TU MÜNCHEN
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Solutions for exercise 7 20 December 2019

PROBABILITY THEORY AND BAYESIAN NETWORKS

Problem 7.1:

a. The missing probability can be computed knowing that the sum of all the elements of a full joint probability distribution of a set of random variables is 1.
$\mathbf{P}(R = wet, e, \neg a) = 1 - (0.0607 + 0.0449 + 0.0084 + 0.3605 + 0.0240 + 0.0851 + 0.0654 + 0.0152 + 0.1435 + 0.0400 + 0.0022) = 0.1501$
(Note that $A = true$ is abbreviated with a, and $E = true$ with e).

b. The prior probability distribution of R is obtained by summing over all the possible combinations of the other random variables. This means that we simply have to sum over the columns of the table:
$\mathbf{P}(R) = [P(R = dry), P(R = wet), P(R = snow/ice)]$ where
$P(R = dry) = 0.0607 + 0.3605 + 0.0851 + 0.1435 = 0.65,$
$P(R = wet) = 0.0449 + 0.1501 + 0.0654 + 0.0400 = 0.30,$
$P(R = snow/ice) = 1 - (0.65 + 0.30) = 0.05.$
The same is done for E:
$\mathbf{P}(E) = [P(e), P(\neg e)]$ where
$P(e) = 0.0607 + 0.0449 + 0.0084 + 0.3605 + 0.1501 + 0.0240 = 0.6486,$
$P(\neg e) = 1 - P(e) = 0.3514.$

c. The solution is obtained using the definition of conditional probability:
$P(\neg e | a, R = wet) = \frac{P(R=wet,\neg e,a)}{P(a,R=wet)} = \frac{P(R=wet,\neg e,a)}{P(e,a,R=wet)+P(\neg e,a,R=wet)} = \frac{0.0654}{0.0449+0.0654} \approx 0.5933.$

d. Each random variable is a node of the Bayesian network (BN). The random variables R and E can be considered as independent. Additionally, they directly influence A and thus they are represented as its parents in the BN. Usually we let the causes precede effects for the placement of the nodes.* As usual, we write in the conditional probability tables only the values for E = true and A = true.

By applying the chain rule, it can be written the following: $P(A,R,E) = P(A|R,E) \cdot P(R|E) \cdot P(E)$. Since R and E are independent, $P(R|E) = P(R)$. Therefore, $P(A|R,E) = \dfrac{P(A,R,E)}{P(R) \cdot P(E)}$. (Note that snow/ice is abbreviated with s/i.)

$x_1 = P(a|R = dry, e) = \frac{P(R=dry,e,a)}{P(R=dry,e)} = \frac{P(R=dry,e,a)}{P(R=dry) \cdot P(e)} = \frac{0.0607}{0.65 \cdot 0.6486} = 0.1440$

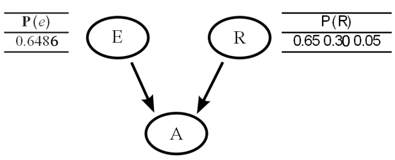
$x_2 = P(a|R = dry, \neg e) = \frac{P(R=dry,\neg e,a)}{P(R=dry,\neg e)} = \frac{P(R=dry,\neg e,a)}{P(R=dry) \cdot P(\neg e)} = \frac{0.0851}{0.65 \cdot 0.3514} = 0.3726$

$x_3 = P(a|R = wet, e) = \frac{P(R=wet,e,a)}{P(R=wet,e)} = \frac{P(R=wet,e,a)}{P(R=wet) \cdot P(e)} = \frac{0.0449}{0.30 \cdot 0.6486} = 0.2309$

$x_4 = P(a|R = wet, \neg e) = \frac{P(R=wet,\neg e,a)}{P(R=wet,\neg e)} = \frac{P(R=wet,\neg e,a)}{P(R=wet) \cdot P(\neg e)} = \frac{0.0654}{0.30 \cdot 0.3514} = 0.6204$

$x_5 = P(a|R = s/i, e) = \frac{P(R=s/i,e,a)}{P(R=s/i,e)} = \frac{P(R=s/i,e,a)}{P(R=s/i) \cdot P(e)} = \frac{0.0084}{0.05 \cdot 0.6486} = 0.2590$

$x_6 = P(a|R = s/i, \neg e) = \frac{P(R=s/i,\neg e,a)}{P(R=s/i,\neg e)} = \frac{P(R=s/i,\neg e,a)}{P(R=s/i) \cdot P(\neg e)} = \frac{0.0152}{0.05 \cdot 0.3514} = 0.8651$



R	E	$\mathbf{P}(a\|R,E)$
dry	t	x_1
	f	x_2
wet	t	x_3
	f	x_4
snow/ice	t	x_5
	f	x_6

Problem 7.2:

a. The answer is no. To calculate this posterior probability we should have a useful prior distribution of the random variables for the color of the taxis to use Bayes' rule.

