Introduction to

Algorithm Design and Analysis

[08] logn search

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In the last class …

- **Selection - warm up**
  - Max and min
  - Second largest
- **Selection - rank k (median)**
  - Expected linear time
  - Worst-case linear time
- **Adversary argument**
  - Lower bound

The Searching Problem

- **Searching v.s. Selection**
  - Search for “Alice” or “Bob”
    - The key itself matters
  - Select the “rank 2” student
    - The partial order relation matters
- **Expected cost for searching**
  - Brute force case: O(n)
  - Ideal case: O(1)
  - Can we achieve O(logn)?

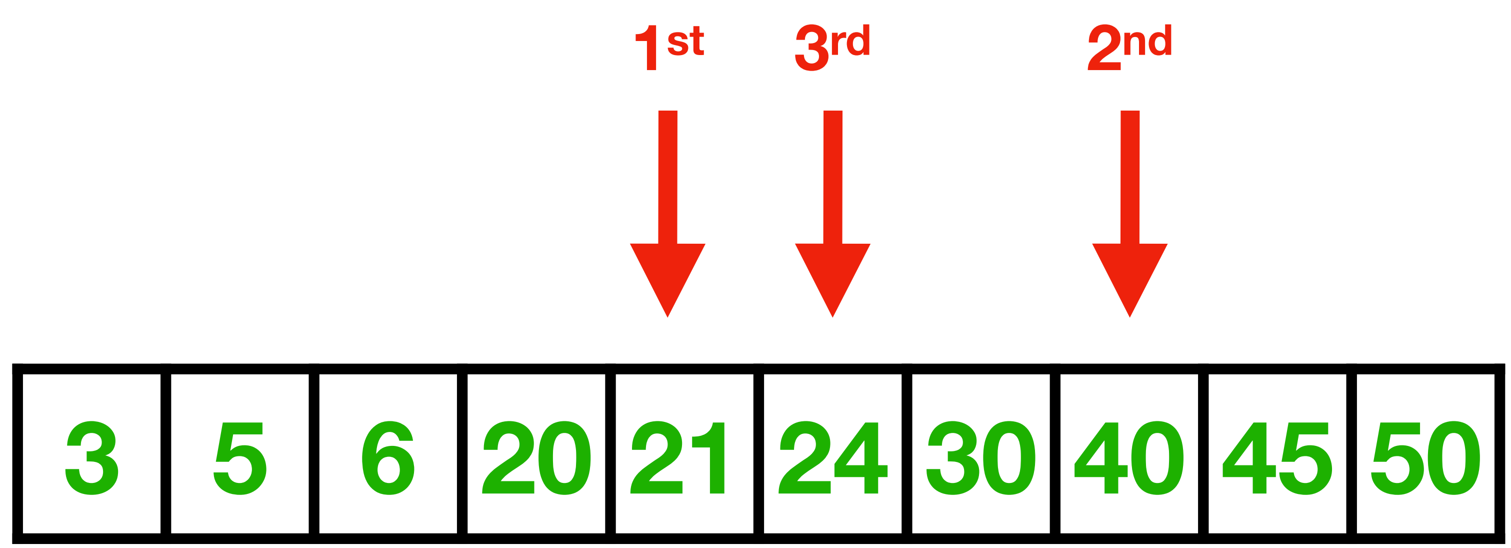
The Searching Problem

- **Essential of searching**
  - How to organize the data to enable efficient search
  - logn search
    - Each search cuts off half of the search space
    - How to organize the data to enable logn search
- **logn search techniques**
  - Warmup
    - Binary search over sorted sequences
  - Balanced Binary Search Tree (BST)
    - Red-black tree

Binary Search by Example

- **Binary search for "24"**
  - Divide the search space
  - Cut off half the space after each search

