دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

دانشکده مهندسی مکانیک
دانشگاه صنعتی امیرکبیر

Lecture 3_2:
Manipulator Kinematics
Inverse Kinematics

Advanced Robotics
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Outlines

- Introduction
- Solvability
- Algebraic & Geometric
- Pieper’s Solution
- Example of Inverse Manipulator Kinematics

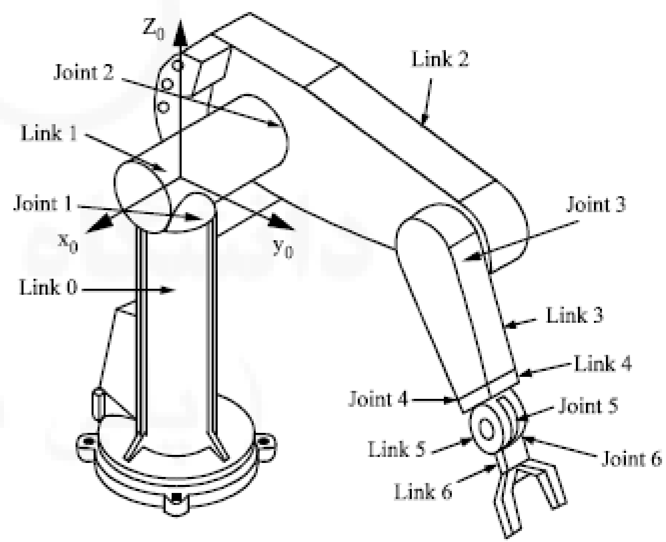
Introduction

- **Forward Kinematic:** Describe the **position** and **orientation** of the manipulator end-effector (EE) relative to the base frame as a function of joint variables. (Application:…)

- $\boldsymbol{X} = \boldsymbol{f}(\theta)$

- **Inverse Kinematics:** Given the **desired** position and orientation of the end-effector relative to the base, compute the set of **joint variables** which will achieve this desired result. (Application:…)

- $\theta = \boldsymbol{f}^{-1}(\boldsymbol{X})$



Solvability

- Inverse Kinematic (IK) mapping is a nonlinear mapping.
- Given the **desired** numerical value of ${}^{0}T_{n}$, find $\theta_1, \ldots, \theta_n$?
- It is a **Nonlinear** Problem.

$${}^{0}T_{n}|_{Numerical} = {}^{0}T_{n}|_{Parametric} \quad \Rightarrow \quad \text{Finding } \theta_1, \ldots, \theta_n$$

❖ **Example:** PUMA-560 Robot. Given ${}^{0}T_{6}$; Find $\theta_1, \ldots, \theta_6$?!!

$$\begin{aligned}
r_{11} &= C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + S_1(S_4C_5C_6 + C_4S_6), \\
r_{21} &= S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - C_1(S_4C_5C_6 + C_4S_6), \\
r_{31} &= -S_{23}(C_4C_5C_6 - S_4S_6) - C_{23}S_5C_6, \\
&\ldots \\
&\ldots \qquad \textit{Equation}: (3.14)
\end{aligned}$$

Solvability

- For a 6-DOF manipulator, we have:

$$ {}^{0}T_6\Big|_{Parametric} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- 12-Equations & 6-Unknowns?
- From 9-Equations of the Rotation Matrix, only 3-Equations are independent.
- Therefore, we have 6-independent non-linear equations and 6-unknowns. Is it solvable?

- **Nonlinear equations:**
  - Existence of solution
  - Multiple solution
  - Method of solution

Solvability

- **Existence of solution**

- The existence relates to the manipulator's workspace.

- **Workspace:** The volume of space which the end-effector of a robot *can reach*.

- For a solution to exist the point should be in manipulator's workspace.

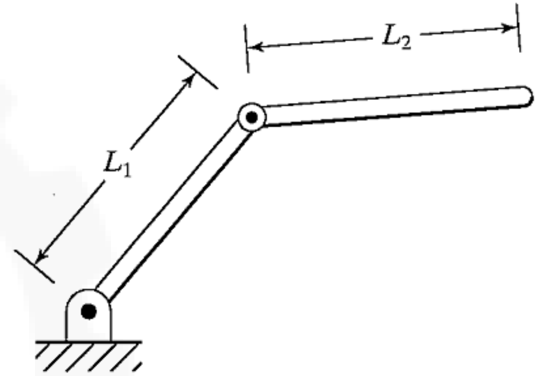
  - **Reachable Workspace:** The volume of space which the end-effector of a robot can reach with at least one orientation.

  - **Dexterous Workspace:** The volume of space which the end-effector of a robot can reach with all orientations.

- **Note:** The dexterous workspace is a subset of the reachable workspace

Solvability

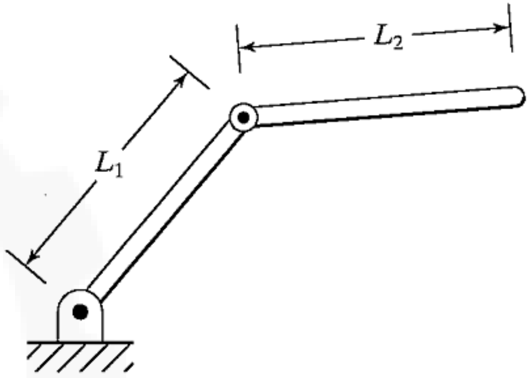
- **Existence of solution**
- **Example 1:** A two-link planar manipulator



- If $l_1 = l_2$,
  - Reachable workspace is …
  - Dexterous workspace is …

Solvability

- **Existence of solution**
- **Example 1:** A two-link planar manipulator

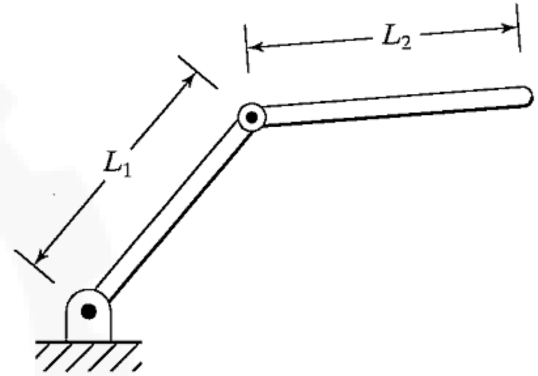


- If $l_1 = l_2$,
  - Reachable workspace is a disc of radius $2l_1$.
  - Dexterous workspace is a single point (the origin).
- If $l_1 \neq l_2$,
  - Reachable workspace …
  - Dexterous workspace …

Solvability

- **Existence of solution**

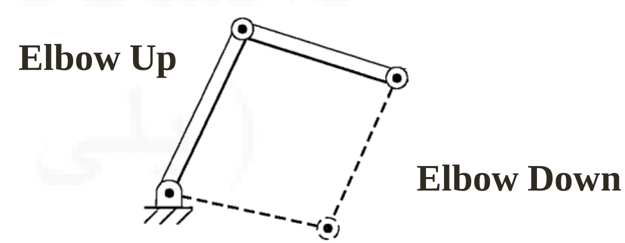
- **Example 1:** A two-link planar manipulator



- If $l_1 = l_2$,
  - Reachable workspace is a disc of radius $2l_1$.
  - Dexterous workspace is a single point (the origin).

- If $l_1 \neq l_2$,
  - Reachable workspace is a ring with outer radius $l_1 + l_2$ and the inner radius is $|l_1 - l_2|$.
  - Dexterous workspace: Empty.

