دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

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Lecture 2:
Spatial Descriptions and Transformations

Advanced Robotics

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Outlines

- Descriptions: Positions, Orientations and Frames
- Mappings: Changing Description from Frame to Frame
- Operators
- Transformation Arithmetic
- More on Representation of Orientation
  - Euler Angles
  - Fixed Angles
  - Equivalent Angle-Axis
  - Euler Parameters
- Transformation of Free Vectors

Descriptions: Positions, Orientations and Frames

- To define position and orientation, we must define coordinate systems and conventions for its representation.

- There is a **universe coordinate system** to which everything can be referenced.

- A description is used to specify attributes of various objects.

- Objects are parts, tools, and the manipulator itself.

- **Description:**
  - Positions
  - Orientations
  - An entity that contains both of these descriptions: the **frame**

Descriptions: Positions, Orientations and Frames

Description of a Position

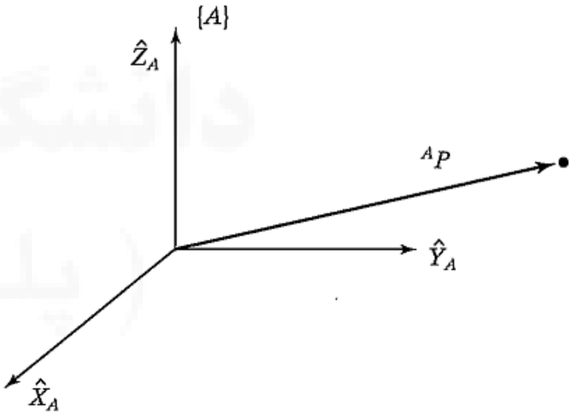
- Once a coordinate system is established, locate any point with a 3×1 position vector.

- Vectors must be tagged identifying which coordinate system they are defined within, e,g, ${}^{A}P$.

- ${}^{A}P$ have numerical values that indicate distances along the axes of {A} (Projection).

$$
{}^{A}P = \begin{bmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{bmatrix}
$$

- ${}^{A}p_x = {}^{A}P \,.\, \hat{X}_A$
- ${}^{A}p_y = {}^{A}P \,.\, \hat{Y}_A$
- ${}^{A}p_z = {}^{A}P \,.\, \hat{Z}_A$



Descriptions: Positions, Orientations and Frames

❑ Description of a Position

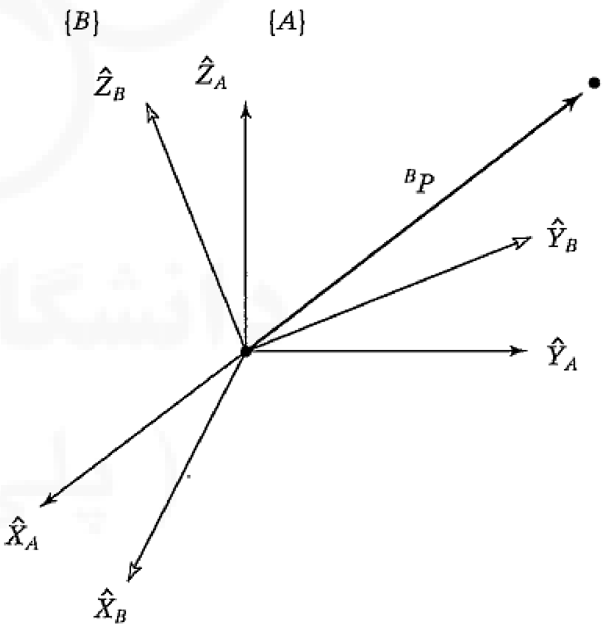
- **Vector Norm:**

$$\|{}^{A}P\| = ({}^{A}P\,.\,{}^{A}P)^{1/2} = \left({}^{A}p_x{}^{2} + {}^{A}p_y{}^{2} + {}^{A}p_z{}^{2}\right)^{1/2}$$

- It is invariant of the frame.

$$\|{}^{A}P\| = ({}^{A}P\,.\,{}^{A}P)^{1/2} = ({}^{B}P\,.\,{}^{B}P)^{1/2} = \|{}^{B}P\|$$

❖ ***Q:*** Condition ?



Descriptions: Positions, Orientations and Frames

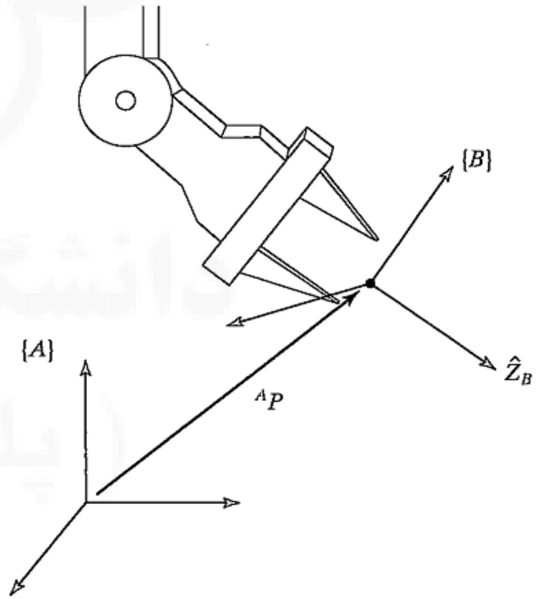
Description of an Orientation

- **Positions of points** are described with **vectors** and **orientations of bodies** are described with an attached **coordinate system.**

- Attach a coordinate system to the body and then give a description of this coordinate system relative to the reference system.

- One way to describe the coordinate system {B}:
  - Write its unit vectors $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$ in terms of the coordinate system {A}.

- They are called ${}^A\hat{X}_B$, ${}^A\hat{Y}_B$, ${}^A\hat{Z}_B$.



Descriptions: Positions, Orientations and Frames

- **Description of an Orientation**

- Each component is the dot product of a pair of unit vectors.

$$^{A}\hat{X}_B = \begin{bmatrix} \hat{X}_B.\hat{X}_A \\ \hat{X}_B.\hat{Y}_A \\ \hat{X}_B.\hat{Z}_A \end{bmatrix} = \begin{bmatrix} Cos\left(\hat{X}_B,\hat{X}_A\right) \\ Cos\left(\hat{X}_B,\hat{Y}_A\right) \\ Cos\left(\hat{X}_B,\hat{Z}_A\right) \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

$$^{A}\hat{Y}_B = \begin{bmatrix} \hat{Y}_B.\hat{X}_A \\ \hat{Y}_B.\hat{Y}_A \\ \hat{Y}_B.\hat{Z}_A \end{bmatrix} = \begin{bmatrix} Cos\left(\hat{Y}_B,\hat{X}_A\right) \\ Cos\left(\hat{Y}_B,\hat{Y}_A\right) \\ Cos\left(\hat{Y}_B,\hat{Z}_A\right) \end{bmatrix} = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix}$$

$$^{A}\hat{Z}_B = \begin{bmatrix} \hat{Z}_B.\hat{X}_A \\ \hat{Z}_B.\hat{Y}_A \\ \hat{Z}_B.\hat{Z}_A \end{bmatrix} = \begin{bmatrix} Cos\left(\hat{Z}_B,\hat{X}_A\right) \\ Cos\left(\hat{Z}_B,\hat{Y}_A\right) \\ Cos\left(\hat{Z}_B,\hat{Z}_A\right) \end{bmatrix} = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$$

- Stack these three unit vectors together as the columns of a 3×3 matrix.

$$^{A}R_B = \left[^{A}\hat{X}_B \quad ^{A}\hat{Y}_B \quad ^{A}\hat{Z}_B\right] = \begin{bmatrix} \hat{X}_B.\hat{X}_A & \hat{Y}_B.\hat{X}_A & \hat{Z}_B.\hat{X}_A \\ \hat{X}_B.\hat{Y}_A & \hat{Y}_B.\hat{Y}_A & \hat{Z}_B.\hat{Y}_A \\ \hat{X}_B.\hat{Z}_A & \hat{Y}_B.\hat{Z}_A & \hat{Z}_B.\hat{Z}_A \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- It is called **Rotation Matrix** (**R**).
- Components of rotation matrices are direction cosines.

