Домашняя работа №3-4

Вариант №8

№1

Задание:

$$u(M) = xe^y + ye^x - z^2$$

$$M_1(3;0;2) \quad M_2(4,1,3)$$

Решение:

$$u'_x = e^y + ye^x$$

$$u'_y = xe^y + e^x$$

$$u'_z = -2z$$

$$\text{grad}(u) = (e^y + ye^x; xe^y + e^x; -2z)$$

$$\text{grag}(u(M_1)) = (1+0; 3+e^3; -4) =$$

$$= (1; 3+e^3; -4)$$

$$\overrightarrow{M_1M_2} = (4-3; 1-0; 3-2) = (1;1;1)$$

$$(\cos\alpha; \cos\beta; \cos\gamma) = \frac{\overrightarrow{M_1M_2}}{|\overrightarrow{M_1M_2}|} = \left(\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}\right)$$

$$\frac{\partial u(M_1)}{\partial(\overrightarrow{M_1M_2})} = \text{grad}(u(M_1)) \cdot (\cos\alpha; \cos\beta; \cos\gamma) =$$

$$= 1 \cdot \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{e^3}{\sqrt{3}} - \frac{4}{\sqrt{3}} = \frac{e^3}{\sqrt{3}}$$

Ответ: а) $(1; 3+e^3; -4)$

б) $\dfrac{e^3}{\sqrt{3}}$

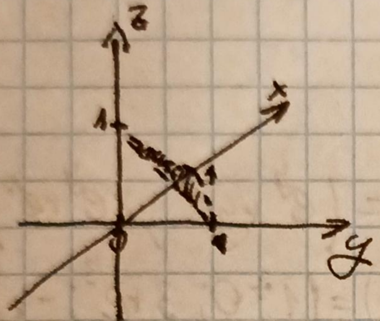
№2

Задание:

$$\iint_S xz\,dx\,dy + xy\,dy\,dz + yz\,dx\,dz$$

S – верхняя часть плоскости $x+y+z=1$, отсечённой координатными плоскостями

Решение:



$$\iint_S \underset{1}{xz\,dx\,dy} + \underset{2}{xy\,dy\,dz} + \underset{3}{yz\,dx\,dz} = $$

1 $\iint_S xz\,dx\,dy = \int_0^1\int_0^{1-x} x(1-x-y)\,dy\,dx =$

$\boxed{z=1-x-y}$

$$= \int_0^1 \left(yx - yx^2 - x\frac{y^2}{2}\right)\Big|_0^{1-x} dx =$$

$$= \int_0^1 (1-x)x - (1-x)x^2 - x\frac{(1-x)^2}{2}\,dx =$$

$$= \left(\left(\frac{x^2}{2}-\frac{x^3}{3}\right) - \left(\frac{x^3}{3}-\frac{x^4}{4}\right) - \frac12\left(\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{x^4}{4}\right)\right)\Big|_0^1 =$$

$$= \left(\frac12-\frac13\right) - \left(\frac13-\frac14\right) - \frac12\left(\frac12-\frac23+\frac14\right) =$$

$$= \frac{3-2}{6} - \frac{4-3}{12} - \frac12\left(\frac{6-8+3}{12}\right) = \frac16 - \frac1{12} - \frac1{24} =$$

$$= \frac{4-2-1}{24} = \frac1{24}$$

(2) $\iint_S xy\,dy\,dz = \int_0^1\int_0^{1-y}(1-y-z)\,y\,dz\,dy =$

$\boxed{x = 1-y-z}$

$$= \int_0^1 \left(yz - y^2z - y\frac{z^2}{2}\right)\Big|_0^{1-y} dy =$$

$$= \int_0^1 y(1-y) - y^2(1-y) - y\frac{(1-y)^2}{2}\,dy =$$

$$= \int_0^1 y - y^2 - y^2 - y^3 - \frac{y}{2} + y^2 + \frac{y^3}{2}\,dy =$$

$$= \int_0^1 \frac{1}{2}y - y^2 + \frac{1}{2}y^3\,dy = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{6-8+3}{24} = \frac{1}{24}$$