The sequence is already sorted



Binary Search Generalized

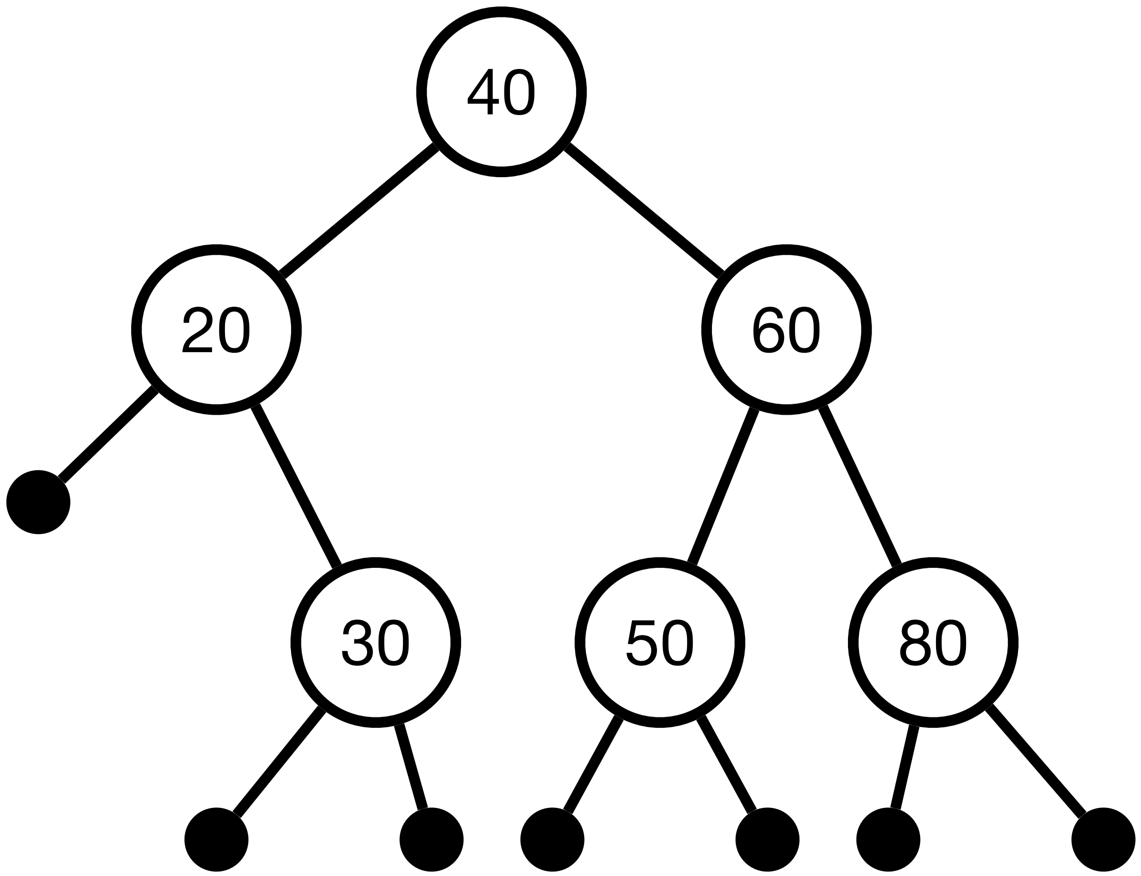
- **Peak-number**
  - Uni-modal array
- **Least number not in the array**
  - Sorted array of natural numbers
- **A[i]=i**
  - Sorted array of integers

Balanced Binary Search Tree

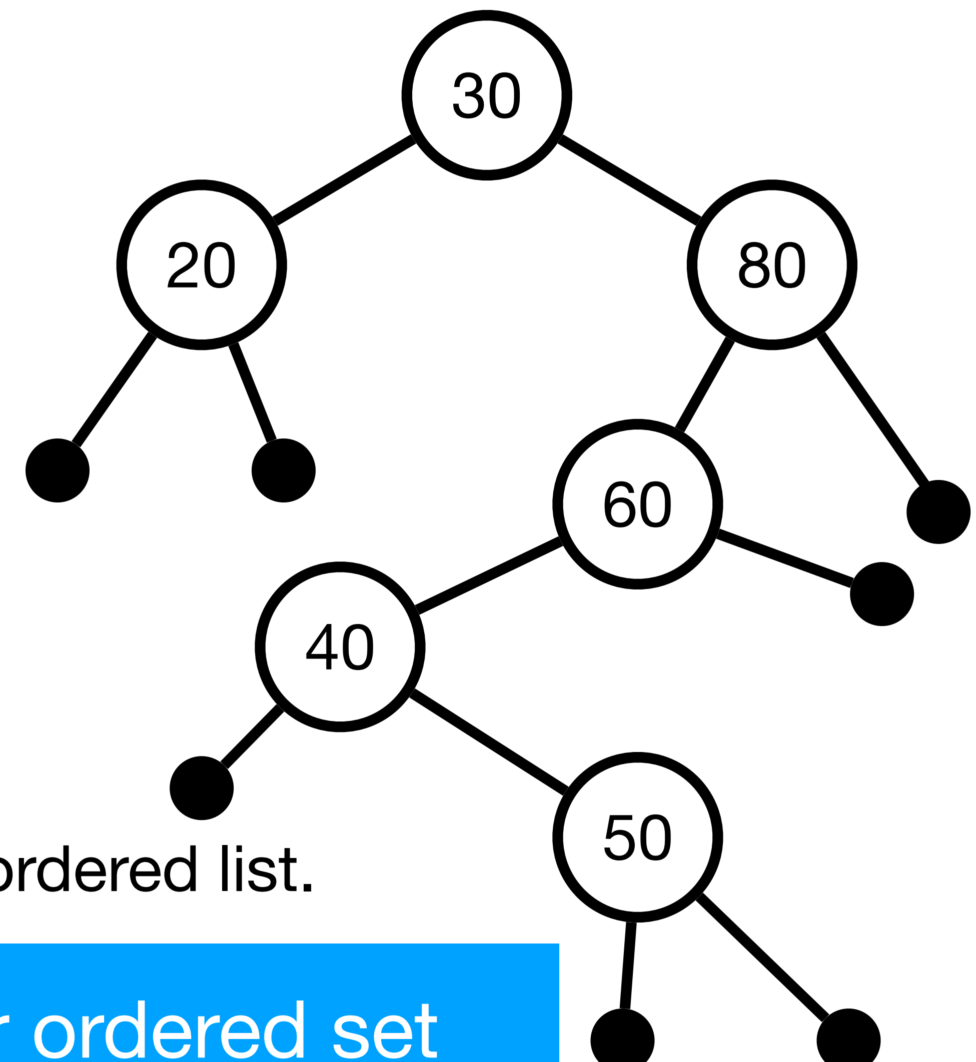
- **Binary search tree (BST)**
  - Definitions and basic operations
- **Definition of Red-Black Tree (RBT)**
  - Black height
- **RBT operations**
  - Insertion into a red-black tree
  - Deletion from a red-black tree