Descriptions: Positions, Orientations and Frames

❑ Properties of Rotation Matrix

1) **Orthonormal**

- The columns all have unit magnitude, and, these unit vectors are orthogonal.

$$\left\| {}^{A}\hat{X}_{B} \right\| = \left\| {}^{A}\hat{Y}_{B} \right\| = \left\| {}^{A}\hat{Z}_{B} \right\| = 1$$

$${}^{A}\hat{X}_{B} \,.\, {}^{A}\hat{Y}_{B} = {}^{A}\hat{Y}_{B} \,.\, {}^{A}\hat{Z}_{B} = {}^{A}\hat{X}_{B} \,.\, {}^{A}\hat{Z}_{B} = 0$$

2) The rows of the matrix are the unit vectors of {A} expressed in {B}.

$${}^{A}R_{B} = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} \hat{X}_{B}.\hat{X}_{A} & \hat{Y}_{B}.\hat{X}_{A} & \hat{Z}_{B}.\hat{X}_{A} \\ \hat{X}_{B}.\hat{Y}_{A} & \hat{Y}_{B}.\hat{Y}_{A} & \hat{Z}_{B}.\hat{Y}_{A} \\ \hat{X}_{B}.\hat{Z}_{A} & \hat{Y}_{B}.\hat{Z}_{A} & \hat{Z}_{B}.\hat{Z}_{A} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{bmatrix}$$

$$= \begin{bmatrix} {}^{B}\hat{X}_{A} & {}^{B}\hat{Y}_{A} & {}^{B}\hat{Z}_{A} \end{bmatrix}^{T} = {}^{B}R_{A}^{T}$$

- The description of frame {A} relative to {B} (${}^{B}R_{A}$) is the transpose of ${}^{A}R_{B}$.

$${}^{B}R_{A} = {}^{A}R_{B}^{T}$$

Descriptions: Positions, Orientations and Frames

❑ Properties of Rotation Matrix

3) From linear algebra, the inverse of a orthonormal matrix is equal to its transpose.

$$ {}^{A}R_{B}^{T} = {}^{A}R_{B}^{-1} $$

- To demonstrate:

$$ {}^{A}R_{B}^{T}\;{}^{A}R_{B} = \begin{bmatrix} {}^{A}\hat{X}_{B}^{T} \\ {}^{A}\hat{Y}_{B}^{T} \\ {}^{A}\hat{Z}_{B}^{T} \end{bmatrix} \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = I_3 $$

- where I_3 is the 3×3 identity matrix.

4)

$$ det({}^{A}R_{B}) = 1 $$

❖ ***Q:*** Why?

5) It can be expressed by Only 3 independent numbers.

❖ ***Q:*** Why

Descriptions: Positions, Orientations and Frames

Description of a Frame

- For a frame, both position and orientation should be determined.

- For convenience, the point whose position described is chosen as the **origin** of the body-attached frame.

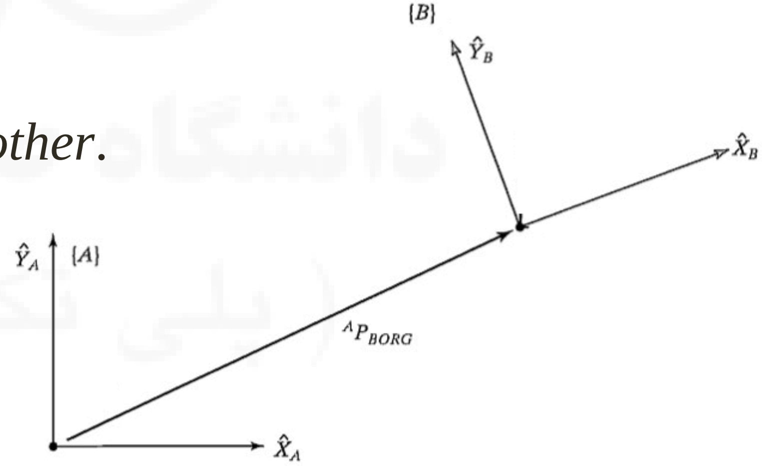
- The description of a frame: a **position vector** and a **rotation matrix.**

- Frame {B} is described by ${}^{A}R_{B}$ and ${}^{A}P_{BORG}$.

$$\{B\} = \{{}^{A}R_{B}\,, {}^{A}P_{BORG}\}$$

- ${}^{A}P_{BORG}$ is the vector that locates the origin of the frame {B}.

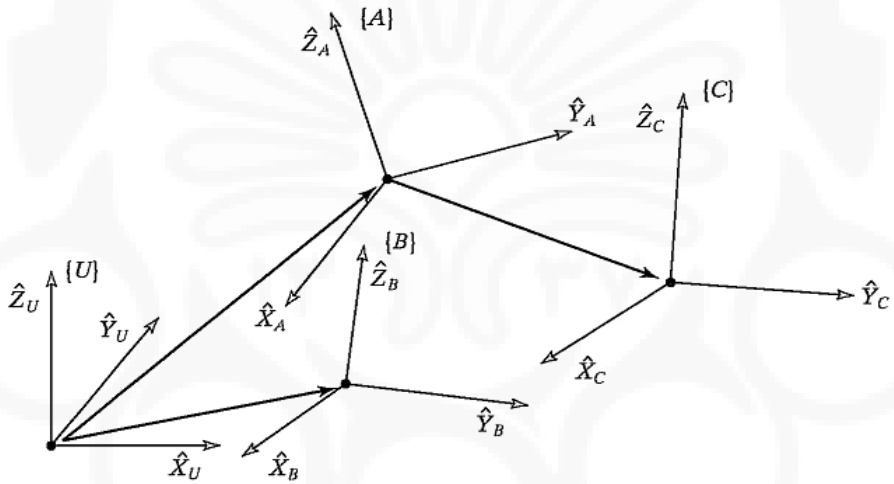
- A frame can be used as a *description of one coordinate system relative to another*.



Descriptions: Positions, Orientations and Frames

❑ Description of a Frame

- Frames {A} and {B} are known relative to the universe frame.
- Frame {C} is known relative to frame {A}.



- **Position** and **orientation** can be represented as frames:
  ➢**Position:** a frame with identity rotation-matrix and position-vector which locates the point
  ➢**Orientation:** a frame whose position-vector was the zero vector.

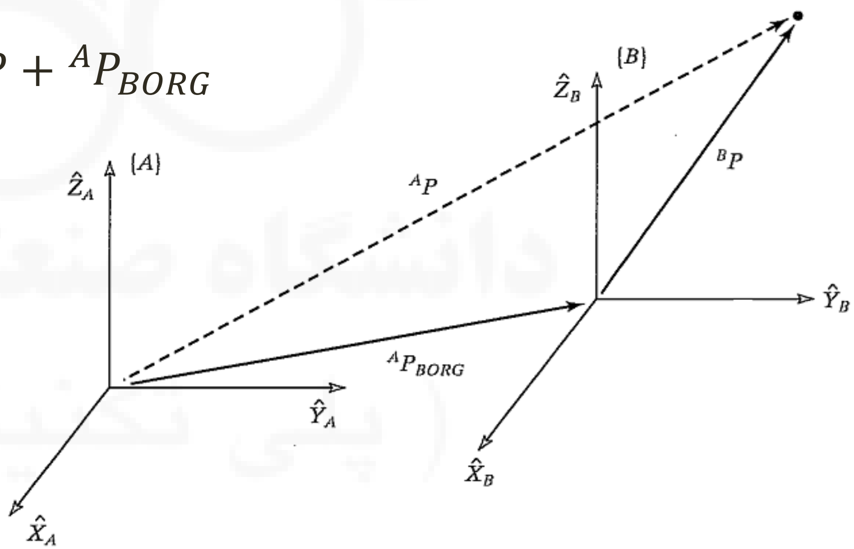
Mappings: Changing Description from Frame to Frame

- **Mapping** between frames can be done by:
  - ➢Translation
  - ➢Rotation
  - ➢General Transformation

❑ Translation

- A position defined by the vector ${}^{B}P$.
- {A} has the same orientation as {B}
- {A} differs only by a translation, i.e. ${}^{A}P_{BORG}$.
- Express this point in space in terms of frame {A}, i.e. ${}^{A}P$.

$$ {}^{A}P = {}^{B}P + {}^{A}P_{BORG} $$



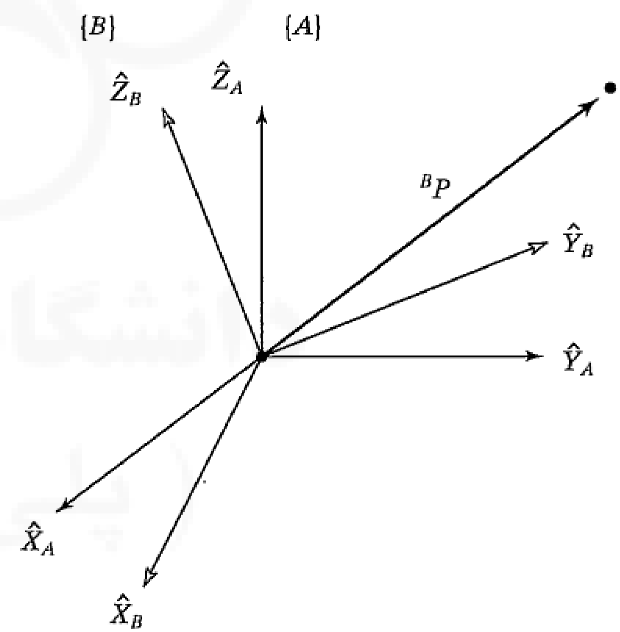
Mappings: Changing Description from Frame to Frame

❑ Rotation

- A position defined by the vector ${}^{B}P$.
- The orientation of {B} is known relative to {A} i.e. ${}^{A}R_{B}$.
- The origins of the two frames are coincident.
- Express this point in space in terms of frame {A}, i.e. ${}^{A}P$.

