16.1

Q.1 Expand The given function in a Laurent Series

Sol. $$f(z) = \frac{e^{-z}}{z^3}$$

Solution: Given $f(z) = \frac{e^{-z}}{z^3}$

$$\Rightarrow \frac{1}{z^3}\left[1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right]$$

$$\frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{2!z} - \frac{1}{3!}z + \cdots$$

Q#2 $f(z) = z\cos\frac{1}{z}$

Solution As ~~f(z)~~ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

$$\cos\frac{1}{z} = 1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \frac{1}{6!}\frac{1}{z^6} + \cdots$$

$$z\cos\frac{1}{z} = z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \frac{1}{6!}\frac{1}{z^5} + \cdots$$

Q#3 $f(z) = \frac{e^z}{z^2 - z^3}$

Solution: Here $f(z) = \frac{e^z}{z^2 - z^3}$

$$\Rightarrow \frac{e^z}{z^2 - z^3} = \frac{e^z}{z^2(1-z)}$$

$z^2(1-z)=0 \Rightarrow z=0$ & $z=1$

two singularities at $z=0$ & $z=1$

First we expand the given function for $z=0$

Then we have

$$\frac{1}{z^2}\left[\frac{e^z}{1-z}\right]$$

Now we expand $\frac{e^z}{1-z}$ by maclaurin series

$$g(z)=\frac{e^z}{1-z} \qquad g(0)=1$$

$$g'(z)=\frac{(1-z)e^z+e^z}{(1-z)^2} \qquad g'(0)=2$$

$$g'(z)=\frac{2e^z}{(1-z)^2}-\frac{ze^z}{(1-z)^2}$$

$$g''(z)=\frac{(1-z)^2 2e^z+2(1-z)2e^z}{(1-z)^4}-\cancel{(1-z)^2(e^z+ze^z)}-$$

$$-\frac{(1-z)^2\left[e^z+ze^z\right]+2(1-z)ze^z}{(1-z)^4}$$

$$g''(0)=2+2-1 \Rightarrow g''(0)=3$$

$$\frac{e^{z}}{z^{2}-z^{3}} = \frac{1}{z^{2}}\left[\frac{e^{z}}{1-z}\right]$$

$$= \frac{1}{z^{2}}\left[1 + 2\frac{z}{1!} + \frac{3z^{2}}{2!} + \cdots\right]$$

$$\frac{e^{z}}{z^{2}-z^{3}} = \frac{1}{z^{2}} + \frac{2}{z} + \frac{3}{2!} + \cdots$$

Now Laurent Series at $z_0 = 1$ Then we have

As we know That Laurent Series at $z = z_0$ we have

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots$$

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots$$

$$g(z) = \frac{e^{z}}{z^{2}} \qquad g(1) = \frac{e}{1} = e$$

$$g'(z) = \frac{z^{2}e^{z} - 2ze^{z}}{z^{2}} = \frac{e-2e}{1} = -e$$

$$g''(z) = e^{z} - 2e^{z}/z$$

$$g'(z) = e^z - 2\left(\frac{z e^z - e^z}{z^2} \right)$$

$$g'(1) = e - 2(0) = e$$

$$g'' = e^z - \frac{2e^z}{z} + 2e^z/z^2$$

$$g'''(z) = e^z - 2\left(\frac{z e^z - e^z}{z^2} \right) + 2\left(\frac{z^2 e^z - 2z e^z}{z^4} \right)$$

$$g'''(1) = e - 2(0) + 2(-1)e = -e$$

Thus

$$\frac{e^z}{z^2} = e + (z-1)\frac{(-e)}{1!} + \frac{(z-1)^2}{2!}(e) + \frac{(z-1)^3}{3!}(-e)$$

$$\frac{e^z}{(1-z)z^2} = \left[\frac{-e^z}{z^2(z-1)} \right]$$

$$= -\left[\frac{e}{z-1} - \frac{e}{1!} + \frac{(z-1)}{2!}e - \frac{(z-1)^2}{3!}e \right]$$

$$= \frac{-e}{z-1} + \frac{e}{1!} - \frac{(z-1)e}{2!} + \frac{(z-1)^2}{3!}e$$

Q# 04 If $f(z) = \frac{z^2-4}{(z-1)^2}$ $z_0 = 1$

Then expand the given function in Laurent Series at $z_0 = 1$

Solution $f(z) = \frac{z^2-4}{(z-1)^2}$

Here $g(z) = z^2 - 4$ then

$g(1) = 1 - 4 = -3$

$g'(z) = 2z \Rightarrow g'(1) = 2$

$g''(z) = 2 \Rightarrow g''(1) = 2$

$g'''(z) = 0$ $g'''(1) = 0$

$$\frac{1}{(z-1)^2}\left[z^2-4\right] = \frac{1}{(z-1)^2}\left[-3 + \frac{(z-1)}{1!}(2) + \frac{(z-1)^2}{2!}(2) + 0\right]$$

$$\frac{z^2-4}{(z-1)^2} = \frac{-3}{(z-1)^2} + \frac{2}{(z-1)} + 1$$

Zeros and order of zeros

Q1 $f(z) = (z^4-1)^4$

For zero of $f(z)$ we take $f(z) = 0$

so $(z^4-1)^4 = 0 \Rightarrow z^4-1 = 0$

$z^4-1 = (z^2-1)(z^2+1) = 0$

$z = \pm 1, \pm i$

Thus zeros of $(z^4-1)^4$ are ± 1 & $\pm i$

for order of zeros we have

$f'(z) = 4(z^4-1)^3 \cdot 4z^3$

$f'(1) = 4(1-1)4(1) = 0$

$f''(z) = 16 \times 3z^2(z^4-1)^3 + 48\, z^3(z^4-1)^2 4z$

$f''(1) = 0$ Similarly $f'''(1) = 0$

& $f^{iv}(1) \neq 0$ So this type of problem we

Take without power we have

$f(z) = z^4 - 1$

$\Rightarrow f'(z) = 4z^3$

$\Rightarrow f'(1) = 4 \quad f'(-1) = -4$

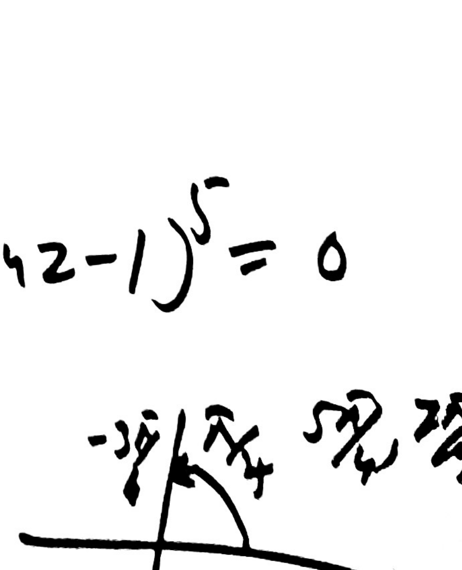
$f'(i) = -4i \quad f'(-i) = 4i$

Since all the zero of order 1 But power is 4 so these zero are ~~power~~ order of 4.

Q# 2 $\quad f(z) = (\sin z - 1)^5$

For zero we have $f: (\sin z - 1)^5 = 0$

$\sin z - 1 = 0 \Rightarrow \sin z = 1$



$z = (4k+1)\pi/4 \quad k \in Z$

order is 5.

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{g(z)} = \frac{p(z_0)}{g'(z_0)}$$

If $f(z)$ has a pole of mth order at $z = z_0$ the residue is given by

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}$$