Binary Search Tree Revisited

Good balancing
$\Theta(\log n)$



Poor balancing
$\Theta(n)$



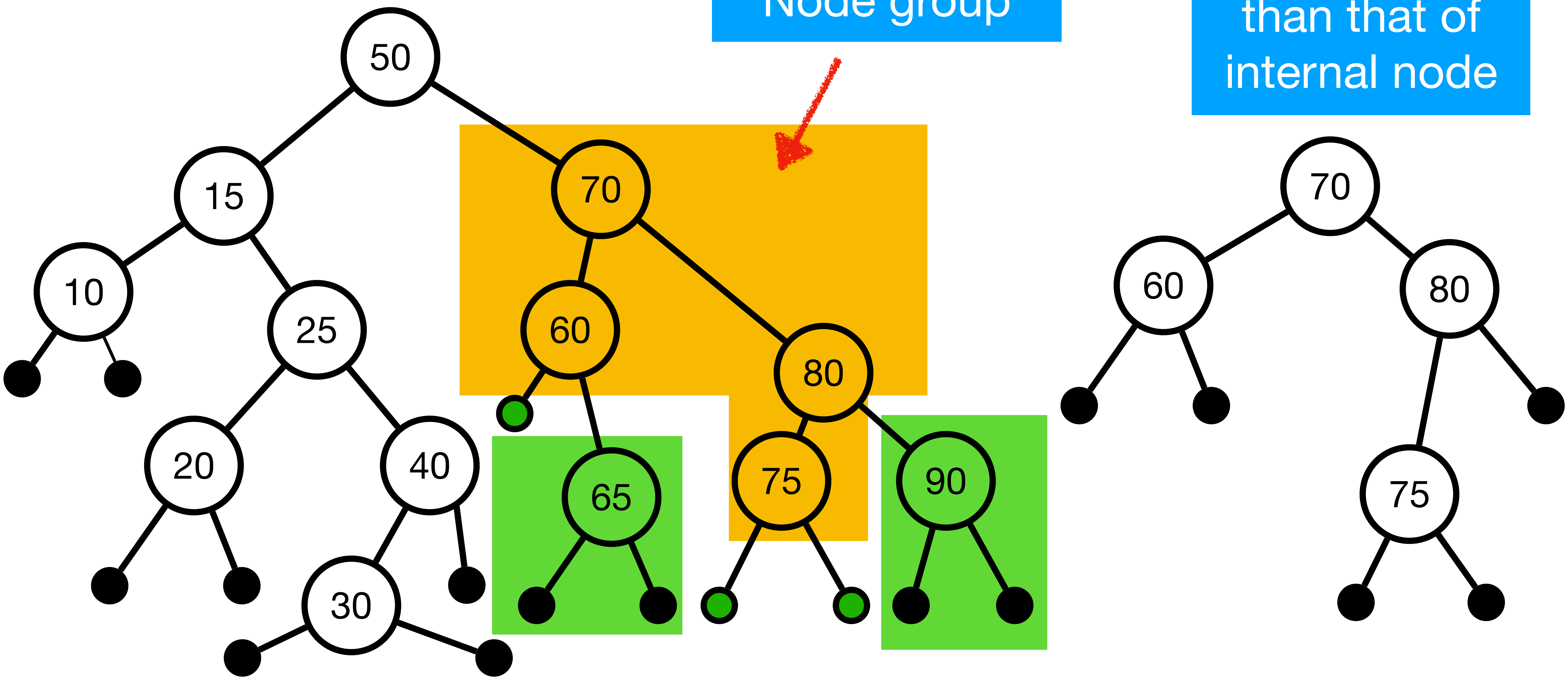
In a properly drawn tree, pushing forward to get the ordered list.