- The components of ${}^{A}P$ may be calculated by the projection as:

$$ {}^{A}P = \begin{bmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{bmatrix} $$



- ${}^{A}p_x = {}^{B}\hat{X}_A . {}^{B}P$
- ${}^{A}p_y = {}^{B}\hat{Y}_A . {}^{B}P$
- ${}^{A}p_z = {}^{B}\hat{Z}_A . {}^{B}P$

❖ ***Q:*** Why?

Mappings: Frame to Frame

❑ Rotation

$$^{A}P = \begin{bmatrix} ^{A}p_x \\ ^{A}p_y \\ ^{A}p_z \end{bmatrix}$$

- $^{A}p_x = {}^{B}\hat{X}_A . {}^{B}P$
- $^{A}p_y = {}^{B}\hat{Y}_A . {}^{B}P$
- $^{A}p_z = {}^{B}\hat{Z}_A . {}^{B}P$

- Expressing in the matrix form:

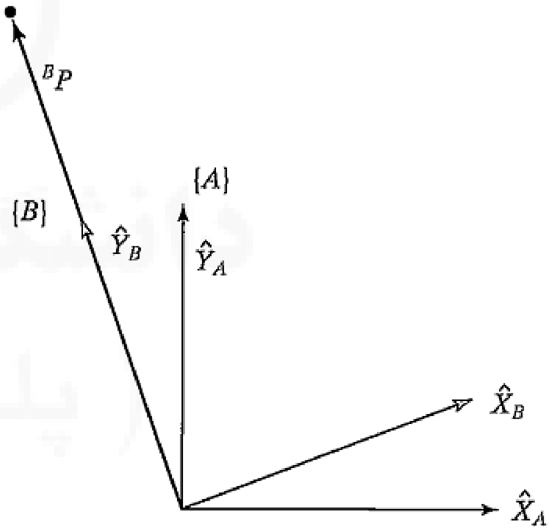
$$^{A}P = \begin{bmatrix} ^{A}p_x \\ ^{A}p_y \\ ^{A}p_z \end{bmatrix} = \begin{bmatrix} ^{B}\hat{X}_A^T \\ ^{B}\hat{Y}_A^T \\ ^{B}\hat{Z}_A^T \end{bmatrix} {}^{B}P$$

- Note that the rows of rotation matrix $^{A}R_B$ are $^{B}\hat{X}_A^T$, $^{B}\hat{Y}_A^T$ and $^{B}\hat{Z}_A^T$.

$$^{A}P = {}^{A}R_B {}^{B}P$$

Mappings: Changing Description from Frame to Frame

- **Rotation**
- Example:
- Frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by θ degrees.
- ${}^{B}P$ is given.
- Find ${}^{A}P$?



Mappings: Changing Description from Frame to Frame

General Transformation

- A position defined by the vector ${}^{B}P$.
- The orientation of {B} is known relative to {A} i.e. ${}^{A}R_{B}$.
- The vector that locates {B}'s origin is called ${}^{A}P_{BORG}$.
- Express this point in space in terms of frame {A}, i.e. ${}^{A}P$.

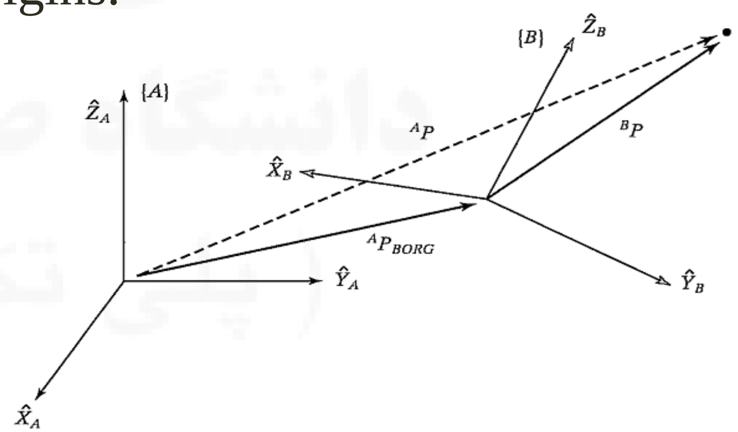
- Assume an intermediate frame {C}:
  - the same origin of {B} and the same orientation of {A}.

- Describe ${}^{B}P$ in the intermediate frame.

$$ {}^{C}P = {}^{C}R_{B}\,{}^{B}P = {}^{A}R_{B}\,{}^{B}P $$

- Then account the translation between origins.

- ${}^{A}P = {}^{A}R_{B}\,{}^{B}P + {}^{A}P_{BORG}$



Mappings: Changing Description from Frame to Frame

General Transformation

- General transform into a single matrix form.

$$ {}^{A}P = {}^{A}T_{B}\ {}^{B}P $$

- Define a 4×4 matrix operator and use 4×1 position vectors.

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}R_{B} & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} $$

- So,

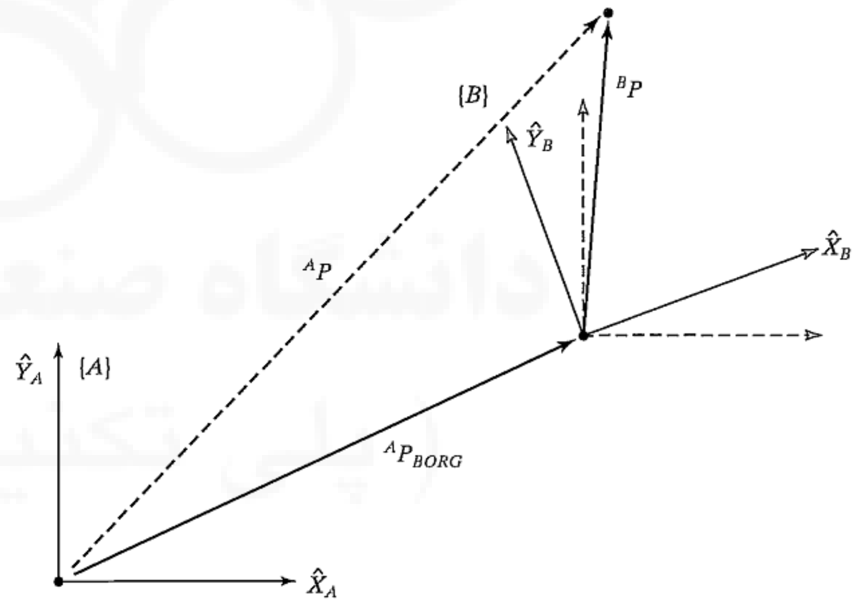
$$ {}^{A}P = {}^{A}R_{B}\ {}^{B}P + {}^{A}P_{BORG} $$
$$ 1 = 1 $$

- ${}^{A}T_{B}$ is called a **Homogeneous Transformation Matrix.**

- The description of frame {B} relative to (A} is ${}^{A}T_{B}$.

Mappings: Changing Description from Frame to Frame

- **General Transformation**
- Example:
- Frame {B} is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees, translated 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$.
- where ${}^{B}P = [3 \quad 7 \quad 0]^T$.
- Find ${}^{A}P$?



