Name : SHAH RAZA

REGISTRATION NO : 18PWCSE1658

EXAM : COMPLEX VARIABLE

(Q1) (a) Find all roots in the complex plane $\sqrt[5]{-2}$.

Solution:

$$z = -2 + 0i$$

$$r = \sqrt{(-2)^2} = \sqrt{4} = 2$$

$x < 0$ and $y = 0$

so $\theta = \pi + 2k\pi$

$\theta = \pi(1+2k) \qquad k \in \mathbb{Z}$

$$-2 = 2\left(\cos(1+2k)\pi + i\sin(1+2k)\pi\right)$$

$$(-2)^{1/5} = (2)^{1/5}\left(\cos(1+2k)\pi + i\sin(1+2k)\pi\right)^{1/5}$$

$$= (2)^{1/5}\left(\cos\frac{(1+2k)\pi}{5} + i\sin\frac{(1+2k)\pi}{5}\right)$$

Put $k = 0$

$$(2)^{1/5}\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right) = 2^{1/5}\,\text{cis}\,\frac{\pi}{5}$$

Put $k = 1$

$$(2)^{1/5}\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right) = 2^{1/5}\,\text{cis}\,\frac{3\pi}{5}$$

Put $k = 2$

$$2^{1/5}\left(\cos\pi + i\sin\pi\right) = 2^{1/5}\,\text{cis}\,\pi$$

Put $K=3$

$$2^{1/5}\left(\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}\right)=2^{1/5}\,cis\,\frac{7\pi}{5}$$

Put $K=4$

$$2^{1/5}\left(\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}\right)=2^{1/5}\,cis\,\frac{9\pi}{5}$$

Put $K=5$

$$2^{1/5}\left(\cos\frac{11\pi}{5}+i\sin\frac{11\pi}{5}\right)=2^{1/5}\,cis\,\frac{11\pi}{5}$$

Thus the roots of $\sqrt[5]{-2}$ are:

$$\boxed{2^{1/5}\,cis\,\pi/5,\ 2^{1/5}\,cis\,\frac{3\pi}{5},\ 2^{1/5}\,cis\,\pi,\ 2^{1/5}\,cis\,\frac{7\pi}{5},\ 2^{1/5}\,cis\,\frac{9\pi}{5}}$$

(b) Find "a" so that the given function is harmonic and find a harmonic conjugate

$$u=e^{4x}\cos ay$$

solution:

$$u=e^{4x}\cos ay$$

As the given function is harmonic

so $$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$$

$$\frac{du}{dx}=4\cos ay\,e^{4x}$$

$$\frac{d^2u}{dx^2} = 16e^{4x}\cos ay$$

$$\frac{du}{dy} = -e^{4x}a\sin ay$$

$$\frac{d^2u}{dy^2} = -a^2e^{4x}\cos ay$$

$$16e^{4x}\cos ay - a^2e^{4x}\cos ay = 0$$

$$e^{4x}\cos ay\,(16 - a^2) = 0$$

$$16 - a^2 = 0$$

$$a^2 = 16$$

$$a = 4$$

Thus $u = e^{4x}\cos 4y$

$$\frac{dV}{dy} = \frac{du}{dx}$$

$$\frac{dV}{dy} = 4e^{4x}\cos 4y$$

$$V = \frac{\cancel{4}e^{4x}\sin 4y}{\cancel{4}} = e^{4x}\sin 4y + h(x)$$

Now $\frac{dV}{dx} = -\frac{du}{dy}$

$$\cancel{4e^{4x}\sin 4y} + h'(x) = -(\cancel{-4e^{4x}\sin 4y})$$

$$h'(x) = 0$$

$h(x) = C$ so $\boxed{V = e^{4x}\sin 4y + C}$

Scanned with CamScanner

(c) Find all solutions in the complex plane
$\cosh z = 0$

Solution:

$$\cosh z = \cosh(x + iy) = 0$$

$$\cosh x \cos y + i \sinh x \sin y = 0$$

$$\cosh x \cos y = 0$$
$$\cos y = 0$$
$$y = (2k+1)\pi/2$$

$$\sinh x \sin y = 0$$
$$\sinh x = 0$$
$$x = 0$$
$$z = 0 + i(2k+1)\pi/2$$

$$z = i(2k+1)\pi/2$$

(Q2)(a) Find the line integral $\int_C Re(z^2)\,dz$
C is the shortest path from 0 to 0+i

Solution:

$(0,0)\ ,\ (0,1)$

$$\frac{x-0}{0-0} = \frac{y-0}{1-0} = t$$

$x = 0 \ ; \ y = t$

$z = x + iy = 0 + it$

$z^2 = x^2 - y^2 + 2ixy = 0 - t^2 + 0 = -t^2$

$Re(z^2) = -t^2$

$dz = i$

when $z = 0 + 0i \Rightarrow t = 0$

when $z = 0 + i \Rightarrow t = 1$

$$\int_C Re(z^2)\,dz = \int_0^1 -t^2 (i)\,dt$$

$$= -i \int_0^1 t^2\,dt$$

$$= -i \frac{t^3}{3}\Big|_0^1$$

$$= -\frac{1}{3} i - 0 = \boxed{-\frac{1}{3} i}$$

Scanned with CamScanner

(Q2)(b) Find the integral $\oint_C \frac{\sinh 2z}{z^2-3z}\,dz$

C consists of $|z|=1$ (cw)

Solution:

$$\oint_C \frac{\sinh 2z}{z^2-3z}\,dz$$

$$\oint_C \frac{\sinh 2z}{z(z-3)}\,dz$$

On $z=0$ there is singularity so,

$$\oint_C \frac{\frac{\sinh 2z}{z-3}}{z}\,dz$$

$$F(z) = \frac{\sinh 2z}{z-3}, \quad z_0 = 0$$

$$\oint_C f(z)\,dz = 2\pi i f(z_0)$$

$$= 2\pi i\left(\frac{\sinh(2\times 0)}{0-3}\right)$$

$$= 2\pi i\left(\frac{0}{-3}\right)$$

$$= 2\pi i(0)$$

$$\boxed{\oint_C f(z)\,dz = 0}$$

(Q3)(c) Find the integral $\oint_C \frac{\sin 6z}{(z-4)^3} dz$

C consists of $|z|=5$ (CCW) and $|z-3|=\frac{3}{2}$ (CW).

Solution:

The integral $\frac{\sin 6z}{(z-4)^3}$ is not analytic

at $z=4$

Region does not

contain $z=4$

so by using Cauchy's

integral formula

$$\oint_C \frac{\sin 6z}{(z-4)^3} dz = 0$$