# Rainbow Perfect Matchings

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## *Algorithm*

1. Fix prime $p \gg n$ [1]
2. For each colour $C \in [n]$
   Construct $n * n$ matrix $m_C$
   For each $uv \in E$ with $c(uv) = C$
   Pick random integer $r \in (0, p]$
   Set $m_C[u, v] = r$
3. Set $B = \sum_{i=0}^{n-1} m_i$
4. Compute $d_B = \det(B) \bmod p$
5. If $d_B = 0$ return "no"
6. Else set $sum = 0$
7. For each $X \subset [n]$
   Initialize $M = \mathbf{0}$ [2]
   For each $C \in X$
   Set $M = M + m_C$
   Set $sum = sum + (-1)^{n-|X|-1} \cdot \det(M) \bmod p$
8. If $d_B - sum = 0$ return "no" else return "yes"

[1] In our case, we selected $p = 32749$.

[2] $\mathbf{0}$ is the $n * n$ all-zeroes matrix.

Our algorithm modifies the given Algorithmic Piece 1 on step 2, by creating $n$ matrices, one for each colour. But in step 3 we combine them into the biadjacency matrix $B$ (called $A_G$ in the assignment) and do the same end condition for its determinant.

For Algorithmic Piece 2, we have modified the pseudo code to be defined via matrix sums instead. We construct the biadjacency matrix for the current set of colours by simple addition, and compute the determinant for every such matrix. We are not including $\det(B)$ in $sum$, so to get the signs right we subtract an extra 1 in the exponent for $-1$.[3] This is really just an optimization, as we could remove steps 3 and 4 and have Algorithmic Piece 2 more or less intact. Our algorithm is faster for "no"-instances, however. With this change in mind, the logic is the same as in the assignment for why a non-rainbow perfect matching will eliminate itself in the calculation of $sum$.

[3] Another way to fix this would be to compute $d_B + sum$ in step 7 of the algorithm, but we thought this was cleaner.

## *Running Time*

In our calculations, we define the basic, constant-time operations to be scalar additions and scalar multiplications. Any other operation we assume to take no time.

We are only interested in a running time bound with respect to $n$, hence we assume a worst-case $m = n^2$, and do not include $m$ in our $O$-notation.

Step by step, we pay as follows:

1. No time.
2. $n$ repetitions of

   No time.

   Worst case $n$ edges of colour $C$ from all $n$ nodes $\Rightarrow T(n) = n^2$ repetitions of

   No time.

   No time.
3. $T(n) = n^3$ additions $\Rightarrow T(n) = n^3$
4. $T(n) = \frac{2}{3} \cdot n^3 \Rightarrow T(n) = \frac{2}{3}n^3$ [4]
5. No time.
6. No time.
7. There are $2^n$ subsets to a set of $n$ elements $\Rightarrow T(n) = 2^n - 2$ (the empty set and the full set) repetitions of

   No time.

   On average for each $X$, $T(n) = \frac{1}{2}n$ [5] repetitions of $n^2$ additions $\Rightarrow T(n) = n^2$

   1 addtition, worst case $n-2$ multiplications of $-1$, 1 multiplication and $T(n) = \frac{2}{3}n^3$ cost for determinant calculation gives $T(n) = 1 + (n-2) + 1 + \frac{2}{3}n^3 = \frac{2}{3}n^3 + n$
8. 1 addtition $\Rightarrow T(n) = 1$

[4] Knowing our determinant calculation implements Gaussian elimination, this is the number of operations required. Normally, this would have exponential bit complexity, but our calculations are made modulus $p$, and are thus constant in bit complexity .

[5] This follows from how frequent a given size of an element in the power set of $[n]$ is .

This results in final running time of

$$T(n) = 0 + n(0 + n^2(0+0)) + n^3 + \frac{2}{3}n^3 + 0 + 0 + (2^n - 2)(0 + \frac{1}{2}n(n^2) + \frac{2}{3}n^3 + n) + 1$$

$$= \frac{5}{3}n^3 + (2^n - 2)(\frac{7}{6}n^3 + n) + 1$$

$$= \frac{5}{3}n^3 + 1 - \frac{7}{3}n^3 - 2n + 2^n(\frac{7}{6}n^3 + n)$$

$$= 1 - 2n - \frac{2}{3}n^3 + 2^n(\frac{7}{6}n^3 + n)$$

The dominant factor here is, as expected, exponential in $n$. Thus we do have $O^*(2^n)$. We could also present a running time bound of $O(n^3 \cdot 2^n)$, in which we consider all factors with an exponential in $n$ relevant, and take the largest-degree polynomial of this factor as our bound.

The exponential $2^n$ factor come from the fact that we are traversing the power set of an $n$-sized set. The plynomial $n^3$ factor comes from our determinant calculation, which runs this fast. Thus, the $O(n^3 \cdot 2^n)$ really summarizes our algorithm in a nice way; we do calculate a determinant for a matrix corresponding to every element of the power set of $[n]$.

## *Failure bound*

Since we are working with random numbers, there is always some cases in which these random numbers doesn't work as we expect them to. In particular, two random numbers may become the same. This will provide our algorithm with false negatives, which's frequency we want to investigate.