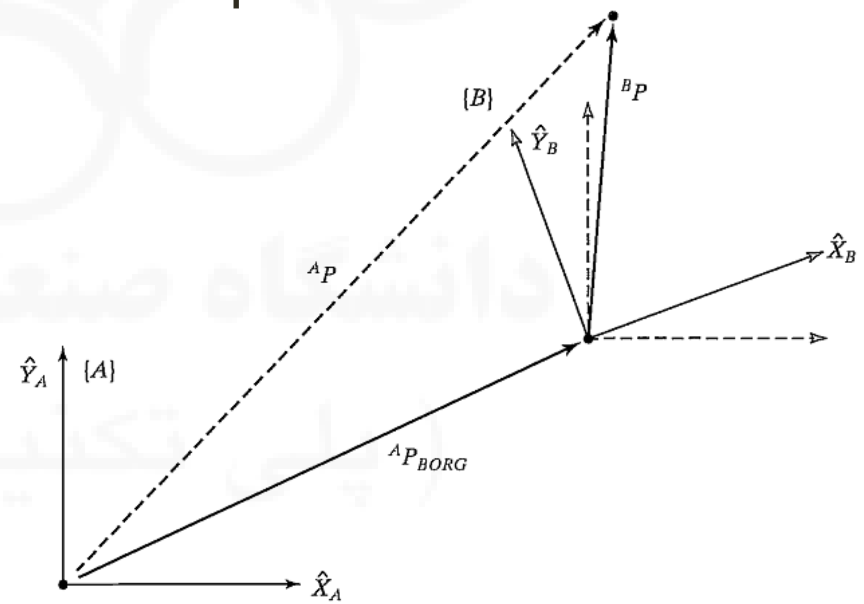
Mappings: Changing Description from Frame to Frame

- **General Transformation**
- **Special Transformations**
  - Translation:

$$ {}^{A}T_{B} = T_{Trans} = \left[\begin{array}{ccc|c} & I_{3\times 3} & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

  - Rotation:

$$ {}^{A}T_{B} = T_{Rot} = \left[\begin{array}{ccc|c} & & & 0 \\ & {}^{A}R_{B} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$



Operators

- Mapping concept can be used as **operators.**
  - Translation
  - Rotation
  - Transformation

Translation

- Moving a point $^{A}P_1$ in space a finite distance along a given vector direction ^{A}Q.

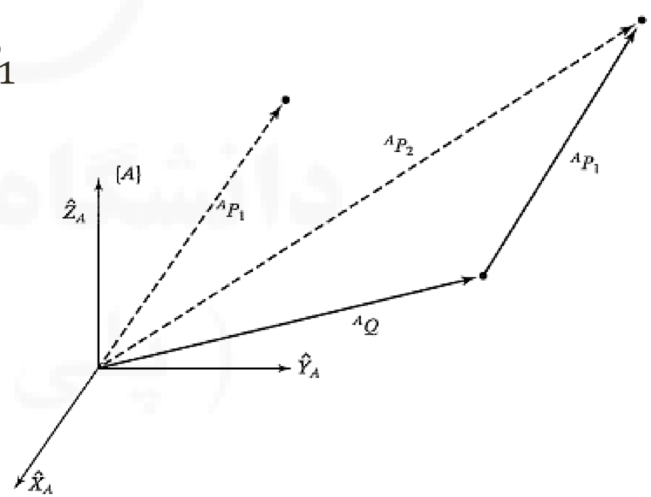
$$^{A}P_2 = {}^{A}P_1 + {}^{A}Q$$

- Frame is invariant.

- Translational Operator:

$$^{A}P_2 = T_{Trans}(Q)\, {}^{A}P_1$$

❖ ***Q:*** $T_{Trans}(Q) = ?$



Operators

❑ Translation

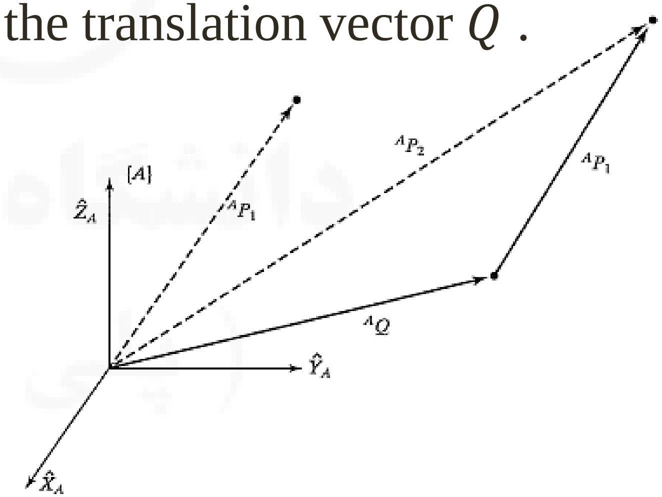
- It is accomplished with the same mathematics as mapping the point to a second frame.

- When a vector is moved "**forward**" relative to a frame = the frame is moved "**backward**“.

$$T_{Trans}(Q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- where q_x, q_y and q_z are the components of the translation vector Q .

❖ ***Q:*** Sign of q_i?



Operators

Rotation

- Rotating a vector ${}^{A}P_1$ to a new vector ${}^{A}P_2$, by means of a rotation, R.

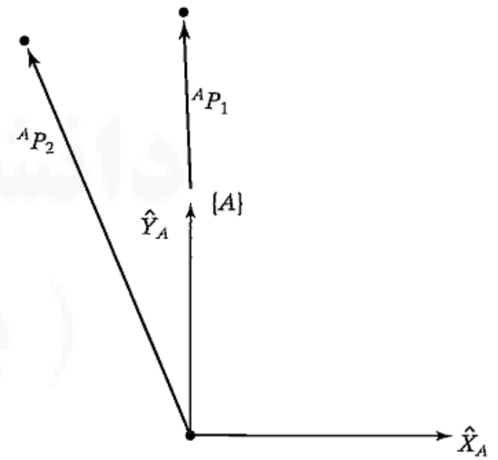
$$ {}^{A}P_2 = R\,{}^{A}P_1 $$

- The rotation matrix that rotates vectors through some rotation, R = the rotation matrix that describes a frame rotated by R relative to the reference frame.

$$ {}^{A}P_2 = R_K(\theta)\,{}^{A}P_1 $$

- "$R_K(\theta)$" performs a rotation about the axis direction K by θ degrees.

❖ ***Q:*** Assume $K = \hat{Z}$, what is $R_K(\theta)$?



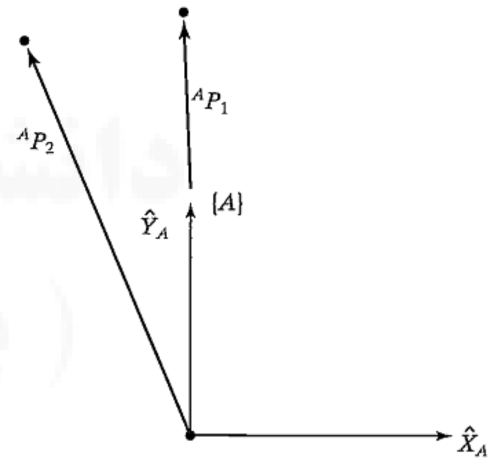
Operators

- **Rotation**
- **Rotational Operator:**

$$^{A}P_2 = T_{Rot}(\theta)\ ^{A}P_1$$

- $$T_{Rot}(R) = \left[\begin{array}{ccc:c} & R_K(\theta) & & 0 \\ & & & 0 \\ & & & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$$

- $$T_{Rot}(R) = \begin{bmatrix} cos\ \theta & -sin\ \theta & 0 & 0 \\ sin\ \theta & cos\ \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Operators

❑ Transformation

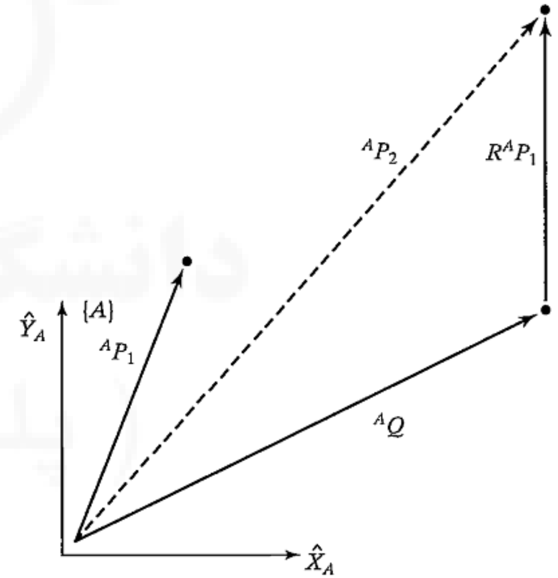
- Rotating and translating a vector ${}^{A}P_1$ to compute a new vector ${}^{A}P_2$:

$${}^{A}P_2 = T\ {}^{A}P_1$$

- The transform that rotates by R and translates by Q
= the transform that describes a frame rotated by R and translated by Q relative to the reference frame.

$$T = \left[\begin{array}{ccc|c} & & & q_x \\ & R_K(\theta) & & q_y \\ & & & q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

❖ ***Q:*** Is its sequence important?



