PYU33C01 – Assignment 4:
Throwing Darts and the Poisson Distribution

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1 Introduction

In this report we demonstrate how the Poisson distribution emerges naturally from a simulated random process, throwing darts onto a divided dart board. Firstly, we plot the Poisson distribution, with a range of $\langle n \rangle = 1, 5$, and 10, and use this to verify the normalisation condition, mean and variance. Subsequently, a Monte Carlo simulation ran for T number of trials approximated the darts being thrown onto an equally divided dart board. This produced the histograms, $H(n)$, which were used to obtain $P_{sim}(n)$ and shown to approximate the Poisson distribution, $P(n)$. The simulated distributions were then compared with the analytical distributions using logarithmic scales to determine the lowest probabilities probed by the simulation. Lastly, by varying the number of trials T, and divided regions L, we examine how the convergence of P_{sim} towards the Poisson Poisson distribution. This shows the transition from discrete binomial sampling of dart throws to the continuous Poisson distribution in the limit $L \to \infty$ and $N \to \infty$.

1.1 Theoretical Background

Each throw of a dart can be seen as an independent *Bernoulli trial*, with two possible outcomes for a given region of the dart board, dart lands on this region(success), or dart misses this region(failure). The probability of success is given by $p = \frac{1}{L}$, meaning after N independent trials the probability that exactly n darts land in a given region follows from the *binomial distribution*:

$$P(n) = \binom{N}{n} p^n (1-p)^{N-n}. \tag{1}$$

Where N is the number of darts thrown and p is the probability of a dart hitting a given region, $\frac{1}{L}$. The Poisson distribution will emerge from this process when there has been many darts thrown, each with very small probability of success to land in a given region. This can be seen by taking the appropriate limits:

$$N \to \infty, \quad L \to \infty \implies p \to 0, \quad \text{with } \langle n \rangle = Np = \text{constant}. \tag{2}$$

Starting from the binomial distribution,

$$P(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}, \tag{3}$$

Substituting $p = \frac{\langle n \rangle}{N}$ and expanding the $(1-p)^{N-n}$ term using the limit definition of the exponential function:

$$(1-p)^{N-n} = \left(1 - \frac{\langle n \rangle}{N}\right)^{N-n} \approx e^{-\langle n \rangle} \left(1 - \frac{\langle n \rangle}{N}\right)^{-n} \xrightarrow[N\to\infty]{} e^{-\langle n \rangle}. \tag{4}$$

For large N and finite n, the binomial coefficient simplifies as

$$\frac{N!}{(N-n)!} \approx N^n, \tag{5}$$

so that

$$P(n) \approx \frac{N^n}{n!}\left(\frac{\langle n\rangle}{N}\right)^n e^{-\langle n\rangle} = \frac{\langle n\rangle^n}{n!}e^{-\langle n\rangle}. \tag{6}$$

This leads us to the desired *Poisson distribution*,

$$P(n) = \frac{\langle n\rangle^n}{n!}e^{-\langle n\rangle}. \tag{7}$$

1.2 Key Properties of the Poisson Distribution

The Poisson distribution has the following moments:

- Normalization, μ_0: $\sum_{n=0}^{\infty} P(n) = 1$
- Mean, μ_1: $\langle n\rangle = \sum_{n=0}^{\infty} nP(n)$
- Variance, μ_2: $\sigma^2 = \langle n\rangle \implies \sigma = \sqrt{\langle n\rangle}$

2 Methodology

2.1 Plotting Poisson Distributions

Three Poisson distributions were plotted for $\langle n\rangle = 1, 5, 10$ using Equation 7. To avoid overflows due to large factorials, the probability was computed in the log space:

$$\log P(n) = n\log(\langle n\rangle) - \langle n\rangle - \log(n!) \tag{8}$$

where $\log(n!) = \log\Gamma(n+1)$ was obtained using the gamma function.

2.2 Verification of Poisson Distribution Moments

For $N = 50$ and $\langle n\rangle = 1, 5, 10$, the following moments were calculated:

- $\sum_{n=0}^{N} P(n) - \mu_0$
- $\sum_{n=0}^{N} nP(n) - \mu_1$
- $\sum_{n=0}^{N} n^2P(n) - \mu_0$

leading to a variance of $\sigma = \mu_2 - \mu_1^2$ and standard deviation a $\sigma = \sqrt{\sigma^2}$

2.3 Monte Carlo Simulation

The throwing of darts was simulated using the following algorithm:

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-Initialize a histogram H(n) with zeros
-For each trial t = 1 to T:
----Create an array B of length L initialized to zero
----Throw N darts by generating N random integers between 0 and L − 1
----For each dart, increment the corresponding element in array B
----Count how many regions have exactly n darts and add to H(n)
Normalize:  P_sim(n) = H(n)/(LT)
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Initial parameters: $N = 50$ darts, $L = 100$ regions, $T = 10$ trials.

