Introduction to

Algorithm Design and Analysis

[13] Undirected Graph

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In the last class …

- **Directed Acyclic Graph**
  - Topological order
  - Critical path analysis

- **Strongly Connected Component (SCC)**
  - Strong connected component and condensation
  - Finding SCC based on DFS

DFS on Undirected Graph

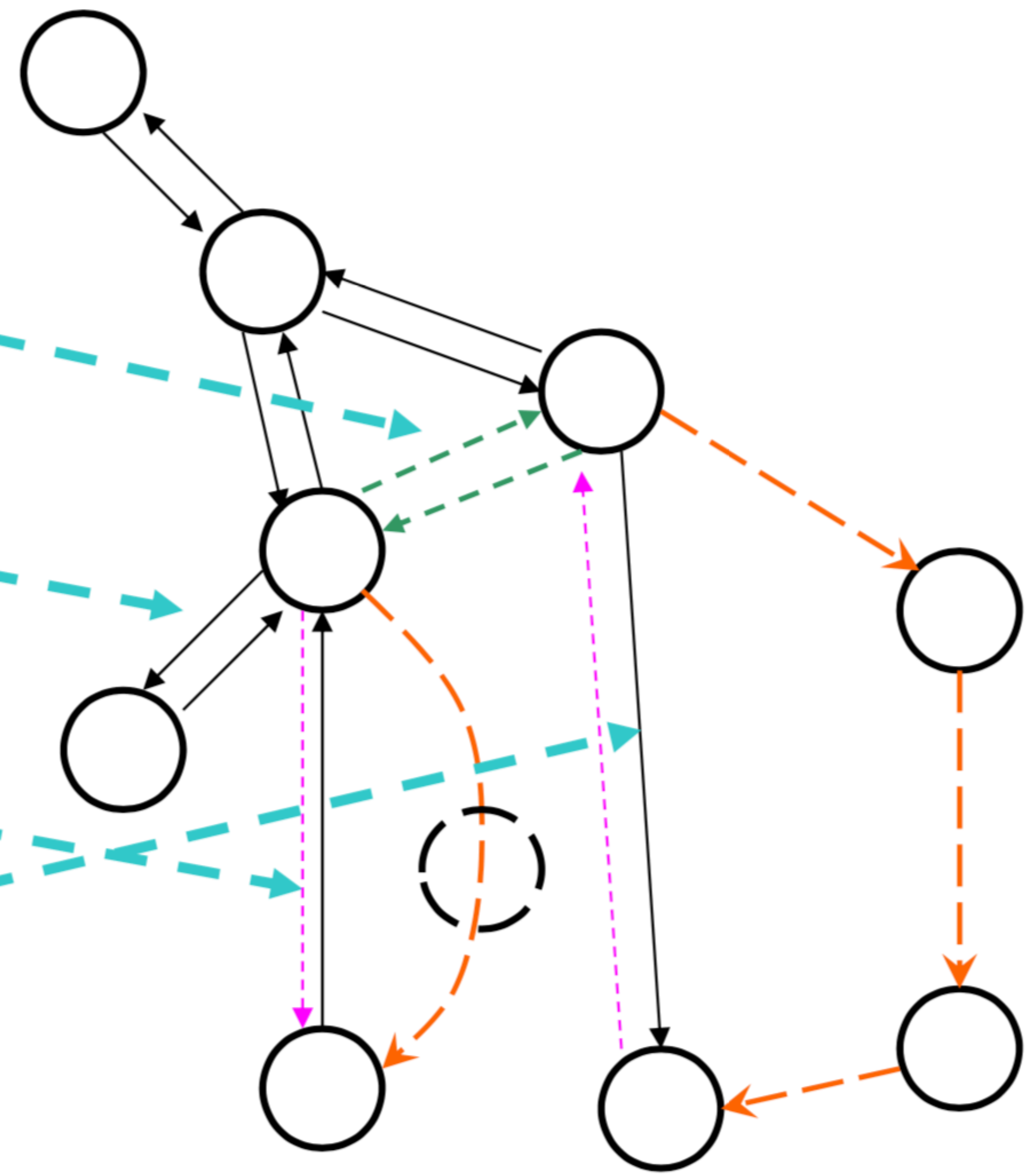
- **Undirected Graph**
  - Symmetric Digraph
  - Undirected Graph DFS Skeleton
- **Biconnected Components**
  - Articulation Points
  - Bridge
- **Other undirected graph problems**
  - Orientation of an undirected graph
  - Simplified Minimum Spanning Tree

What is Different for “Undirected”

- **Characteristics of undirected graph traversal**
  - One edge may be traversed for two times in opposite directions.
- **For an undirected graph, DFS provides an orientation for each of its edges**
  - Oriented in the direction in which they are first encountered.

Edges in DFS

- **Cross edge**
  - Not existing
- **Back edge**
  - Back to the direct parent: second encounter
  - Otherwise: **first encounter**
- **Forward edge**
  - Always second encounter, ***and first time as back edge***



Modifications to the DFS Skeleton

- All the second encounter are bypassed.
- So, the only substantial modification is for the possible back edges leading to an ancestor, but not direct parent.
- We need know the parent, that is, the direct ancestor, for the vertex to be processed.

DFS Skeleton for Undirected Graph

- void dfsSweep(intList[] adjVertices, int n, …)
- int ans;
- <Allocate color array and initialize to white>
- for each vertex v of G, in some order
- if(color[v]==white)
- Int vAns=dfs(adjVertices, color, v, -1,…);
- <Process vAns>
- //continue loop
- return ans;

DFS Skeleton for Undirected Graph

- int dfs(intList[] adjVertices, int[] color, int v, int p,…)
-     int w; intList remAdj; int ans; color[v]=gray;
-     <Preorder processing of vertex v>
-     remAdj=adjVertices[v];
-     while(remAdj != nil)
-         w=first(remAdj);
-         if(color[w]==white)
-             <Exploratory processing for tree edge vw>
-             dfs(adjVertices, color, w, v, …);
-             <Backtrack processing for tree edge vw, using wAns>
-         else if(color[w]==gray && w!=p)
-             <Checking for nontree edge vw>
-        remAdj=rest(remAdj);
-     <Postorder processing of vertex v, including final computation of ans>
-     color[v]=black;
-     return ans;

