NONLINEAR CONTROL BASED ON T-S FUZZY MODEL

Consider T-S fuzzy systems

If $z_1(t)$ is $\tilde{A}_{i1}$ and $z_2(t)$ is $\tilde{A}_{i2}$ and … and $z_q(t)$ is $\tilde{A}_{iq}$, then

$$\begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \tag{1}$$

For $i = 1, 2, \cdots, N$, where $A_i \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}$ and $C_i \in \mathbf{R}^{p\times n}$. N is the number of the subsystems and we call system (1) the ith subsystem. $z(t)=\begin{bmatrix} z_1(t) & z_2(t) & \cdots & z_q(t) \end{bmatrix}^T$ is a vector of premise variables. $\tilde{A}_{ij}(i = 1, \cdots N; j = 1, \cdots, q)$ are fuzzy sets. Denote $\mu_{\tilde{A}_{ij}}(\cdot)$ the membership function determining the membership of $z_j(t)$ in the fuzzy set $\tilde{A}_{ij}$.

A T-S fuzzy system actually represents a nonlinear system. Fuzzy system (1) can be represented by equivalently

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{N} h_i(z(t))[A_i x(t) + B_i u(t)] \\ y(t) = \sum_{i=1}^{N} h_i(z(t)) C_i x(t) \end{cases} \tag{2}$$

where

$$h_i(z(t)) = \frac{\prod_{j=1}^{q} \mu_{\tilde{A}_{ij}}(z_j(t))}{\sum_{i=1}^{N} \prod_{j=1}^{q} \mu_{\tilde{A}_{ij}}(z_j(t))} \tag{3}$$

We call $h_i(z(t))$ the fuzzy weights. Obviously, we have $\sum_{i=1}^{N} h_i(z(t)) = 1$.

Consider affine nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{3}$$

where $f(x) \in \mathbf{R}^n$ and $g(x) \in \mathbf{R}^{n\times m}$ are two nonlinear function.

Theorem 1: Suppose that $f(x)$ is continuous and differentiable, then for any $\varepsilon > 0$,

there exists a T-S fuzzy model of system (1) or (2) such that

$$\left\| f(x) + g(x)u - \sum_{i=1}^{N} h_i (A_i x + B_i u) \right\| \le \varepsilon$$

for any x and u in a suitable neighbourhood.

The computation of matrix A_i and B_i

Step 1: Choose the operating points of $x_0^i (i = 1, \cdots, N)$. Usually, the equilibrium point x_0 : $f(x_0) = 0$ should be considered.

Step 2: (1) For the equilibrium point x_0,

$$A_i = \left. \frac{\partial f(x)}{\partial x} \right|_{x=0} \quad \text{and} \quad B_i = g(x_0) \tag{4}$$

where

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}$$

(2) for other operating points:

$$a_{l,i} = \nabla f_l(x_0^i) + \frac{f_l(x_0^i) - x_0^{iT} \nabla f_l(x_0^i)}{\left\| x_0^i \right\|^2} x_0^i, \; (l = 1, 2, \cdots, n) \tag{5}$$

$$A_i = \begin{bmatrix} a_{1,i}^T \\ \vdots \\ a_{n,i}^T \end{bmatrix} \quad \text{and} \quad B_i = g(x_0^i)$$

where

$$\nabla f_l(x) = \begin{bmatrix} \dfrac{\partial f_l(x)}{\partial x_1} \\ \dfrac{\partial f_l(x)}{\partial x_2} \\ \vdots \\ \dfrac{\partial f_l(x)}{\partial x_n} \end{bmatrix}$$