2.4 Log-Scale Analysis

The data obtained from the analytical Poisson distribution and the simulated darts throwing experiment was plotted twice, first in a linear scale and second in a logarithmic scale. This was used to identify the minimum probability values that the simulated experiment could probe.

2.5 Varying Trial Numbers

The simulation was ran for a range of numbers of trials, $T = 10, 100, 1000, 10000$ with $L = 100$, and then again with $L = 5$. This demonstrated the effects of violating the assumptions set out in the limits taken in (2), as for $L = 5$, the probability of success on a given trial is not sufficiently small to lead to the Poisson distribution. This leads to a slower convergence in the agreement between the simulated distribution $P_{sim}(n)$ and analytical Poisson distribution, $P(n)$.

3 Results

3.1 Task 1: Poisson Distributions

Figure 1 shows the Poisson distribution foe three different means. As $\langle n \rangle$ increases, the distributions broadens and shifts right, whilst the peak probability diminishes.

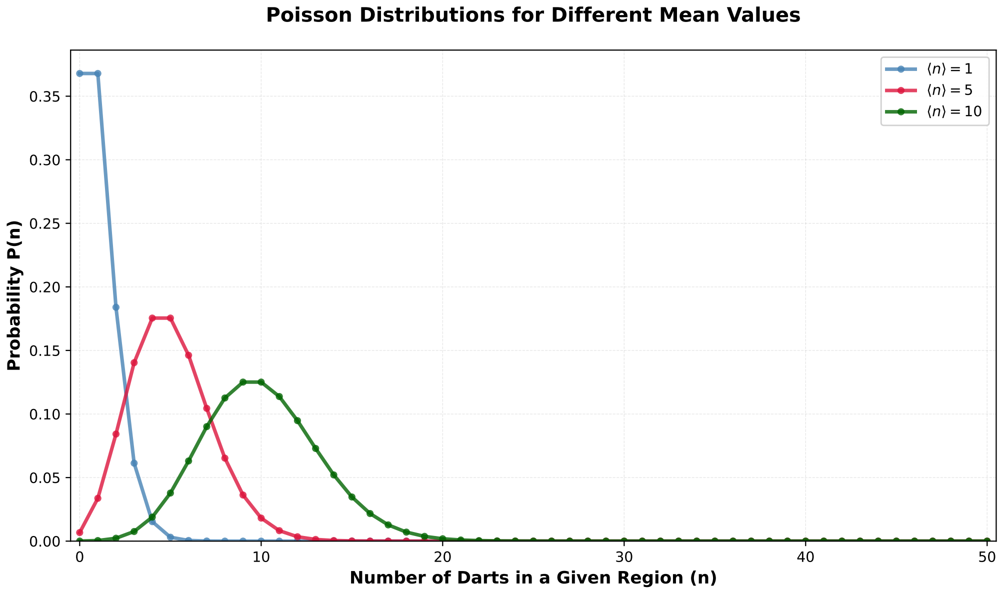

Figure 1: Poisson distributions for $\langle n \rangle = 1, 5, 10$. The distributions show characteristic shape with peak near the mean value and exponential decay.

3.2 Task 2: Distribution Properties

Table 1 presents the computed moments and properties of the Poisson distribution for $N = 50$:

- All distributions are properly normalized (sum ≈ 1.000)
- The calculated mean matches the expected value $\langle n \rangle$
- The variance approximately equals the mean, confirming the Poisson property $\sigma^2 = \langle n \rangle$

Table 1: Statistical moments and properties of Poisson distributions for different mean values with $N = 50$.

$\langle n \rangle$	$\sum P(n)$	$\sum n \cdot P(n)$	$\sum n^2 \cdot P(n)$	Variance	Std Dev
1	1.000000	1.000000	2.000000	1.000000	1.000000
5	1.000000	5.000000	30.000000	5.000000	2.236068
10	1.000000	10.000000	110.000000	10.000000	3.162278

The agreement affirms that for $N = 50$, the contributions from the tails of the distribution are negligible in this case.