Each node has a key, belonging to a linear ordered set
An inorder traversal produces a sorted list of the keys

Node Group

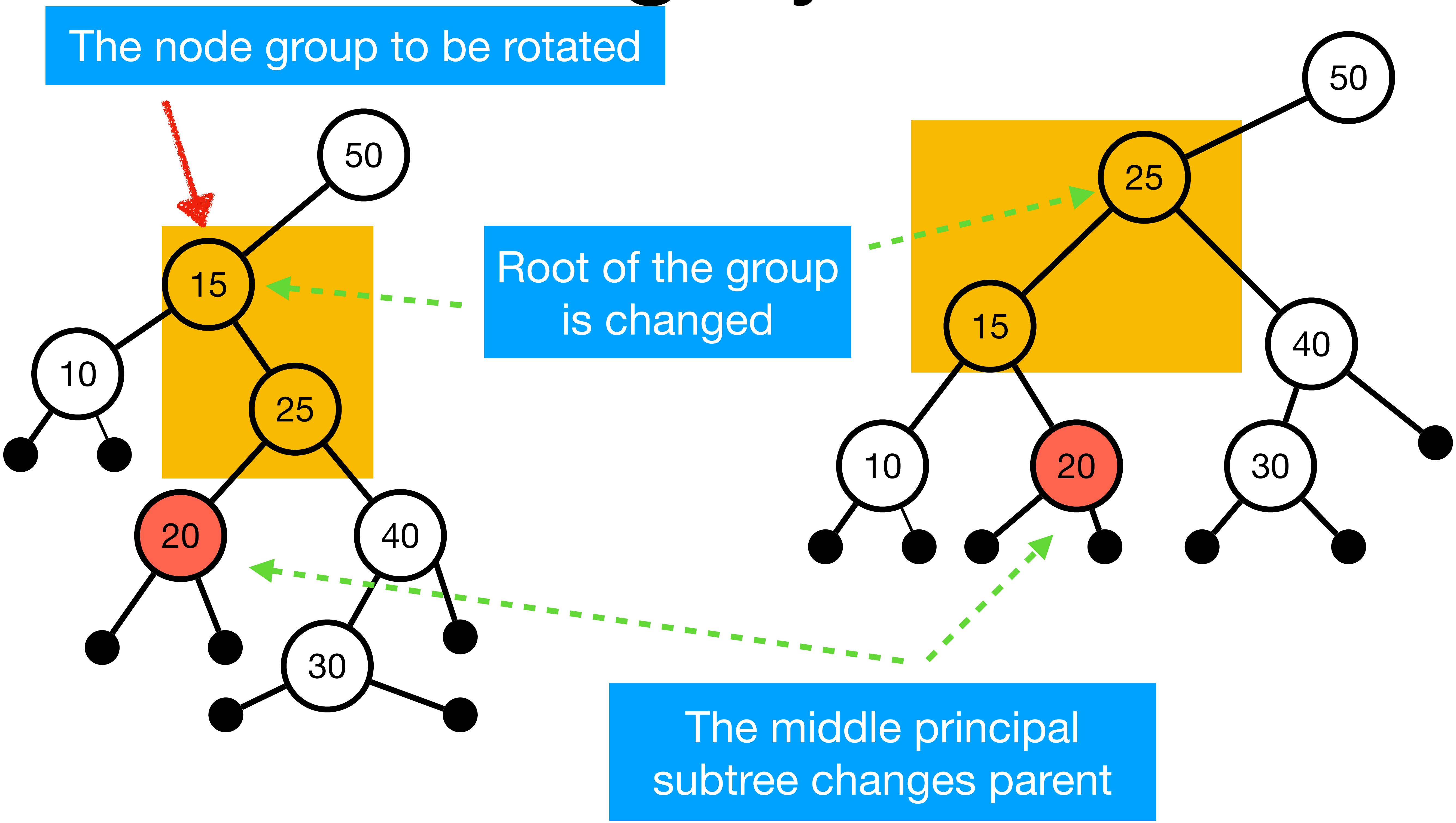
As in 2-tree, the number of external node is one more than that of internal node

Node group



5 principal subtrees

Balancing by Rotation



Red-Black Tree: Definition

- **If T is a binary search tree in which each node has a color, red or black, and all external nodes are black, then T is a red-black tree if and only if:**
  - [Color constraint] No red node has a red child
  - [Black height constraint] The black length of all external paths from a given node u is the same (the black height of u)
  - The root is black.
- **Almost-red-black tree (ARB tree)**
  - Root is red, satisfying the other constraints.