First, we conclude that if there is exactly one Perfect Rainbow Matching, there will be no false negative. This follows because $p$ is a prime, and $d_B - sum$ will consist of only one monomial $x$, which is a product of random numbers and hence it can not be prime, so $x$ mod $p \neq 0$.

Second, if we have $k > 1$ Perfect Rainbow Matchings, we will have $k$ monomials in $d_B - sum$, each of degree $d$ such that $2 \leq d \leq n$, depending on how many edges the PRM represented by $k_i$ shares with its counterpart $k_j$. Note that the minimum bound on $d$ can not be 1, since we are not allowing any two edges $uv$ and $(uv)'$ to have the same colour. The sum of the monomials may be equal to zero, in fact the probability that they are follows directly from the Schwartz-Zippel lemma as[6]

[6]

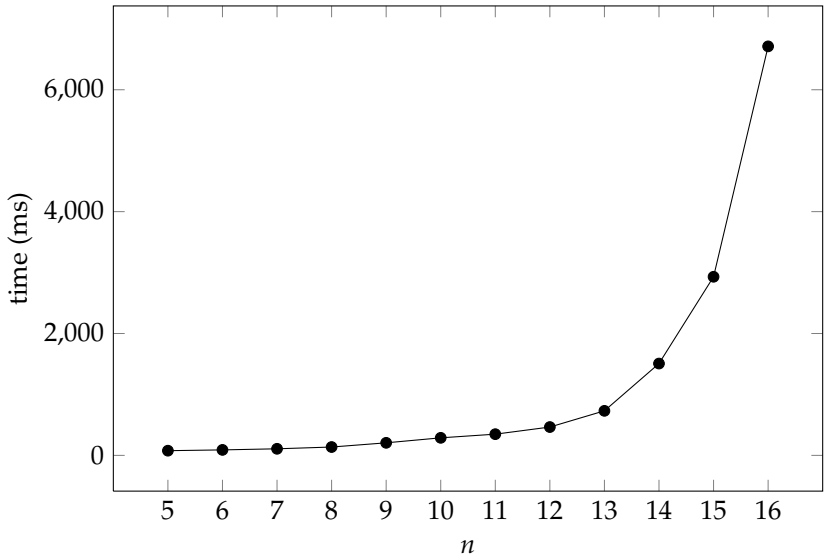
$$Pr[f(r_1, r_2, \ldots, r_m) = 0] \leq \frac{d}{|S|}$$

where $f$ is the sum of our $k$ monomials, with random numbers $r_1$ through $r_m$ (in general) taken from the finite interval $[1, p-1] \Rightarrow |S| = p - 1 - 1 + 1 = p - 1$. In the worst case, we have $d = n$ which gives us the probability of $f = 0$ as

$$\frac{d}{|S|} \leq \frac{n}{p-1}$$

## *Benchmark*

We perfrom two benchmarking tests on our PRM implementation. First, we run the algorithm for all the provided yes and no instances (total 160 instances) and measure the time (in milliseconds) it took to complete. We plot the average time of the 10 $n$-sized instances (yes and no instances show no difference in time) to $n$.




Clearly, this is in alignment with the results from the running time analysis, pointing towards an exponential time algorithm.

Now we want to investigate how fool-proof our algorithm is. So for various values of $p$, we run through the yes instances provided and count the number of false negatives. Our first intension was to plot this, but the experiment did not produce enough false negatives for this to be interesting. For $p = 23$, we had a mere 4 false negatives for all the 80 instances[7], indicating failure probability of about 5%, which is in accordance with Schwartz-Zippel's bound of about 52%[8].

[7] We did not make sure that the number of PRM in these instances were in fact more than 1.
[8] Average size of $n$ is 11,45.