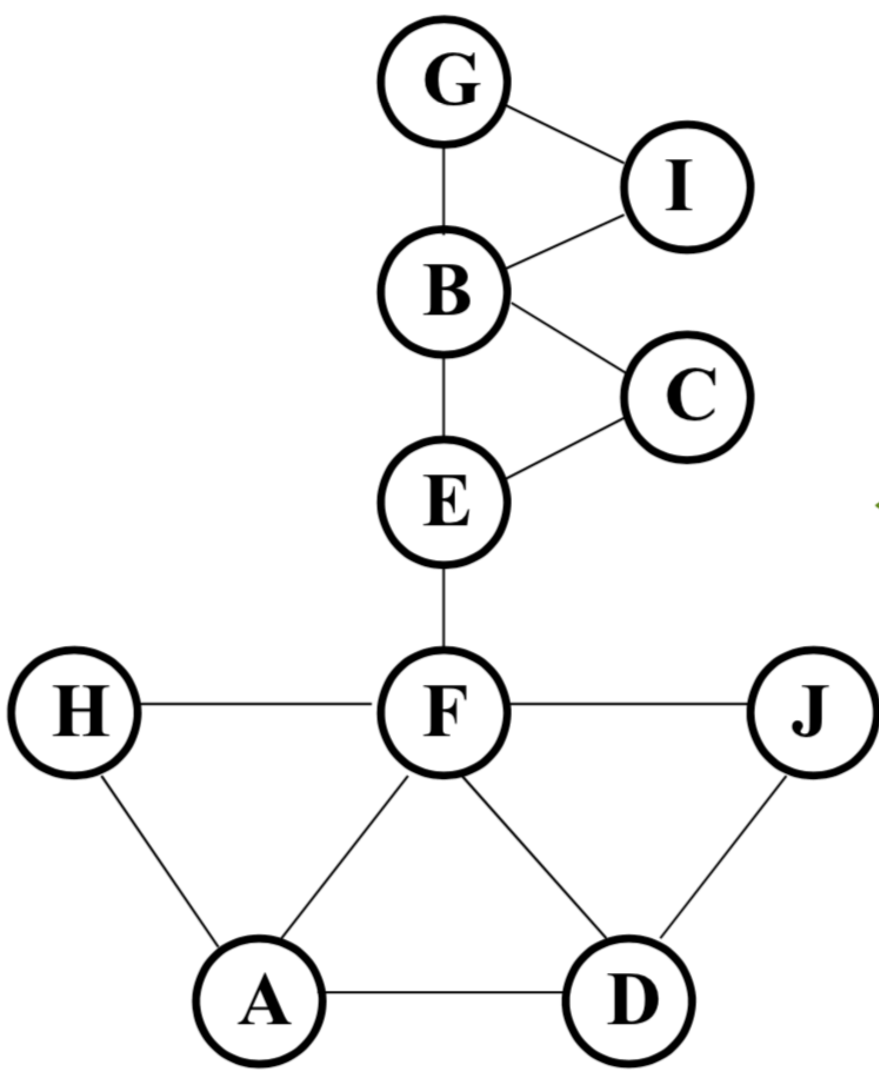
Complexity of Undirected DFS

- **$\Theta(m+n)$**
  - If each inserted statement for specialized application runs in constant time
  - The same with directed graph DFS
- **Extra space $\Theta(n)$**
  - For array color, or activation frames of recursion

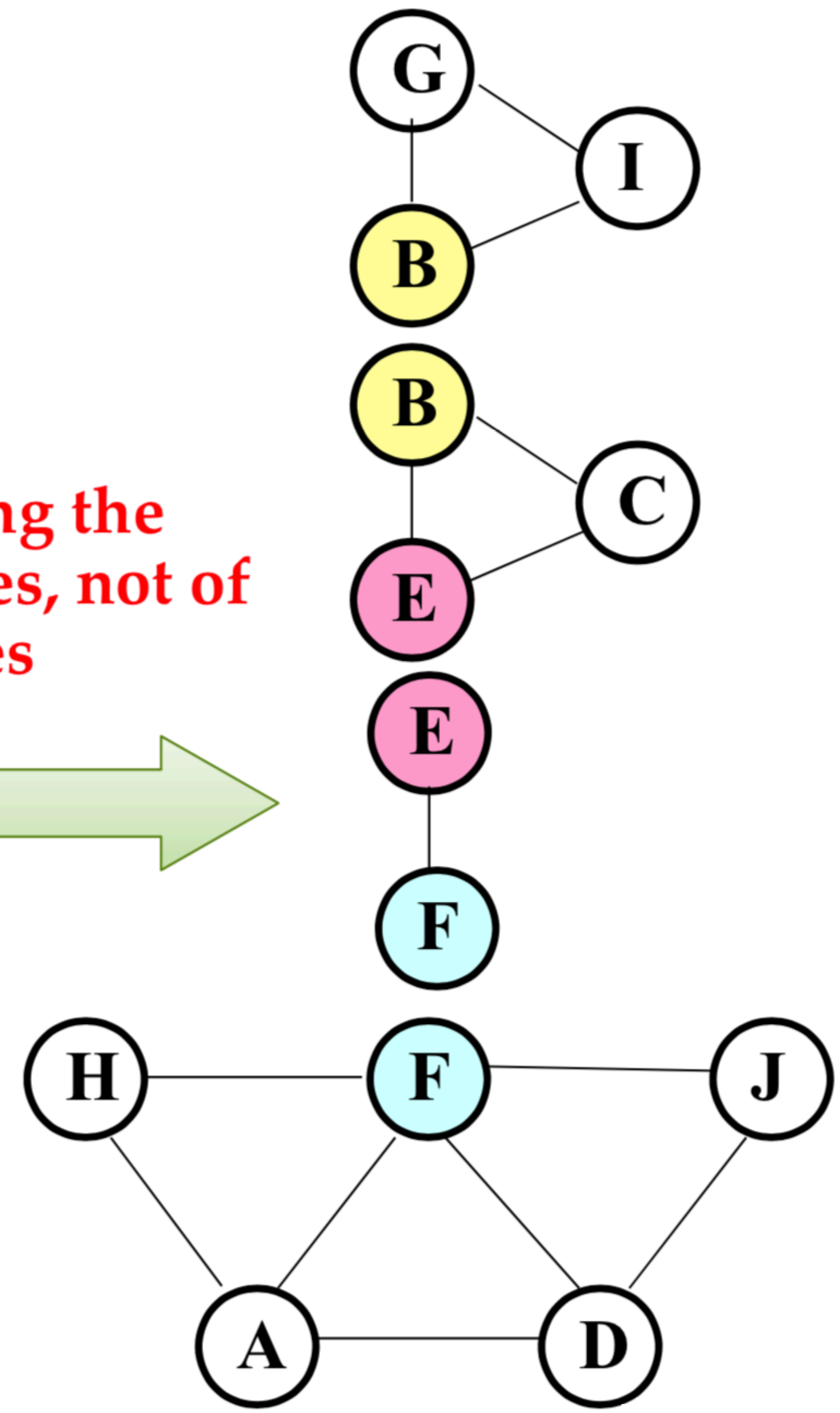
Biconnected Graph

- **Being connected**
  - Tree: acyclic, least (cost) connected
  - Node/edge connected: fault-tolerant connection
- **Articulation point (2-node connected)**
  - v is an articulation point if deleting v leads to disconnection
- **Bridge (2-edge connected)**
  - uv is a bridge if deleting uv leads to disconnection

