PYU33C01, S. Hutzler, 2025

Assignment 4: Throwing Darts and the Poisson Distribution

The following problem is adapted from Harvey Gould and Jan Tobochnik, An Introduction to Computer Simulations Methods, Addison-Wesley Publishing Company, 1996 (2nd edition).

Suppose we randomly throw N darts at a dart board that has been divided into L equal size regions. The probability that a dart hits a given region for any one throw is $p = 1/L$. If we count the number of darts in the different regions, we would find that many regions are empty, some regions have one dart, and other regions have more than one dart. What is the probability that a given region has a particular number of darts?

We usually choose N (the number of darts) to be much greater than one, and p (the probability that a dart strikes a given region) to be much less than one. The conditions $N \gg 1$ and $p \ll 1$, and the independence of the events (the landing of a dart in a particular region) satisfy the requirement for a *Poisson distribution*,

$$P(n) = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \tag{1}$$

where n is the number of darts in a given region and $\langle n \rangle$ is the mean number, $\langle n \rangle = \sum_{n=0}^{N} nP(n)$.

1. Write Python code to plot three Poisson distributions, eqn(1), for $\langle n \rangle = 1, 5, 10$.

2. Set $N = 50$ (to avoid large numbers for the factorial) and write code to compute the sums $\sum_{n=0}^{N} P(n)$, $\sum_{n=0}^{N} nP(n)$, and $\sum_{n=0}^{N} n^2 P(n)$ for the three different values of $\langle n \rangle$ given above. Verify that $P(n)$ is normalized. What are the values of the standard deviation and variance? Present all your results in a table.

The *simulation* of the dart throw problem is straightforward. Imagine that the L regions of the dart board are labeled and correspond to the L elements in the one-dimensional array B. Throwing a dart at random at the board is equivalent to choosing an integer at random between 1 and L using a random number generator.

Randomly "throw N darts" and count how many elements in the array B end up having n darts (n can range from 0 to N). Calls this distribution $H_1(n)$. Then repeat the simulation and add the results to your previous histogram data. That is, if $H_1(n)$

contains the data from your first simulation, and $H_2(n)$ the data from the second simulation, then $H(n) = H_1(n)+H_2(n)$ for both simulations combined. Summing up these distribution over a number of T such simulation runs (Trials), $H(n) = \sum_{i=1}^{T} H_i(n)$, one obtains the *normalized distribution*

$$P_{sim}(n) = \frac{H(n)}{\sum_{n=0}^{N} H(n)} = \frac{H(n)}{LT}. \tag{2}$$

3. Write a Python program to simulate the dart problem. Throw $N = 50$ darts at random in one trial (i.e. choose $N = 50$ integer random numbers) and initially set the number of regions to $L = 100$. Run $T = 10$ such "experiments" and determine $H(n)$, the number of regions that have n darts, and the mean number of darts per region, $\langle n \rangle$. Normalize $H(n)$ to determine the probability distribution, $P_{sim}(n)$ of eqn.(2). Plot $P_{sim}(n)$ as obtained from your data and compare it with the Poisson distribution of eqn.(1), where you need to use your value of $\langle n \rangle$ from your simulation as input.

4. Plot both distributions using a log-scale for the vertical axis. Down to what values of $P(n)$ does your numerical data probe the Poisson distribution?

5. Repeat the simulations for a different number of trials, $T = 100$, 1000, and 10000, and record the smallest values of $P(n)$ that you can simulate.

6. Repeat these calculations for $L = 5$, $N = 50$ and $T = 10, 1000, 10000$. What do you observe?

Submit a report (as pdf) **via Blackboard** which details your findings, include relevant figures. Also submit your Python code as a *.py* file which should also print your name and student number onto the screen when executed.