Grundlagen der Künstlichen Intelligenz

Solution 10: Making Simple Decisions 24th January 2020

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Problem 10.1:

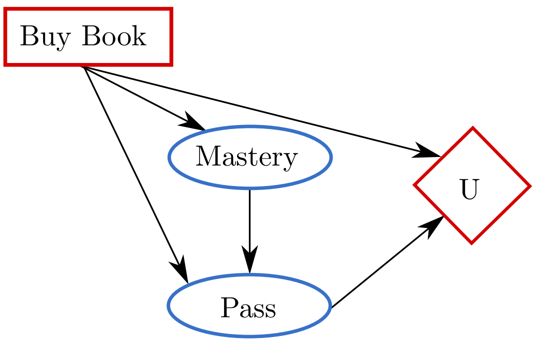
(Taken from **Russell and Norvig 2010** Exercise 16.15)

$$\begin{aligned} P(p|b,m) &= 0.9 \\ P(p|b,\neg m) &= 0.5 \\ P(p|\neg b,m) &= 0.8 \\ P(p|\neg b,\neg m) &= 0.3 \\ P(m|b) &= 0.9 \\ P(m|\neg b) &= 0.7 \end{aligned}$$

We have the following utilities:

$$U(b) = -100€, \quad U(\neg b) = 0€ \quad U(p) = 2000€, \quad U(\neg p) = 0€$$

Problem 10.1.1: Draw the decision network for the problem.



Problem 10.1.2: Compute the expected utility of buying the book and of not buying it.

General formula for expected utility:

$$EU(a|\mathbf{e}) = \sum_{s'} P(\texttt{Result}(a) = s'|a, \mathbf{e})U(s')$$

First of all, the probabilities of passing the exam after buying the book and not buying the book must be calculated. The probability of passing the exam after buying the book:

$$\begin{aligned}
P(p|b) &= \frac{\sum_{M\in\{m,\neg m\}} P(p,b,M)}{P(b)} \\
&= \frac{\sum_{M\in\{m,\neg m\}} P(p|b,M)P(M|b)P(b)}{P(b)} \\
&= \sum_{M\in\{m,\neg m\}} P(p|b,M)P(M|b) \\
&= P(p|b,m)P(m|b) + P(p|b,\neg m)P(\neg m|b) \\
&= 0.9 \cdot 0.9 + 0.5 \cdot 0.1 \\
&= 0.86
\end{aligned}$$

The probability of passing the exam without buying the book:

$$\begin{aligned}
P(p|\neg b) &= \sum_{M\in\{m,\neg m\}} P(p|\neg b,M)P(M|\neg b) \\
&= P(p|\neg b,m)P(m|\neg b) + P(p|\neg b,\neg m)P(\neg m|\neg b) \\
&= 0.8 \cdot 0.7 + 0.3 \cdot 0.3 \\
&= 0.65
\end{aligned}$$

The expected utility of buying the book:

$$\begin{aligned}
EU(b) &= \sum_{P\in\{p,\neg p\}} P(P|b)\,U(P,b) \\
&= P(p|b)\,U(p,b) + P(\neg p|b)\,U(\neg p,b) \\
&= 0.86 \cdot (2000€ - 100€) + 0.14 \cdot (-100€) \\
&= 1620€
\end{aligned}$$

The expected utility of not buying the book:

$$\begin{aligned}
EU(\neg b) &= \sum_{P\in\{p,\neg p\}} P(P|\neg b)\,U(P,\neg b) \\
&= P(p|\neg b)\,U(p,\neg b) + P(\neg p|\neg b)\,U(\neg p,\neg b) \\
&= 0.65 \cdot (2000€) + 0.35 \cdot (0€) \\
&= 1300€
\end{aligned}$$

Problem 10.1.3: What should Sam do?
Sam should definitely buy the book!

