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(Q1.A)

$$f(z) = \frac{\cosh z}{(z-\pi i)^2}, \quad z_0 = \pi i$$

solution:

As πi is singularity
so $g(z) = \cosh z$

$g(z_0) = \cosh(\pi i) = -1$
$g'(z_0) = \sinh(\pi i) = 0$
$g''(z_0) = \cosh(\pi i) = -1$
$g'''(z_0) = \sinh(\pi i) = 0$
$g''''(z_0) = \cosh(\pi i) = -1$

$$f(z) = \frac{1}{(z-\pi i)^2}\left[\cosh z\right]$$

$$= \frac{1}{(z-\pi i)^2}\left[-1 - \frac{1}{2!}(z-\pi i)^2 - \frac{1}{4!}(z-\pi i)^4 - \dots\right]$$

$$\frac{\cosh z}{(z-\pi i)^2} = \frac{-1}{(z-\pi i)^2} - \frac{1}{2!} - \frac{1}{4!}(z-\pi i)^2 - \dots$$

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(Q1.B)

$$f(z) = \sinh z$$

Solution:

To find zeros of $f(z)$ we take $f(z) = 0$

$$\sinh z = 0$$

$$\frac{e^z - e^{-z}}{2} = 0$$

$$e^z - e^{-z} = 0$$

$$e^z - \frac{1}{e^z} = 0$$

$$e^{2z} - 1 = 0$$

$$e^{2z} = 1$$

$$2z = i(0 + 2\pi k)$$

$$z_0 = i\pi k$$

Now For order of zero

$$f'(z) = \cosh z$$

$$f'(z_0) = \cosh(i\pi k) \neq 0$$

So order is 1

(Q2)(A) Solve the integral (ccw)

$$\oint_C \frac{z+1}{z^4-3z^3}\,dz \qquad C: |z| = \frac{1}{2}$$

Solution:

$$\oint_C \frac{z+1}{z^3(z-3)}\,dz$$

Here we haw two singularities
$z=0$, $z=3$
since 3 is outside the circle, we will only take $z=0$ into consideration.

According to Laurent series

$$\frac{(z+1)}{(z-3)} = \left[-\frac{1}{3}-\frac{4z}{9}-\frac{4z^2}{27}-\frac{4z^3}{81}-\cdots\right]$$

$$\frac{1}{z^3}\left[\frac{z+1}{z-3}\right] = \frac{1}{z^3}\left[-\frac{1}{3}-\frac{4z}{9}-\frac{4z^2}{27}-\frac{4z^3}{81}-\cdots\right]$$

$$= \left[-\frac{1}{3z^3}-\frac{4}{9z^2}-\frac{4}{27z}-\frac{4}{81}-\cdots\right]$$

we can see that the residue is $-\frac{4}{27}$

CCW yields:

$$\oint_C \frac{z+1}{z^4-3z^3}\,dz = 2\pi i \operatorname*{Res}_{z\to 0} f(z)$$

$$= 2\pi i\left(\frac{-4}{27}\right)$$

$$\boxed{\oint_C \frac{z+1}{z^4-3z^3} = \frac{-8}{27}\pi i}$$

(Ans)

(Q2)(B) Solve the integral $\int_0^{2\pi} \frac{1+\sin\theta}{1+\cos\theta}\,d\theta$

solution:

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$= \frac{1}{2z}\left(z^2+1\right)$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$= \frac{1}{2zi}\left(z^2-1\right)$$

$$\sin\theta = \frac{z^2-1}{2zi}$$

and $d\theta = \frac{dz}{iz}$

Replacing the values in the integral

$$\int_0^{2\pi} \frac{1+\sin\theta}{1+\cos\theta}\, d\theta = \oint_c \left[\frac{\left(1+\frac{z^2-1}{2zi}\right)}{\left(1+\frac{z^2+1}{2z}\right)}\right] \frac{dz}{iz}$$

$$= \oint_c \left(\frac{z^2+2zi-1}{\cancel{2z}i}\right) \times \left(\frac{\cancel{2z}}{z^2+2z+1}\right) \frac{dz}{iz}$$

$$= \oint_c \frac{z^2+2zi-1}{i^2(z^2+2z+1)z}\, dz$$

$$= -1 \oint_c \frac{z^2+2zi-1}{z(z^2+2z+1)}\, dz$$

Here c is a unit circle $|z| = 1$

And we have two singularitise

$z(z^2+2z+1)$

$z = 0 \quad ; \quad (z+1)^2 = 0$

$z = -1$

we will only take $z=0$

Residue at $z=0$

$$f(z) = \frac{1}{(1-1)!} \lim_{z \to z_0} \left[\frac{d^{1-1}}{dz^{1-1}} \left[(z-0)^1 \left[\frac{-(z^2+2zi-1)}{z(z+1)^2}\right]\right]\right]$$

$$= \frac{1}{0!} \lim_{z \to z_0} \left[\frac{\cancel{z}(-(z^2+2zi-1))}{\cancel{z}(z+1)^2}\right]$$

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$$= \lim_{z \to z_0} \left[\frac{-(z^2 - 2zi - 1)}{(z+1)^2} \right]$$

$$= \frac{-(0-0-1)}{(0+1)^2}$$

$$= \frac{-(-1)}{1}$$

$$\operatorname{Res}_{z=0} f(z) = 1$$

$$\oint_C \frac{1+\sin\theta}{1+\cos\theta}\, d\theta = 2\pi i \operatorname{Res}_{z=0} f(z)$$

$$\boxed{\oint_C \frac{1+\sin\theta}{1+\cos\theta}\, d\theta = 2\pi i(1)}$$

(Ans).