b. To answer this question we have to compute the probability that the taxi of the accident has a particular color given our observation (posterior). We first define two Boolean random variables: B which is true if the taxi is blue (abbreviated with b) and false if the taxi is green ($\neg b$) and OB which is true if the observed color is blue (abbreviated with ob) and false if it is green ($\neg ob$). Remember that all the taxis are green or blue and that we swear that the taxi was blue. Actually, our reliability is only 75% for discriminating blue and green taxis. We can write this as:
$P(ob|b) = 0.75$ and $P(\neg ob|b) = 0.25$,
$P(\neg ob|\neg b) = 0.75$ and $P(ob|\neg b) = 0.25$.
We also know the prior distribution of B:
$P(b) = \frac{1}{10}$ and $P(\neg b) = \frac{9}{10}$.
In order to compute the posterior probability $P(b|ob)$ we use Bayes' rule with normalization.
$P(b|ob) = \alpha P(ob|b) P(b)$
$P(\neg b|ob) = \alpha P(ob|\neg b) P(\neg b)$
$\alpha [P(ob|b) P(b) + P(ob|\neg b) P(\neg b)] = 1$
$\alpha = 1/(\frac{3}{4}\frac{1}{10}+\frac{1}{4}\frac{9}{10}) = \frac{10}{3}$
Result:
$P(b|ob) = \frac{10}{3}\frac{3}{4}\frac{1}{10} = \frac{1}{4}$.
This means that the probability that the taxi was blue given our observation is 25%. It is more likely that the taxi was green!

Problem 7.3:

a. We first define the random variables: FC that is true if we pick a fake coin (FC = true is abbreviated with fc) and Heads that is true if we flip and get head (Heads = true is abbreviated with heads)
The required probability can be written as
$P(fc|heads) = \alpha P(heads|fc) P(fc)$.
To obtain α we also need
$P(\neg fc|heads) = \alpha P(heads|\neg fc) P(\neg fc)$.
Considering the information in the text of the exercise we can write:
$P(heads|fc) = 1$,
$P(heads|\neg fc) = \frac{1}{2}$,
$P(fc) = \frac{1}{n}$.
We can now calculate α:
$\alpha [P(heads|fc) P(fc) + P(heads|\neg fc) P(\neg fc)] = 1$
$\alpha = 1/(1\cdot\frac{1}{n}+\frac{1}{2}\cdot(1-\frac{1}{n})) = 2\frac{n}{n+1}$
Result:
$P(fc|heads) = 2\frac{n}{n+1}\frac{1}{n} = \frac{2}{n+1}$.

b. The required probability is $P(fc|heads_1,\ldots heads_k)$. We can compute this using Bayes' rule and normalization: $P(fc|heads_1,\ldots heads_k) = \alpha P(heads_1,\ldots heads_k|fc) P(fc)$.
Knowing that a fake coin has heads both sides, we can write:

$P(heads_1, \ldots heads_k | fc) = 1,$
and than
$P(fc | heads_1, \ldots heads_k) = \alpha P(heads_1, \ldots heads_k | fc) P(fc) = \alpha \frac{1}{n}.$
The result of a flip is conditionally independent from the other results given that the coin is not fake. We can thus write:
$P(\neg fc | heads_1, \ldots heads_k) = \alpha \prod_{i=1}^{k} P(heads_i | \neg fc) P(\neg fc) = \alpha \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{n}\right).$
$\alpha \left[\frac{1}{n} + \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{n}\right)\right] = 1$
$\alpha = 1 / \left[\frac{1}{n} + \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{n}\right)\right] = \frac{1}{\left[1 + \left(\frac{1}{2}\right)^k (n-1)\right] \frac{1}{n}}.$
Result:
$P(fc | heads_1, \ldots heads_k) = \frac{1}{\left[1 + \left(\frac{1}{2}\right)^k (n-1)\right] \frac{1}{n}} \frac{1}{n} = \frac{2^k}{2^k + n - 1}.$

c. The required probability is $P(\neg fc, heads_1, \ldots heads_k)$:
$P(\neg fc, heads_1, \ldots heads_k) = P(heads_1, \ldots heads_k | \neg fc) P(\neg fc) = \prod_{i=1}^{k} P(heads_i | \neg fc) P(\neg fc) = \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{n}\right).$

Problem 7.4:

a. We first define two Boolean random variables: CP indicates whether the robot is in a correct pose ($CP = true$ abbreviated with cp) or not ($\neg cp$) and PM that indicates whether a measurement for correct pose is positive ($PM = true$ abbreviated with pm) or not ($\neg pm$). We can write what we know about the reliability of each measurement as:
$P(pm | cp) = 0.80,$
$P(\neg pm | \neg cp) = 0.80.$
We also know the prior probability of a correct pose of the robot:
$P(cp) = 0.45$ and thus $P(\neg cp) = 0.55.$
The posterior probability that the robot is in a good pose given that we have a positive measurement can be computed using Bayes' rule with normalization as
$P(cp | pm) = \alpha P(pm | cp) P(cp),$
$P(\neg cp | pm) = \alpha P(pm | \neg cp) P(\neg cp).$
Considering that $P(pm | \neg cp) = 1 - P(\neg pm | \neg cp) = 0.2$, we can finally answer the question:
$\alpha [P(pm | cp) P(cp) + P(pm | \neg cp) P(\neg cp)] = 1$
$\alpha = 1/(0.8 \cdot 0.45 + 0.2 \cdot 0.55) \approx 2.1277$
Result:
$P(cp | pm) \approx 2.1277 \cdot 0.8 \cdot 0.45 \approx 0.7660.$