Example 1: Consider free-body of a cart with an inverted pendulum system

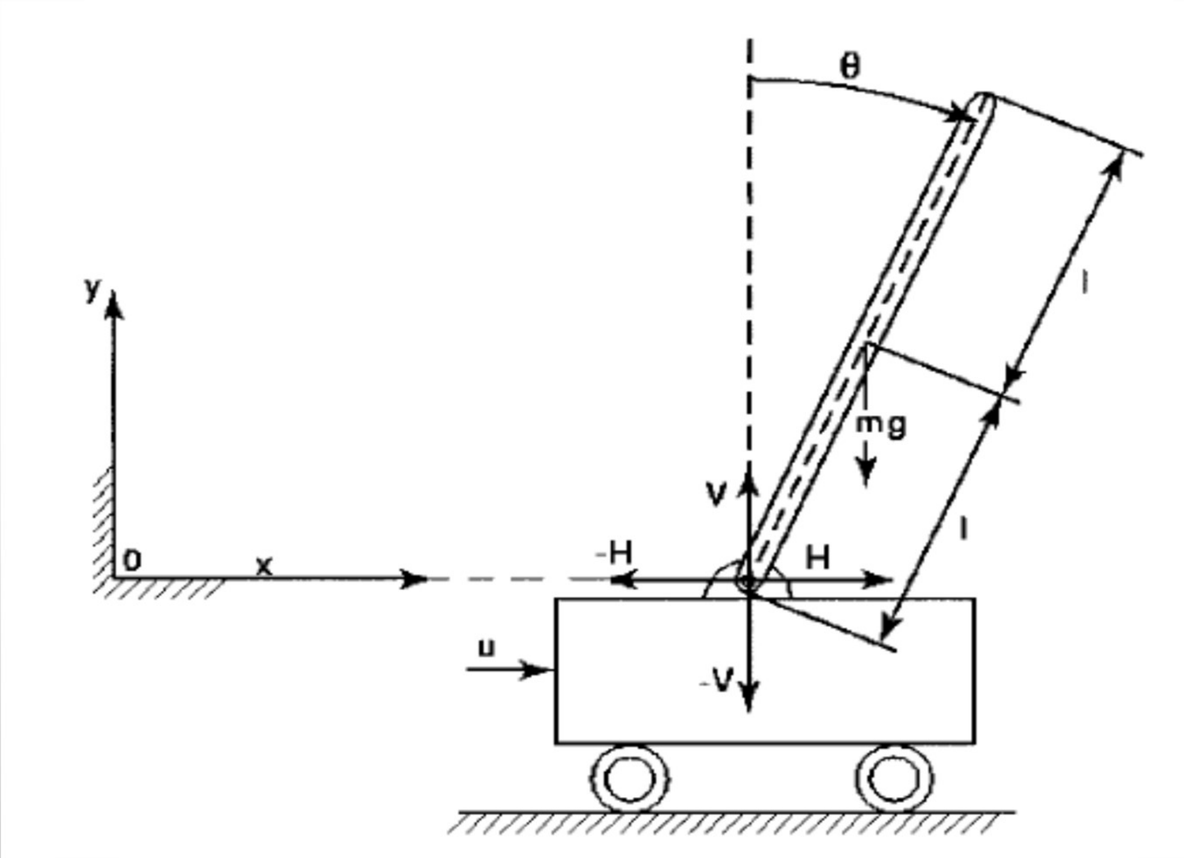

Fig. 1: Free-bosy of a cart with an inverted pendulum system

The dynamic system of the inverted pendulum system can be described in form

$$\dot{x} = f(x) + g(x)(u + \eta(x)) \qquad (6)$$

with

$$f(x) = \begin{bmatrix} x_2 \\ \dfrac{g \sin x_1}{4l/3 - mla\cos^2 x_1} \\ x_4 \\ \dfrac{-(1/2)mag\sin(2x_1)}{4/3 - ma\cos^2 x_1} \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \dfrac{-a\cos x_1}{4l/3 - mla\cos^2 x_1} \\ 0 \\ \dfrac{4a/3}{4/3 - ma\cos^2 x_1} \end{bmatrix} \quad \text{and}$$

$$\eta(x) = f_c + mlx_2^2 \sin x_1$$

For the nonlinear system (6), we propose the following two rules to construct a T-S fuzzy system:

Rule1: If $x_1(t)$ is about 0, then

$$x = A_1 x + B_1 u$$

Rule 2: If $x_1(t)$ is about $\pm\pi/4$, then

$$x = A_2 x + B_2 u$$

That is we choose two operating points:

$$x_0^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \quad \text{and} \quad x_0^2 = \begin{bmatrix} \pm\pi/4 & 0 & 0 & 0 \end{bmatrix}^T$$

Next, we compute the matrices of A_1, A_2, B_1 and B_2 based on (4) and (5)

For the operating point $x_0^2 = \begin{bmatrix} \pm\pi/4 & 0 & 0 & 0 \end{bmatrix}^T$, based on (5), we have

$$\nabla f_1(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \nabla f_2(x) = \begin{bmatrix} \dfrac{\partial f_2}{\partial x_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \nabla f_3(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \nabla f_4(x) = \begin{bmatrix} \dfrac{\partial f_2}{\partial x_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$a_{1,2} = \nabla f_1(x_0^2) + \frac{f_1(x_0^2) - x_0^{2T} \nabla f_1(x_0^2)}{\left\| x_0^2 \right\|^2} x_0^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$$

Since,

$$a_{2,2}=\nabla f_2(x_0^2)+\frac{f_2(x_0^2)-x_0^{2T}\nabla f_2(x_0^2)}{\left\|x_0^2\right\|^2}x_0^2$$

$$f_2(x)=\frac{g\sin x_1}{4l/3-mla\cos^2 x_1}$$

The computation results are:

$$A_1=\begin{bmatrix}0 & 1 & 0 & 0\\ 17.2941 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -1.7249 & 0 & 0 & 0\end{bmatrix}, A_2=\begin{bmatrix}0 & 1 & 0 & 0\\ 14.3077 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -0.9723 & 0 & 0 & 0\end{bmatrix}, B_1=\begin{bmatrix}0\\ -0.1765\\ 0\\ 0.1176\end{bmatrix}, B_2=\begin{bmatrix}0\\ -0.1147\\ 0\\ 0.1081\end{bmatrix}$$

We use the membership functions of the form

$$h_1(x_1(t)) = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}} \quad \text{and} \quad h_2(x_1(t)) = 1 - h_1(x_1(t))$$

Then the T-S fuzzy model consisting two subsystems for nonlinear system (6) can be represented as

$$\begin{aligned} \dot{x} &= h_1(x_1)(A_1 x + B_1(u + \eta)) + h_2(x_1)(A_2 x + B_2(u + \eta)) \\ &= \sum_{i=1}^{2} h_i(x_1)(A_i x + B_i(u + \eta)) \end{aligned} \tag{A1}$$

Nonlinear Control Scheme Based on T-S Fuzzy Model

Consider a homogenous linear system

$$\dot{x}(t) = Ax(t) \tag{7}$$

where $x \in \mathbf{R}^n$ is sate vector and $A \in \mathbf{R}^{n\times n}$ is a square constant matrix.

Definition 1: If all the eigenvalues of matrix A have negative real part, then we call matrix A is a (asymptotical) stable matrix, and the system (7) is asymptotically stable at equilibrium state $x = 0$.

Theorem 1: Square matrix A is stable if and only if for any symmetrical positive definite matrix Q, the following Lyapunov (matrix) equation

$$A^T P + PA = -Q \tag{8}$$

has a symmetrical positive definite matrix solution of P.

An equivalent statement of Theorem 1 is: Square matrix A is stable if and only if Lyapunov inequality $A^T P + PA < 0$ has a symmetrical positive definite matrix solution of P.

Now consider a T-S fuzzy homogenous model described by

$$\dot{x}(t) = \sum_{i=1}^{N} h_i(z(t)) A_i x(t) \tag{9}$$

Theorem 2: A sufficient condition for the T-S fuzzy system (9) being asymptotically stable at 0 is that there exists a common symmetric positive definite matrix P such that

$$A_i^T P + PA_i < 0 \tag{10}$$

holds for $i = 1, 2, \cdots, N$.