Articulation Points



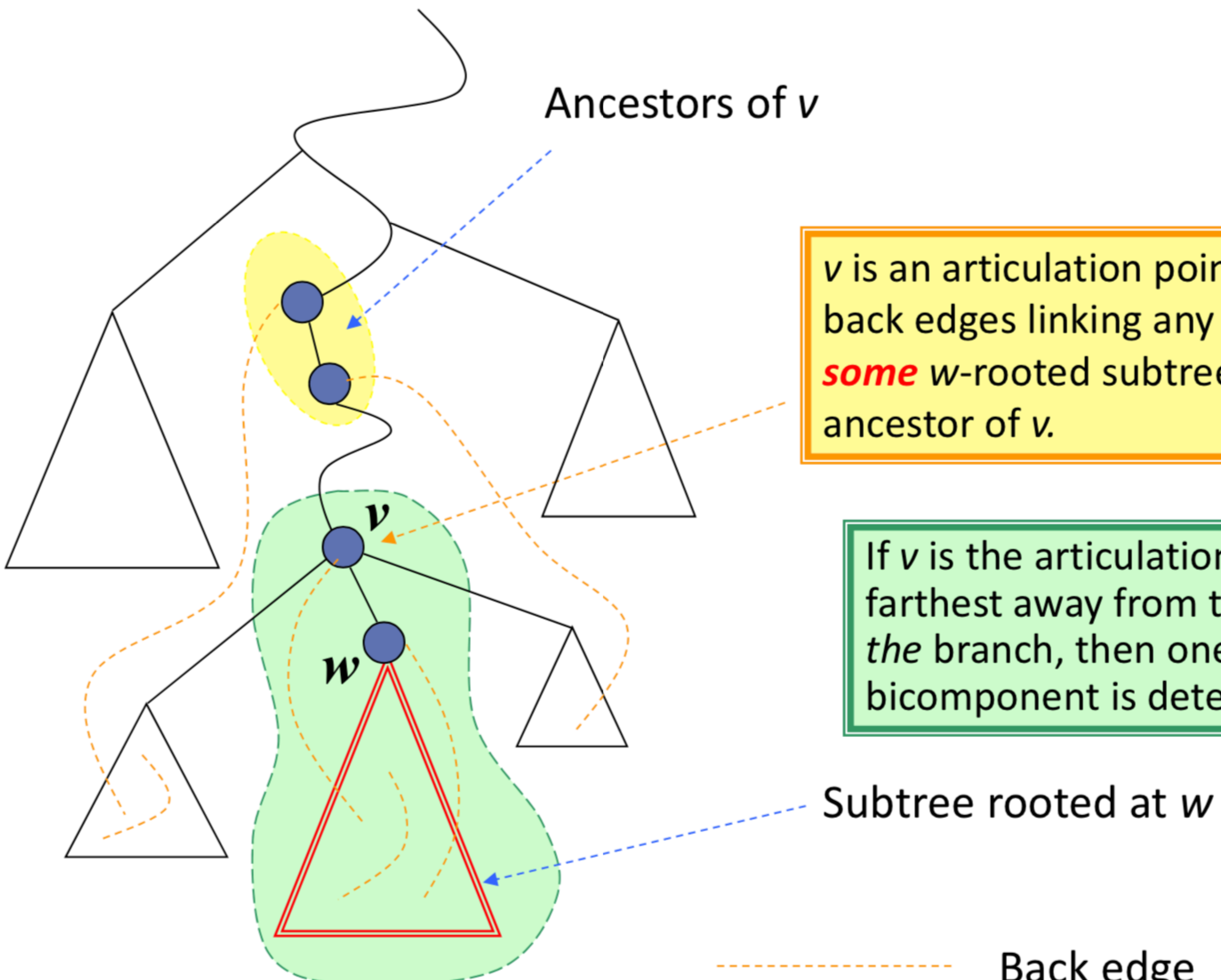
Partitioning the set of edges, not of the vertices



Definition Transformation

- **“Short definition”**
  - Deleting v leads to disconnection
- **“Long definition”**
  - If there exist nodes w and x, such that v is in every path from w to x (w and x are vertices different from v)
- **“Long definition” or “DFS definition”**
  - No back edges linking any vertex in some w-rooted subtree and any ancestor of v

Articulation Point Algorithm



v is an articulation point iff no back edges linking any vertex in ***some*** *w*-rooted subtree and any ancestor of *v.*