(3) $\iint_S yz\,dy\,dx = \int_0^1\int_0^{1-x} y(1-y-x)\,dy\,dx =$

$\boxed{z = 1-y-x}$

$$= \int_0^1 \left(\frac{y^2}{2} - \frac{y^3}{3} - x\frac{y^2}{2}\right)\Big|_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} -$$

$$- x\frac{(1-x)^2}{2}\,dx =$$

$$= \int_0^1 \frac{1-2x+x^2}{2} - \frac{1-2x+x^2}{3} - \frac{x-2x^2+x^3}{2}\,dx =$$

$$= \frac{1}{6}\int_0^1 3-6x+3x^2-2+4x-2x^2-3x+3x^2-3x^3\,dx =$$

$$= \frac{1}{6}\int_0^1 1-5x+4x^2-3x^3 = \frac{1}{6}\left(1-\frac{5}{2}+\frac{4}{3}-\frac{3}{4}\right) =$$

$$= \frac{1}{6}\cdot\frac{12-30+16-9}{12} = \frac{-37}{6\cdot 12}$$

$\cancel{③ \iint\limits_S yz\, dy\, dz \quad \boxed{z = 1 - y - x}}$

$$③ \iint\limits_S yz\, dx\, dz = \int_0^1 \int_0^{1-x} (1 - x - z) z \, dz\, dx = \quad \boxed{y = 1 - x - z}$$

Случай идентичен пункту ① с точностью до замены переменной $(y = z)$

$$= \frac{1}{24}$$

$$\boxed{=} \ \frac{1}{24} \cdot 3 = \frac{1}{8}$$

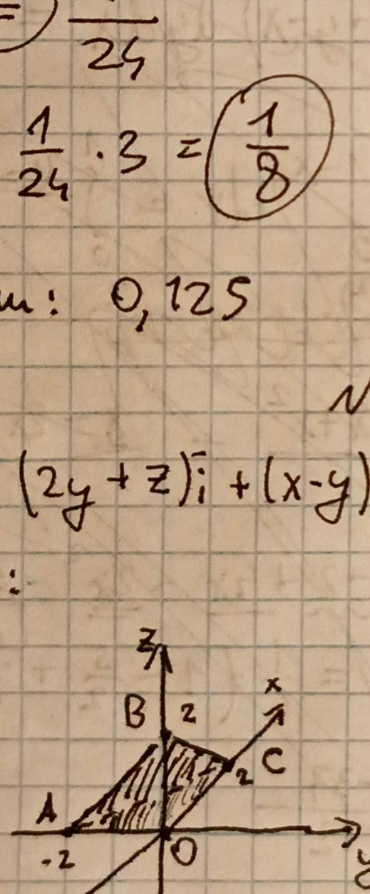
Ответ: 0,125

№3

$$a(M) = (2y + z)\bar{i} + (x - y)\bar{j} - 2z\bar{k} \qquad g: x - y + z = 2$$

Решение:

$$\bar{a} \cdot \bar{n} = a_x \cos\alpha + a_y \cos\beta + a_z \cos\gamma$$



Рассчитаем для (ABC)

$$g: y = x + z - 2$$

$$\frac{\partial y}{\partial x} = 1 \quad \frac{\partial y}{\partial z} = 1 \qquad \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + 1 + \left(\frac{\partial y}{\partial z}\right)^2} = \sqrt{3}$$

$$\cancel{\cos\alpha = \frac{-\frac{\partial y}{\partial x}}{\sqrt{3}} \qquad \cos\beta = \frac{1}{-\sqrt{3}} \qquad \cos\gamma = \frac{\frac{\partial y}{\partial z}}{\sqrt{3}}}$$

$$\cancel{\bar{a}\cdot\bar{n} = (2y+z)\left(-\frac{1}{\sqrt{3}}\right) + (x-y)\left(\frac{1}{-\sqrt{3}}\right) - 2z\left(\frac{-1}{\sqrt{3}}\right) =}$$