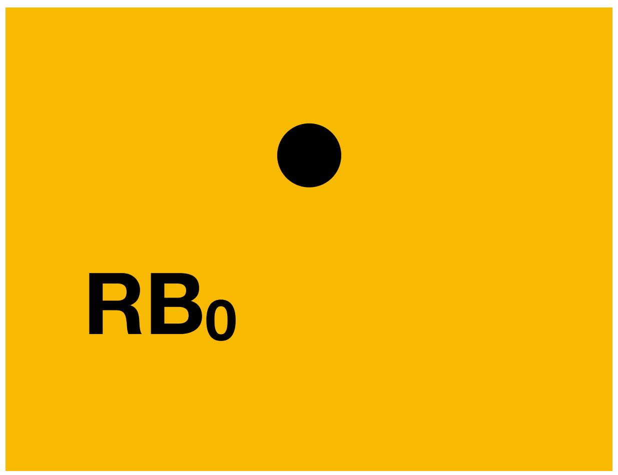
Balancing is under control

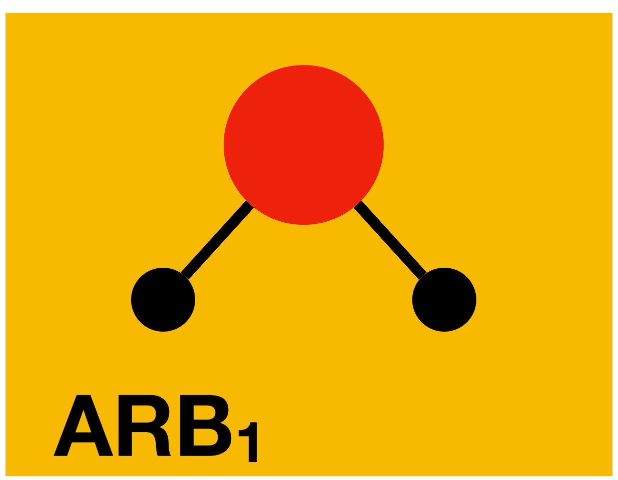
Recursive Definition of RBT

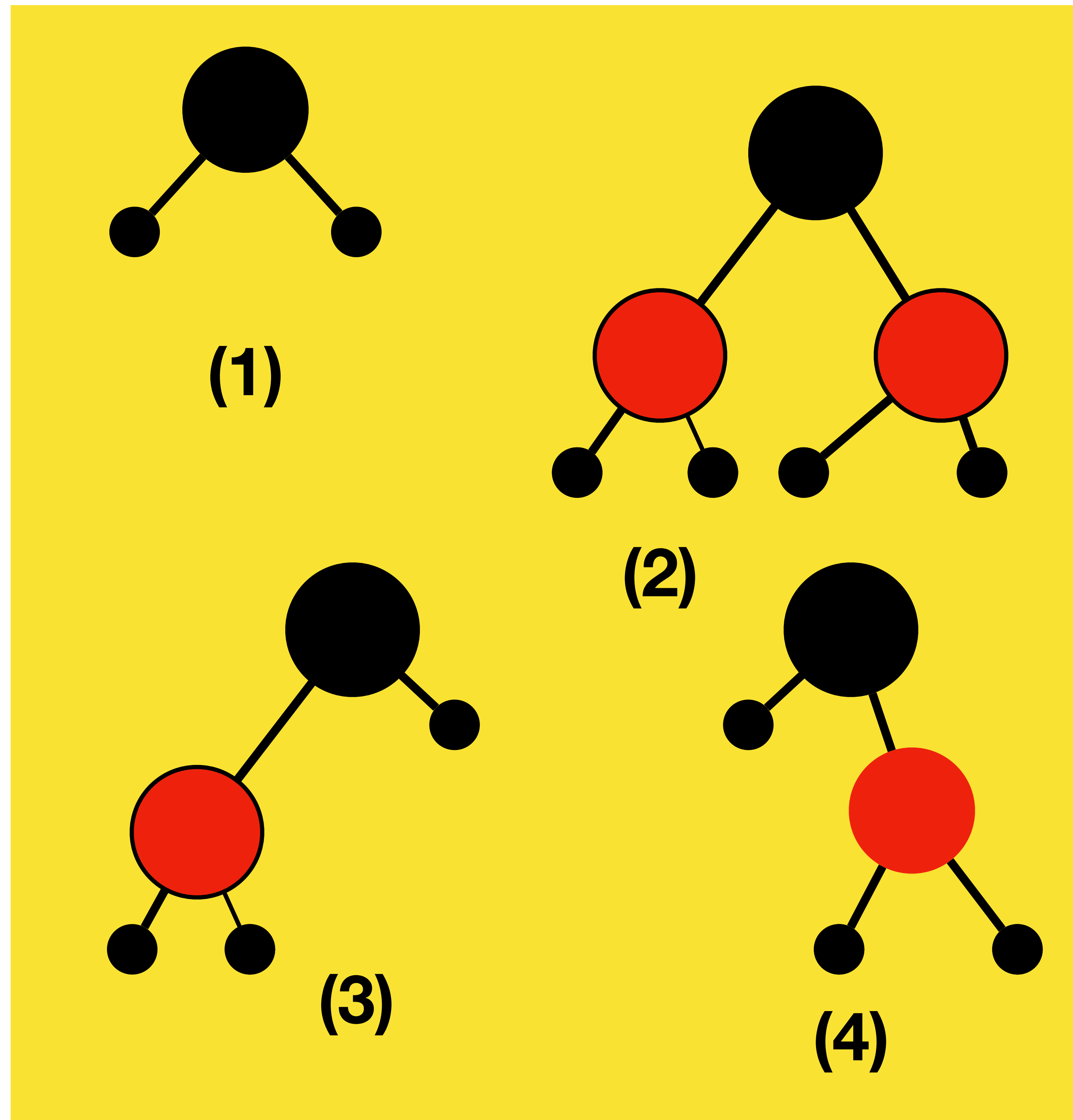
(A red-black tree of black height h is denoted as RB_h)