3.3 Task 3: Simulation Comparison

Figure 2 compares the simulated probability distribution $P_{\mathrm{sim}}(n)$ with the analytical Poisson distribution for $N = 50$, $L = 100$, and $T = 10$ trials, giving $\langle n \rangle = 0.5$.

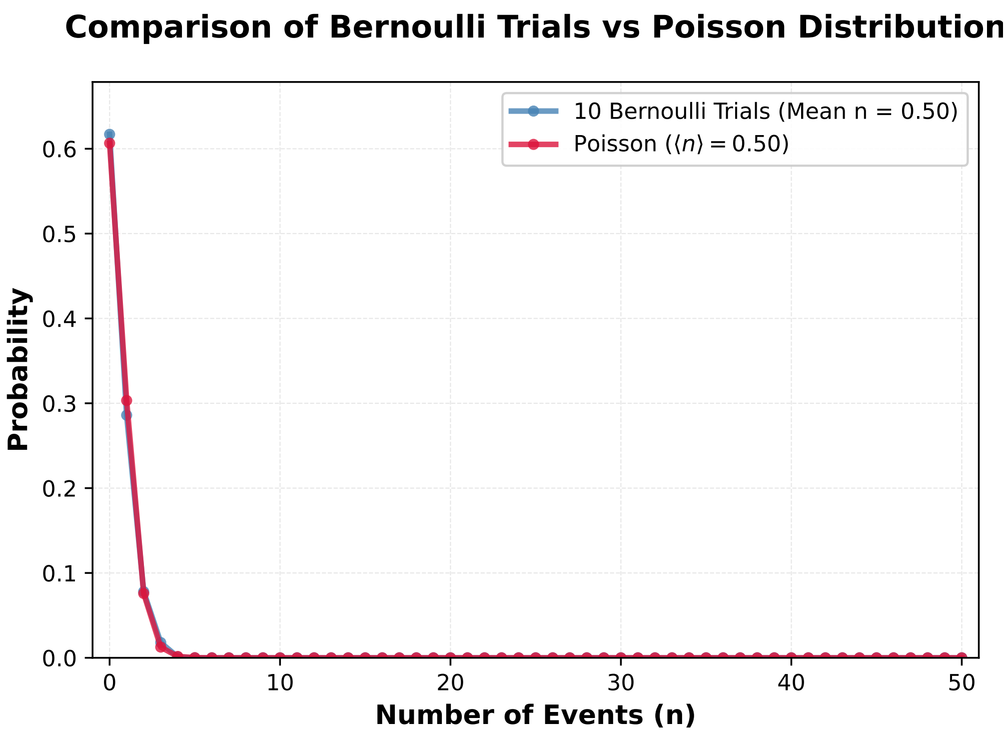

Figure 2: Comparison of simulated dart-throwing distribution (blue) with analytical Poisson distribution (red) for $T = 10$ trials. Despite limited statistics, the simulation shows reasonable agreement with theory.

The plot of $P_{sim}(n)$ agrees strongly with the Poisson curve, despite the limited number of trials ($T = 10$).