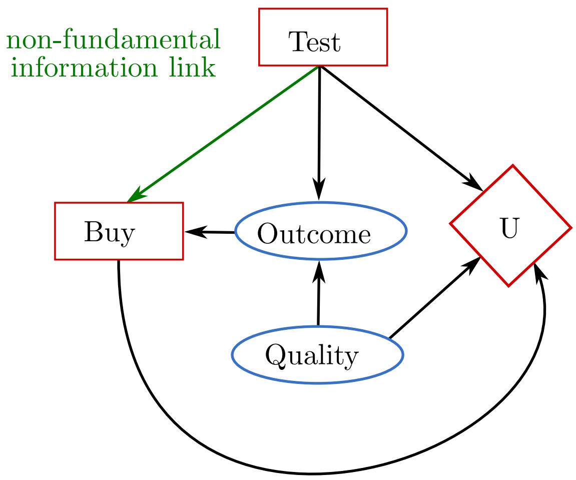
Problem 10.2:
(Taken from **Russell and Norvig 2010** Exercise 16.17)

$$\begin{aligned} P(q) &= 0.7 \\ P(\neg q) &= 0.3 \end{aligned}$$

We have the following utilities:

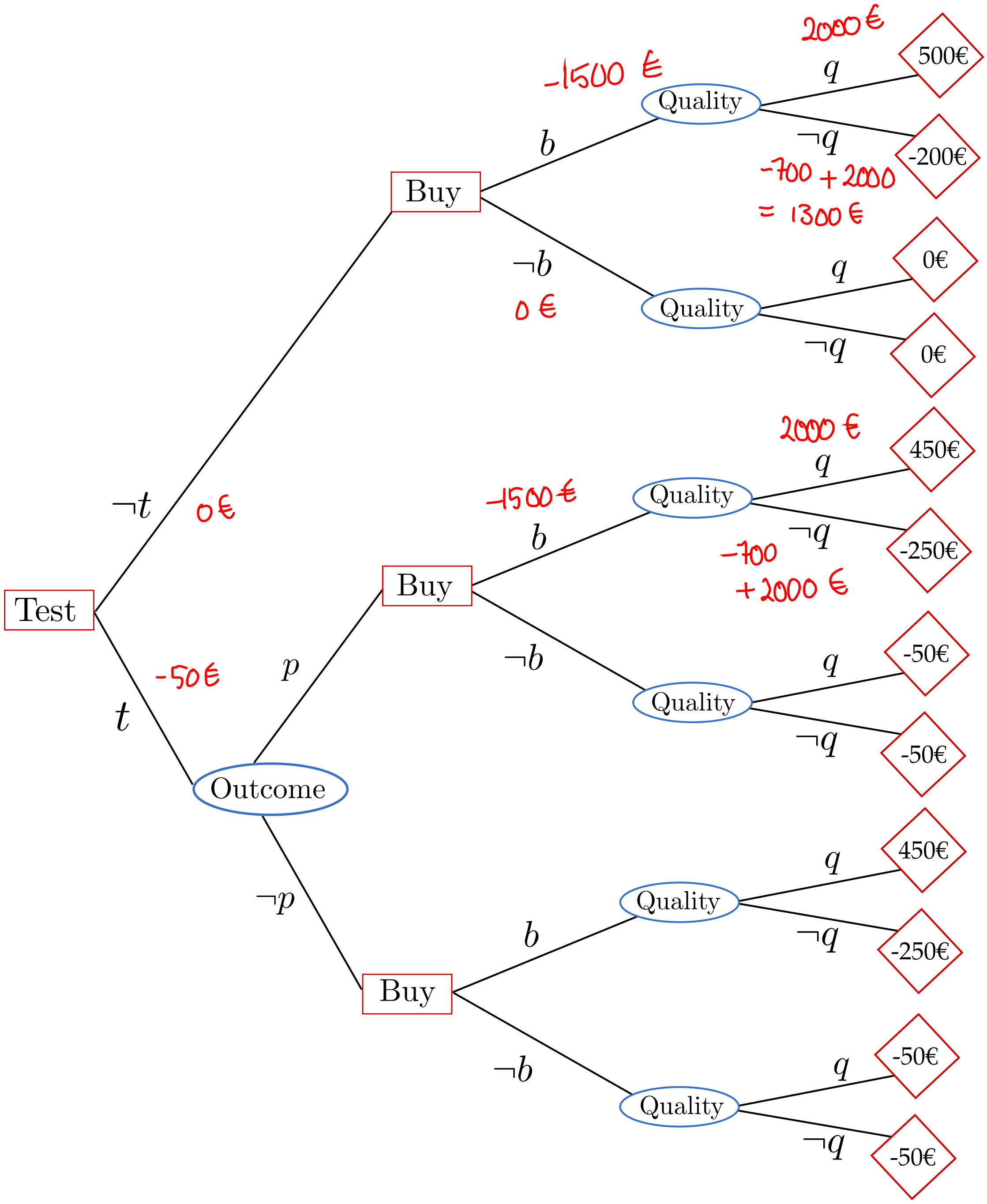
$U(b) = -1500$€, $U(q) = 2000$€ $U(\neg q) = 2000 - 700 = 1300$€, $U(t) = -50$€
$U(\neg t) = 0$€