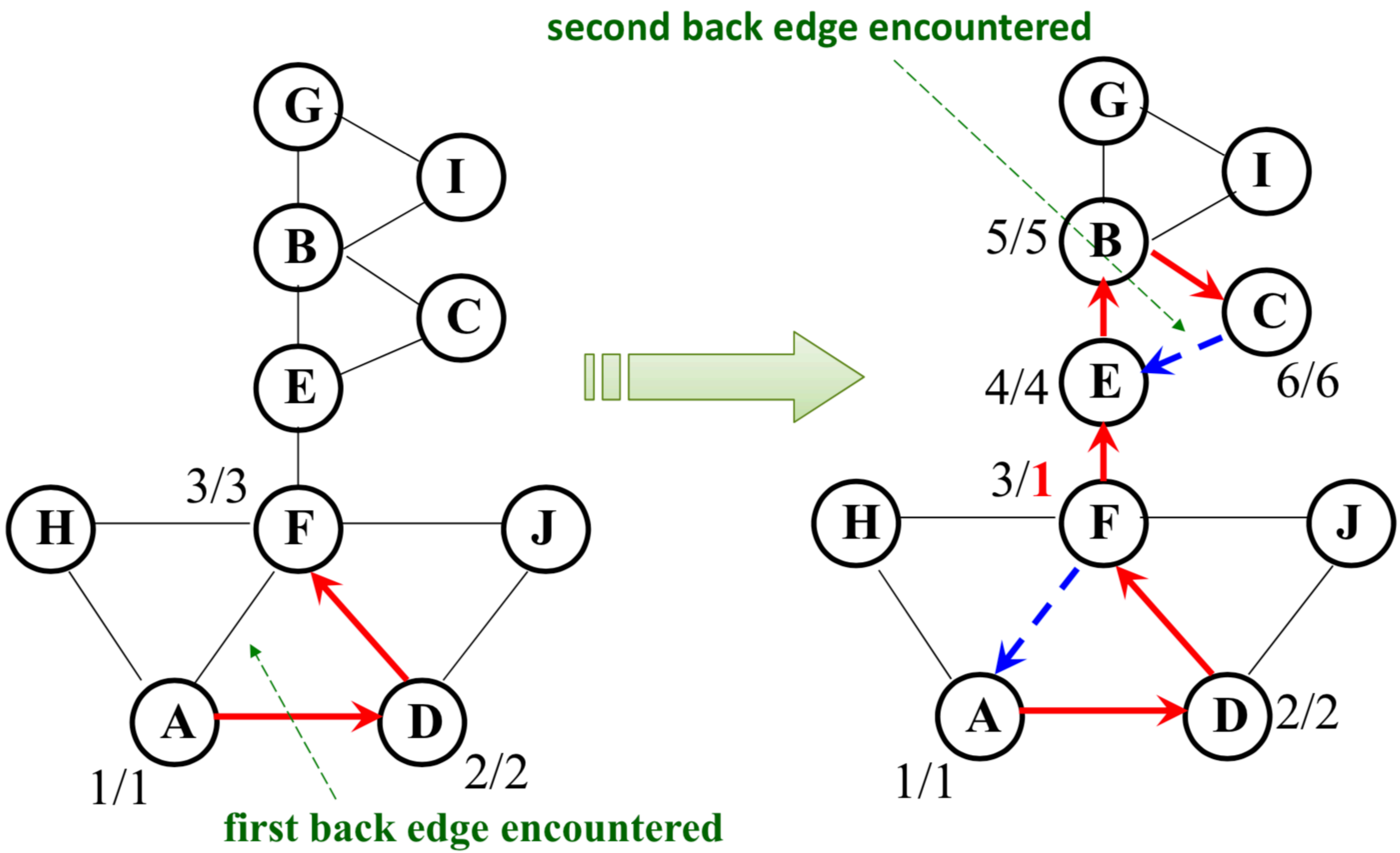
If *v* is the articulation point farthest away from the root on *the* branch, then one bicomponent is detected.

Updating the value of back

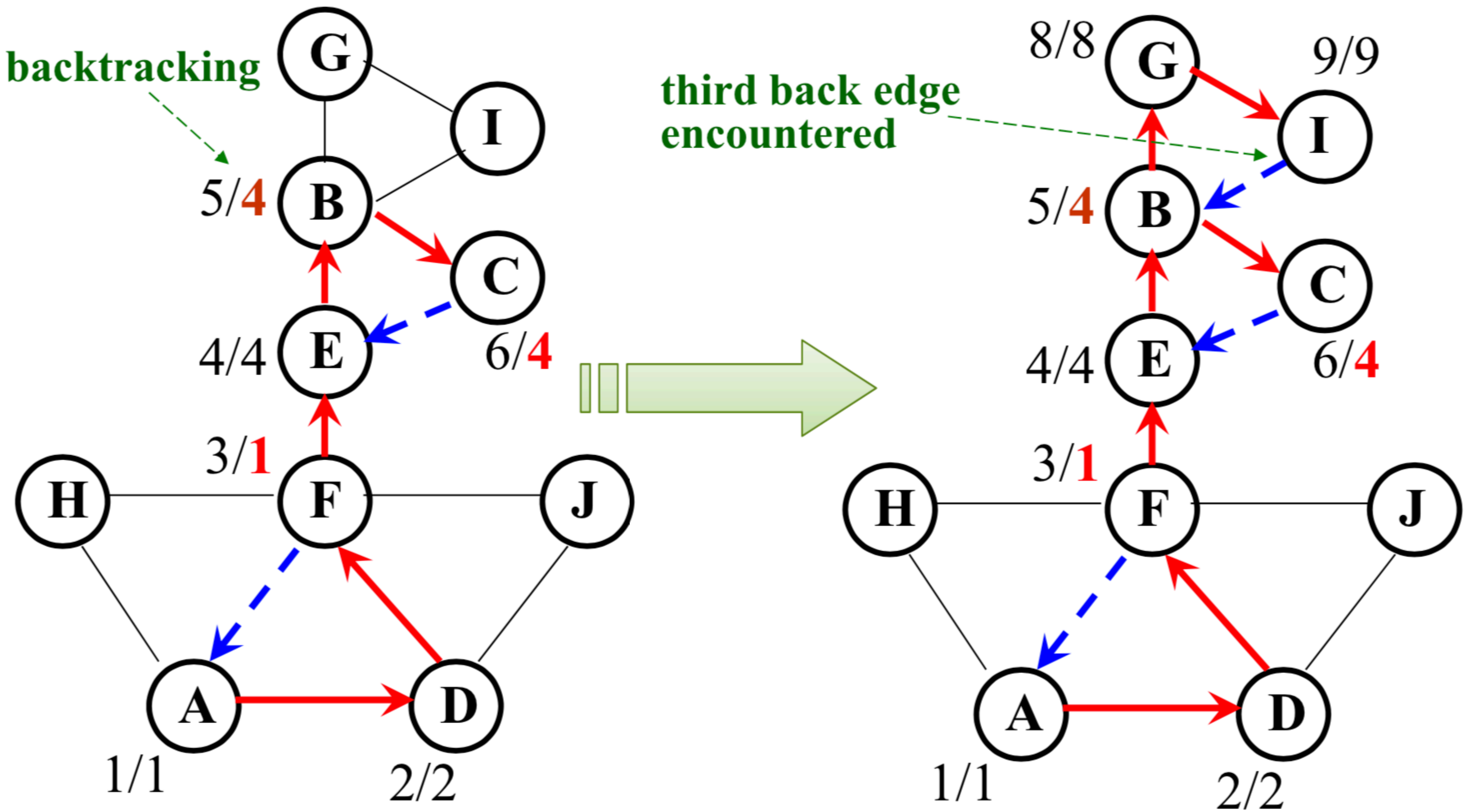
- v first discovered
  - back=discoverTime(v)
- Trying to explore, but a back edge vw from v encountered
  - back=min(back, discoverTime(w))
- Backtracking from w to v
  - back=min(back, wback)

The back value of v is the smallest discover time a back edge “sees” from any subtree of v.