Remark: A necessary condition for the existence of a common symmetrical positive definite P satisfying (10) is that each A_i is asymptotically stable.

Theorem 3: There exists a common symmetric positive definite matrix P such that (10) holds, then the matrices

$$\sum_{k=1}^{s} A_{i_k}$$

are asymptotically stable, where $i_k \in \{1, 2, \cdots, N\}$ and $s = 1, 2, \cdots, N$.

Example: Given a homogenous T-S fuzzy system (9) with two subsystems:

$$A_1 = \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}$$

Obviously, both A_1 and A_2 are asymptotically stable, but there is no common positive definite matrix P such that (10) holds because

$$A_1 + A_2 = \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -4 \end{bmatrix}$$

is unstable since its eigenvalues are {1.1231, -7.1231}.

Now consider the T-S fuzzy system (2), we plan to design a state feedback control scheme such that the close-loop system is asymptotically stable.

Assumption 1: The pair $\{A_i, B_i\}(i = 1, 2, \cdots, N)$ is controllable. (i.e. every subsystem of the T-S fuzzy system is controllable.)

Under Assumption1, the state feedback controller is designed as

$$u = \sum_{i=1}^{N} h_i(z(t)) K_i x \qquad (11)$$

where the control gains K_i are chosen such that $A_i - B_i K_i \ (i = 1, \cdots, N)$ is stable.

Substituting (11) into the state equation in (2) leads to the closed-loop system:

$$\dot{x} = \sum_{i=1}^{N} h_i(z(t))[A_i x + B_i \sum_{j=1}^{N} h_j(z(t)) K_j x] = \sum_{i=1}^{N} h_i(z(t))[\sum_{j=1}^{N} h_j(z(t)) A_i x + \sum_{j=1}^{N} h_j(z(t)) B_i K_j x]$$

$$= \sum_{i=1}^{N} h_i(z(t)) \sum_{j=1}^{N} h_j(z(t))(A_i + B_i K_j)x = \sum_{i=1}^{N}\sum_{j=1}^{N} h_i(z(t)) h_j(z(t))(A_i + B_i K_j)x$$

That is, the closed-loop system is

$$\dot{x} = \sum_{i=1}^{N}\sum_{j=1}^{N} h_i h_j (A_i + B_i K_j)x \tag{12}$$

Theorem 4: A sufficient condition for the closed-loop system (12) to be asymptotically stable at $x = 0$ is that there exists a symmetric positive definite matrix P such that the following conditions are satisfied:

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0, (i = 1, 2, \cdots, N) \tag{12}$$

and

$$G_{ij}^T P + P G_{ij} < 0, (i < j \le N) \tag{13}$$

where $G_{ij} = (A_i + B_i K_j) + (A_j + B_j K_i)$

Example 2: Consider the inverted pendulum nonlinear system (6) and the corresponding T-S fuzzy model (A1), and based on the T-S fuzzy model (A1), we design the state feedback as

$$u = h_1 K_1 + h_2 K_2 \tag{14}$$

The two gains K_1 and K_2 are chosen such that the eigenvalues of $A_1 + B_1 K_1$ and $A_2 + B_2 K_2$ are placed to

$$a_1 = \begin{bmatrix} -1.0970 & -2.1263 & -2.5553 & -3.9090 \end{bmatrix}^T$$

and

$$a_2 = \begin{bmatrix} -1.5794 & -1.6857 & -2.7908 & -3.6895 \end{bmatrix}^T$$

respectively. Now by MATLAB code:

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a1=[-1.0970 -2.1263 -2.5553 -3.9090]'; a2=[-1.5794 -1.6857 -2.7908 -3.6895]';
K1=place(A1, B1, a1); K2=place(A2, B2, a2);
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We obtain

$$K_1 = \begin{bmatrix} 294.8755 & 73.1208 & 13.4726 & 27.3362 \end{bmatrix}^T$$

and

$$K_2 = \begin{bmatrix} 440.3915 & 118.5144 & 19.1611 & 35.5575 \end{bmatrix}^T$$

