Discrete Mathematics 1

Second Midterm Exam

Exercise 1.

(a) What is an open variable in a formula? Provide an example of a bound variable in a formula. (**2 points**)

(b) Let $P(x, y)$ be a formula depending on the variables x, y from the sets X, Y. Is $(\forall x \in X)(\exists y \in Y)P(x, y)$ equivalent to $(\exists y \in Y)(\forall x \in X)P(x, y)$? Motivate your answer. (**3 points**)

(c) Let P, Q be formulas with variables x, y from some universe. Show that

$$\neg(\forall x)P(x) \equiv (\exists x)(\forall y)\neg Q(x, y),$$

where $P(x) = (\exists y)Q(x, y)$. (**4 points**)

Exercise 2.

(a) *Let $x, y \in \mathbb{N}$. If x, y are both odd, then their sum is even.*

Which kind of statement is this: universal or existential? If possible, prove it using a contrapositive proof and a contradiction proof. (**3 points**)

(b) Explain what the pigeon-hole principle is and provide an example of application. Can this principle be used to prove a universal statement? Motivate your answer. (**3 points**)

(c) What is the induction principle useful for? What is the role played by natural numbers? (**3 points**)

Exercise 3.

(a) Provide the theoretical formulation of the induction principle and use it to prove the most common formulation of it. Which of them uses the notion of inductive set and which one is expressed in terms of logic (formulas, variables, quantifiers, etc.)? Motivate your answer. (**4 points**)

(b) What is the main difference between the standard induction and the strong induction? Provide a formulation of the latter. (**3 points**)

(c) Use induction to show that the Cartesian product of $n \in \mathbb{N}$ countable sets is countable. (**4 points**)

Exercise 4.

(a) Define countable sets. Which cardinals do countable sets give rise to? Is it possible to obtain uncountable sets from a countable set? Motivate your answer. (**3 points**)

(b) Is there any partial ordering on cardinals? Is this total? Motivate your answer by mentioning the well-known statements that are involved here. (**4 points**)

(c) Define the sum and multiplication of cardinals. Why $2\alpha < 2^\alpha + \alpha$ (where α is cardinal) and $\aleph_0 \cdot 53 = \aleph_0 + 97$? (**4 points**)

Exercise 5. Consider the sets

$$S_1 = \{n \in \mathbb{N} \mid 3 \leq n \leq 15\}$$
$$S_2 = \{n \in S_1 \mid n = 2k + 1 \text{ for some } k \in \mathbb{Z}\}.$$

Let $P(n)$ be the statement "$n \in S_1$" and $Q(n)$ be the statement "$n \in \mathbb{Z}^{\text{odd}}$".

1. Write an equivalent statement to "$n \in S_2$", using statements $P(n)$ and $Q(n)$. (**4 points**)
2. Write "$P(n)$ and $\neg Q(n)$" in words. For which numbers is this statement true? (**4 points**)

Exercise 6. Let $p > 1$ be an integer. Prove that $x \equiv y \pmod p$ if and only if x and y have the same remainder in the division by p. (**8 points**)

Exercise 7*. Prove that $1 + 3n < n^2$ for every positive integer $n \geq 4$. (**12 points**)

Exercise 8*. If x is a non-zero integer, then x^{2n} is positive for every positive integer n. (**12 points**)

Exercise 9. Prove that the open intervals $(\frac{1}{2}, \frac{3}{4})$ and $(\sqrt{2}, 5)$ have the same cardinality. Furthermore, say what cardinal they represent. (**10 points**)

Exercise 10. Prove that the set $S = \{2^n + 3^m \mid n, m \in \mathbb{N}\}$ is countable. (**10 points**)

***Use induction!**