- Inside the reachable workspace there are two possible orientations of the end-effector.
- On the boundaries the reachable workspace the workspace there is only one possible orientation (So, …).

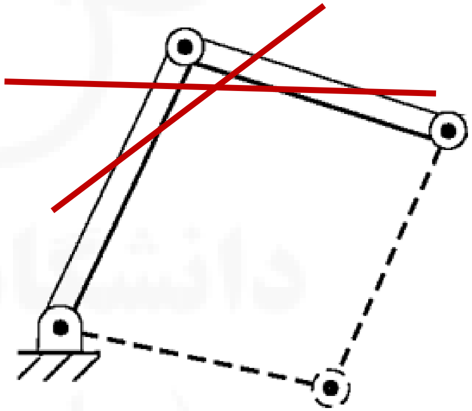


Solvability

- **Existence of solution**
- **Joint Limitation**
- If joints have **mechanical limitation** (e.g. those cannot rotate 360-degree), the workspace may be reduced.

- For example if θ_1 has 360-degree motion, but θ_2's motion is limited to $0 \leq \theta_2 \leq 180$, what is the reachable workspace?

- The reachable workspace has the same volume, but only one orientation is attainable.

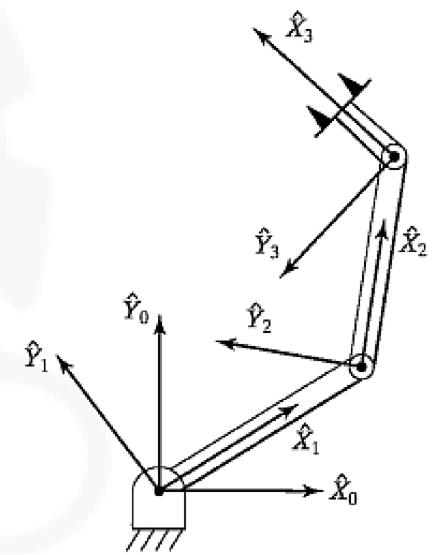


- Example of workspace reduction: …

Solvability

- **Existence of solution**

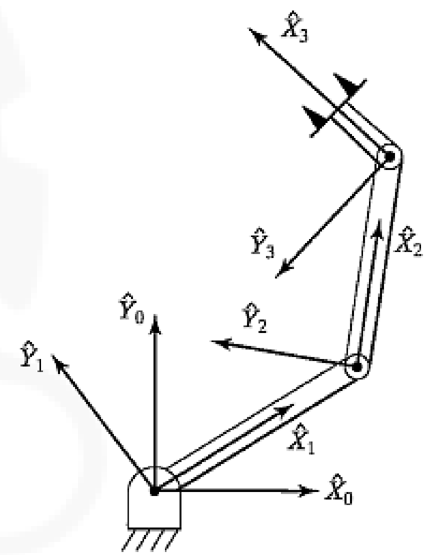
- **Example 2:** A RRR planar manipulator
- The Link 3 is a wrist.
- (Assume $l_1 > l_2$ & l_3 is negligible)
- Reachable workspace = ?
- Dexterous workspace = ?



Solvability

- **Existence of solution**

- **Example 2:** A RRR planar manipulator
- The Link 3 is a wrist.
- (Assume $l_1 > l_2$ & l_3 is negligible)
- Reachable workspace = ?
- Dexterous workspace = ?



- Reachable workspace is a ring with outer radius $(l_1 + l_2)$ and the inner radius is $(l_1 - l_2)$.
- Dexterous workspace = Reachable workspace

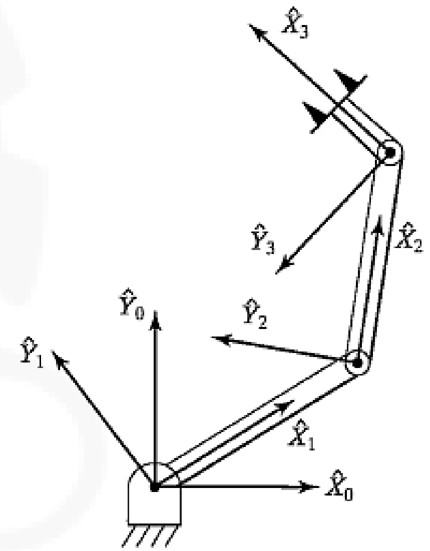
- **Note:** A wrist can expand the Dexterous workspace.

  This is an important property of **Decoupled Manipulator.**

Solvability

- **Existence of solution**

- **Example 3:** A RRR planar manipulator
- (Assume $l_1 > l_2 > l_3 \quad \& \quad l_1 > l_2 + l_3$)
- Reachable workspace = ?
- Dexterous workspace = ?



Solvability

- **Remark:**
  - **Reachable Workspace:** The volume of space which the end-effector of a robot can reach with at least one orientation.
  - **Dexterous Workspace:** The volume of space which the end-effector of a robot can reach with all orientations.

- **Note:**
- To solve the manipulator inverse kinematics, if the goal is …
  - Desired position:
    - That is enough to be located in the **Reachable Workspace.**
  - Desired position & orientation:
    - If it is located in the **Dexterous Workspace**, the solution always exist.
    - If it is located in the **Reachable Workspace**, the solution may exist.

Solvability

- **Multiple solution**

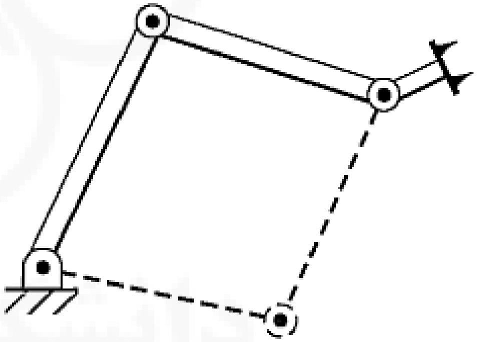
- A manipulator may reach any position and orientation in the interior of its workspace with different configurations (**multiple solution)**.

- **Example:** RRR planar manipulator with a wrist

  - Large dexterous workspace in the plane.

  - Any position in the interior of its workspace can be reached with any orientation.

  - Multiple solution could be found.

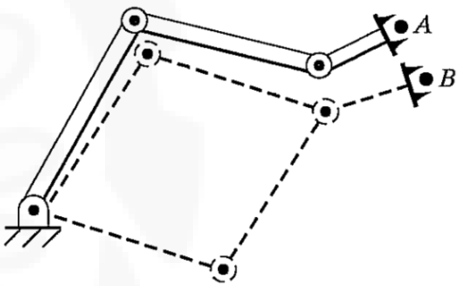


- **Note:** Multiple solution can cause problems, because the system has to be able to choose one.

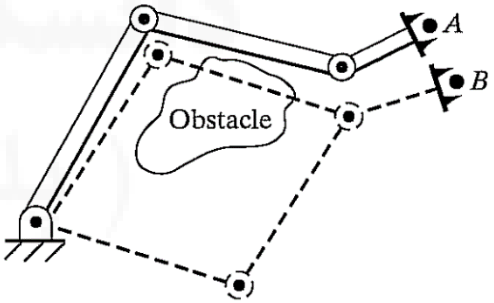
Solvability

Multiple solution

- Assume the manipulator is at point A and we wish to move it to point B.