Let $G_{12} = A_1 + B_1K_2 + A_2 + B_2K_1$ and using LMI toolbox in MABLAB to solve the following LMIs

$$(A_1 + B_1K_1)^T P + P(A_1 + B_1K_1) < 0$$

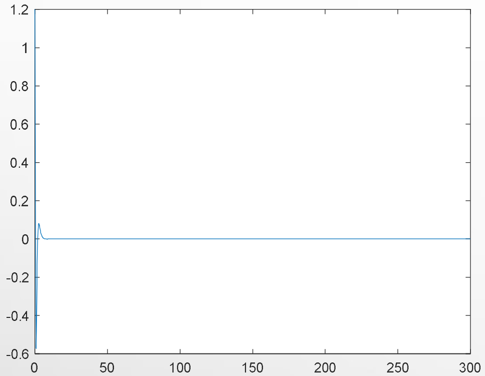
$$(A_2 + B_2K_2)^T P + P(A_2 + B_2K_2) < 0$$

$$G_{12}^T P + PG_{12} < 0$$

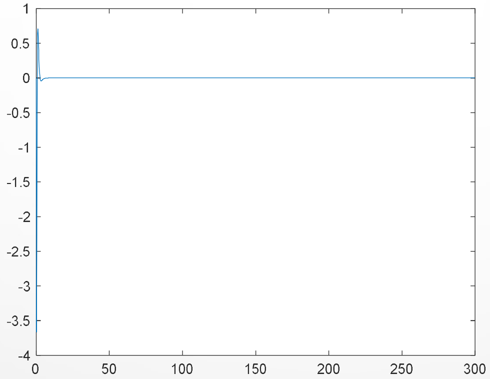
provides a common solution of P:

$$P = \begin{bmatrix} 54.9580 & 15.6219 & 6.9389 & 12.1165 \\ 15.6219 & 4.5429 & 2.1011 & 3.5488 \\ 6.9389 & 2.1011 & 1.3972 & 1.7978 \\ 12.1165 & 3.5488 & 1.7978 & 2.9375 \end{bmatrix}$$

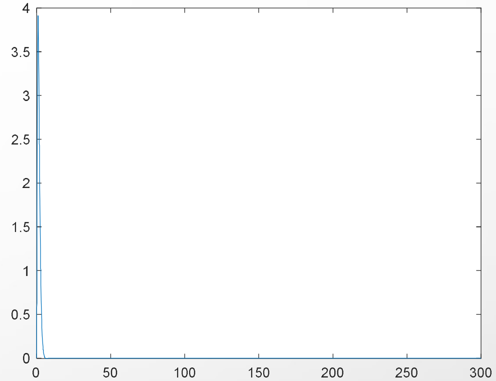
So, the controller (14) can be applied to both the fuzzy model and the nonlinear system.