b. We introduce an additional random variable for the result of the second measurement $PM2$ (abbreviated with $pm2$ when true and $\neg pm2$ when false). What we need is the probability that the robot is in a good pose given two positive measurements. This can be written as:
$P(cp | pm, pm2) = \alpha P(pm, pm2 | cp) P(cp).$
The two measurements are not independent in general because they both depend from the pose of the robot. We can assume that given a fixed pose of the robot, the two measurements are independent (conditional independence). This simplifies the previous equation and lets us write:
$P(cp | pm, pm2) = \alpha P(pm | cp) P(pm2 | cp) P(cp),$
$P(\neg cp | pm, pm2) = \alpha P(pm | \neg cp) P(pm2 | \neg cp) P(\neg cp).$
$\alpha [P(pm | cp) P(pm2 | cp) P(cp) + P(pm | \neg cp) P(pm2 | \neg cp) P(\neg cp)] = 1$
$\alpha = 1/(0.8 \cdot 0.8 \cdot 0.45 + 0.2 \cdot 0.2 \cdot 0.55) \approx 3.2258$
Result:
$P(cp | pm, pm2) \approx 3.2258 \cdot 0.8 \cdot 0.8 \cdot 0.45 \approx 0.9290.$

The answer is yes, the robot can now collect a sample.

Problem 7.5:

a. This is true.
$P(a,b,c) = P(a|b,c)P(b|c)P(c) = P(b|a,c)P(a|c)P(c)$.
If $P(a|b,c) = P(b|a,c)$:
$\cancel{P(a|b,c)}P(b|c)\cancel{P(c)} = \cancel{P(b|a,c)}P(a|c)\cancel{P(c)}$
$P(b|c) = P(a|c)$. □

b. This is not always true.
Let us consider e.g. the case that: $P(a) = \frac{1}{K_a}$, $P(b) = \frac{1}{K_b}$, $P(c) = \frac{1}{K_c}$, with $K_i > 0$ for $i = a,b,c$.
If $P(a,b,c) > \frac{1}{K_a K_b K_c}$,
$P(a,b,c) = P(a|b,c)P(b|c)P(c) > \frac{1}{K_a K_b K_c}$, thus
$P(b|c) > \frac{1}{K_a K_b K_c} \frac{1}{P(c)P(a|b,c)}$.
Now imposing $P(a|b,c) = P(a)$ we get:
$P(b|c) > \frac{1}{\cancel{K_a} K_b \cancel{K_c}} \frac{1}{\cancel{P(c)}\cancel{P(a)}}$,
$P(b|c) > \frac{1}{K_b}$, and thus $P(b|c) \neq P(b)$. □

c. This is not always true.
Let us consider three Boolean random variables A, B and C and assume independence between A and B.
For compactness we use the notation a for $A = true$ and $\neg a$ for $A = false$, the same for the other random variables.
Consider the case that the random variable C assumes the following values:

- True if $< a,b >$ or $< \neg a, \neg b >$;
- False if $< \neg a, b >$ or $< a, \neg b >$.

$P(c|a,b) = 1,\ P(\neg c|a,b) = 0$;
$P(c|\neg a,\neg b) = 1,\ P(\neg c|\neg a,\neg b) = 0$;
$P(c|\neg a,b) = 0,\ P(\neg c|\neg a,b) = 1$;
$P(c|a,\neg b) = 0,\ P(\neg c|a,\neg b) = 1$.
We can now calculate the prior probability of c as:
$P(c) = P(c|a,b)P(a,b) + P(c|\neg a,\neg b)P(\neg a,\neg b) + P(c|\neg a,b)P(\neg a,b) + P(c|a,\neg b)P(a,\neg b)$.
Exploiting the independence between A and B we can write:
$P(c) = P(a)P(b) + P(\neg a)P(\neg b)$.
Let us introduce numbers:
$P(a) = \frac{1}{2}$,
$P(b) = \frac{1}{2}$,
$P(c) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
In this case we obtain:
$P(a,c) = P(c|a,b)P(a)P(b) + P(c|a,\neg b)P(a)P(\neg b) = \frac{1}{4}$,
$P(b,c) = P(c|a,b)P(a)P(b) + P(c|\neg a,b)P(\neg a)P(b) = \frac{1}{4}$,
$P(a,b,c) = P(c|a,b)P(a)P(b) = \frac{1}{4}$.
It is now possible to show that in this case the statement of this exercise is not true:
$P(a|b,c) = \frac{P(a,b,c)}{P(b,c)} = 1 \neq P(a|c) = \frac{P(a,c)}{P(c)} = \frac{1}{2}$. □