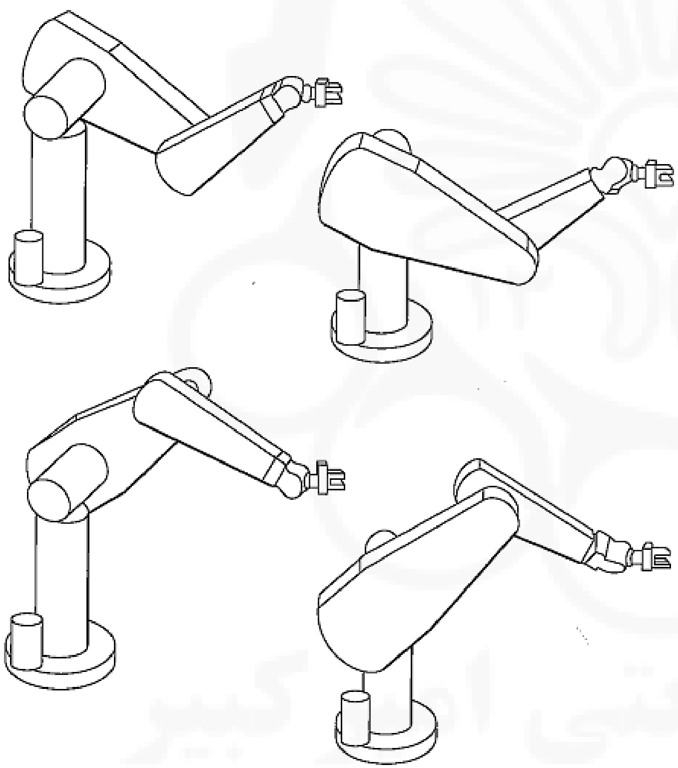


- Prominent points to choose a solution:
  - A very reasonable choice would be the closest solution.
    The solution that minimizes the amount that each joint is required to move.
  - Moving smaller joints rather than moving the large joints.
  - Presence of obstacles might force a "farther" solution to be chosen in cases where the "closer" solution would cause a collision.

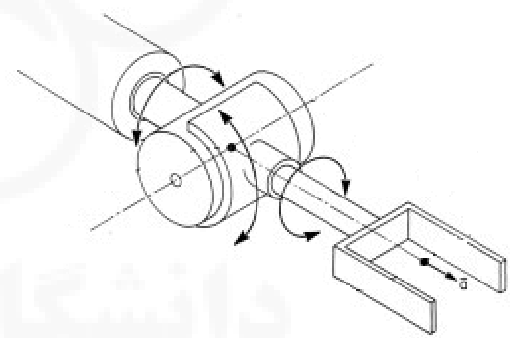


Solvability

- **Multiple solution**
- **Example:** PUMA 560
- How many solutions are there for a given position and orientation?



$$\theta_4' = \theta_4 + 180°,$$
$$\theta_5' = -\theta_5,$$
$$\theta_6' = \theta_6 + 180°.$$



- It can reach certain goals with 8-different solutions.
- Due to the joint limitations, some solutions might not be accessible.

Solvability

Multiple solution

- **The number of solutions depends upon:**
  - The number of joints
  - The allowable ranges of motion of the joints
  - Link parameters

- The more Non-Zero link parameters, the more Solutions to reach a certain goal.

- Up to 16-solutions are possible for a completely general revolute arm with 6-DOF.

a_i	Number of solutions
$a_1 = a_3 = a_5 = 0$	$\leqslant 4$
$a_3 = a_5 = 0$	$\leqslant 8$
$a_3 = 0$	$\leqslant 16$
All $a_i \neq 0$	$\leqslant 16$

Solvability

- **Methods of Solutions**

- Unlike linear equations, no general algorithms exist for solving a set of nonlinear equations.

- A manipulator is said to be **solvable** if the set of *all* joint variable associated to a given position and orientation can be determined.

- Two forms of solution strategies exist:

- **1- Closed-form-Solutions:**
  Solution method is based on analytical expressions.

- **2- Numerical Solutions:**
  Due to their iterative nature, they are too slow, and therefore not a useful approach in solving robot kinematics.

Solvability

- **Methods of Solutions**
- **Notes:**
- All systems with revolute and prismatic joints having 6 DOF in a single series chain are solvable.

- Only in special cases it could be solved analytically.

- Robots with analytical solutions have several intersecting joints or many α_i are 0 or ±90 degrees.

- It is very important to design a manipulator such that a closed form solution exists.

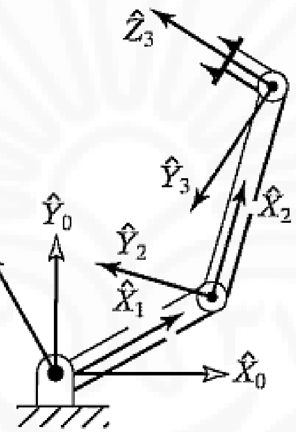
Solvability

- **Methods of Solutions**
- **Theorem:**
  - A **sufficient condition** that a manipulator with six revolute joints have a closed-form solution is that three neighboring joint axes **intersect** at a point.
- Included in this family of manipulators are those with three consecutive parallel axes, because they meet at the point at infinity.
- Closed-form solutions exist for decoupled manipulator (three joints intersect).
- **Closed-form solutions:**
  - Algebraic solution
  - Geometric solution
  - Algebraic solution by Reduction to Polynomial
  - …

← Outline

Algebraic & Geometric

- **Algebraic Solution**
- **Example:** RRR Planar Manipulator



i	$\alpha_i - 1$	$a_i - 1$	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Algebraic & Geometric

❑ Algebraic Solution

$$
\left.{}^{0}T_{3}\right)_{desired} = \begin{bmatrix} c\varphi & -s\varphi & 0 & x \\ s\varphi & c\varphi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{0}T_{3} = \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\left.\begin{array}{lll}
* \quad c\varphi = c_{123} & & (1) \\
* \quad s\varphi = s_{123} & & (2) \\
* \quad x = l_1c_1 + l_2c_{12} & & (3) \\
* \quad y = l_1s_1 + l_2s_{12} & & (4)
\end{array}\right\} \text{۴ معادله غیرخطی} \longrightarrow \theta_1, \theta_2, \theta_3
$$

$$
\begin{cases} c_{12} = c_1c_2 - s_1s_2 \\ s_{12} = s_1c_2 + c_1s_2 \end{cases}
$$

Algebraic & Geometric

Algebraic Solution

- Finding θ_2:

$$(3)^2+(4)^2 \longrightarrow x^2+y^2=l_1^2+l_2^2+2l_1l_2c_2$$

$$\Rightarrow \quad c_2=\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}$$

— شرط وجود جواب $\longrightarrow -1<c_2<1$

if $c_2>1$ [illegible] $\longrightarrow$

$$s_2=\pm\sqrt{1-c_2^2} \xrightarrow[\text{Two Argument Arc tangent}]{} \theta_2=\mathrm{Atan2}(s_2,c_2)$$

elbow down $\leftarrow \theta_2>0$

elbow up $\leftarrow \theta_2<0$

۲ مقدار برای θ_2 موجود است

Algebraic & Geometric

- **Algebraic Solution**
  - Finding θ_1:

$$③\quad x = k_1 c_1 - k_2 s_1$$
$$④\quad y = k_1 s_1 + k_2 c_1$$

$$\begin{cases} k_1 = l_1 + l_2 c_2 \\ k_2 = l_2 s_2 \end{cases}$$

$$x = k_1 c_1 - k_2 s_1 = r\cos(\theta_1 + \gamma)$$
$$y = k_1 s_1 + k_2 c_1 = r\sin(\theta_1 + \gamma)$$

$$r = \sqrt{k_1^2 + k_2^2} \qquad \gamma = \text{Atan2}(k_2, k_1)$$

$$\begin{cases} x/r = \cos(\theta_1 + \gamma) \\ y/r = \sin(\theta_1 + \gamma) \end{cases} \implies \theta_1 + \gamma = \text{Atan2}\left(\frac{y}{r}, \frac{x}{r}\right) = \text{Atan2}(y, x)$$