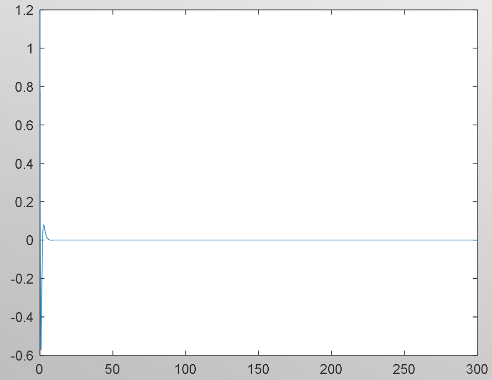
Response of state x1



Response of state x2

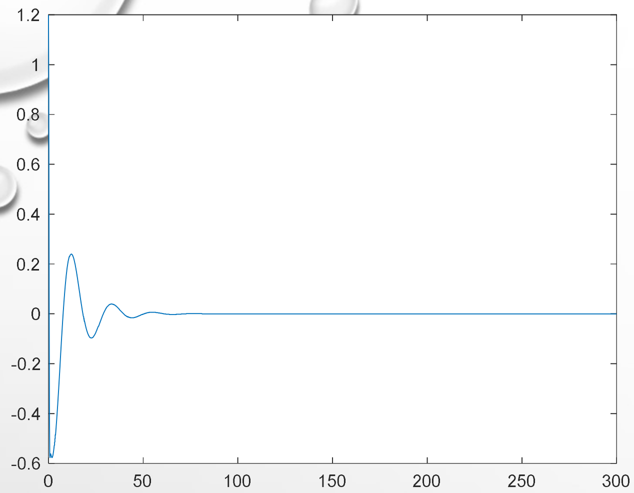


Response of state x3

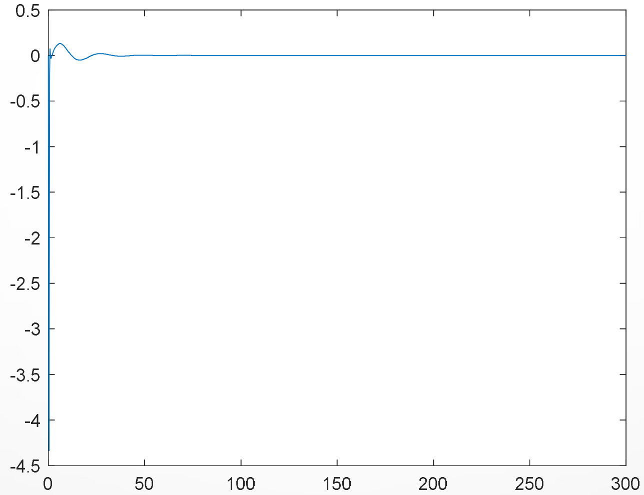


Response of state x4

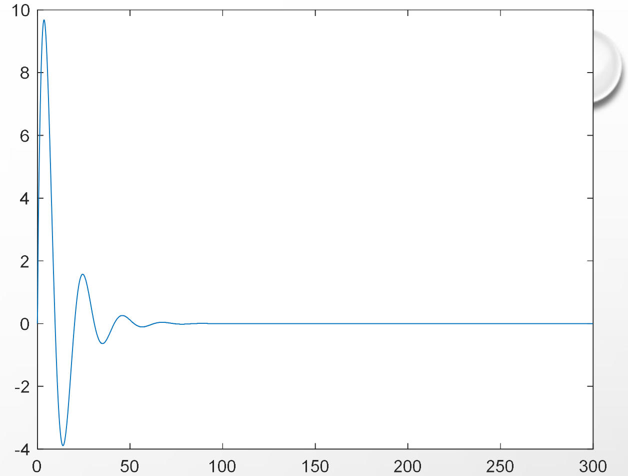
Fig. 2: The controller applied to the T-S fuzzy model



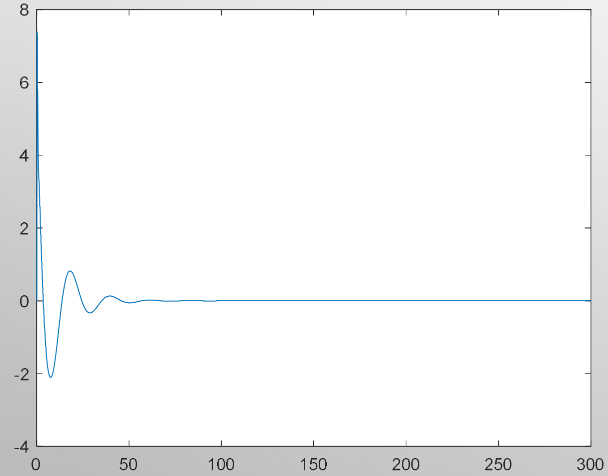
Response of state x1



Response of state x2



Response of state x3



Response of state x4

Fig. 3: The controller applied to the nonlinear system