Problem 10.2.1: Draw the decision network that represents this problem.



Note: A information link is fundamental if its removal would change the partial ordering. The *no forgetting assumption* means that all past decisions and revealed variables are available at the current decision. When the *no forgetting assumption* hold, we do not need to draw non-fundamental information links.

Problem 10.2.2: Draw the decision tree that represents this problem.



Note: The utility depends on the decision T and B as well as on the quality Q of the car: $U(T, B, Q)$, e.g. $U(t, b, \neg q) = -50€ - 1500€ + (2000 - 700)€ = -250€$.

Problem 10.2.3: Calculate the expected utility of buying c1, given no test.

$$\begin{aligned}
EU(b, \neg t) &= \sum_{Q \in \{q, \neg q\}} P(Q|b, \neg t) U(Q, b, \neg t) \\
&= \sum_{Q \in \{q, \neg q\}} P(Q) U(Q, b, \neg t) \\
&= P(q)U(q, b, \neg t) + P(\neg q)U(\neg q, b, \neg t) \\
&= 0.7 \cdot (2000€ - 1500€ - 0€) + 0.3 \cdot (1300€ - 1500€ - 0€) \\
&= 290€
\end{aligned}$$

Tests can be described by the probability that the car will pass or fail the test ($P \in \{p, \neg p\}$) given that the car is in good or bad shape. We have the following information:

$$\begin{aligned}
P(p) &= 0.67 & P(q|p) &= 0.84 \\
P(\neg p) &= 0.33 & P(\neg q|p) &= 0.16 \\
P(q|\neg p) &= 0.42 & P(\neg q|\neg p) &= 0.58
\end{aligned}$$

Important

Problem 10.2.4: Calculate the value of information of the test, and derive an optimal conditional plan for the buyer. Start with determining the optimal decisions whether to buy the car given no test, a pass or a fail.

The general formula for the value of information is:

$$\boxed{VOI_{\mathbf{e}}(E_j) = \left(\sum_k P(E_j = e_{jk}|\mathbf{e}) MEU(\alpha_{e_{jk}}|\mathbf{e}, E_j = e_{jk}) \right) - MEU(\alpha|\mathbf{e}).}$$

The basic idea of the expected value of information is that we compute the expected value of the best action given the (costless!) information minus the expected value of the best action without information. Thus, the expected value of information helps to determine how much one should pay for an additional piece of information:

- $VOI_{\mathbf{e}}(E_j) > cost_{E_j}$: Buying the new information helps to gain money,
- $VOI_{\mathbf{e}}(E_j) = cost_{E_j}$: We do not loose or gain money by buying the new information,
- $VOI_{\mathbf{e}}(E_j) < cost_{E_j}$: We loose money by buying the new information.

