Appendix A
Controllability of the Linearized VSG Model

Consider the linearized VSG model

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{B}\Delta\boldsymbol{u}, \qquad \Delta\boldsymbol{x} \in \mathbb{R}^5,\ \Delta\boldsymbol{u} \in \mathbb{R}^2, \tag{1}$$

with the shorthand parameters $a := D_pK_p$, $b := K_pk_{pv}$, $c := K_pk_{p\delta}$, $d := D_qK_q$, $e := D_qK_q + K_qk_{qv}$, $f := K_qk_{q\delta}$, and system matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & e & f \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{2}$$

It is clear that $\boldsymbol{A}^3 = 0$, hence the controllability matrix

$$\boldsymbol{\mathcal{C}} = [\,\boldsymbol{B} \ \ \boldsymbol{AB} \ \ \boldsymbol{A}^2\boldsymbol{B}\,]$$

is sufficient to decide controllability. Direct multiplication gives

$$\boldsymbol{AB} = \begin{bmatrix} a & b \\ d & e \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{A}^2\boldsymbol{B} = \begin{bmatrix} c & 0 \\ f & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

Therefore

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} 1 & 0 & a & b & c & 0 \\ 0 & 1 & d & e & f & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

Among all 5×5 minors of $\boldsymbol{\mathcal{C}}$, the only nonzero one is the minor constructed from the first five columns:

$$\det\big(\boldsymbol{\mathcal{C}}\big) = \det \begin{bmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = bf - ce, \tag{4}$$

Hence $\operatorname{rank}\mathcal{C} = 5$ if and only if $bf - ce \neq 0$, which is the necessary and sufficient condition for controllability. Writing this condition back in the original parameters yields

$$bf - ce = K_pK_q(k_{pv}k_{q\delta} - k_{p\delta}(D_q + k_{qv})) \neq 0, \tag{5}$$

In particular, when $K_p, K_q \neq 0$, the model is controllable iff

$$k_{pv}k_{q\delta} \neq k_{p\delta}(D_q + k_{qv}), \tag{6}$$

Using the small-signal coefficients from (**??**), (**??**), (**??**), and (**??**) with $D > 0$, all factors $1/D$ cancel in (6). For $k_{p\delta} \neq 0$, (6) is equivalent to

$$D_q \neq D_q^\star \ \triangleq\ \frac{k_{pv}k_{q\delta}}{k_{p\delta}} - k_{qv}, \tag{7}$$

After substitution and simplification,

$$D_q^\star \ = \ \frac{(R_g^2 + X_g^2)\left(V_g - 2V_0\cos\delta_0\right)}{R_g\sin\delta_0 + X_g\cos\delta_0}, \tag{8}$$

Writing $Z_g \triangleq \sqrt{R_g^2 + X_g^2}$ and $\theta_g \triangleq \operatorname{atan2}(X_g, R_g)$, note that $R_g\sin\delta_0 + X_g\cos\delta_0 = Z_g\sin(\delta_0 + \theta_g)$, hence

$$D_q^\star \ = \ Z_g\,\frac{V_g - 2V_0\cos\delta_0}{\sin(\delta_0 + \theta_g)}, \tag{9}$$

If $k_{p\delta} = 0$: (6) reduces to $k_{pv}k_{q\delta} \neq 0$ with $k_{q\delta}$ is always nonzero as long as the network impedance is not zero. Therefore the remaining condition is simply $k_{pv} \neq 0$ and is equivalent to $2R_gV_0 \neq \pm Z_gV_g$ Cite documents. This condition is unlikely to occur in usual operating conditions.

If $k_{p\delta} \neq 0$:

- If δ_0 approaches $-\theta_g$, $\sin(\delta_0 + \theta_g) \to 0$ and $|D_q^\star|$ blows up: controllability then holds for any practical finite D_q.
- In typical operation $V_g \approx V_0$ and δ_0 small, so $V_g - 2V_0\cos\delta_0 < 0$. With standard $D_q > 0$, the inequality $D_q \neq D_q^\star$ is automatically satisfied.

Therefore, (**??**) is always controllable if $2R_gV_0 \neq \pm Z_gV_g$.