- **Definition**
  - An external node is an RB_0 tree, and the node is black.
  - A binary tree is an ARB_h ($h \geq 1$) tree if:
    - Its root is red, and
    - Its left and right sub trees are each an RB_{h-1} tree.
  - A binary tree is an RB_h ($h \geq 1$) tree if:
    - Its root is black, and
    - Its left and right sub trees are each either an RB_{h-1} tree or an ARB_h tree.

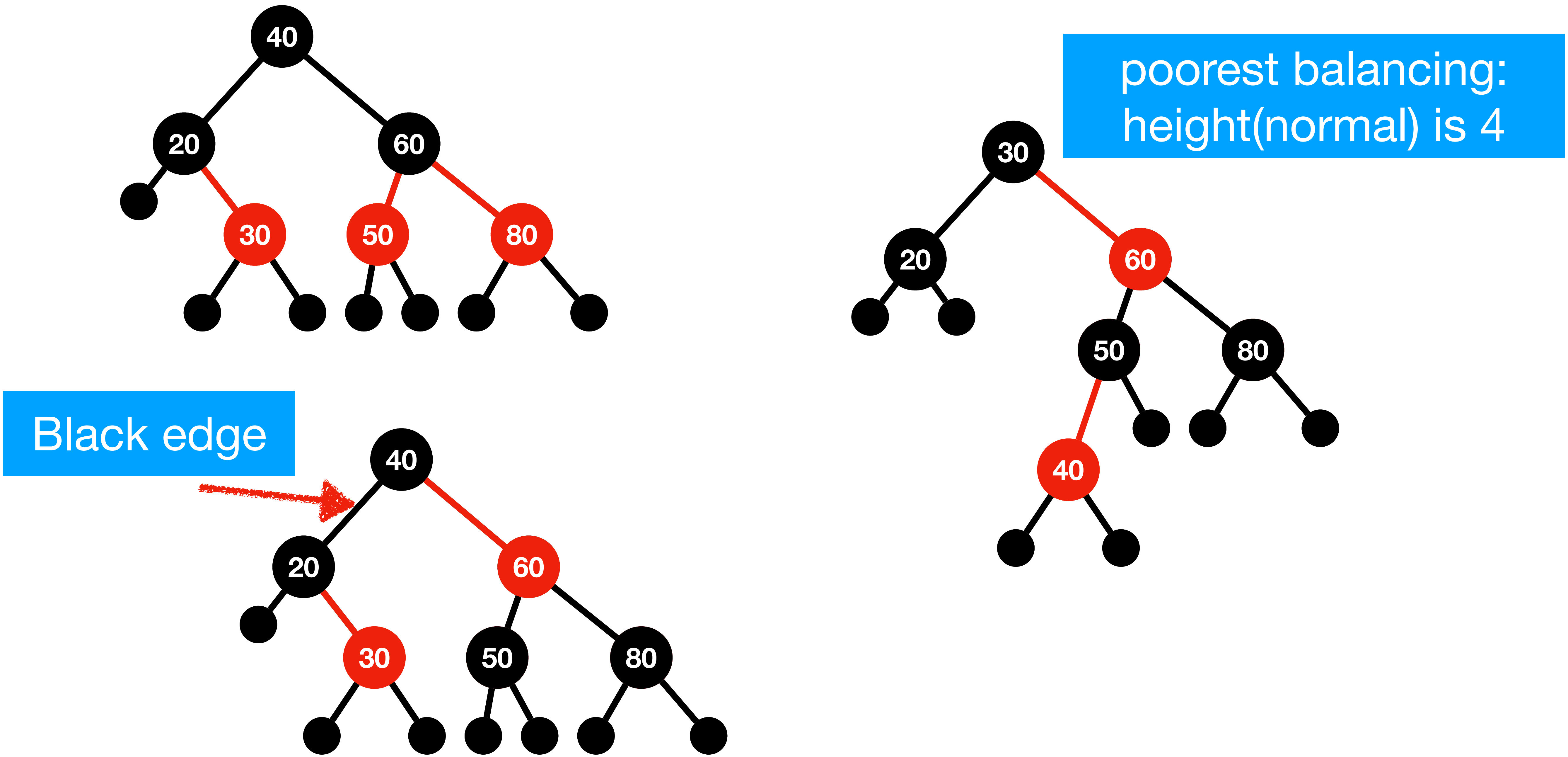
RB_i and ARB_i



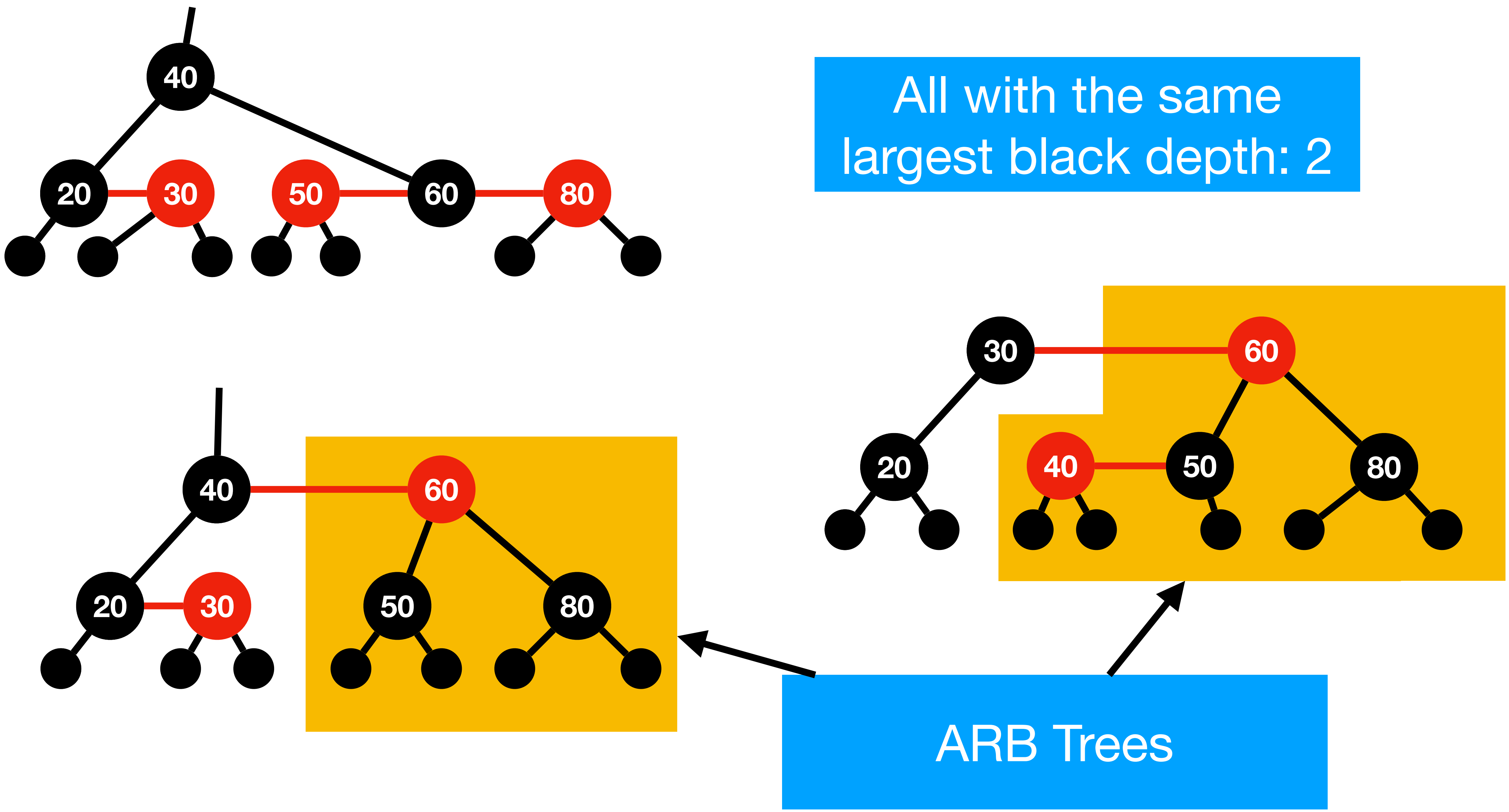




Red-Black Tree with 6 Nodes



Black-depth Convention



Properties of Red-Black Tree

- **The black height of any RB_h tree or ARB_h tree is well-defined and is h.**
- **Let T be an RB_h tree, then:**
    - T has at least 2^h-1 internal black nodes.
    - T has at most 4^h-1 internal nodes.
    - The depth of any black node is at most twice its black depth.
- **Let A be an ARB_h tree, then:**
    - A has at least 2^h-2 internal black nodes.
    - A has at most $(4^h)/2-1$ internal nodes.
    - The depth of any black node is at most twice its black depth.