Transformation Arithmetic

Compound Transformations

- Frame {C} is known relative to frame {B}, and frame {B} is known relative to frame {A}

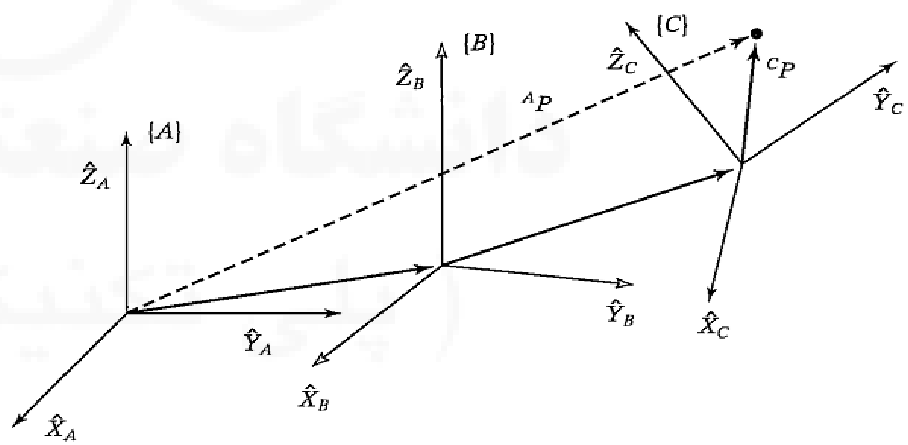
$$^{A}P = {}^{A}T_{B}\,{}^{B}T_{C}\,{}^{C}P$$

$$^{A}T_{C} = {}^{A}T_{B}\,{}^{B}T_{C}$$

$$^{A}T_{C} = \left[\begin{array}{ccc:c} & {}^{A}R_{C} & & {}^{A}P_{CORG} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$$

- We have $^{A}R_{B}$, $^{B}R_{C}$, $^{A}P_{BORG}$, $^{B}P_{CORG}$, So

- $^{A}T_{C} = \left[\begin{array}{ccc:c} & ? & & ? \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$



Transformation Arithmetic

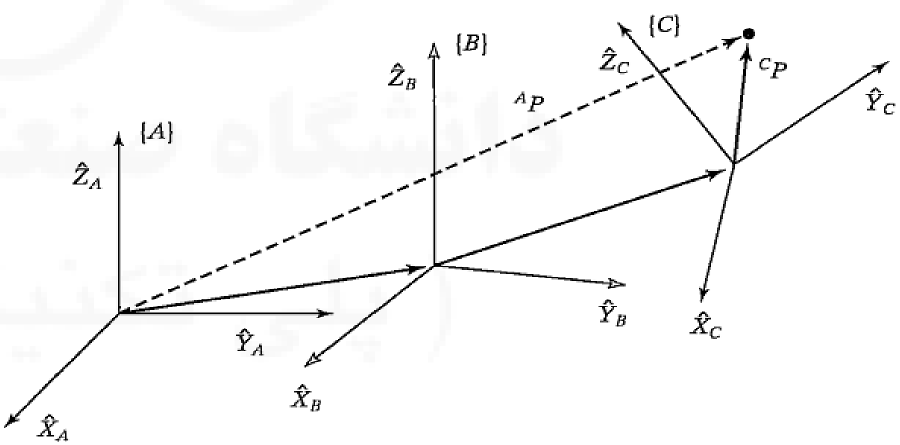
❑ Compound Transformations

$$
{}^{A}T_{C} = \left[\begin{array}{ccc|c} & {}^{A}R_{B}\,{}^{B}R_{C} & & {}^{A}R_{B}\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

- Computation:

$$
\begin{array}{ll} {}^{A}P = {}^{A}T_{B}\,({}^{B}T_{C}\,{}^{C}P) & 32\, Multiplication + 24\, Addition \\ {}^{A}P = ({}^{A}T_{B}\,{}^{B}T_{C})\,{}^{C}P & 80\, Multiplication + 60\, Addition \end{array}
$$

- However, the second one is better when you want to do the transformation of many vectors many times.



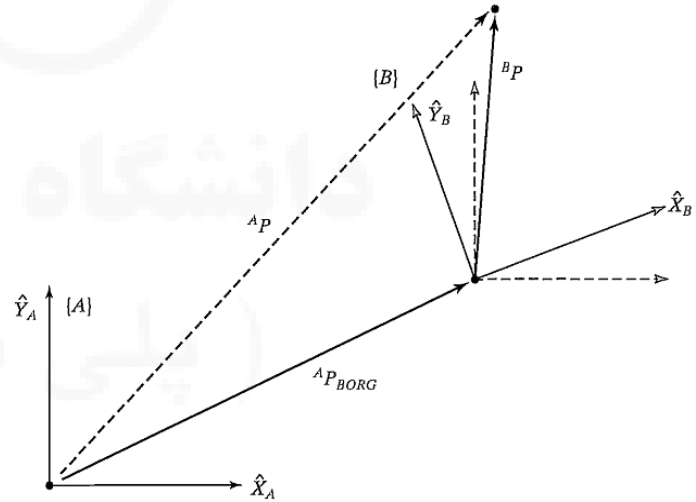
Transformation Arithmetic

❑ Inverting a Transform

- Frame {B} with respect to a frame {A}, ${}^{A}T_{B}$, is known (${}^{A}R_{B}$ & ${}^{A}P_{BORG}$).

- Invert this transform to get a description of {A} relative to {B}, (${}^{B}T_{A} = ?$)

$${}^{A}P = {}^{A}T_{B}\ {}^{B}P$$



Transformation Arithmetic

❑ Inverting a Transform

- Frame {B} with respect to a frame {A}, ${}^{A}T_{B}$, is known (${}^{A}R_{B}$ & ${}^{A}P_{BORG}$)

- Invert this transform to get a description of {A} relative to {B}, (${}^{B}T_{A} = ?$)

$$ {}^{A}P = {}^{A}T_{B}\, {}^{B}P $$

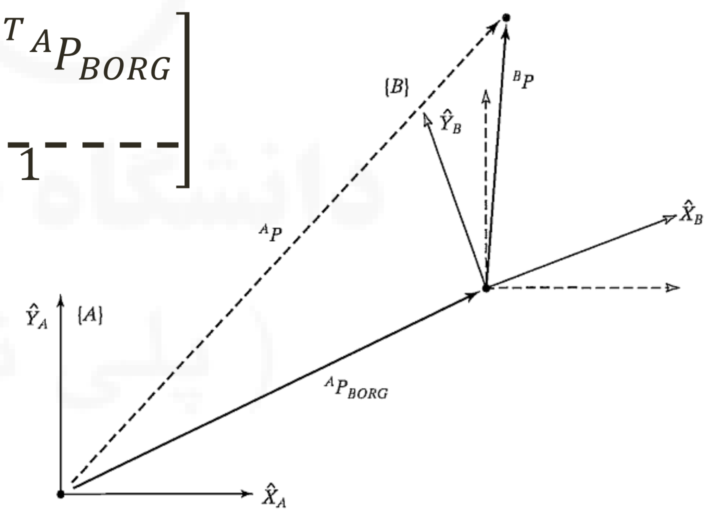
$$ {}^{A}P = {}^{A}R_{B}\, {}^{B}P + {}^{A}P_{BORG} $$
$$ {}^{A}P - {}^{A}P_{BORG} = {}^{A}R_{B}\, {}^{B}P $$
$$ {}^{A}R_{B}^{\;-1}\left({}^{A}P - {}^{A}P_{BORG}\right) = {}^{B}P $$
$$ {}^{A}R_{B}^{\;T}\, {}^{A}P - {}^{A}R_{B}^{\;T}\, {}^{A}P_{BORG} = {}^{B}P $$

$$ {}^{B}T_{A} = \left[\begin{array}{ccc:c} & {}^{A}R_{B}^{\;T} & & -{}^{A}R_{B}^{\;T}\, {}^{A}P_{BORG} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] $$

❖ ***Q:*** Geometrical Description = ?

- Note that

$$ {}^{B}T_{A} = {}^{A}T_{B}^{\;-1} $$



More on Representation of Orientation

- Remember rotation matrix:

$$^{A}R_{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- The nine elements are not all independent (six dependencies).

- Imagine R as three columns, as originally introduced:

$$^{A}R_{B} = [\,^{A}\hat{X}_{B} \quad ^{A}\hat{Y}_{B} \quad ^{A}\hat{Z}_{B}]$$

- These three vectors are the unit axes of some frame written in terms of the reference frame.