3.4 Task 4: Log-Scale Exploration

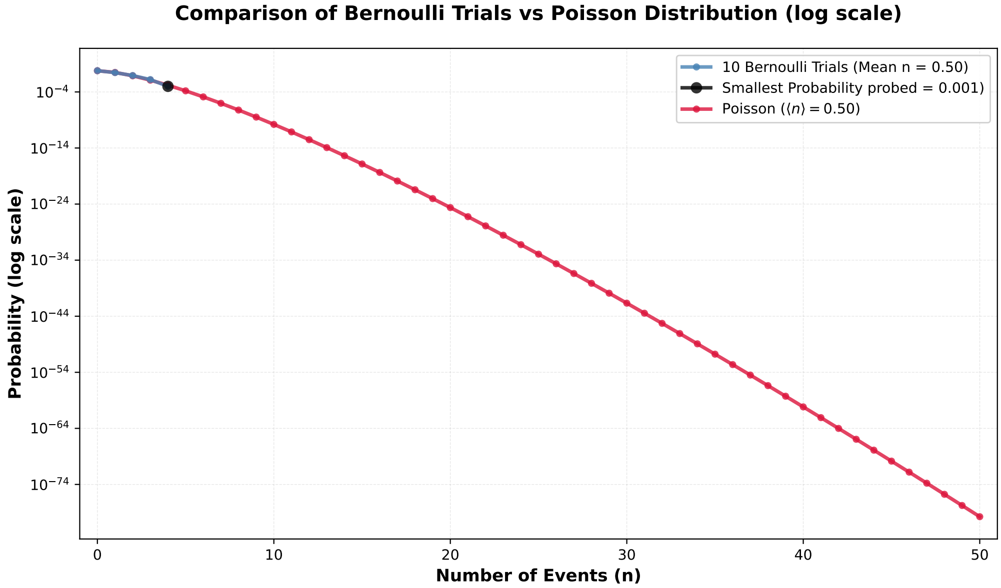

Figure 3: Simulated distribution (blue) and the Poisson distribution (red) on a logarithmic scale for $T = 10$ trials and $L = 100$. The smallest probability probed by the simulation is seen to be 0.001(black).

The log-scale plot shows the range of probabilities accessible with this simulation. For $T = 10$ trials with $L = 100$ regions, the smallest probability that could be probed was approximately $P(n) \approx 10^{-3}$ to 10^{-4}.

3.5 Task 5: Effect of Increasing Trials

Table 2 shows the smallest probability values obtained for various numbers of trials, with $L = 100$. Five independent runs were performed for each value of T in an attempt to account statistical variation.

Table 2: Smallest non-zero probabilities P_{min} probed for $L = 100$ and $N = 50$ across different numbers of trials T. Five independent runs illustrate statistical variation.

Run	$T = 100$	$T = 1000$	$T = 10000$
1	1.0×10^{-4}	1.0×10^{-5}	2.0×10^{-6}
2	1.0×10^{-4}	1.3×10^{-4}	1.0×10^{-6}
3	1.2×10^{-3}	2.6×10^{-4}	8.0×10^{-6}
4	1.5×10^{-3}	1.4×10^{-4}	1.8×10^{-5}
5	1.0×10^{-4}	2.0×10^{-5}	1.0×10^{-6}

The smallest probability probes appears to scale approximately as $\frac{1}{LT}$, showing that increased trials allows for further exploration of rare events.

3.6 Task 6: Small Region Number ($L = 5$)

Table 3 presents results for $L = 5$ regions, where the Poisson approximation begins to break down since $p = 1/5 = 0.2$ is no longer $\ll 1$.

Table 3: Smallest probabilities probed for $L = 5$, $N = 50$ across different trial numbers.

Run	$T = 10$	$T = 1000$	$T = 10000$
1	2.0×10^{-2}	2.0×10^{-4}	2.0×10^{-5}
2	2.0×10^{-2}	2.0×10^{-4}	2.0×10^{-5}
3	2.0×10^{-2}	2.0×10^{-4}	2.0×10^{-5}
4	2.0×10^{-2}	8.0×10^{-4}	6.0×10^{-5}
5	2.0×10^{-2}	2.0×10^{-4}	4.0×10^{-5}