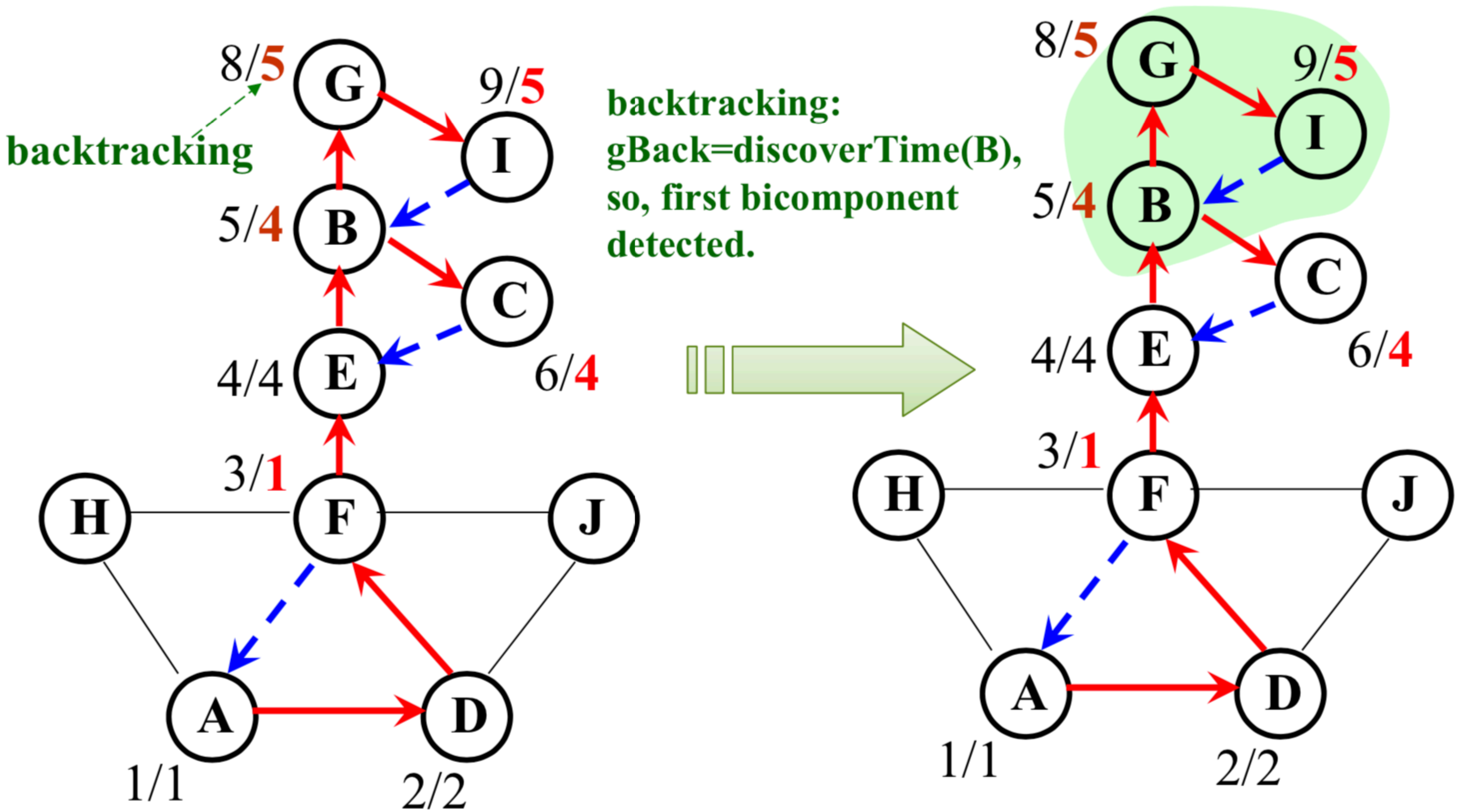
Example


second back edge encountered
G
I
B
5/5
C
E
4/4
6/6
3/3
3/1
H
F
J
A
D
1/1
2/2
first back edge encountered

Example


backtracking
G
I
5/4
B
C
4/4
E
6/4
3/1
H
F
J
A
D
1/1
2/2
8/8
G
9/9
third back edge
encountered
I
5/4
B
C
4/4
E
6/4
3/1
H
F
J
A
D
1/1
2/2

Example


backtracking
8/5
9/5
G
I
5/4
B
C
E
4/4
6/4
3/1
H
F
J
A
D
1/1
2/2
backtracking:
gBack=discoverTime(B),
so, first bicomponent
detected.
8/5
9/5
G
I
5/4
B
C
E
4/4
6/4
3/1
H
F
J
A
D
1/1
2/2