$$\cancel{= -\frac{2y+z}{\sqrt{3}} + \frac{x-y}{-\sqrt{3}} + \frac{2z}{\sqrt{3}} =}$$

$$\cancel{= \frac{1}{\sqrt{3}}\left(-2y - z - x + y + 2z\right) =}$$

$$\cancel{= \frac{1}{\sqrt{3}}(-y + z - x}$$

$$\cos\alpha = \frac{\frac{\partial y}{\partial x}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \cos\beta = -\frac{1}{\sqrt{3}}$$

$$\cos\gamma = \frac{\frac{\partial y}{\partial z}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\bar{a}\cdot\bar{n} = (2y+z)\left(\frac{1}{\sqrt{3}}\right) + (x-y)\left(-\frac{1}{\sqrt{3}}\right) - 2z\left(\frac{1}{\sqrt{3}}\right) =$$

$$= \frac{1}{\sqrt{3}}\left(2y + z - x + y - 2z\right) = \frac{1}{\sqrt{3}}(3y - z - x)$$

$$Q = \iint_S \bar{a}\cdot\bar{n}\, dS = \cancel{\iint_{D_{xz}} \frac{1}{\sqrt{3}}(3y - z - x)\sqrt{3}\, dxz}$$

$$= \iint_{D_{xz}} \frac{1}{\sqrt{3}}\left(3(x+z-2) - z - x\right)\sqrt{3}\, dx\, dz =$$

$$= \int_0^2 \int_0^{2-z} 2x + 2z - 6\, dx\, dz = \int_0^2 \left(2\frac{x^2}{2} + 2zx - 6x\right)\Big|_0^{2-z} dz =$$

$$= \int_0^2 (2-z)^2 + 2z(2-z) - 6(2-z)\, dz =$$

$$= \int_0^2 4 - \cancel{4z} + z^2 + \cancel{4z} - 2z^2 - 12 + 6z\, dz =$$

$$\int_0^2 -z^2 + 6z - 8\, dz = -\frac{2^3}{3} + 6\frac{2^2}{2} - 8\cdot 2 =$$

$$= -\frac{8}{3} + 12 - 16 = -\frac{20}{3}$$

Рассмотрим (AOC):

$$\bar{a}\cdot\bar{n} = (2y+z)\cdot 0 + (x-y)\cdot 0 - 2z\cdot(-1)$$

$$Q = \iint\limits_S \bar{a}\cdot\bar{n}\,dS = \cancel{\iint\limits_{D_{yx}} 2z\,dy\,dx} \iint\limits_{D_{yx}} 2\cdot 0\,dy\,dx = 0$$

Рассмотрим (BOC):

$$\bar{a}\cdot\bar{n} = (2y+z)\cdot 0 + (x-y)\cdot 1 - 2z\cdot 0$$

$$Q = \iint\limits_S \bar{a}\cdot\bar{n}\,dS = \iint\limits_{D_{zx}} x-\cancel{y}\,dz\,dx = \int_0^2\int_0^{2-x} x\,dz\,dx =$$

$$= \int_0^2 x(2-x)\,dx = 2\frac{2^2}{2} - \frac{2^3}{3} = 4 - \frac{8}{3} = \frac{4}{3}$$

Рассмотрим (AOB):

$$\bar{a}\cdot\bar{n} = (2y+z)(-1) + (x-y)\cdot 0 - 2z\cdot 0$$

$$Q = \iint\limits_S \bar{a}\cdot\bar{n}\,dS = \iint\limits_{D_{yz}} (2y+z)(-1)\,dy\,dz =$$

$$= \int_{-2}^{0}\int_0^{2+y} -2y - z\,dz\,dy = \int_{-2}^{0} -2y(2+y) - \frac{4+4y+y^2}{2}\,dy =$$