Algebraic & Geometric

❑ Algebraic Solution

$$\gamma = A\tan 2(k_2, k_1)$$

$$\theta_1 + \gamma = A\tan 2\left(\frac{y}{r}, \frac{x}{r}\right) = A\tan 2(y, x)$$

$$\Rightarrow \theta_1 = A\tan 2(y, x) - A\tan 2(k_2, k_1)$$

- Finding θ_3:

$$\theta_1 + \theta_2 + \theta_3 = A\tan 2(s\alpha, c\alpha) = \alpha$$

$$\Rightarrow \theta_3 = \alpha - [\theta_1 + \theta_2]$$

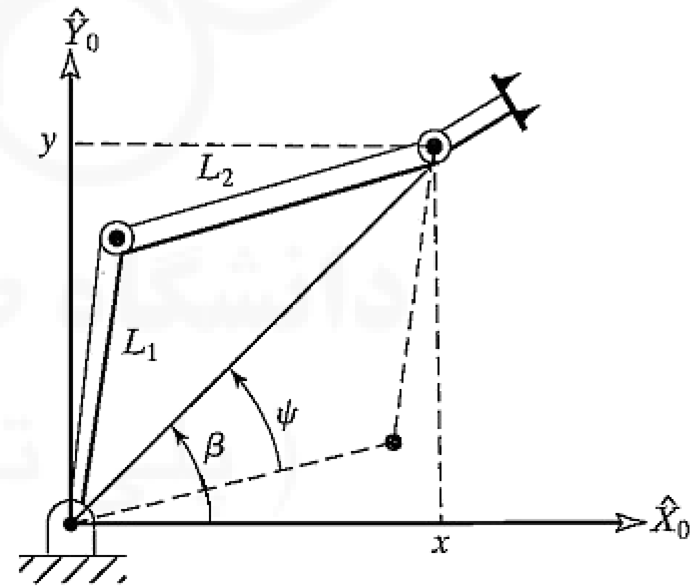
Algebraic & Geometric

- **Geometric Solution**
- Decompose the spatial geometry of the arm into several plane-geometry problems.

- Joint angles can then be solved for by using the tools of plane geometry.

- For many manipulators (particularly when the $\alpha_i = 0$ or ± 90) this can be done quite easily.

- **Example:** RRR Planar Manipulator



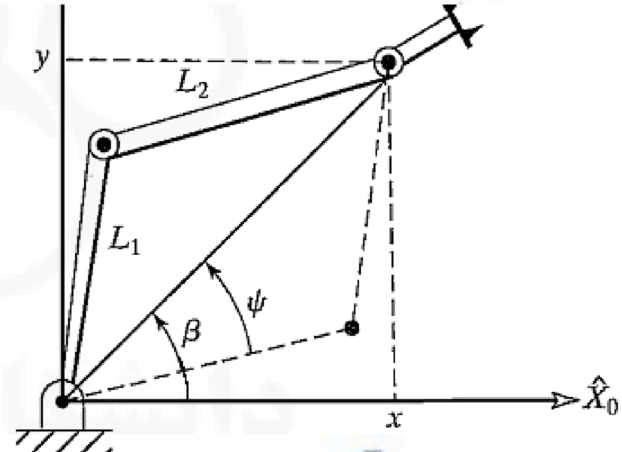
Algebraic & Geometric

- **Geometric Solution**

Elbow down

Law of Cosines: $x^2+y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \underbrace{\cos(\pi-\theta_2)}_{-\cos\theta_2}$

$$\Rightarrow \quad C_2 = \frac{x^2+y^2-l_1^2-l_2^2}{2 l_1 l_2} \longrightarrow \boxed{\theta_2 \checkmark}$$



★ وجود پاسخ چک شود

★ $0 < \theta_2 < \pi$ ـــــ چون برای cos کسینوس دائم حل می‌کنیم باید $0 < \theta_2 < \pi$ باشد

★ جواب دیگر $\theta_2' = -\theta_2$

Algebraic & Geometric

Geometric Solution

$$\beta = \text{Atan2}(y, x)$$

$$l_2^2 = l_1^2 + (x^2+y^2) - 2 l_1 \sqrt{x^2+y^2}\,\cos\psi$$

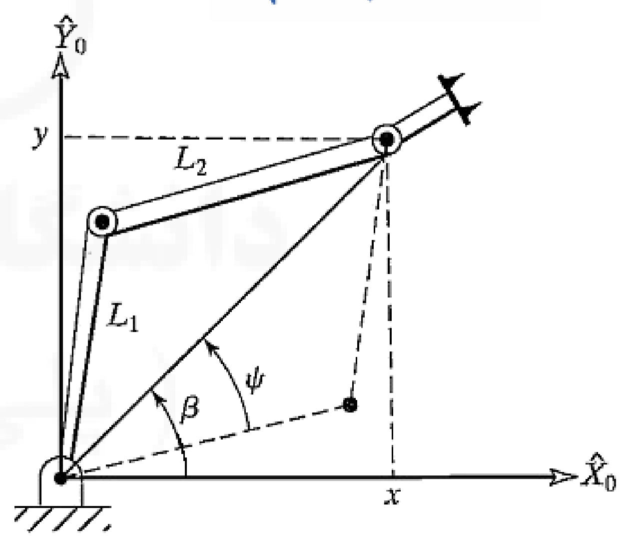
$$\Rightarrow \cos\psi = \frac{l_1^2 + x^2 + y^2 - l_2^2}{2 l_1 \sqrt{x^2+y^2}} \rightarrow \boxed{\psi \checkmark}$$

$$0 \le \psi \le \pi \quad \star$$

$$\boxed{\theta_1 = \beta \pm \psi}$$

elbow up $\approx \theta_2 < 0$

elbow down $\approx \theta_2 > 0$

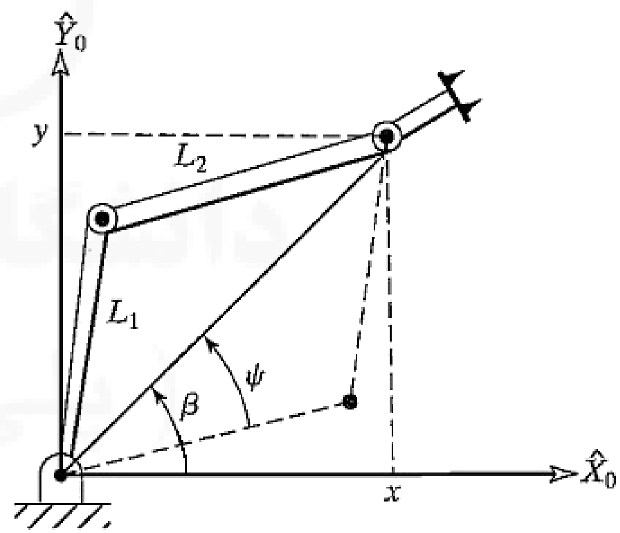


Algebraic & Geometric

❑ Geometric Solution

درنهایت

$$\theta_1+\theta_2+\theta_3=\varphi \Rightarrow \boxed{\theta_3=\varphi-[\theta_1+\theta_2]}$$