Keeping the Track of Backing

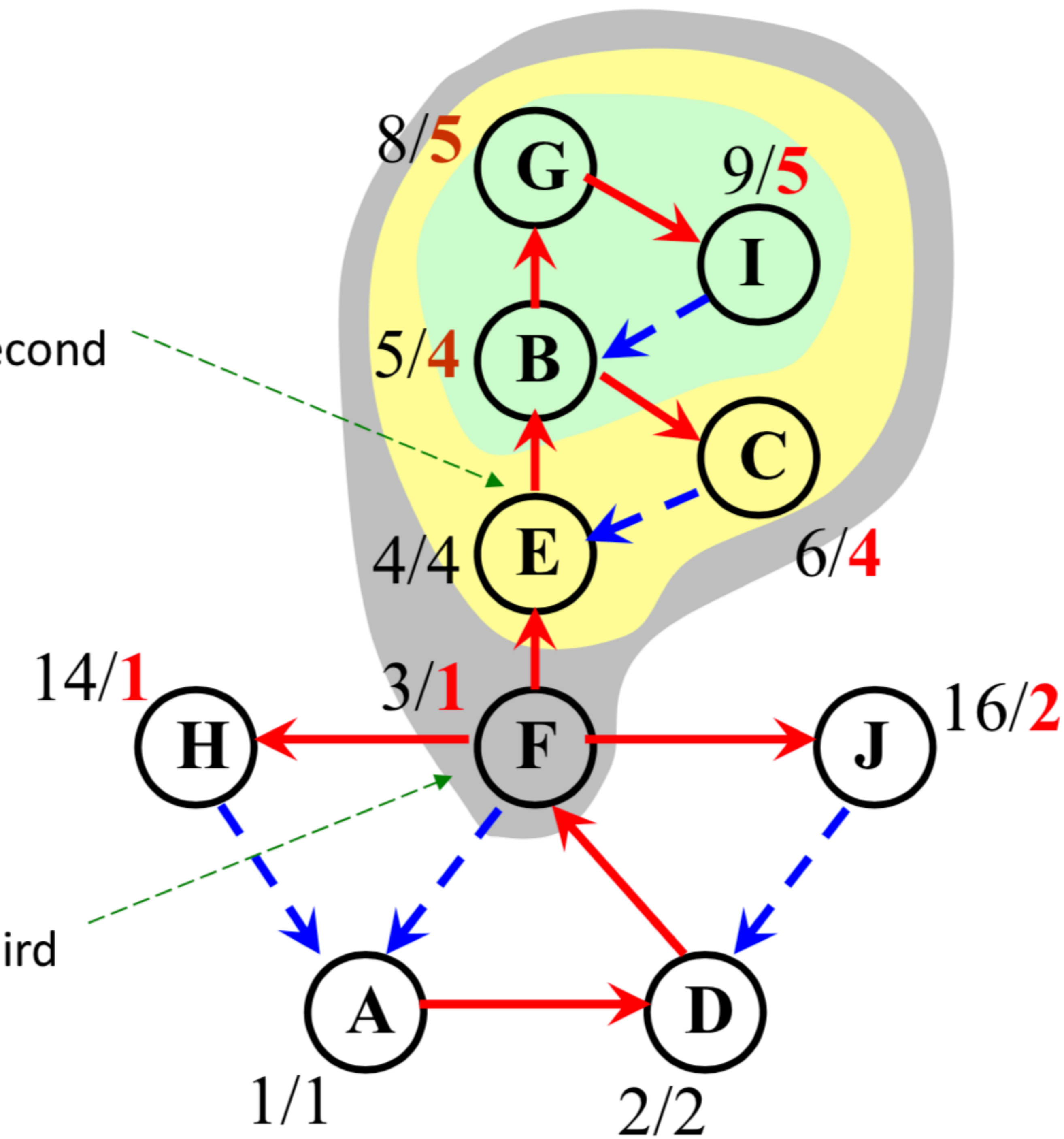
- **Tracking data**
  - For each vertex v, a local variable back is used to store the required information, as the value of discoverTime of some vertex.
- **Testing for bicomponent**
  - At backtracking from w to v, the condition implying a bicomponent is:
    - wBack $\geq$ discoverTime(v)

(where wback is the returned back value for w)

When back is no less than the discover time of v, there is at least one subtree of v connected to other part of the graph only by v.

Example

Backtracking from B to E:
bBack=discoverTime(E), so, the second bicomponent is detect



Backtracking from E to F:
eBack>discoverTime(F), so, the third bicomponent is detect

Articulation Point Algorithm

Algorithm 12: ARTICULATION-POINT-DFS(v)

```
v.color := GRAY ;
time := time + 1 ;
v.discoverTime := time ;
v.back := v.discoverTime ;
foreach neighbor w of v do
    if w.color = WHITE then
        w.back := ARTICULATION-POINT-DFS(w) ;
        if w.back ≥ v.discoverTime then
            Output v as an articulation point ;
        v.back := min{v.back, w.back} ;
    else
        if vw is BE then                          /* w 是 v 非父节点的祖先节点 */
            v.back := min{v.back, w.discoverTime} ;
return back ;
```

Correctness

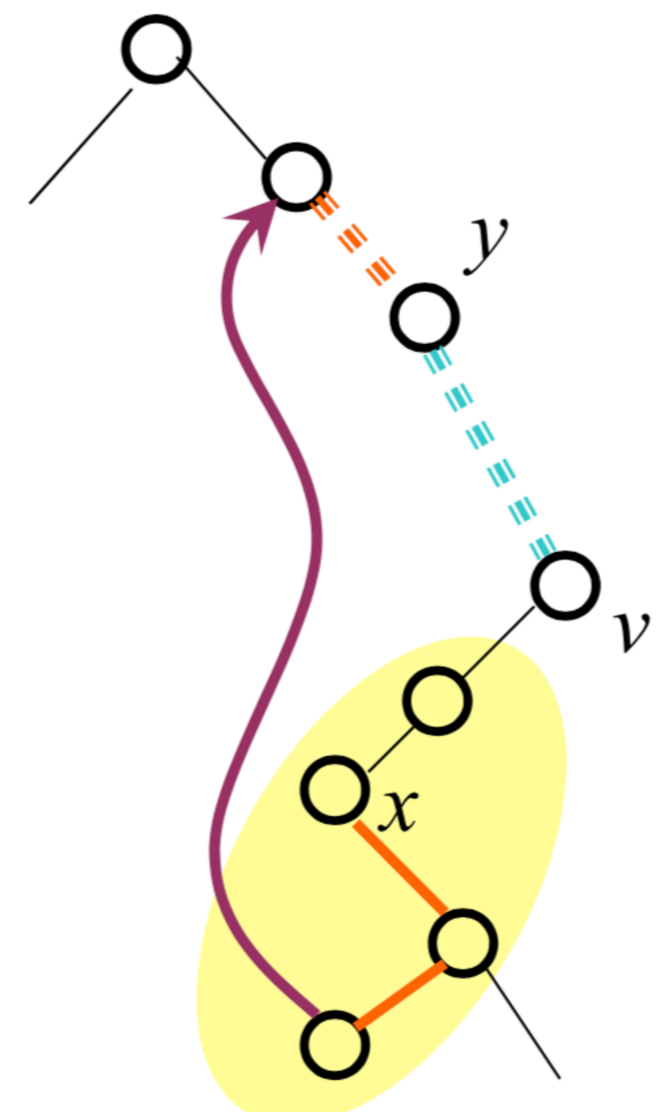
- **We have seen that:**
  - If v is the articulation point farthest away from the root on the branch, then one bicomponent is detected.
- **So, we need only prove that:**
  - In a DFS tree, a vertex (not root) v is an articulation point if and only if (1) v is not a leaf; (2) some subtree of v has no back edge incident with a proper ancestor of v.