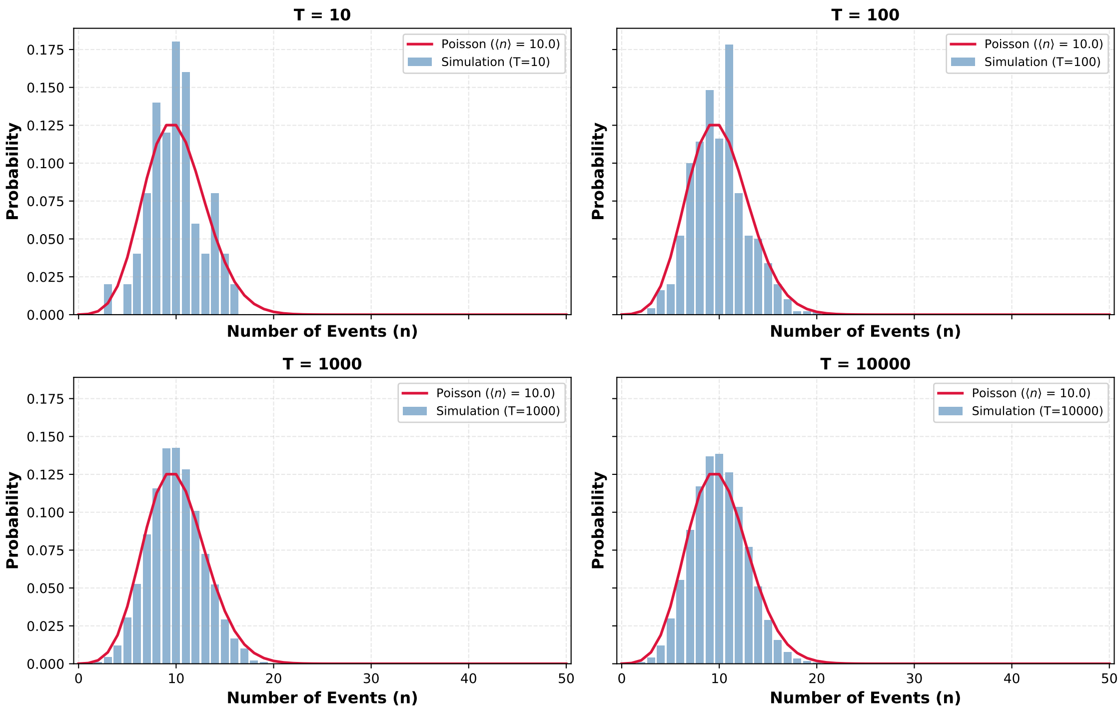

Figure 4: Simulated probability distributions $P_{\text{sim}}(n)$ for $N = 50$ darts and $L = 5$ regions, shown for increasing numbers of trials $T = 10$, 100, 1000, and 10000. The Poisson distribution with $\langle n \rangle = 10$ is overlaid in each subplot. As T increases, statistical noise decreases and $P_{\text{sim}}(n)$ converges, however deviation from the Poisson remain due to the large value of $p = 1/L = 0.2$, violating the small-p assumption.

Figure 4 demonstrates that the smallest probabilities reached when $L = 5$ are larger than that of when $L = 100$, which corroborates the smallest accessible probability scaling with $\frac{1}{LT}$. This distribution also shows that as the assumption, $p \ll 1$ breaks down, so does the convergence to a Poisson distribution, as when $T = 10000$ there is still significant deviation in $P_{sim}(n)$ to $P(n)$.

4 Discussion

4.1 Interpretation of Results

The Monte Carlo simulation successfully approximates the Poisson distribution with Bernoulli trials under the appropriate assumptions(2). When $L = 100$ and $N = 50(p = 0.01)$ our assumptions are sufficiently met, and the resulting simulated distribution agrees strongly with the Poisson distribution. However, as the validity of these assumptions weakens(p not $\ll 1$), so does the convergence of $P_{sim}(n)$ to $P(n)$. This was seen in the case where $L = 5$ and $N = 50$, as the probability of success in each Bernoulli trial, $p = 0.2$, was not sufficiently small for the required limits(2) to hold.

The smallest probability that can be reliably probed by this method of simulated experiment was observed to scale on the order $\frac{1}{LT}$, with L being the number of equal divisions in the dart board, and T being the number of Bernoulli trials.

4.2 Numerical Considerations

Computing factorials when n is large produced overflows due to limited memory. This causes errors in calculations of probabilities, and needed to be accounted for when programming functions to evaluate the Poisson distribution, $P(n)$, for a given number of darts per region. This was done by calculating probabilities in the log space, and using the gamma function, $\Gamma(n+1) = n!$. This allowed for numerical stability across all values of n.

5 Conclusion

This report demostrates the emergence of the Poisson distribution from the Binomial distribution, under certain assumptions($N \to \infty$ and $p \ll 1$), by conducting a simulated Monte Carlo experiment with Bernoulli trials. The simulated darts throwing experiment showed that when the number of darts thrown is sufficiently large($N = 50$) relative to the probability of each a dart hitting a given region($p = 0.01$), the distribution of the number of darts in any given region is given by the Poisson distribution. The moments of the Poisson distribution were also found initially to ensure the implementation of the Poisson distribution was correct.

Monte Carlo simulation with $N = 50$, $L = 100$, and $T = 10$ trials reproduces the Poisson distribution with reasonable accuracy, improving systematically as T increases. The probabilities accessible to this simulation were found to scale on the order of $\frac{1}{LT}$. When $L = 5$ the condition $p \ll 1$ is violated ($p = 0.2$), and deviations from the Poisson distribution become observable.

5.1 AI Usage Declaration

Claude AI was used to enhance the visualization aesthetics of the plots, including color schemes, marker styles, gridlines, and overall figure presentation. Claude AI also provided guidance on implementing the log-space calculation to avoid numerical overflow, formatting output tables.