Algebraic & Geometric

Algebraic Solution by Reduction to Polynomial

- Transcendental equations are often difficult to solve.
- Even when there is only one variable (θ), it generally appears as $sin\,\theta$ and $cos\,\theta$.
- Change of variable using the tangent of half-angle.
- Convert transcendental equations into polynomial equations in u .
- Polynomials up to degree four possess closed-form solutions.

$$u = tan\frac{\theta}{2}$$

$$cos\,\theta = \frac{1-u^2}{1+u^2}$$

$$sin\,\theta = \frac{2u}{1+u^2}$$

Algebraic & Geometric

❑ Algebraic Solution by Reduction to Polynomial

$$a\cos\theta + b\sin\theta = c$$

$u = tg\frac{\theta}{2}$

$$a\left(\frac{1-u^2}{1+u^2}\right) + b\,\frac{2u}{1+u^2} = c \quad \Rightarrow \quad a(1-u^2) + 2bu = c(1+u^2)$$

$$\Rightarrow \quad (a+c)u^2 - 2bu + (c-a) = 0$$ (چندجمله‌ای)

$$u = \frac{b \pm \sqrt{b^2+a^2-c^2}}{a+c} \quad \Rightarrow \quad \theta = 2\,tg^{-1}\left\{\frac{b \pm \sqrt{b^2+a^2-c^2}}{a+c}\right\}$$

★ اگر u مختلط شود، پاسخ حقیقی برای معادله غیرجبری اولیه وجود ندارد

Algebraic & Geometric

❑ Algebraic Solution by Reduction to Polynomial

$$u = \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a+c} \Rightarrow \theta = 2\,tg^{-1}\left\{\frac{b \pm \sqrt{b^2+a^2-c^2}}{a+c}\right\}$$

$$\text{if } (a+c) = 0 \quad \begin{cases} u = \frac{0}{0} \\ u = \infty = \frac{2b}{0} \Rightarrow \theta = 180^\circ \end{cases}$$

* اگر $a+c=0$ ← یکی از جوابها 180° است جواب دیگر نیز پاسخ مهم است ←

- ....

Algebraic & Geometric

❑ Algebraic Solution by Reduction to Polynomial

$$a\cos\theta + b\sin\theta = c$$

روش دوم حل :

این روش در مورد تغییر چندجمله‌ای نیست ولی راه حل موجود برای یک [illegible] لازم است

$$a\cos\theta + b\sin\theta = c$$

$$a\cos\theta + b\sin\theta = \rho\cos(\theta-\psi) \qquad \rho = \sqrt{a^2+b^2}$$

$$\psi = tg^{-1}\,b/a$$

$$\Rightarrow \quad c = \rho\cos(\theta-\psi) \quad \Rightarrow \quad \cos(\theta-\psi) = \frac{c}{\rho} \quad \text{I}$$

$$\rightarrow \quad \sin(\theta-\psi) = \pm\sqrt{1-\frac{c^2}{\rho^2}} \quad \text{II}$$

$$\text{I}, \text{II} \Rightarrow \quad \theta-\psi = \operatorname{Atan2}\left(\pm\sqrt{1-\frac{c^2}{\rho^2}}, \frac{c}{\rho}\right)$$

$$\theta = \operatorname{Atan2}\left(\pm\sqrt{1-\frac{c^2}{\rho^2}}, \frac{c}{\rho}\right) + tg^{-1}\,b/a$$

Pieper's Solution

- Although a completely general robot with six degrees of freedom does not have a closed-form solution, certain important special cases can be solved.

- **Pieper** studied manipulators with six degrees of freedom in which three consecutive axes intersect at a point (Included three consecutive parallel axes).

- This section outlines the method he developed for the case of all six joints revolute, with the last three axes intersecting (**Decoupled Manipulator**).

- It can be also applied to prismatic joints, as well to many industrial robot.

Pieper's Solution

- Consider a 6 DOF manipulator with all 6 joint revolute and the last three intersect (such as PUMA).

- The origin of frames {4}, {5}, {6} located at the point of intersection.

$$ {}^{0}P_{4org} = {}^{0}T_1 \, {}^{1}T_2 \, {}^{2}T_3 \, {}^{3}P_{4org} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} $$

- ${}^{3}P_{4org}$ is in fact the position part of ${}^{3}T_4$

$$ {}^{i-1}T_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}P_{4org} = {}^{0}T_1 \, {}^{1}T_2 \, {}^{2}T_3 \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} $$

Pieper's Solution

$$^{0}P_{4org} = {}^{0}T_1 \, {}^{1}T_2 \, {}^{2}T_3 \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix}$$

- Substituting for $^{2}T_3$:

$$^{0}P_{4org} = {}^{0}T_1 \, {}^{1}T_2 \, {}^{2}T_3 \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} = {}^{0}T_1 \, {}^{1}T_2 \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} f_1 &= a_3 c_3 + d_4 s\alpha_3 s_3 + a_2, \\ f_2 &= a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2, \\ f_3 &= a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2. \end{aligned}}$$

Pieper's Solution

- Similarly, substituting for ${}^1T_2$:

$$ {}^0P_{4org} = {}^0T_1\,{}^1T_2 \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^0T_1 \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ 1 \end{bmatrix} $$

$$ \boxed{\begin{aligned} g_1 &= c_2 f_1 - s_2 f_2 + a_1 \\ g_2 &= s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 \\ g_3 &= s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 \end{aligned}} $$

- Similarly, substituting for ${}^0T_1$:

$$ {}^0P_{4org} = {}^0T_1 \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

Pieper's Solution

- Consider:

$$^0P_{4org} = \begin{bmatrix} c_1g_1 - s_1g_2 \\ s_1g_1 + c_1g_2 \\ g_3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} g_1 &= c_2f_1 - s_2f_2 + a_1 \\ g_2 &= s_2c\alpha_1f_1 + c_2c\alpha_1f_2 - s\alpha_1f_3 - d_2s\alpha_1 \\ g_3 &= s_2s\alpha_1f_1 + c_2s\alpha_1f_2 + c\alpha_1f_3 + d_2c\alpha_1 \end{aligned}}$$

- Consider the squared norm of $^0P_{4org}$:
$r = x^2 + y^2 + z^2$

- Also,

$$\begin{aligned} r &= g_1^2 + g_2^2 + g_3^2 \\ r &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 + 2a_1(c_2f_1 - s_2f_2) \end{aligned}$$

- Take r and z simultaneously into account:

$$\boxed{\begin{aligned} r &= (k_1c_2 + k_2s_2)2a_1 + k_3 \\ z &= (k_1s_2 - k_2c_2)s\alpha_1 + k_4 \end{aligned}}$$

$$\boxed{\begin{aligned} k_1 &= f_1 \\ k_2 &= -f_2 \\ k_3 &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 \\ k_4 &= f_3c\alpha_1 + d_2c\alpha_1 \end{aligned}}$$

Pieper's Solution

$$\begin{aligned} r &= (k_1 c_2 + k_2 s_2) 2a_1 + k_3 \\ z &= (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4 \end{aligned}$$

$$\begin{aligned} k_1 &= f_1 \\ k_2 &= -f_2 \\ k_3 &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 \\ k_4 &= f_3 c\alpha_1 + d_2 c\alpha_1 \end{aligned}$$

- Consider the solution for θ_3 :

1) If $a_1 = 0$, then $r = k_3$, where r is known. The right-hand side (k_3) is a function of θ_3 only. A polynomial approach can be used for θ_3 .