Characteristics of Articulation Point

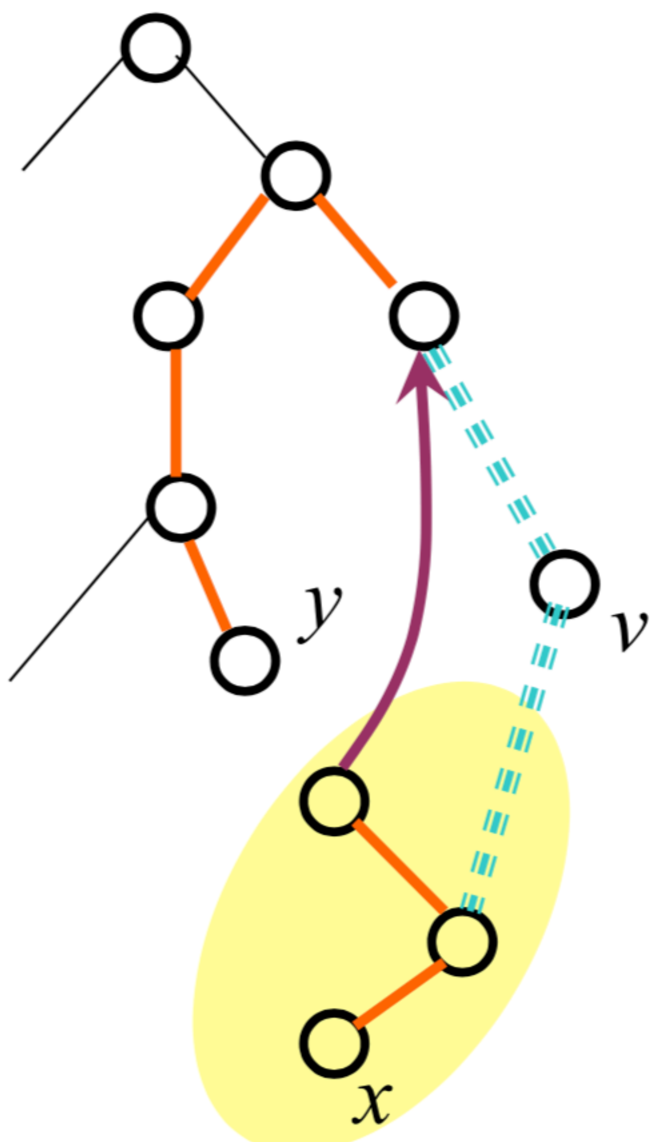
- **In a DFS tree, a vertex (not root) v is an articulation point if and only if (1) v is not a leaf; (2) some subtree of v has no back edge incident with a proper ancestor of v.**
- **<= Trivial**
- **=>**
  - By definition, v is on every path between some x, y (different from v).
  - At least one of x, y is a proper descendent of v (otherwise, x<->root<->y not containing v).
  - By contradiction, suppose that every subtree of v has a back edge to a proper ancestor of v, we can find a xy-path not containing v for all possible cases (only 2 cases)

Case 1

Case 1.2: another is an ancestor of *v*



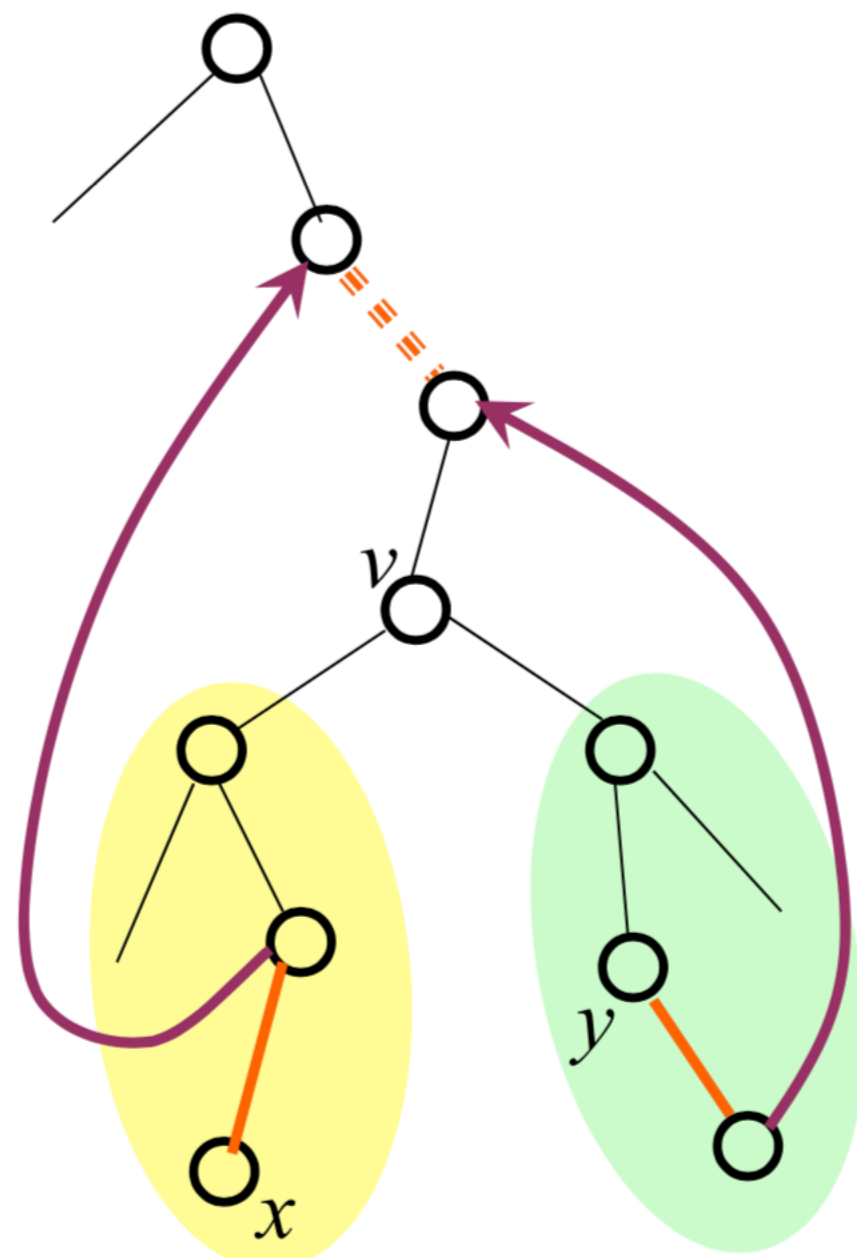
suppose that **every** subtree of *v* has a back edge to a proper ancestor of *v*, and, exactly one of *x*, *y* is a descendant of *v*.



Case 1.1: another is not an ancestor of *v*

Case 2

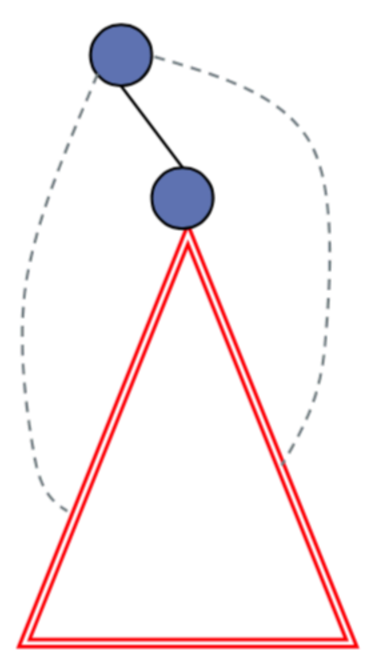
suppose that **every** subtree of *v* has a back edge to a proper ancestor of *v,* and, both *x*, *y* are descendants of *v*.