$$= \int_{-2}^{0} -4y - 2y^2 - 2 - 2y - \frac{y^2}{2}\, dy =$$

$$= 4\cdot\frac{(-2)^2}{2} + 2\cdot\frac{(-2)^3}{3} + 2(-2) + 2\cdot\frac{(-2)^2}{2} + \frac{1}{2}\,\frac{(-2)^3}{3} =$$

$$\cancel{= 8 + 2\frac{8}{3} - 4 + 4 - \frac{4}{3} = \ldots}$$

$$\cancel{= 8 - \frac{14}{3} + \frac{4}{3}}$$

$$= 8 - \frac{16}{3} \cancel{-4} \cancel{+4} - \frac{4}{3} = \frac{24}{3} - \frac{20}{3} = \frac{4}{3}$$

$$-\frac{20}{3} + \frac{4}{3} + \frac{4}{3} + 0 = -\frac{12}{3} = -4$$

Ответ: -4 (по опр.)

~~По формуле Остроградского-Гаусса~~

CF $\iint\limits_S \operatorname{rot}\bar a\cdot\bar n\, dS = -\iint\limits_{D_{yz}} \operatorname{rot}\bar a_x\, dydz + \iint\limits_{D_{xz}} \operatorname{rot}\bar a_y\, dxdz -$

$- \iint\limits_{D_{xy}} \operatorname{rot}\bar a_z\, dxdy$

$$\operatorname{rot}\bar a = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y+z & x-y & -2z \end{vmatrix} = \bar i\frac{\partial}{\partial y}(-2z) + \bar j\frac{\partial}{\partial z}(2y+z)$$

$$+ \bar k\frac{\partial}{\partial x}(x-y) - \bar k\frac{\partial}{\partial y}(2y+z) - (x-y)\frac{\partial}{\partial z}\bar i - \bar j\frac{\partial}{\partial x}(-2z) =$$

$$= \bar i\left(\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(x-y)\right) + \bar j\left(\frac{\partial}{\partial z}(2y+z) - \frac{\partial}{\partial x}(-2z)\right) + \bar k\cdot$$

$$\cdot\left(\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial y}(2y+z)\right) = \bar i\cdot 0 + \bar j(1) + \bar k\cdot(1-2) =$$

$= \bar{j} - \bar{k}$

$C = 0 + \iint\limits_{D_{xz}} dx\,dz + \iint\limits_{D_{xy}} dx\,dy = \int_0^2 \int_0^{2-x} \ldots$

$= \int_0^2 \int_0^{2-x} dz\,dx + \int_0^2 \int_{x-2}^{x} dy\,dx = \int_0^2 2-x\,dx + \int_0^2 2-x\,dy =$

$= 4 - \frac{2^3}{2} + 4 - \frac{2^2}{2} = 8 - 4 = 4$

По теореме Остроградского-Гаусса

$Q = \iiint\limits_T \operatorname{div}\bar{a}\,dv$

$\operatorname{div}\bar{a} = \frac{2y+z}{\partial x} + \frac{x-y}{\partial y} + \frac{-2z}{\partial z} = 0 + (-1) + (-2) = -3$

$Q = -3 \iiint\limits_T dv = -3 \cdot V = -3\left(\frac{1}{3} \cdot 2 \cdot \left(\frac{1}{2} \cdot 2 \cdot 2\right)\right) = -4$

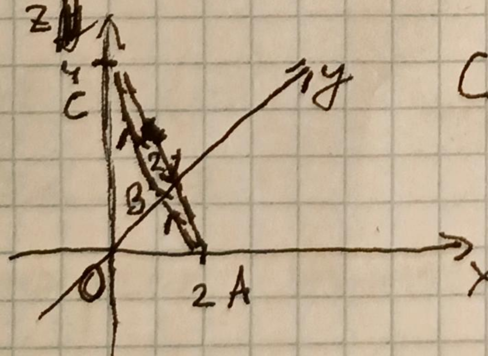
Ответ: -4 (по теор. Остроградского-Гаусса)