2) If $s\alpha_1 = 0$, then $z = k_4$, where z is known. The right-hand side (k_4) is a function of θ_3 only. A polynomial approach can be used for θ_3 .

3) Otherwise, eliminate s_2 and c_2

$$\frac{(r - k_3)^2}{4a_1^2} + \frac{(z - k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2$$

➢ A polynomial approach (4th order) can be used for θ_3 .

$$f_1^2 + f_2^2 + f_3^2 = a_3^2 + d_4^2 + d_3^2 + a_2^2 + 2d_4 d_3 c\alpha_3 + 2a_2 a_3 c_3 + 2a_2 d_4 s\alpha_3 s_3$$

Pieper's Solution

- By finding θ_3, θ_2 can be found.

$$\boxed{\begin{aligned} r &= (k_1c_2 + k_2s_2)2a_1 + k_3 \\ z &= (k_1s_2 - k_2c_2)s\alpha_1 + k_4 \end{aligned}}$$

$$\boxed{\begin{aligned} k_1 &= f_1 \\ k_2 &= -f_2 \\ k_3 &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 \\ k_4 &= f_3c\alpha_1 + d_2c\alpha_1 \end{aligned}}$$

- By finding θ_3 & θ_2, θ_1 can be found

$$\begin{aligned} {}^0P_{4org} &= \begin{bmatrix} c_1g_1 - s_1g_2 \\ s_1g_1 + c_1g_2 \\ g_3 \\ 1 \end{bmatrix} \\ g_1 &= c_2f_1 - s_2f_2 + a_1 \\ g_2 &= s_2c\alpha_1f_1 + c_2c\alpha_1f_2 - s\alpha_1f_3 - d_2s\alpha_1 \\ g_3 &= s_2s\alpha_1f_1 + c_2s\alpha_1f_2 + c\alpha_1f_3 + d_2c\alpha_1 \end{aligned}$$

Pieper's Solution

- To complete the IK problem, θ_4, θ_5 & θ_6 needs to be found.

- **3 Methods** can be applied:

Pieper's Solution

- **1st Method:**

$$ {}^{0}R_{6}\Big|_{Numerical} = {}^{0}R_{6}\Big|_{Parametric} $$

- Having found θ_1, θ_2 & θ_3, we can find ${}^{0}R_{6}|_{Parametric}$ based on θ_4, θ_5 & θ_6.

- It should be solved by algebraic methods to find θ_4, θ_5 & θ_6 .

Pieper's Solution

- **2nd Method:**

- Since they intersect, they just affect the orientation.
- So, they can be obtained from thorientation part.

$$ {}^{0}R_6 = {}^{0}R_3 \; {}^{3}R_6 $$

- Having found θ_1, θ_2 & θ_3, we can find ${}^{0}R_3$ and hence ${}^{3}R_6$:

$$ {}^{3}R_6 = {}^{0}R_3{}^{-1} \; {}^{0}R_6 $$

$$ {}^{3}R_6 = {}^{3}R_0 \; {}^{0}R_6 $$

- It should be solved by algebraic methods to find θ_4, θ_5 & θ_6 .

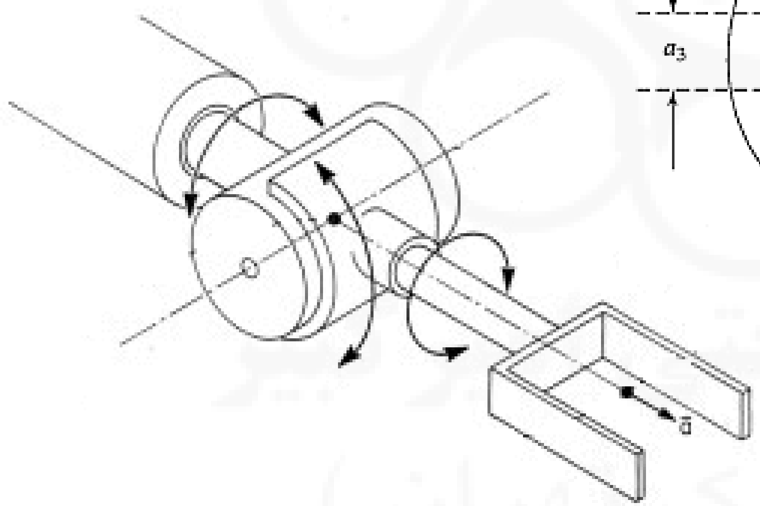
$$ {}^{3}R_6\Big|_{Numerical} = {}^{3}R_6\Big|_{Parametric} $$

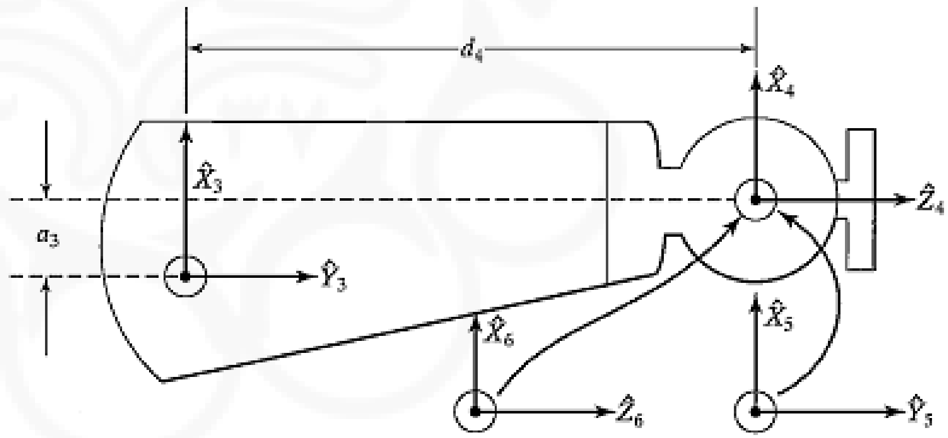
Pieper's Solution

- **3rd Method:**

➢ For any manipulator, a set of proper Euler angles can be defined for θ_4, θ_5 & θ_6 .

➢ For many manipulators, the last three joints can be solved by ZYZ Euler angle.





Pieper's Solution

- **3rd Method:**

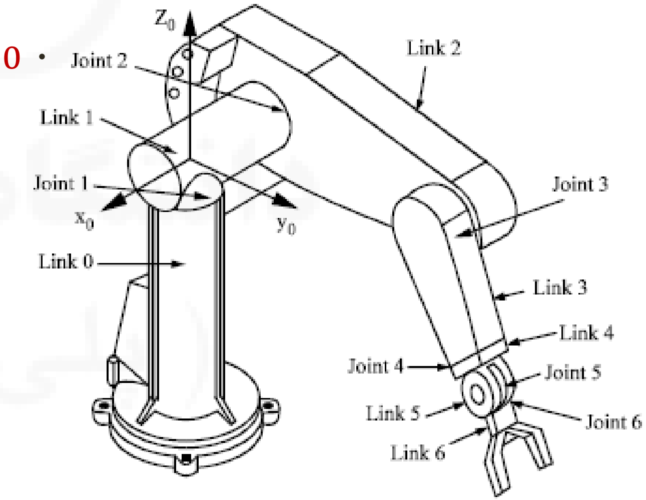
- Assume ${}^{0}R_6|_{Numerical}$ is given.
- Regarding the forward kinematic problem:

$$ {}^{0}R_6 = {}^{0}R_4\ {}^{4}R_6 $$

$$ {}^{0}R_6 = {}^{0}R_4\Big|_{\theta_4=0}\ {}^{4}R_6 $$

$$ {}^{4}R_6 = {}^{0}R_4^{\ -1}\Big|_{\theta_4=0}\ {}^{0}R_6 $$

- Having found θ_1, θ_2 & θ_3, we can find ${}^{4}R_0|_{\theta_4=0}$.
- Therefore, ${}^{4}R_6|_{Numerical}$ can be achieved.



