_1 + c_{221}\dot{\theta}_2^2 + g_1 = \tau_1 \\ m_{21}\ddot{\theta}_1 &+ m_{22}\ddot{\theta}_2 + c_{112}\dot{\theta}_1^2 + g_2 = \tau_2 \end{aligned}$$

Properties of Robot Dynamic Equations

- Structural properties of dynamic equations: useful for control algorithms
  1. **Skew Symmetry and Passivity**
     - **The Skew symmetry property:** The following relationship exists between $M(\theta)$ and $C(\theta, \dot{\theta})$ in (2):
       * The matrix $N(\theta, \dot{\theta}) = \dot{M}(\theta) - 2C(\theta, \dot{\theta})$ is **skew symmetric**, i.e. $n_{jk} = -n_{kj}$.
       * **Proof:** The derivative of m_{kj}:

$$\dot{m}_{kj} = \sum_{i}^{n} \frac{\partial m_{kj}}{\partial \theta_i} \dot{\theta}_i$$

Hence the components of $N(\theta, \dot{\theta}) = \dot{M}(\theta) - 2C(\theta, \dot{\theta})$:

$$\begin{aligned} n_{kj} &= \dot{m}_{kj} - 2c_{kj} \\ &= \sum_{i=1}^{n} \left[\frac{\partial m_{kj}}{\partial \theta_i} - \left\{ \frac{\partial m_{kj}}{\partial \theta_{\cancel{j}}} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \right] \dot{\theta}_i \\ &= \sum_{i=1}^{n} \left[\frac{\partial m_{ij}}{\partial \theta_k} - \frac{\partial m_{ki}}{\partial \theta_j} \right] \dot{\theta}_i \end{aligned}$$

Passivity Property

- The inertia matrix $M(\theta)$ is symmetric ($m_{ij} = m_{ji}$). Hence:

$$n_{jk} = -n_{kj}$$

- Note that in order for $N = \dot{M} - 2C$ to be skew-symmetric, C has to be defined according to (2).

2. **Passivity:**

    - Property of passive systems:
    - The energy dissipated by the system is positive:

$$\int_0^T \dot{\theta}^T(\zeta)\tau(\zeta)d\zeta + \beta \geq 0, \quad \forall T > 0 \tag{3}$$

    - The integral represents the energy dissipated by the system
    - The concept is borrowed from the circuit theory (passive circuits can be built from passive elements).
    - Passive mechanical systems include masses, springs, and dampers.

$$H = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} + P(\theta)$$

Passivity Property

- The derivative of H:

$$\begin{aligned}\dot{H} &= \dot{\theta}^T M(\theta)\ddot{\theta} + \frac{1}{2}\dot{\theta}^T \dot{M}(\theta)\dot{\theta} + \dot{\theta}^T \frac{\partial P}{\partial \theta} = \dot{\theta}^T \left\{\tau - C(\theta,\dot{\theta})\dot{\theta} - g(\theta)\right\} \\ &+ \frac{1}{2}\dot{\theta}^T \dot{M}(\theta)\dot{\theta} + \dot{\theta}^T \frac{\partial P}{\partial \theta}\end{aligned}$$

- Using the fact that $g(\theta) = \frac{\partial P}{\partial \theta}$:

$$\begin{aligned}\dot{H} &= \dot{\theta}^T \tau + \frac{1}{2}\dot{\theta}^T \left\{\dot{M}(\theta) - 2C(\theta,\dot{\theta})\right\}\dot{\theta} \\ &= \dot{\theta}^T \tau\end{aligned}$$

- Integrating both sides of the equations:

$$\int_0^T \dot{\theta}^T(\zeta)\tau(\zeta)d\zeta = H(T) - H(0) \geq -H(0)$$

- Since the total energy is nonnegative, passivity property follows

Property of Dynamic Equation

3. **Bounds on Inertia Matrix**

- Recall that the inertia matrix M is symmetric positive definite
- Let $0 < \lambda_1(\theta) \leq \cdots \leq \lambda_n(\theta)$ denotes the n eigenvalues of $M(\theta)$.
- A property of positive definite matrices:

$$\lambda_1(\theta) I_{n\times n} \leq M(\theta) \leq \lambda_n(\theta) I_{n\times n}$$

- If all joints are revolute, then inertia matrix contains only sine and cosine fcns which are bounded.
- Constants λ_m and λ_M can be found s.t.

$$\lambda_m(\cancel{\theta}) I_{n\times n} \leq M(\theta) \leq \lambda_M(\cancel{\theta}) I_{n\times n} < \infty$$

4. **Linearity in Parameters**

- Equations of motion described in terms of certain parameters, such as link masses, moment of inertias, etc.
- Exact determinations of these parameters are a difficult task
- Equation of motion is linear in these parameters

Property of Dynamic Equations

- There exists an $n \times l$ function $Y(\theta, \dot{\theta}, \ddot{\theta})$ and an $l-$dimensional parameter vector ϕ such that the equation of motions can be written as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = Y(\theta, \dot{\theta}, \ddot{\theta})\phi$$

- $Y(\theta, \dot{\theta}, \ddot{\theta})$ is called the **regressor** and $\phi \in \mathbb{R}^l$ is called **parameter vector**
- Parameters include COM coordinates, total mass, and inertia tensor
- **Example: Two-link planar manipulator**
  - Regroup the parameters in dynamic equations:

$$\begin{aligned} \phi_1 &= m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + I_1 + I_2) \\ \phi_2 &= m_2 l_1 l_{c2} \\ \phi_3 &= m_2 l_{c2}^2 + I_2 \end{aligned}$$

  - The entries of inertia matrix can be written as:

$$\begin{aligned} m_{11} &= \phi_1 + 2\phi_2 c_2 \\ m_{12} &= m_{21} = \phi_3 + \phi_2 c_2 \\ m_{22} &= \phi_3 \end{aligned}$$

The parameters in C depend on the entries of M

Linearity in Parameters

- Gravity terms require additional parameters:

$$\begin{aligned} \phi_4 &= m_1 l_{c1} + m_2 l_1 \\ \phi_5 &= m_2 l_2 \end{aligned}$$

- Then, the gravity term can be written as

$$\begin{aligned} g_1 &= \phi_4 g \cos\theta_1 + \phi_5 g \cos(\theta_1 + \theta_2) \\ g_2 &= \phi_5 \cos(\theta_1 + \theta_2) \end{aligned}$$

- Then, the dynamic equation can be written in regressor form with:

$$\begin{aligned} Y(\theta, \dot\theta, \ddot\theta) &= \begin{bmatrix} \ddot\theta_1 & c_2(2\ddot\theta_1 + \ddot\theta_2) - s_2(\dot\theta_1^2 + 2\dot\theta_1\dot\theta_2) & \ddot\theta_2 & gc_1 & gc_{12} \\ 0 & c_2\ddot\theta_1 + s_2\dot\theta_1^2 & \ddot\theta_1 + \ddot\theta_2 & 0 & gc_{12} \end{bmatrix} \\ \Phi &= \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + I_1 + I_2 \\ m_2 l_1 l_{c2} \\ m_2 l_{c2}^2 + I_2 \\ m_1 l_{c1} + m_2 l_1 \\ m_2 l_2 \end{bmatrix} \end{aligned}$$

- Dynamic equation is parameterized in 5 dimensional parameter space