№4

$a(M) = (x+z)\bar{i} + z\bar{j} + (2x-y)\bar{k}$ $\qquad \ell: 2x+2y+z=4$

Решение

$C = \int\limits_l \bar{a}\,d\bar{r} = \int\limits_l a_x dx + a_y dy + a_z dz$



$C = \int\limits_{ABCA} a_x dx + a_y dy + a_z dz =$

$= \int\limits_{ABCA} (x+z)dx + z\,dy + (2x-y)dz$

Рассмотрим AB: $\begin{cases} z = 0 \\ y = 2 - x \end{cases} \Rightarrow \begin{matrix} dz = 0 \\ dy = -1\,dx \end{matrix}$

~~$C_{AB} = \int_2^0 x\,dx$ + [illegible] $(-1)\,dx$ = [illegible] $\int_2^0 -2x + 2\,dx =$~~

$$C_{AB} = \int_2^0 x\,dx = -2$$

Рассмотрим BC: $\begin{cases} x = 0 \\ z = 4 - 2y \end{cases} \Rightarrow \begin{matrix} dx = 0 \\ dz = -2dy \end{matrix}$

~~$C_{BC} = \int_0^2 z$ [illegible] $dy + (-y)$ [illegible] $(-2)\,dy = \int_2^2 z + 2y\,dy =$~~

$$C_{BC} = \int_2^0 (4-2y)\,dy + (-y)(-2)\,dy = \int_2^0 4 - 2y + 2y\,dy =$$

$$= \int_2^0 4\,dy = -4 \cdot 2 = -8$$

Рассмотрим CA: $\begin{cases} y = 0 \\ z = 4 - 2x \end{cases} \Rightarrow \begin{matrix} dy = 0 \\ dz = -2dx \end{matrix}$

$$C_{CA} = \int_0^2 (x + (4 - 2x))\,dx + 2x(-2)\,dx =$$

$$= \int_0^2 4 - x - 4x\,dx = \int_0^2 4 - 5x = 4 \cdot 2 - 5\,\frac{2^2}{2} = -2$$

$$-2 + (-2) + (-8) = C_{ABCA}$$

Ответ: -12

По теореме Стокса:

$$C = \iint_S rot\,\bar{a} \cdot \bar{n}\, dS = \iint_{D_{yz}} rot\,\vec{a}_x\, dy\,dz + \iint_{D_{xz}} rot\,\bar{a}_y\, dx\,dz +$$

$$+ \iint_{D_{xy}} rot\,\bar{a}_y\, dx\,dy =$$

$$rot\,\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z & z & 2x-y \end{vmatrix} = \bar{i}\frac{\partial}{\partial y}(2x-y) + \bar{j}\frac{\partial}{\partial z}(x+z) +$$

$$+ \bar{k}\frac{\partial}{\partial x} z - \bar{k}\frac{\partial}{\partial y}(x+z) - \bar{i}\frac{\partial}{\partial z} z - \bar{j}\frac{\partial}{\partial x}(2x-y) =$$

$$= \bar{i}(-1-1) + \bar{j}(1-2) + \bar{k}(0-0) = -2i - j$$

$$= \iint_{D_{yz}} -2\,dy\,dz + \iint_{D_{xz}} -1\,dx\,dz = -2\iint_{D_{yz}} dy\,dz -$$

$$- \iint_{D_{xz}} dx\,dz = -2\, S_{\Delta OCB} - S_{\Delta OCA} = -2 \cdot \frac{1}{2} \cdot 4 \cdot 2 -$$

$$- \frac{1}{2} \cdot 4 \cdot 2 = -8 - 4 = -12$$

Ответ: -12

№5

$$u(M) = y^2 z - x^2 \qquad M_0(0;1;1)$$

$$\nabla u = \left(\frac{\partial u}{\partial x}; \frac{\partial u}{\partial y}; \frac{\partial u}{\partial z}\right) = (-2x;\ 2yz;\ y^2)$$

$$\nabla u(M_0) = (0;2;1) \qquad |\nabla u(M_0)| = \sqrt{0^2+2^2+1^2} = \sqrt{5}$$

Ответ: направление $(0;2;1)$, величина $\sqrt{5}$