Pieper's Solution

- **3rd Method:**

➢ If ${}^4R_6$ can be found by a set of ZYZ Euler angles, they will be exactly θ_4, θ_5 & θ_6 .

$$ {}^4R_6\Big|_{Numerical} = R_{ZYZ}(\alpha,\beta,\gamma) $$

$$ R_{ZYZ}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha\, c\beta\, c\gamma - s\alpha\, s\gamma & -c\alpha\, c\beta\, s\gamma - s\alpha\, c\gamma & c\alpha\, s\beta \\ s\alpha\, c\beta\, c\gamma + c\alpha\, s\gamma & -s\alpha\, c\beta\, s\gamma + c\alpha\, c\gamma & s\alpha\, s\beta \\ -s\beta\, c\gamma & s\beta\, s\gamma & c\beta \end{bmatrix} $$

$$ \alpha = \theta_4 $$

$$ \beta = \theta_5 $$

$$ \gamma = \theta_6 $$

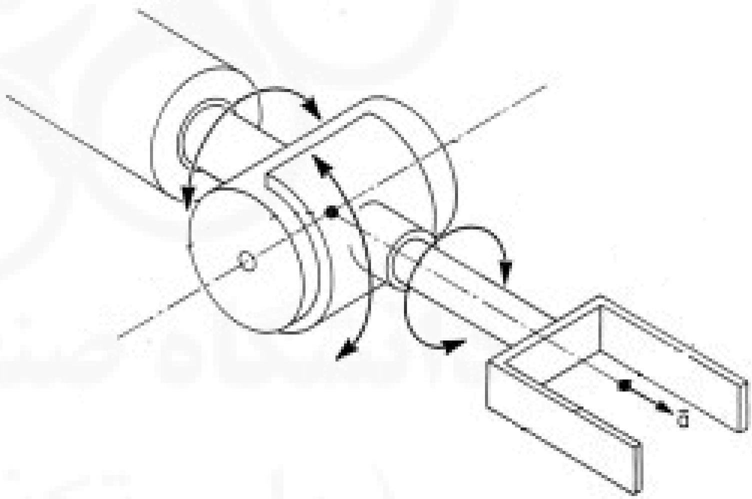
Pieper's Solution

- **Note:** Two solutions always exist for these last three joints, so the total number of solutions will be **twice** the number found for the first three joints.

$$\theta_4' = \theta_4 + 180°$$

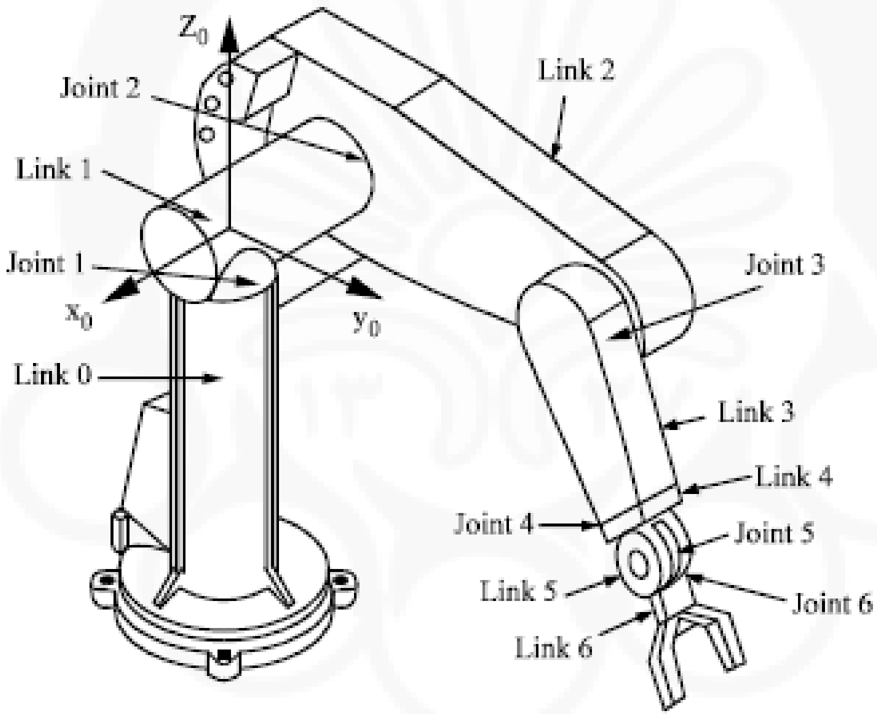
$$\theta_5' = -\theta_5,$$

$$\theta_6' = \theta_6 + 180°$$



Example of Inverse Manipulator Kinematics

- **The Unimation PUMA 560**



- The manipulator solution is done purely algebraically.
- *Pieper's method* can be also used, but we choose an alternative solution (various available methods).

Example of Inverse Manipulator Kinematics

❑ Position $(\theta_1, \theta_2, \theta_3)$

- **Objective:** Solve for θ_i when ${}^0T_6$ is given as numeric values.

$$
\begin{aligned}
{}^0T_6 &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= {}^0T_1(\theta_1)\,{}^1T_2(\theta_2)\,{}^2T_3(\theta_3)\,{}^3T_4(\theta_4)\,{}^4T_5(\theta_5)\,{}^5T_6(\theta_6)
\end{aligned}
$$

- Put the dependence on θ_1 on the left-hand side of the equation by inverting ${}^0T_1$.

$$
[{}^0T_1(\theta_1)]^{-1}\,{}^0T_6 = {}^1T_2(\theta_2)\,{}^2T_3(\theta_3)\,{}^3T_4(\theta_4)\,{}^4T_5(\theta_5)\,{}^5T_6(\theta_6)
$$

$$
\therefore \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1T_6 \qquad \textbf{(1)}
$$

- The advantage is in separating out variables in the search for a solvable equation.