*Given the initial evidence $\mathbf{e}$, the value of the current best action α is

$$\boxed{MEU(\alpha|\mathbf{e}) = \max_a \sum_{s'} P(\mathtt{Result}(a) = s'|a, \mathbf{e}) U(s'),}$$

* and the value of the new best action α_{e_j} (after new evidence $E_j = e_j$) is

$$\boxed{MEU(\alpha_{e_j}|\mathbf{e}, e_j) = \max_a \sum_{s'} P(\texttt{Result}(a) = s'|a, \mathbf{e}, e_j)U(s').}$$

We start with the definition of our actions a, the initial evidence $\mathbf{e}$, the new evidence e_j and the states s':

$$a \in \{b, \neg b\} \qquad \mathbf{e} = \emptyset$$
$$e_j \in \{p, \neg p\} \qquad s' \in \{q, \neg q\}$$

We have the following utilities:
$U(q, b) = 2000€ - 1500€ = 500€$
$U(\neg q, b) = 1300€ - 1500€ = -200€$
$U(q, \neg b) = 0€$
$U(\neg q, \neg b) = 0€$

First, we can show that buying the car is the optimal solution when no evidence is given:

$$\begin{aligned}
MEU(\alpha) &= \max \quad (EU(b); EU(\neg b)) \\
&= \max \quad \big(P(q)U(q, b) + P(\neg q)U(\neg q, b); \\
&\qquad\qquad P(q)U(q, \neg b) + P(\neg q)U(\neg q, \neg b)\big) \\
&= \max \quad (0.7 \cdot 500€ + 0.3 \cdot (-200€); \\
&\qquad\qquad 0.7 \cdot 0€ + 0.3 \cdot 0€) \\
& \max \quad (290€, 0€) \\
&= 290€
\end{aligned}$$

Now, we assume that we are given the information of pass or fail without paying for the test and determine the optimal decisions. We start with the assumption that p is observed and compute the new best action α_p:

$$\begin{aligned}
MEU(\alpha_p|p) &= \max \quad (EU(b|p);\ EU(\neg b|p)) \\
&= \max \quad \big(P(q|p)U(q, b) + P(\neg q|p)U(\neg q, b); \\
&\qquad\qquad P(q|p)U(q, \neg b) + P(\neg q|p)U(\neg q, \neg b)\big) \\
&= \max \quad (0.84 \cdot 500€ + 0.16 \cdot (-200€); \\
&\qquad\qquad 0.84 \cdot 0€ + 0.16 \cdot 0€) \\
& \max \quad (388€, 0€) \\
&= 388€
\end{aligned}$$

Buying the car provides the maximum expected utility when pass is observed.

Then we assume that $\neg p$ is observed and compute the new best action $\alpha_{\neg p}$:

$$
\begin{array}{rcll}
MEU(\alpha_{\neg p}|\neg p) & = & \max & (EU(b|\neg p);\ EU(\neg b|\neg p)) \\
 & = & \max & \big(P(q|\neg p)U(q,b) + P(\neg q|\neg p)U(\neg q,b); \\
 & & & (P(q|\neg p)U(q,\neg b) + P(\neg q|\neg p)U(\neg q,\neg b)\big) \\
 & = & \max & (0.42 \cdot 500€ + 0.58 \cdot (-200€); \\
 & & & 0.42 \cdot 0€ + 0.58 \cdot 0€) \\
 & & \max & (94€, 0€) \\
 & = & 94€ &
\end{array}
$$

Buying the car provides the maximum expected utility when not pass is observed.

Finally, we can calculate the value of information of the test, and derive an optimal conditional plan for the buyer.

$$
\begin{array}{rcl}
VOI_{\mathbf{e}}(E_j) & = & \big(P(p)MEU(\alpha_p|p) + P(\neg p)MEU(\alpha_{\neg p}|\neg p)\big) - MEU(\alpha) \\
 & = & (0.67 \cdot 388€ + 0.33 \cdot 94€) - 290€ \\
 & \approx & 0€
\end{array}
$$

Buying the car without carrying out a test is the best solution, since the test outcome does not change the optimal decision and $VPI_{\mathbf{e}}(E_j) < 50€$. For instance, if the probability $P(q|\neg p)$ is reduced, the test would be more valuable.