MARCEL - Inverse Kinematics

1 Introduction

This document delves into a detailed and comprehensive explanation of the mathematical framework employed by the MARCEL robot (Mecanism Autonom Românesc care este leneș) for effectively controlling its robotic arms.

By delving into the intricate mathematics underpinning MARCEL's control system, this document aims to provide a thorough understanding of the theoretical foundations that enable the robot to perform various tasks with accuracy and reliability.

With all of that said, let's see what this paper is trying to solve: When we are trying to control robot arms (also called manipulators) like these ones, we usually need to solve the following problem: Given a position in space, find the angles that the motors need to turn (also called revolute joints) such that the tip of the hand (the tool frame) reaches that point. This problem can be solved using the concept of Inverse Kinematics.

2 Notations

2.1 Robot structure

The manipulator is made up of 6 revolute joints. The origin of the manipulator is the center of the first joint which sits on the X axis. All of the following joints are linked to one another. The second joint sits on the Y axis. The third joint sits on the Z axis. The first 3 joints assure the position of the manipulator. The forth joint is initially parallel to the first one, it can be called "Elbow". The last 2 joints, wrist rotation and claw are used to position the hand once it has reached a destination.

2.2 Variables & Constants

We'll use the following notations:

θ_1: The azimuthal angle of the first link with respect to O (first joint)

ϕ_1: The polar angle of the first link with respect to O (second joint)

θ_2: The azimuthal angle of the first link with respect to W (third joint)

ϕ_2: The polar angle of the first link with respect to W (forth joint)

θ_3: The angle of the wrist rotation (fifth joint)

ϕ_3: The angle of the claw motor (sixth joint)

a_1: The length of the first link

a_2: The length of the second link

2.3 Points

O: The origin frame of the entire system, also known as the center of the first joint or the "world" frame. $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

W: The elbow frame. $W = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$

P: The destination frame, also known as the "user" frame. It will also be considered the tool frame in this paper. $P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$

2.4 Functions

Let functions Rx, Ry, Rz be the basic 3d rotation matrices:

$$Rx(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$Ry(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Rz(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Solution

3.1 Thinking process

As many other problems, we can solve this one by dividing it into smaller problems. The manipulator can be split into 2 links (a_1 and a_2). The first link goes from O to E and the second one from E to P. We already know the position of O since it is the origin, we also know the point P since it is given to us. The only point that remains unknown is W.

We know that W should be at a distance of a_1 from O, so it sits on the surface of a sphere $\mathscr{S}(O, a_1)$. We also know that W should be at a distance of a_2 from P, so it sits on the surphace of a sphere $\mathscr{S}(P, a_2)$. From these, we can conclude:

$$W \in \mathscr{S}(O, a_1) \ \cap \ \mathscr{S}(P, a_2)$$

And since the intersection of 2 spheres is a circle:

$$W \in \mathscr{C}(M, r)$$

Where:

M: Center of the circle formed by intersecting the 2 spheres.

r: Radius of the circle formed by intersecting the 2 spheres.