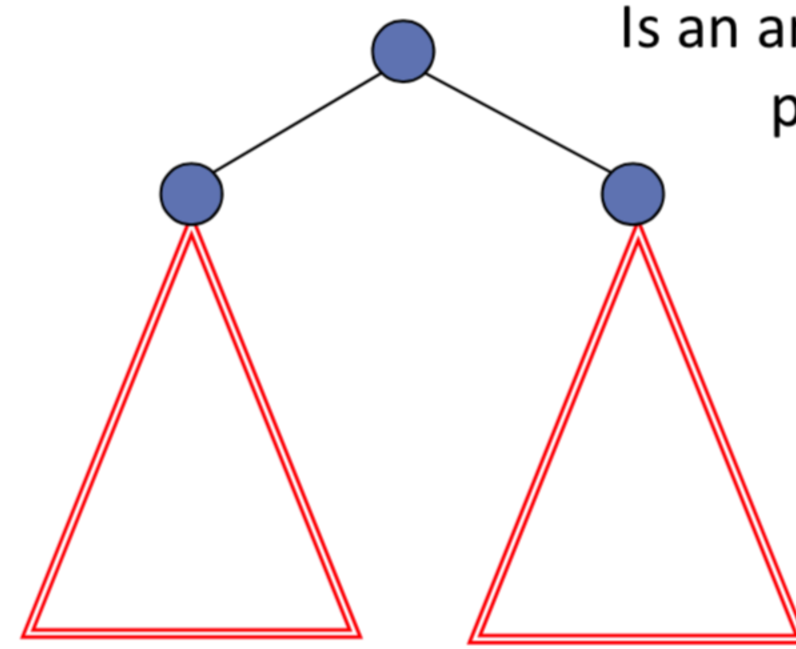


What about the root?

- **One single DFS tree**
    - We only consider each connected component
- **Root AP ≡ Two or more sub-trees**
    - The root is an articulation point

Not an articulation point

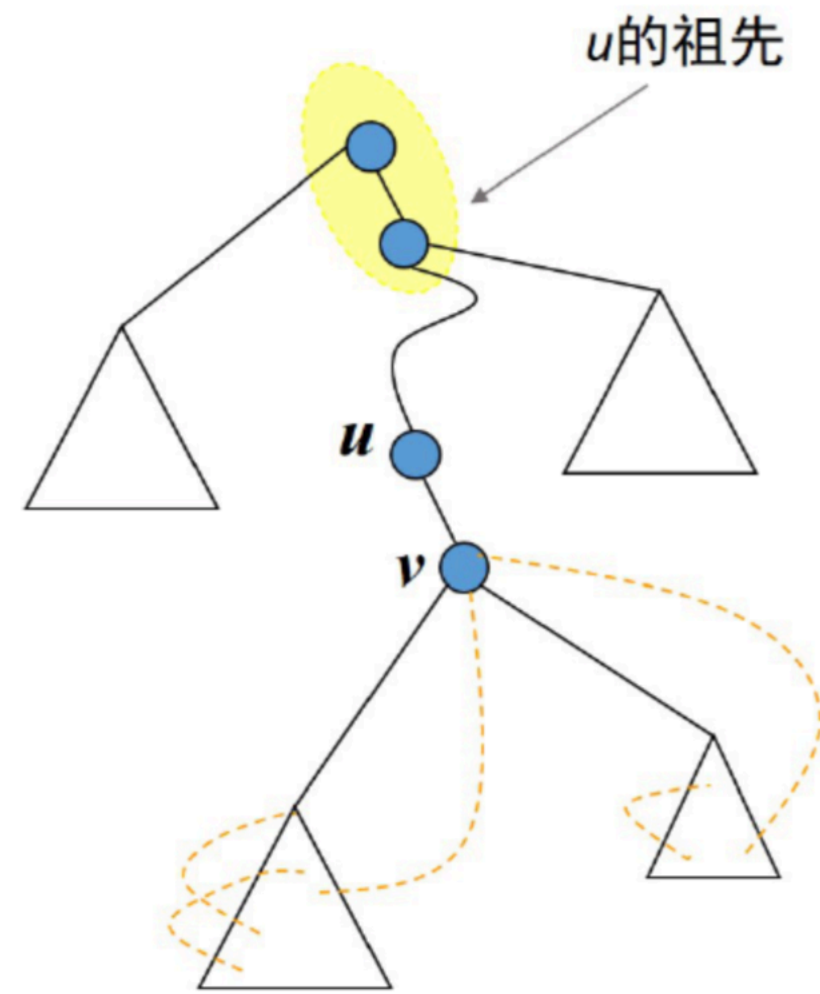




Is an articulation point

Defining the Bridge

- **Short definition**
  - Removing uv leading to disconnection
- **Long definition**
  - Edge uv is a bridge iff node u and v are connected only by uv
- **DFS Definition**
  - Edge uv is a tree edge in DFS
  - Three is no subtree rooted at v to any proper ancestor of v (including u)



Bridge Algorithm

Algorithm 11: BRIDGE-DFS(u)

```
u.color := GRAY ;
time := time + 1 ;
u.discoverTime := time ;
u.back := u.discoverTime ;
foreach neighbor v of u do
    if v.color = WHITE then
        BRIDGE-DFS(v) ;
        u.back := min{u.back, v.back} ;
        if v.back > u.discoverTime then
            Output uv as a bridge ;
    else
        if uv is BE then                              /* v 是 u 非父节点的祖先节点 */
            u.back := min{u.back, v.discoverTime} ;
```

Other Traversal Problems

- **Orientation of an undirected graph**
  - Give each edge a direction
  - Satisfying pre-specified constraints
    - E.g., the “in-degree of each vertex is at least 1”
- **Possible or not?**
  - If possible, how to?
- **As for “in-degree ≥ 1”**
  - Orientation possible iff. the graph has at least a circle
    - Find the end point of some back edge
    - A second DFS from this end point

Other Traversal Problems

MST: Minimum Spanning Tree

- **Get MST in O(m+n) time**
  - Given that edges weights are only 1 and 2
- **Graph traversal is sufficient**
  - DFS over “weight 1 edges” only
  - DFS over “weight 2 edges” only

Thank you!

Q & A