Kamic & Kamic Forte Tablets

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Exam: Differential equation

Term: Mid term

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STANLEY

(Q1) Solve $y''' = x^2+1$

Solution:

$$y''' = x^2+1$$

Integrating x^2+1 w.r.t x

$$= \int (x^2+1)\,dx$$

$$= \frac{x^3}{3} + x + c_1$$

Again

$$= \int \frac{x^3}{3}\,dx + \int x\,dx + c_1\int 1\,dx$$

$$= \frac{x^4}{3\times 4} + \frac{x^2}{2} + c_1 x + c_2$$

$$= \frac{x^4}{12} + \frac{x^2}{2} + c_1 x + c_2$$

Ding one ~~on~~ more time

$$= \int \frac{x^4}{12}dx + \int \frac{x^2}{2}dx + c_1\int x\,dx + c_2\int 1\,dx$$

$$= \frac{x^5}{12\times 5} + \frac{x^3}{2\times 3} + c_1\frac{x^2}{2} + c_2 x + c_3$$

$$y = \frac{x^5}{60} + \frac{x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3$$

where c_1, c_2 and c_3 are arbitrory constants

(Q2) solve $(xy)y' = x^2+y^2$

Solution:

$$y' = \frac{x^2+y^2}{xy}$$

Let $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{x^2+u^2x^2}{ux^2}$$

$$u + x\frac{du}{dx} = u + \frac{1}{u}$$

or $$u\,du = \frac{dx}{x}$$

$$\int u\,du = \int \frac{dx}{x}$$

$$\frac{u^2}{2} = \ln|x| + C$$

$$\frac{y^2}{x^2} = 2\ln|x| + C \qquad \therefore u = \frac{y}{x}$$

$$\boxed{y^2 = 2x^2\ln|x| + Cx^2}$$

(Q3) Solve the Bernoulli's DE:

$$\frac{dy}{dx} + xy = xy^3$$

Solution:

Dividing both sides by y^3

$$y^{-3}\frac{dy}{dx} + xy(y^{-3}) = x$$

$$y^{-3}\frac{dy}{dx} + xy^{-2} = x \quad \text{———} \quad (1)$$

Let $y^{-2} = z$

$$\frac{d(y^{-2})}{dx} = \frac{dz}{dx}$$

$$(-2)\,y^{-2-1}\frac{dy}{dx} = \frac{dz}{dx}$$

$$-2\,y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx} \qquad \text{Put in (1)}$$

$$-\frac{1}{2}\frac{dz}{dx} + xz = x$$

$$\frac{dz}{dx} + (-2x)z = -2x \qquad \text{which is linear DE}$$

So $I.F = e^{\int(-2x)dx} = e^{-x^2}$

$$e^{-x^2}\frac{dz}{dx} + (-2x)z\,e^{-x^2} = -2x\,e^{-x^2}$$

$$\frac{d}{dx}\left[e^{-x^2}.z\right] = -2x\,e^{-x^2}$$

$$\int d\left[e^{-x^2}.z\right] = -\int 2x.e^{-x^2}\,dx$$

Let $x^2 = t$

$2x\,dx = dt$

$$e^{-x^2}.z = -\int e^{-t}\,dt$$

$$= -\frac{e^{-t}}{-1} + c$$

$$e^{-x^2}.z = e^{-t} + c$$

$$e^{-x^2}(y^{-2}) = e^{-x^2} + c$$

$$y^{-2} = e^{x^2}\left[e^{-x^2} + c\right]$$

$$y^{-2} = 1 + c\,e^{x^2}$$

is the General solution

(Q4) Find orthogonal Trajectory of
$y = cx^{-n}$

Solution:

$y = cx^{-n}$ $\therefore$ $c = \frac{y}{x^{-n}} = x^{+n}y$

Step 1: $\frac{dy}{dx} = \frac{d(cx^{-n})}{dx}$

$= c(-n)x^{-n-1}$

Step 2: Replace c

$\frac{dy}{dx} = (x^{n}y)(-nx^{-n-1})$

$= -ny\,x^{n-n-1}$

$= -nx^{-1}y$

Step 3: To Find O.T, replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$

$\frac{-1}{\frac{dy}{dx}} = \frac{-ny}{x}$

$\frac{dy}{dx} = \frac{-1}{\left(\frac{-ny}{x}\right)}$

$\frac{dy}{dx} = \frac{x}{ny}$

$\int y\,dy = \int \frac{x}{h}\,dx$ [by V.S.D.E]

$\frac{y^2}{2} = \frac{1}{h}\int x\,dx$

$= \frac{1}{h}\frac{x^2}{2} + C$

$\frac{y^2}{2} = \frac{x^2}{2h} + C$ is the General solution.