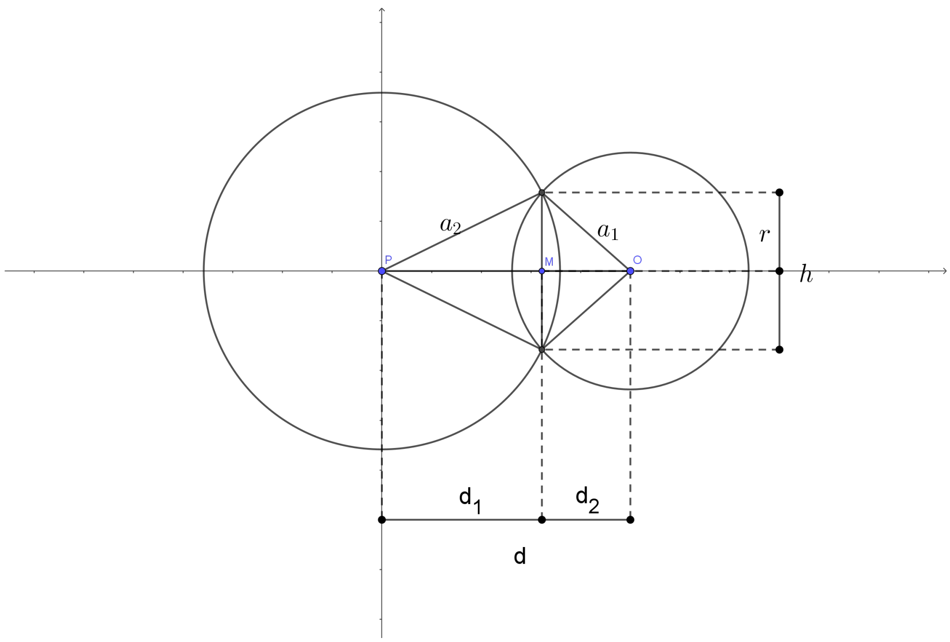

Figure 1: Intersection of spheres in 2D

3.2 Finding point W relative to frame M

This gives us an infinity of points that W could be, so we'll introduce another variable, α, being the angle at which W sits on the circle.

From the angle to point on circle formula, we get:

$$X = r\cos\alpha$$

$$Y = r\sin\alpha$$

Applying this to our system, relative to the M frame (where the orientation of M is given by the plane XY, the circle plane):

$$W_M = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \\ 0 \end{bmatrix}$$

Since α is known, we now need to find the radius of circle $\mathscr{C}(M, r)$. Drawing the schema in 2D can help us see everything better:

h**:** Diameter of the circle.

d**:** Distance between O and P.

$d1$**:** Distance between P and M.

$d2$**:** Distance between O and M.

With that, we can find r:

$$a_1^2 = d_1^2 + r^2$$
$$a_2^2 = d_2^2 + r^2 \implies a_2^2 = (d - d_1)^2 + r^2$$

$$a_1^2 - a_2^2 = d_1^2 + r^2 - (d - d_1)^2 - r^2$$
$$a_1^2 - a_2^2 = d_1^2 - (d - d_1)^2$$

$$a_1^2 - a_2^2 = d_1^2 - (d^2 - 2dd_1 + d_1^2)$$
$$a_1^2 - a_2^2 = d_1^2 - d^2 + 2dd_1 - d_1^2$$
$$a_1^2 - a_2^2 = -d^2 + 2dd_1$$
$$d_1 = \frac{a_1^2 - a_2^2 + d^2}{2d}$$

We can also find d as the distance between O and P:

$$d = \sqrt{p_x^2 + p_y^2 + p_z^2}$$

We now know the position of W relative to the M frame. For our purpose, we need to convert this to the origin.

We can do so by using the transformation matrix of M to O:

$$W_O = T_M^O \cdot W_M$$
$$W_O = (T_O^M)^{-1} \cdot W_M$$

Where the homogeneous transformation matrix is:

$$T_O^M = \begin{bmatrix} R_M & t_M \\ 0 & 1 \end{bmatrix}$$

Since O, M and P are collinear, we can find the translation vector t_M and the rotation matrix R_M like so:

$$t_M = \frac{d_2}{d} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$R_M = R_z(\theta_P)R_y(\phi_P) = \begin{bmatrix} \cos\theta_P & -\sin\theta_P & 0 \\ \sin\theta_P & \cos\theta_P & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi_P & -\sin\phi_P & 0 \\ \sin\phi_P & \cos\phi_P & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_M = \begin{bmatrix} \cos\theta_P \cos\phi_P & -\sin\theta_P & \cos\theta_P \sin\phi_P \\ \sin\theta_P \cos\phi_P & \cos\theta_P & \sin\theta_P \sin\phi_P \\ -\sin\phi_P & 0 & \cos\phi_P \end{bmatrix}$$

Where:

θ_P**:** Azimuthal angle formed by $\overrightarrow{OP}$ with respect to O

ϕ_P**:** Polar angle formed by $\overrightarrow{OP}$ with respect to O

We can get angles θ_P and ϕ_P using the cartesian to polar formulas:

$$\theta_P = \tan^{-1} \frac{p_y}{p_x}$$

$$\phi_P = \cos^{-1} \frac{p_z}{d}$$

3.3 Finding θ_1 and ϕ_1

Now that we know W_O, we can find θ_1 and ϕ_1 by using the cartesian to polar formulas once again:

$$\theta_1 = \tan^{-1} \frac{w_y}{w_x}$$

$$\phi_1 = \cos^{-1} \frac{w_z}{a_1}$$

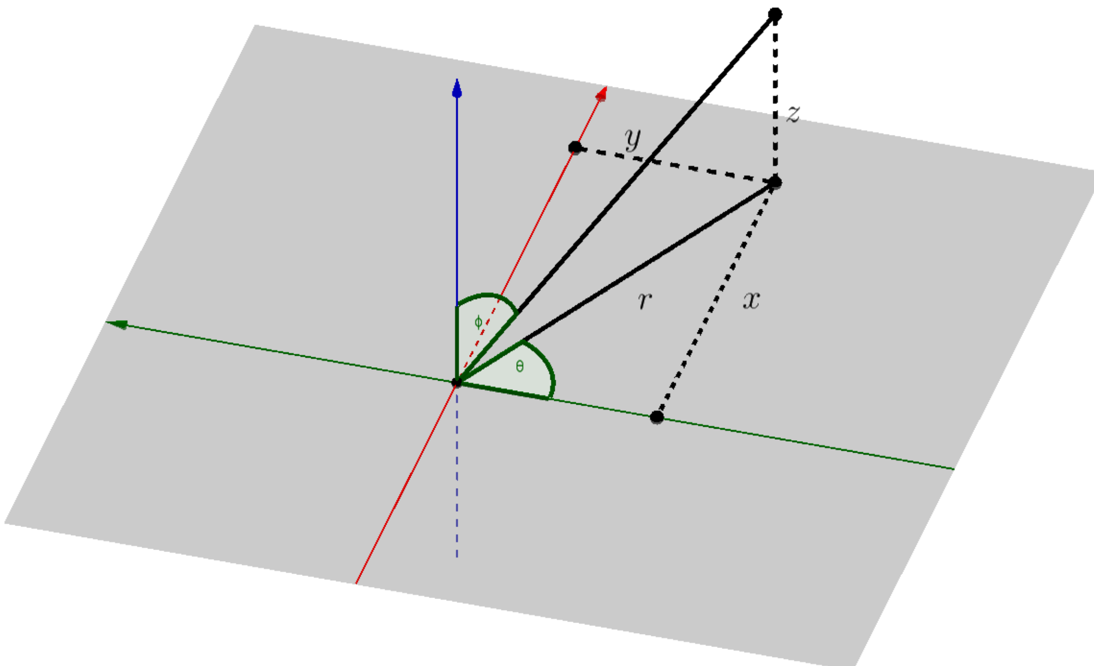

Figure 2: Polar coordinates

3.4 Finding θ_2 and ϕ_2

Since we have the position and the rotation of the wrist, we can now find angles θ_2 and ϕ_2 in a similar manner. Since the angles must be relative to W, we must get P with respect to W:

$$P_W = T_W^O \cdot P$$

$$P_W = (T_O^W)^{-1} \cdot P$$

$$T_O^W = \begin{bmatrix} R_W & t_W \\ 0 & 1 \end{bmatrix}$$

$$R_W = R_z(\theta_1)R_y(\phi_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_W = \begin{bmatrix} \cos\theta_1\cos\phi_1 & -\sin\theta_1 & \cos\theta_1\sin\phi_1 \\ \sin\theta_1\cos\phi_1 & \cos\theta_1 & \sin\theta_1\sin\phi_1 \\ -\sin\phi_1 & 0 & \cos\phi_1 \end{bmatrix}$$

With that, we can find θ_2 and ϕ_2 using polar coordinates convertion, like before:

$$\theta_2 = \tan^{-1}\frac{(p_y)_W}{(p_x)_W}$$

$$\phi_2 = \cos^{-1}\frac{(p_z)_W}{a_2}$$

References

- https://mathworld.wolfram.com/Circle-CircleIntersection.html
- https://www.youtube.com/playlist?list=PLT_0lwItn0sDBE98BsbaZezflB96ws12b
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