- Each is a unit vector, and all three must be mutually perpendicular (six constraints).

$$\left\|^{A}\hat{X}_{B}\right\| = \left\|^{A}\hat{Y}_{B}\right\| = \left\|^{A}\hat{Z}_{B}\right\| = 1$$
$$^{A}\hat{X}_{B} \,.\, ^{A}\hat{Y}_{B} = \, ^{A}\hat{Y}_{B} \,.\, ^{A}\hat{Z}_{B} = \, ^{A}\hat{X}_{B} \,.\, ^{A}\hat{Z}_{B} = 0$$

- The representation is *conveniently* specified with **three parameter.**

More on Representation of Orientation

- Rotation matrix is also called **proper orthonormal matrix**, ("proper" refers to $\boldsymbol{det}\left({}^{A}\boldsymbol{R}_{B}\right) = +\mathbf{1}$)

- **Cayley's formula** for orthonormal matrices:
  For any proper orthonormal matrix R, there exists a **skew-symmetric matrix S** such that

$$R = (I_3 - S)^{-1}(I_3 + S)$$

- A skew-symmetric matrix (i.e., $S = -S^T$) is specified by three parameters (s_x, s_y, s_z).

$$S = \begin{bmatrix} 0 & -s_x & s_y \\ s_x & 0 & -s_z \\ -s_y & s_z & 0 \end{bmatrix}$$

- Consequently, any 3×3 rotation matrix can be specified by just three parameters.

← Outline

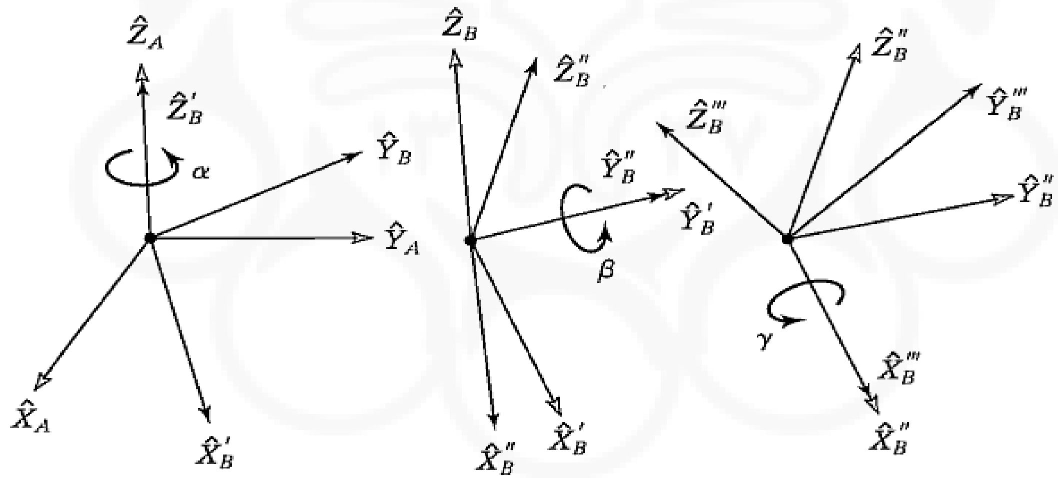
More on Representation of Orientation

- **5 Methods for Representation of Orientation**
  - Direction Cosines (9 Dependent Parameters)
  - Euler angles (3 Parameters)
  - Fixed angles (3 Parameters)
  - Equivalent angle-axis (4 Dependent Parameters)
  - Euler parameters (4 Dependent Parameters)

More on Representation of Orientation

Z-Y-X Euler Angles

- Start with the frame {A}
  - ➢Rotate {A} about $\hat{Z}_A$ by an angle α to get frame $\{B'\}$
  - ➢Rotate $\{B'\}$ about $\hat{Y}_{B'}$ by an angle β to get frame $\{B''\}$
  - ➢Rotate $\{B''\}$ about $\hat{X}_{B''}$ by an angle γ to get frame $\{B\}$

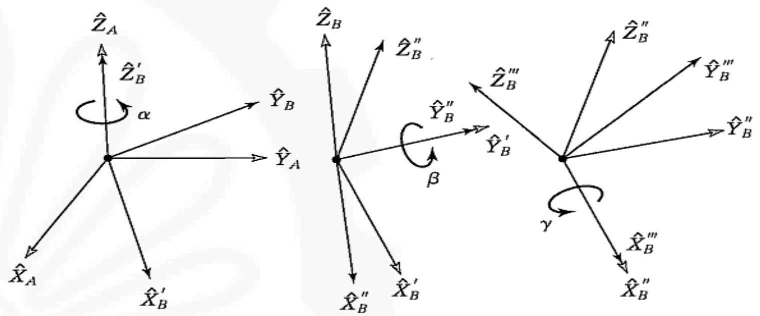


- Each rotation is performed about an axis of the **moving frames.**

❖ ***Q:*** ${}^{A}R_B$=?

More on Representation of Orientation

- **Z-Y-X Euler Angles**



- Using the intermediate frames $\{B'\}$ and $\{B"\}$ in order to give an expression for ${}^{A}R_{B\_ZYX}(\alpha,\beta,\gamma)$.

$${}^{A}R_{B\_ZYX}(\alpha,\beta,\gamma) = {}^{A}R_{B'}\,{}^{B'}R_{B''}\,{}^{B''}R_{B}$$

$${}^{A}R_{B_{ZYX}} = R_Z(\alpha)\,R_Y(\beta)\,R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

- where $c\alpha = cos\ \alpha,\ s\alpha = sin\ \alpha$

$${}^{A}R_{B\_ZYX}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha\, c\beta & c\alpha\, s\beta\, s\gamma - s\alpha\, c\gamma & c\alpha\, s\beta\, c\gamma + s\alpha\, s\gamma \\ s\alpha\, c\beta & s\alpha\, s\beta\, s\gamma + c\alpha\, c\gamma & s\alpha\, s\beta\, c\gamma - c\alpha\, s\gamma \\ -s\beta & c\beta\, s\gamma & c\beta\, c\gamma \end{bmatrix}$$

More on Representation of Orientation

- **Z-Y-X Euler Angles**
- The **Inverse** Problem
- Extracting equivalent Z-Y-X Euler angles from a given rotation matrix.

- If ${}^{A}R_{B\_ZYX}(\alpha,\beta,\gamma)$ is equated to the given rotation matrix:

$$\begin{bmatrix} c\alpha\, c\beta & c\alpha\, s\beta\, s\gamma - s\alpha\, c\gamma & c\alpha\, s\beta\, c\gamma + s\alpha\, s\gamma \\ s\alpha\, c\beta & s\alpha\, s\beta\, s\gamma + c\alpha\, c\gamma & s\alpha\, s\beta\, c\gamma - c\alpha\, s\gamma \\ -s\beta & c\beta\, s\gamma & c\beta\, c\gamma \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- 9 equations & 3 unknowns.
- Due to six dependencies, 3 equations & 3 unknowns.

$$\beta = Atan2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

- $Atan2(y,x)$ is a two-argument arc tangent function.

- Although a *second solution* exists for β, we always compute the single solution for which $-90.0° \le \beta \le 90.0°$ to have a **one-to-one mapping.**

More on Representation of Orientation

- **Z-Y-X Euler Angles**
- The **Inverse** Problem

$$\begin{bmatrix} c\alpha\, c\beta & c\alpha\, s\beta\, s\gamma - s\alpha\, c\gamma & c\alpha\, s\beta\, c\gamma + s\alpha\, s\gamma \\ s\alpha\, c\beta & s\alpha\, s\beta\, s\gamma + c\alpha\, c\gamma & s\alpha\, s\beta\, c\gamma - c\alpha\, s\gamma \\ -s\beta & c\beta\, s\gamma & c\beta\, c\gamma \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\beta = Atan2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

- As long as $c\beta \neq 0$,

$$\alpha = Atan2\,(r_{21}/c\beta, r_{11}/c\beta)$$
$$\gamma = Atan2\,(r_{32}/c\beta, r_{33}/c\beta)$$

- ***Q:*** What about for $\beta = \pm 90.0°$ $(c\beta = 0)$?!!!

More on Representation of Orientation

- **Z-Y-X Euler Angles**
- The **Inverse** Problem
- **Singularity of the Inverse Problem:**

$$\alpha = Atan2\,(r_{21}/c\beta, r_{11}/c\beta)$$
$$\gamma = Atan2\,(r_{32}/c\beta, r_{33}/c\beta)$$

- If $\beta = \pm 90.0°$ ($c\beta = 0$), the solution degenerates.