Example of Inverse Manipulator Kinematics

- Equating the (2,4) elements from both sides

$$[{}^{0}T_{1}(\theta_1)]^{-1}\,{}^{0}T_{6} = {}^{1}T_{2}(\theta_2)\,{}^{2}T_{3}(\theta_3)\,{}^{3}T_{4}(\theta_4)\,{}^{4}T_{5}(\theta_5)\,{}^{5}T_{6}(\theta_6)$$

$$\therefore \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{1}T_{6}$$

$${}^{1}T_{6} = {}^{1}T_{3}\,{}^{3}T_{6} = \begin{bmatrix} {}^{1}r_{11} & {}^{1}r_{12} & {}^{1}r_{13} & {}^{1}p_x \\ {}^{1}r_{21} & {}^{1}r_{22} & {}^{1}r_{23} & {}^{1}p_y \\ {}^{1}r_{31} & {}^{1}r_{32} & {}^{1}r_{33} & {}^{1}p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
{}^{1}r_{11} &= c_{23}\left(c_4c_5c_6 - s_4s_6\right) - s_{23}s_5s_6 \\
{}^{1}r_{21} &= -s_4c_5c_6 - c_4s_6 \\
{}^{1}r_{31} &= -s_{23}\left(c_4c_5c_6 - s_4s_6\right) - c_{23}s_5c_6 \\
{}^{1}r_{12} &= -c_{23}\left(c_4c_5s_6 + s_4c_6\right) + s_{23}s_5s_6 \\
{}^{1}r_{22} &= s_4c_5s_6 - c_4c_6 \\
{}^{1}r_{32} &= s_{23}\left(c_4c_5c_6 + s_4c_6\right) + c_{23}s_5s_6 \\
{}^{1}r_{13} &= -c_{23}c_4s_5 - s_{23}c_5 \\
{}^{1}r_{23} &= s_4s_5 \\
{}^{1}r_{33} &= s_{23}c_4s_5 - c_{23}c_5 \\
{}^{1}p_x &= a_2c_2 + a_3c_{23} - d_4s_{23} \\
{}^{1}p_y &= d_3 \\
{}^{1}p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}
\end{aligned}$$

Example of Inverse Manipulator Kinematics

- Equating the (2,4) elements from both sides:

$$-s_1 p_x + c_1 p_y = d_3 \qquad \textbf{(2)}$$

- To solve, make the polar substitutions:

$$\begin{aligned} p_x &= \rho \cos\phi, \quad p_y = \rho \sin\phi \\ \rho &= \sqrt{p_x^2 + p_y^2}, \quad \phi = Atan2(p_x, p_y) \end{aligned}$$

- Therefore

$$\begin{aligned} c_1 s_\phi - s_1 c_\phi &= \frac{d_3}{\rho} \\ \sin(\phi - \theta_1) &= \frac{d_3}{\rho} \\ \cos(\phi - \theta_1) &= \pm\sqrt{1 - \frac{d_3^2}{\rho^2}} \\ \phi - \theta_1 &= Atan2(\frac{d_3}{\rho}, \pm\sqrt{1 - \frac{d_3^2}{\rho^2}}) \\ \theta_1 &= Atan2(p_y, p_x) - Atan2(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}) \end{aligned}$$

- Two possible solutions for θ_1.

Example of Inverse Manipulator Kinematics

- Now that θ_1 is known, the left-hand side of (1) is known.

- Equate both the (1,4) and (3,4) elements:

$$\begin{aligned} c_1 p_x + s_1 p_y &= a_3 c_{23} - d_4 s_{23} + a_2 c_2 \\ -p_z &= a_3 s_{23} + d_4 c_{23} + a_2 s_2 \end{aligned}$$

- Square equations and add the resulting equations:

$$\begin{aligned} a_3 c_3 - d_4 s_3 &= K \\ K &= \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2} \end{aligned} \qquad \textbf{(3)}$$

- Dependence on θ_2 has been removed.

- The Equation (3) is of the same form as (2).

$$\theta_3 = Atan2(a_3, d_4) - Atan2(K, \pm\sqrt{a_3^2 + d_4^2 - K^2})$$

- Two different solutions for θ_3.

Example of Inverse Manipulator Kinematics

- Rewrite (1) so that all the left-hand side is a function of only knowns and θ_2 .

$$[{}^0T_3(\theta_2)]^{-1}\,{}^0T_6 = {}^3T_4(\theta_4)\,{}^4T_5(\theta_5)\,{}^5T_6(\theta_6)$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3T_6 \qquad \textbf{(4)}$$

- Equating both the (1,4) and (2,4) elements:

$$\begin{aligned} c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 &= a_3 \\ -c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 &= d_4 \end{aligned}$$

- These equations can be solved simultaneously for s_{23} and c_{23} .

$$\begin{aligned} s_{23} &= \frac{(-a_3 - a_2c_3)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4)}{p_z^2 + (c_1p_x + s_1p_y)^2} \\ c_{23} &= \frac{(a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)}{p_z^2 + (c_1p_x + s_1p_y)^2} \end{aligned}$$

Example of Inverse Manipulator Kinematics

$$
\begin{aligned}
s_{23} &= \frac{(-a_3 - a_2c_3)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4)}{p_z^2 + (c_1p_x + s_1p_y)^2} \\
c_{23} &= \frac{(a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)}{p_z^2 + (c_1p_x + s_1p_y)^2}
\end{aligned}
$$

- The denominators are equal and positive, so

$$
\begin{aligned}
\theta_{23} = \; & Atan2[(-a_3 - a_2c_3)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4) \\
& (a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)]
\end{aligned}
$$

- Four values of θ_{23}, according to the four possible combinations of solutions for θ_1 and θ_3.

- Four possible solutions for θ_2.

$$\theta_2 = \theta_{23} - \theta_3$$

← Outline

Example of Inverse Manipulator Kinematics

- **Orientation ($\theta_4, \theta_5, \theta_6$)**

- The entire left side of (4) is known. Equating both the (1,3) and (3,3) elements:

$$\begin{aligned} r_{13}c_1c_{23} + r_{23}s_1c_{23} - r_{33}s_{23} &= -c_4s_5 \\ -r_{13}s_1 + r_{23}c_1 &= s_4s_5. \end{aligned}$$

- As long as $s_5 \neq 0$,

$$\theta_4 = Atan2(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23})$$

- When $\theta_5 = 0$, the manipulator is in a **singular** configuration (Joint axes 4 and 6 line up).

- All that can be solved is the sum or difference of θ_4 and θ_6.

- This situation is detected by checking whether both arguments of the $Atan2$ in are near zero.

- Then, θ_4 is chosen arbitrarily, and θ_6 will be computed accordingly.

Example of Inverse Manipulator Kinematics

- Rewrite (4) so that all the left-hand side is a function of only knowns and θ_4.

$$[{}^0_4T(\theta_4)]^{-1}\,{}^0_6T = {}^4_5T(\theta_5)\,{}^5_6T(\theta_6)$$

(5)

$$[{}^0_4T(\theta_4)]^{-1} = \begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Equating both the (1,3) and (3,3) elements

$$\begin{aligned} r_{13}(c_1c_{23}c_4 + s_1s_4) + r_{23}(s_1c_{23}c_4 - c_1s_4) - r_{33}(s_{23}c_4) &= -s_5 \\ r_{13}(-c_1s_{23}) + r_{23}(-s_1s_{23}) + r_{33}(-c_{23}) &= c_5 \end{aligned}$$

- θ_5 can be solved

$$\theta_5 = Atan2(s_5, c_5)$$

Example of Inverse Manipulator Kinematics

- Compute ${}^{1}T_5^{-1}$ and write (5) in the form:

$$({}^0_5T)^{-1}{}^0_6T = {}^5_6T(\theta_6)$$

- Equating both the (3,1) and (1,1) elements:

$$\begin{aligned}
\theta_6 &= Atan2(s_6, c_6) \\
s_6 &= -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_31(s_{23}s_4) \\
c_6 &= -r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 \\
&- c_1s_4)c_5 - s_1s_{23}s_5] - r_31(s_{23}c_4c_5 + c_{23}s_5)
\end{aligned}$$

- There are four more solutions obtained by "**flipping**" the wrist (flipped solution).

$$\theta_4' = \theta_4 + 180°$$

$$\theta_5' = -\theta_5,$$

$$\theta_6' = \theta_6 + 180°$$

- All eight solutions have been computed.

The END

- **References:**

1) .