(Q3) (a) Solve the improper integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$$

Solution:

Find roots of z^4+1

$$z^4+1=0$$

$$z^4 = -1$$

$$z = (-1)^{1/4}$$

$$(-1)^{1/n} = r(\cos\theta + i\sin\theta)^{1/n}$$

Hence

$$r = \sqrt{(-1)^2+0^2}$$

$$r = 1$$

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For θ, as $x < 0$ & $y = 0$

$$\theta = \pi$$

$$\theta = \pi + 2k\pi$$

Putting values

$$(-1)^{1/4} = 1\left(\cos\pi(2k+1) + i\sin(2k+1)\pi\right)^{1/4}$$

$$= 1\left(\cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}\right)$$

$$= e^{\left(\frac{2k+1}{4}\right)\pi i}$$

$$z_1 = e^{\pi i/4}, \quad z_2 = e^{3\pi i/4}, \quad z_3 = e^{-3\pi i/4}, \quad z_4 = e^{-\pi i/4}$$

Now

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)^d}\right]_{z=z_1}$$

$$= \left[\frac{1}{4z^3}\right]_{z=z_1}$$

$$= \frac{1}{4}e^{-3\pi i/4}$$

$$= -\frac{1}{4}e^{\pi i/4}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)^d}\right]_{z=z_2}$$

$$= \left[\frac{1}{4z^3}\right]_{z=z_2}$$

$$= \frac{1}{4}e^{-3\pi i/4}$$

$$= \frac{1}{4}e^{-\pi i/4}$$

Here $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{2\pi i}{4}\left(e^{\pi i/4} - e^{-\pi i/4}\right)$

$= -\frac{2\pi i}{4} \, 2i \sin\frac{\pi}{4}$

$= \pi \sin\frac{\pi}{4}$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}}$$ (Ans)

(Q3) (b)

Solution:

$r_1 = 1cm$, $r_2 = 5cm$

$U_1 = 100$ volts , $U_2 = 0$ volts

As $\phi(x,y) = a\ln r + b$

where $r = \sqrt{x^2+y^2}$

when $r_1 = 1$, $U_1 = 100$

$a\ln 1 + b = 100$

$b = 100$

When $r_2 = 5$, $U_2 = 0$

$a\ln 5 + b = 0$

$a\ln 5 + 100 = 0$

$a\ln 5 = -100$

$a = \frac{-100}{\ln 5}$

$a = -62.133$

Therefore, potential

$$\phi(x,y) = -62.133 \ln r + 100$$

$$\phi(x,y) = -62.133 \ln\sqrt{x^2+y^2} + 100$$

The complex potential is

$$\phi(x,y) = Re\, f(z)$$

$$\phi(x,y) = \frac{-100}{\ln 5} \ln\sqrt{x^2+y^2} + 100$$

As $\ln z = \ln|z| + i \operatorname{Arg} z$

$= \ln r + i \operatorname{Arg} z$

$$= \frac{-100 \ln z}{\ln 5} = \frac{-100 \ln r}{\ln 5} + i\frac{(-100)}{\ln 5} \operatorname{Arg} z$$

$$F(z) = -\frac{100}{\ln 5} \ln z + 100$$

(Q4.A)

$f(x) = |x| \quad -1 < x < 1, \quad p = 2$

Solution:

By definition $|x|$ is even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$p = 2L = 2$

$\boxed{L = 1}$

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$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} |x|\, dx$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$= \int_{-1}^{1} |x| \cos n\pi x\, dx$$

$$\boxed{a_n = \frac{-4}{n^2}}$$

$$f(x) = \frac{1}{2} + \frac{-4}{\pi^2}\left(\cos\pi x + \cos 2\pi x + \cos 3\pi x + \dots\right)$$

(Q4)(b) Find the fourier transform of $f^{50}(x)$.
where $f(x) = \begin{cases} e^{-4x} & x > 0 \\ 0 & x < 0 \end{cases}$

Solution:
As e^{-ax} if $x > 0$

$$\Rightarrow \frac{1}{\sqrt{2\pi}(a+j\omega)}$$

Here $f^{50}(x)$ so,

As $\frac{d^n}{dx^n} f(x) \longleftrightarrow (j\omega)^n X(\omega)$

$$\boxed{f^{50}(x) = (j\omega)^{50}\left(\frac{1}{2\pi(a+j\omega)}\right)}$$

(Ans)