- For $\beta = +90.0°$:

$$\begin{bmatrix} c\alpha\, c\beta & c\alpha\, s\beta\, s\gamma - s\alpha\, c\gamma & c\alpha\, s\beta\, c\gamma + s\alpha\, s\gamma \\ s\alpha\, c\beta & s\alpha\, s\beta\, s\gamma + c\alpha\, c\gamma & s\alpha\, s\beta\, c\gamma - c\alpha\, s\gamma \\ -s\beta & c\beta\, s\gamma & c\beta\, c\gamma \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} 0 & sin(\gamma - \alpha) & cos(\gamma - \alpha) \\ 0 & cos(\gamma - \alpha) & -sin(\gamma - \alpha) \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- In those cases, only the $(\alpha \pm \gamma)$ can be computed.

More on Representation of Orientation

- **Z-Y-X Euler Angles**
- The **Inverse** Problem
- **Singularity of the Inverse Problem:**
- For $\beta = +90.0°$:

$$\begin{bmatrix} 0 & sin(\gamma-\alpha) & cos(\gamma-\alpha) \\ 0 & cos(\gamma-\alpha) & -sin(\gamma-\alpha) \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

- One possible convention is to choose $\alpha = 0.0$ in these cases and compute γ.

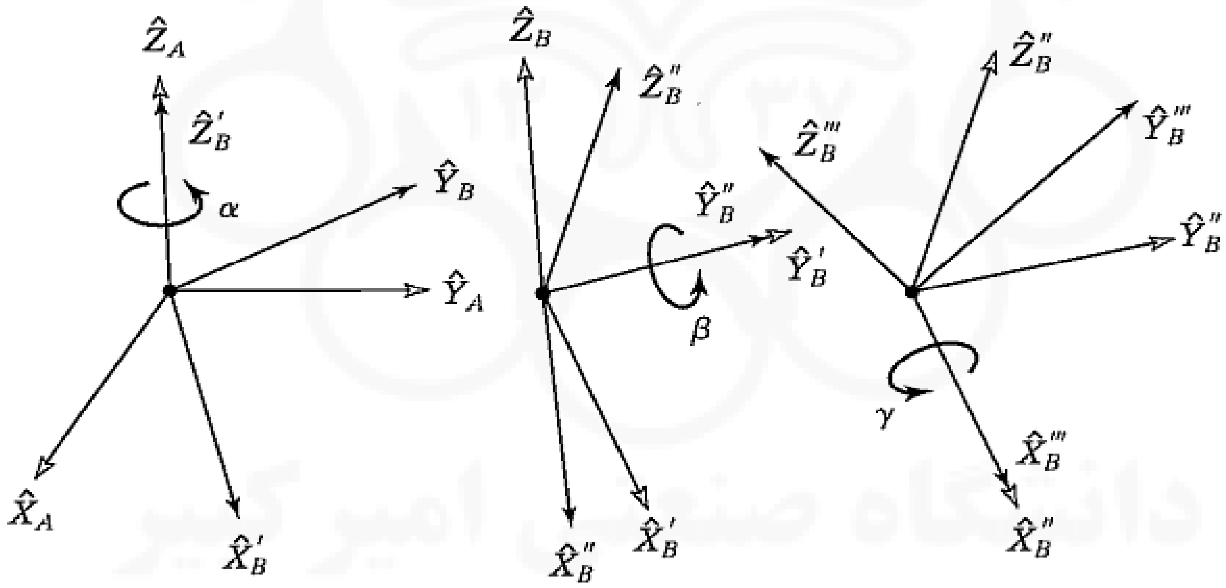
- If $\beta = \pm 90.0°$, then a solution can be calculated to be

$\beta = +90.0°$ $\alpha = 0.0$ $\gamma = Atan2\,(r_{12}, r_{22})$	$\beta = -90.0°$ $\alpha = 0.0$ $\gamma = -Atan2\,(r_{12}, r_{22})$

More on Representation of Orientation

- ❑ **Z-Y-X Euler Angles**
- ▪ The **Inverse** Problem
- ▪ **Singularity of the Inverse Problem:**

❖ ***Q:*** What is the physical interpretation of the IK singularity?

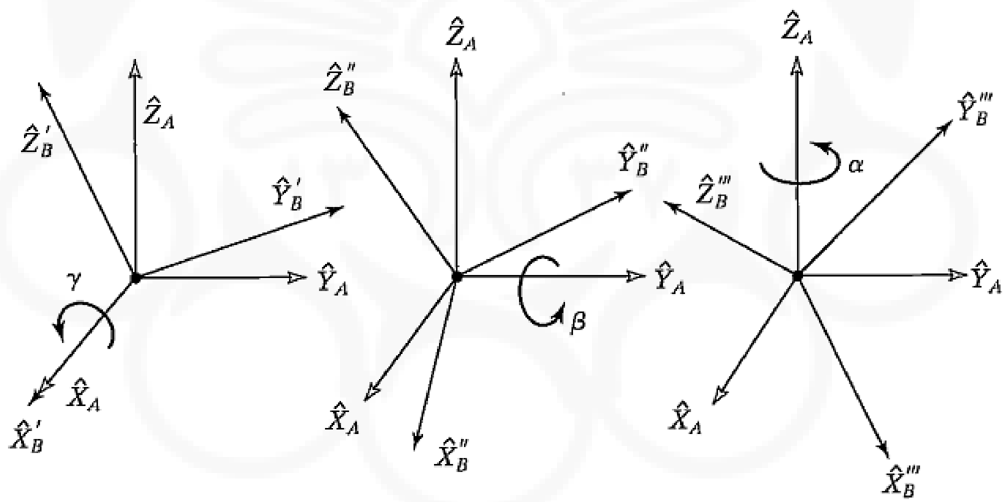


← Outline

More on Representation of Orientation

X-Y-Z Fixed Angles

- Start with the frame {A}
  - Rotate {A} about $\hat{X}_A$ by an angle γ to get frame $\{B'\}$
  - Rotate $\{B'\}$ about $\hat{Y}_A$ by an angle β to get frame $\{B''\}$
  - Rotate $\{B''\}$ about $\hat{Z}_A$ by an angle α to get frame $\{B\}$



- Each of the three rotations takes place about an axis in the **fixed reference** frame {A}.
- This convention is referred to as **roll, pitch, yaw** angles.

❖ ***Q:*** ${}^{A}R_{B}$=?

More on Representation of Orientation

X-Y-Z Fixed Angles

- The composition rule <u>cannot</u> be applied here, similarity transformation can be used instead.

Similarity Transformation

- A rotation matrix (as a coordinate transformation) may be viewed as changing basis from one frame to another.

- A general linear transformation is transformed from one frame to another using similarity transformation.

- M is a linear transformation in frame $\{0\}$ and N is the representation of M in frame $\{1\}$.

$$N = ({}^0R_1)^{-1}\, M\, {}^0R_1$$

More on Representation of Orientation

X-Y-Z Fixed Angles

- Using the intermediate frames $\{B'\}$ and $\{B"\}$ in order to give an expression for ${}^{A}R_{B\_XYZ}(\gamma, \beta, \alpha)$.

$$ {}^{A}R_{B} = {}^{A}R_{B'}\, {}^{B'}R_{B''}\, {}^{B''}R_{B} $$

$$ {}^{A}R_{B'} = R_X(\gamma) $$
$$ {}^{B'}R_{B''} = ({}^{A}R_{B'})^{-1}\, R_Y(\beta)\, ({}^{A}R_{B'}) $$
$$ {}^{B''}R_{B} = ({}^{A}R_{B''})^{-1}\, R_Z(\alpha)\, ({}^{A}R_{B''}) $$

- Therefore,

$$ {}^{A}R_{B'} = R_X(\gamma) $$

$$ {}^{A}R_{B''} = {}^{A}R_{B'}\, {}^{B'}R_{B''} = {}^{A}R_{B'}\, ({}^{A}R_{B'})^{-1}\, R_Y(\beta)\, ({}^{A}R_{B'}) = R_Y(\beta)\, R_X(\gamma) $$

$$ {}^{A}R_{B} = {}^{A}R_{B''}\, {}^{B''}R_{B} = {}^{A}R_{B''}\, ({}^{A}R_{B''})^{-1}\, R_Z(\alpha)\, ({}^{A}R_{B''}) = R_Z(\alpha)\, R_Y(\beta)\, R_X(\gamma) $$