## *Implementation*

Below follows our implementation of our Perfect Rainbow Matching algorithm. It is only sparsely commented, but we expect its behaviour to be fairly straight-forward. We have tried to follow our pseudo-code verbatim, putting aside Java programming practices to some extent.

```
package p1;

import java.io.FileNotFoundException;
import java.io.FileReader;
import java.util.ArrayList;
import java.util.HashSet;
import java.util.Random;
import java.util.Scanner;

/**
 * For all user interaction. Reads input, posts error messages, etc.
 *
 * @author dt08mr7
 * @author dt08ml5
 *
 */
public class Main {
    private static final int PRIME = 32749; // largest prime smaller than 2^15

    public static void main(String[] args) throws FileNotFoundException {
        // Time measurement
        long start = System.currentTimeMillis();

        // 1
        int prime = PRIME;

        Scanner scan = new Scanner(new FileReader(args[0]));
        int n = scan.nextInt();
        init_inverses(prime);
        int m = scan.nextInt();

        ArrayList<HashSet<Integer>> colourSets =
                new ArrayList<HashSet<Integer>>();
        ArrayList<int[][]> colourMatrices = new ArrayList<int[][]>();
        HashSet<Integer> allColours = new HashSet<Integer>();
        for (int i = 0; i < n; i++) {
            colourMatrices.add(new int[n][n]);
            ArrayList<HashSet<Integer>> newColourSets =
                    new ArrayList<HashSet<Integer>>();
            for (HashSet<Integer> set : colourSets) {
                HashSet<Integer> combination = new HashSet<Integer>(set);
                combination.add(i);
                newColourSets.add(combination);
            }
            HashSet<Integer> aloneColourSet = new HashSet<Integer>();
            aloneColourSet.add(i);
            allColours.add(i);
            colourSets.add(aloneColourSet);
            colourSets.addAll(newColourSets);
        }
        colourSets.remove(allColours);
```

```
// 2
Random rand = new Random();
for (int i = 0; i < m; i++) {
    int from = scan.nextInt();
    int to = scan.nextInt();
    int colour = scan.nextInt();
    colourMatrices.get(colour)[from][to] = rand.nextInt(prime-1) + 1;
}

// 3
int[][] allColourMatrix = new int[n][n];
for (int i = 0; i < n; i++) {
    for (int j = 0; j < n; j++) {
        for (int k = 0; k < n; k++) {
            allColourMatrix[j][k] = (allColourMatrix[j][k] +
                                colourMatrices.get(i)[j][k]) % prime;
        }
    }
}

// 4
int dB = 0;
dB = determinant(allColourMatrix, prime);

// 5
if (dB == 0) {
    System.out.println("Running time (ms): " +
                       (System.currentTimeMillis() - start));
    System.out.println("No");
    return;
}

// 6
int sum = 0;

// 7
for (HashSet<Integer> X : colourSets) {
    int[][] M = new int[n][n];
    for (int i : X) {
        for (int j = 0; j < n; j++) {
            for (int k = 0; k < n; k++) {
                M[j][k] =
                    (M[j][k] + colourMatrices.get(i)[j][k]) % prime;
            }
        }
    }
    sum += (int) Math.pow(-1, (n - X.size() - 1)) *
                                                determinant(M, prime);
}
```

```
    // 8
    /*
     * Since the % operator doesn't work as expected for negative numbers
     * in Java, we manually modulo the sum to the correct interval
     */
    while (sum < 0) {
        sum += prime;
    }
    sum = sum % prime;
    System.out.println("Running time (ms): " +
                       (System.currentTimeMillis() - start));
    if (dB - sum == 0)
        System.out.println("No");
    else
        System.out.println("Yes");
}

// Determinant calculation taken from Andreas

static int inv[];

public static int modexp(int a, int e, int p) {
    if (e == 1) {
        return a;
    } else {
        int sq = modexp(a, e / 2, p);
        sq = (sq * sq) % p;
        if ((e & 1) == 1) {
            sq = (sq * a) % p;
        }
        return sq;
    }
}

public static void init_inverses(int p) {
    int i;
    inv = new int[p];
    for (i = 1; i < p; i++) {
        inv[i] = modexp(i, p - 2, p);
    }
}

public static int determinant(int[][] A, int p) {
    int n = A[0].length;
    int i, j, k;
    int rs[], vis[];
    int d = 1;
    int sgn = 0;
    rs = new int[n];
    vis = new int[n];
    for (i = 0; i < n; i++) {
```

```
            rs[i] = 0;
        }
        for (i = 0; i < n; i++) {
            int who = -1;
            for (j = 0; j < n; j++)
                if (rs[j] == 0 && A[j][i] > 0) {
                    rs[j] = i + 1;
                    who = j;
                    d = (d * A[j][i]) % p;
                    break;
                }
            if (who == -1) {
                d = 0;
                break;
            }
            for (j = 0; j < n; j++)
                if (rs[j] == 0 && A[j][i] > 0) {
                    d = (d * inv[A[who][i]]) % p;

                    for (k = n - 1; k >= i; k--) {
                        A[j][k] = (A[who][i] * A[j][k] -
                                    A[j][i] * A[who][k] + p * p) % p;
                    }
                }
        }
        if (d > 0) {
            sgn = n;
            for (i = 0; i < n; i++) {
                vis[i] = 0;
            }
            for (i = 0; i < n; i++) {
                if (vis[i] == 0) {
                    sgn--;
                    j = i;
                    while (vis[j] == 0) {
                        vis[j] = 1;
                        j = rs[j] - 1;
                    }
                }
            }

            if ((sgn & 1) == 1)
                d = p - d;
        }
        return d;
    }
}
```

## *References*

[1] Wikipedia, the Free Encyclopedia, *Power set*, http://en.wikipedia.org/wiki/Power_set#Relation_to_binomial_theorem, 10 October 2012 05:20

[2] Wikipedia, the Free Encyclopedia, *Gaussian elimination*, http://en.wikipedia.org/wiki/Gaussian_elimination#Analysis, 29 September 2012 03:39

[3] Wikipedia, the Free Encyclopedia, *SchwartzâĂŞZippel lemma*, http://en.wikipedia.org/wiki/Schwartz%E2%80%93Zippel_lemma#Schwartz.E2.80.93Zippel_lemma, 3 October 2012 15:22