(Q5) Using appropriate method, solve the ODE

$$4y'' + \frac{24y'}{x} + 25yx^{-2} = 0$$

$$y = 15 \text{ at } x = 1$$

$$\frac{dy}{dx} = 1 \text{ at } x = 1$$

Solution:

$$4\frac{d^2y}{dx^2} + 24x^{-1}\frac{dy}{dx} + 25x^{-2}y = 0$$

$$\left(4\frac{d^2}{dx^2} + 24x^{-1}\frac{d}{dx} + 25x^{-2}\right)y = 0 \quad \text{—(A)}$$

Since (A) is homogeneous Cauchy-Euler D.E, so

Set $x = e^t \Rightarrow \ln x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx}$

Now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} \Rightarrow x\frac{d}{dx} = \frac{d}{dt} = \Delta$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dt} \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{dy}{dt}\left(\frac{-1}{x^2}\right) + \frac{1}{x} \cdot \frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\left(\frac{d}{dt}\cdot\frac{d}{dt} - \frac{d}{dt}\right)y$$

$$x^2\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{d}{dt} - 1\right]y$$

$$x^2\frac{d^2}{dx^2} = \frac{d}{dt}\left[\frac{d}{dt} - 1\right]$$

$$x^2\frac{d^2}{dx^2} = \Delta(\Delta - 1)$$

Multiply (A) by x^2 on both sides

$$\left(4x^2\frac{d^2}{dx^2} + 24x\frac{dy}{dx} + 25\right)y = 0$$

$$(4\,\Delta(\Delta-1) + 24\Delta + 25)\,y = 0$$

$$4(\Delta^2 - \Delta) + 24\Delta + 25 = 0$$

$$4\Delta^2 - 4\Delta + 24\Delta + 25 = 0$$

$$4\Delta^2 + 20\Delta + 25 = 0$$

$$4\Delta^2 + 10\Delta + 10\Delta + 25 = 0$$

$$2\Delta(2\Delta + 5) + 5(2\Delta + 5) = 0$$

$$(2\Delta+5)(2\Delta+5) = 0$$

$2\Delta + 5 = 0$; $\Delta = -5/2, \frac{-5}{2}$

$$2\Delta = -5$$

$$\Delta = -5/2$$

Roots are real and repeated so

$$y_c = (c_1 + c_2 t)\, e^{\lambda t}$$

$$y_c = (c_1 + c_2 t)\, e^{-5/2\, t}$$

$$x = e^t \Rightarrow t = \ln x$$

$$(x)^{-5/2} = e^{-5/2\, t}$$

Therefore $y_c = (c_1 + c_2 \ln x)\, x^{-5/2}$ (C_f. solution)

Now $y = 15$ at $x = 1$

so

$$15 = (c_1 + c_2 \ln 1)\,(1)^{-5/2}$$

$$15 = c_1 + 0$$

$$\boxed{c_1 = 15}$$

$$\frac{dy}{dx} = \frac{d}{dx}(c_1 + c_2 \ln x)\, x^{-5/2}$$

$$= x^{-5/2} \frac{d}{dx}(c_1 + c_2 \ln x) + (c_1 + c_2 \ln x) \frac{d x^{-5/2}}{dx}$$

$$= x^{-5/2}\left(c_2 \frac{1}{x}\right) + (c_1 + c_2 \ln x)\left(-\frac{5}{2}\right) x^{-5/2 - 1}$$

$$= x^{-5/2} . x^{-1} c_2 + (c_1 + c_2 \ln x)\left(-\frac{5}{2}\right) x^{-7/2}$$

$$\frac{dy}{dx} = x^{-7/2} c_2 + (c_1 + c_2 \ln x)\left(-\frac{5}{2}\right) x^{-7/2}$$

Given that $\frac{dy}{dx} = 1$ at $x = 1$

so $1 = (1)^{-7/2} C_2 + (C_1 + C_2 \ln 1)\left(-\frac{5}{2}\right)(1)^{-7/2}$

$1 = C_2 + (C_1 + 0)\left(-\frac{5}{2}\right)$

$1 = C_2 + \left(-\frac{5}{2}\right) C_1$

put $C_1 = 15$

$1 = C_2 - \frac{5}{2}(15)$

$= C_2 - \frac{75}{2}$

$C_2 = 1 + \frac{75}{2} = \frac{2+75}{2} = \frac{77}{2}$

$\boxed{C_2 = \frac{77}{2}}$

$y = \left(15 + \frac{77}{2} \ln x\right) x^{-5/2}$ is the particular solution.