More on Representation of Orientation

❑ X-Y-Z Fixed Angles

$$
\begin{aligned}
{}^{A}R_{B\_XYZ}(\gamma,\beta,\alpha\,) &= R_Z(\alpha)\,R_Y(\beta)\,R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
\end{aligned}
$$

- Therefore,

$$
{}^{A}R_{B\_XYZ}(\gamma,\beta,\alpha\,) = \begin{bmatrix} c\alpha\, c\beta & c\alpha\, s\beta\, s\gamma - s\alpha\, c\gamma & c\alpha\, s\beta\, c\gamma + s\alpha\, s\gamma \\ s\alpha\, c\beta & s\alpha\, s\beta\, s\gamma + c\alpha\, c\gamma & s\alpha\, s\beta\, c\gamma - c\alpha\, s\gamma \\ -s\beta & c\beta\, s\gamma & c\beta\, c\gamma \end{bmatrix}
$$

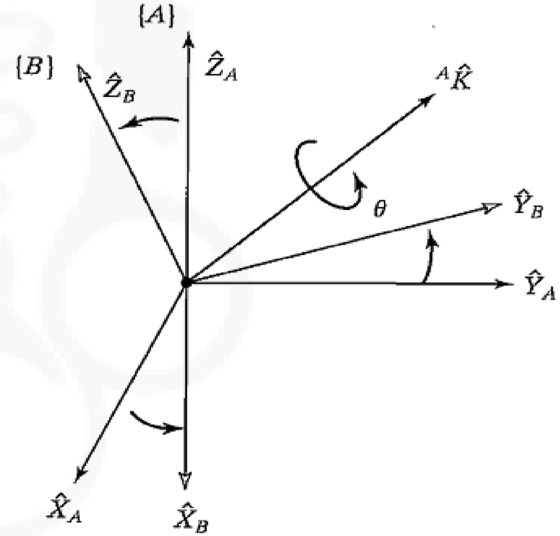
- **Note:** Three rotations taken about **fixed axes** $\left(e.g.\ {}^{A}R_{B\_XYZ}(\gamma,\beta,\alpha\,)\right)$ yield the **same final orientation** as the same three rotations taken **in opposite order** about the axes of the **moving frame** $\left(e.g.\ {}^{A}R_{B\_ZYX}(\alpha,\beta,\gamma)\right)$.

- **24 representations:** 12 Euler angles series + 12 Fixed angles series.

More on Representation of Orientation

❑ Equivalent Angle-Axis

- Start with the frame {A}
  - ➢ Rotate about the unit vector ${}^{A}\hat{K}$ by an angle θ according to the right-hand rule (Based on the *Euler Theorem*).



- ${}^{A}\hat{K}$: Equivalent axis of a finite rotation.
- It may be written as ${}^{A}R_B(\hat{K},\theta)$ or $R_K(\theta)$.
- ${}^{A}\hat{K}$ requires only <u>two parameters</u> caused by its unit length.
- For the general axis of rotation:

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z\, s\theta & k_x k_z v\theta + k_y\, s\theta \\ k_x k_y v\theta + k_z\, s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x\, s\theta \\ k_x k_z v\theta - k_y\, s\theta & k_y k_z v\theta + k_x\, s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

- where $c\theta = cos\ \theta$, $s\theta = sin\ \theta$, $v\theta = 1 - cos\ \theta$ and ${}^{A}\hat{K} = \left[k_x, k_y, k_z\right]^T$

More on Representation of Orientation

❑ Equivalent Angle-Axis

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z\, s\theta & k_x k_z v\theta + k_y\, s\theta \\ k_x k_y v\theta + k_z\, s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x\, s\theta \\ k_x k_z v\theta - k_y\, s\theta & k_y k_z v\theta + k_x\, s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

❖ Example:

- For $\widehat{K}$ as principal axes (e.g. $\hat{K} = \hat{X} = [1 \quad 0 \quad 0]^T$ or $\hat{Y}$ or $\hat{Z}$)

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

$$R_Y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_Z(\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

More on Representation of Orientation

- **Equivalent Angle-Axis**
- The **Inverse** Problem
- Computing $\hat{K}$ and θ from a given rotation matrix.

$$\begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z\, s\theta & k_x k_z v\theta + k_y\, s\theta \\ k_x k_y v\theta + k_z\, s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x\, s\theta \\ k_x k_z v\theta - k_y\, s\theta & k_y k_z v\theta + k_x\, s\theta & k_z k_z v\theta + c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = Acos\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)\ ,\ \hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

- This solution always computes a value of θ between [0-180°].
- For any $({}^{A}\hat{K}, \theta)$, $(-{}^{A}\hat{K}, -\theta)$ results in the same orientation in space.
- For **small** angular rotations, the axis becomes ill-defined.
- If $\theta \to 0°$, the axis becomes completely undefined. ($\theta = 0°$ or $\theta = 180°$)

More on Representation of Orientation

❑ Euler Parameters

- Another representation is by means of four numbers called the Euler parameters.

- In terms of the equivalent axis $\hat{K} = [k_x, k_y, k_z]^T$ and the equivalent angle θ, the Euler parameters are given by

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} sin\frac{\theta}{2} = \hat{K}\, sin\frac{\theta}{2}$$

$$\epsilon_4 = cos\frac{\theta}{2}$$

- These four quantities are not independent (Unit Quaternion)

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$$

- An orientation might be visualized as a point on a unit hypersphere in four-dimensional space.

More on Representation of Orientation

- **Euler Parameters**
- The rotation matrix

$$R_\epsilon = \begin{bmatrix} 1-2\epsilon_2^2-2\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3+\epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2+\epsilon_3\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3-\epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3+\epsilon_1\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_2^2 \end{bmatrix}$$

- The **Inverse** Problem:
- Given a rotation matrix (A), the equivalent Euler parameters are

$$\begin{bmatrix} 1-2\epsilon_2^2-2\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3+\epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2+\epsilon_3\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3-\epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3+\epsilon_1\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_2^2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = A$$

$$\epsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

$$\epsilon = \frac{1}{4\epsilon_4}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

- For a rotation of 180 degrees about some axis, $\epsilon_4 \to 0$ (ill defined).

More on Representation of Orientation

- **Euler Parameters**
- The **Inverse** Problem:
- Remember:

$$\epsilon = \hat{K}\, sin\frac{\theta}{2}$$
$$\epsilon_4 = cos\frac{\theta}{2}$$

- By definition, if $\epsilon_4 = 0$, then $\theta = 180°$ and ϵ is equal to rotation axis, i.e. $\epsilon = \hat{K}$

$$\epsilon_1^2 = \frac{1 + 2r_{11} - tr\, A}{4}$$
$$\epsilon_2^2 = \frac{1 + 2r_{22} - tr\, A}{4}$$
$$\epsilon_3^2 = \frac{1 + 2r_{33} - tr\, A}{4}$$

- **Note:** There is **No Singularity** associated with these parameters.
- As long as the direction cosines are known, we can find the corresponding Euler parameters [1].

Transformation of Free Vectors

- **Line vector:** a vector that is dependent on its line of action, along with direction and magnitude. (*Position* & *Force*)

- **Free vector:** a vector that may be positioned anywhere in space, provided that magnitude and direction are preserved (*Velocity* & *Moments*)

- For free vectors, **only** the rotation matrix relating the two systems is used in transforming.

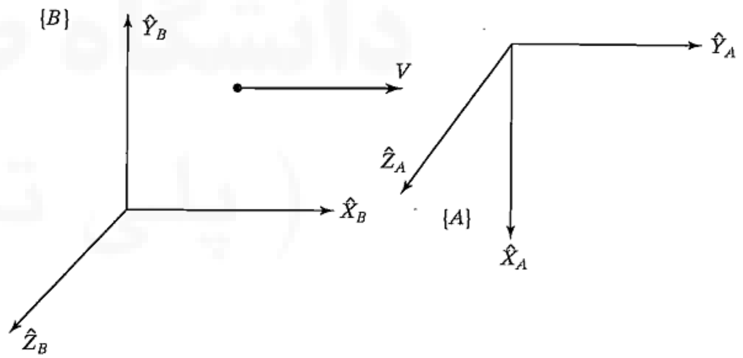
- Position Transformation

$$^{A}P = {^{A}T_{B}}\,{^{B}P}$$

- Velocity Transformation

$$^{A}V = {^{A}R_{B}}\,{^{B}V}$$

- $^{A}P_{BORG}$ which would appear in a position-vector transformation, does not appear in a velocity transform.



The END

- **References:**

[1] www.u.arizona.edu/~pen/ame553/Notes/Lesson%2009.pdf