Well-defined Black Height

- **That "the black height of any RB_h tree or ARB_h tree is well defined" means the black length of all external paths from the root is the same.**
- **Proof: induction on h**
- **Base case: h=0, that is RB_0 (there is no ARB_0)**
- **In ARB_{h+1}, its two subtrees are both RB_h. Since the root is red, the black length of all external paths from the root is h, that's the same as its two subtrees.**
- **In RB_{h+1}:**
  - Case 1: two subtrees are RB_h's
  - Case 2: two subtrees are ARB_{h+1}'s
  - Case 3: one subtree is an RB_h (black height=h), and the another is an ARB_{h+1} (black height=h)

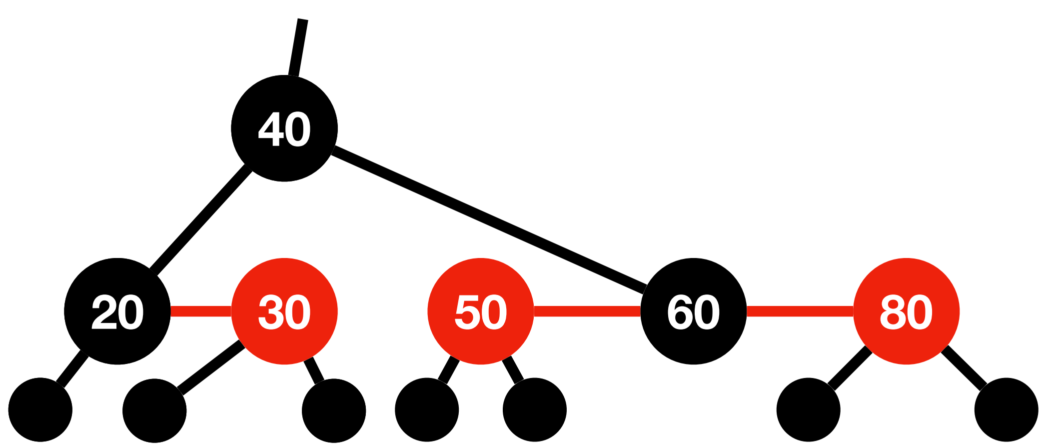
Bound on Depth of Node in RBTree

- **Let T be a red-black tree with n internal nodes. Then no node has black depth greater than log(n+1), which means that the height of T in the usual sense is at most 2log(n+1).**
  - Proof:
  - Let h be the black height of T. The number of internal nodes, n, is at least the number of internal black nodes, which is at least 2^h-1, so h≤log(n+1). The node with greatest depth is some external node. All external nodes are with black depth h. So, the depth is at most 2h.

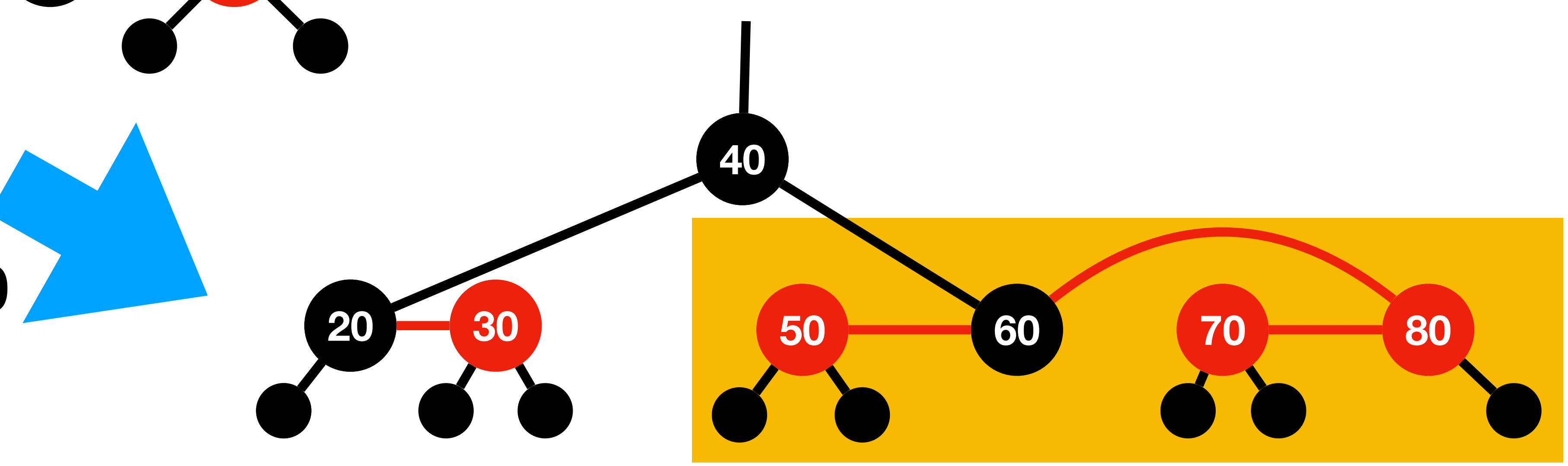
Influences of Insertion to an RBT

- **Black height constraint:**
  - No violation if inserting a red node.
- **Color constraint:**

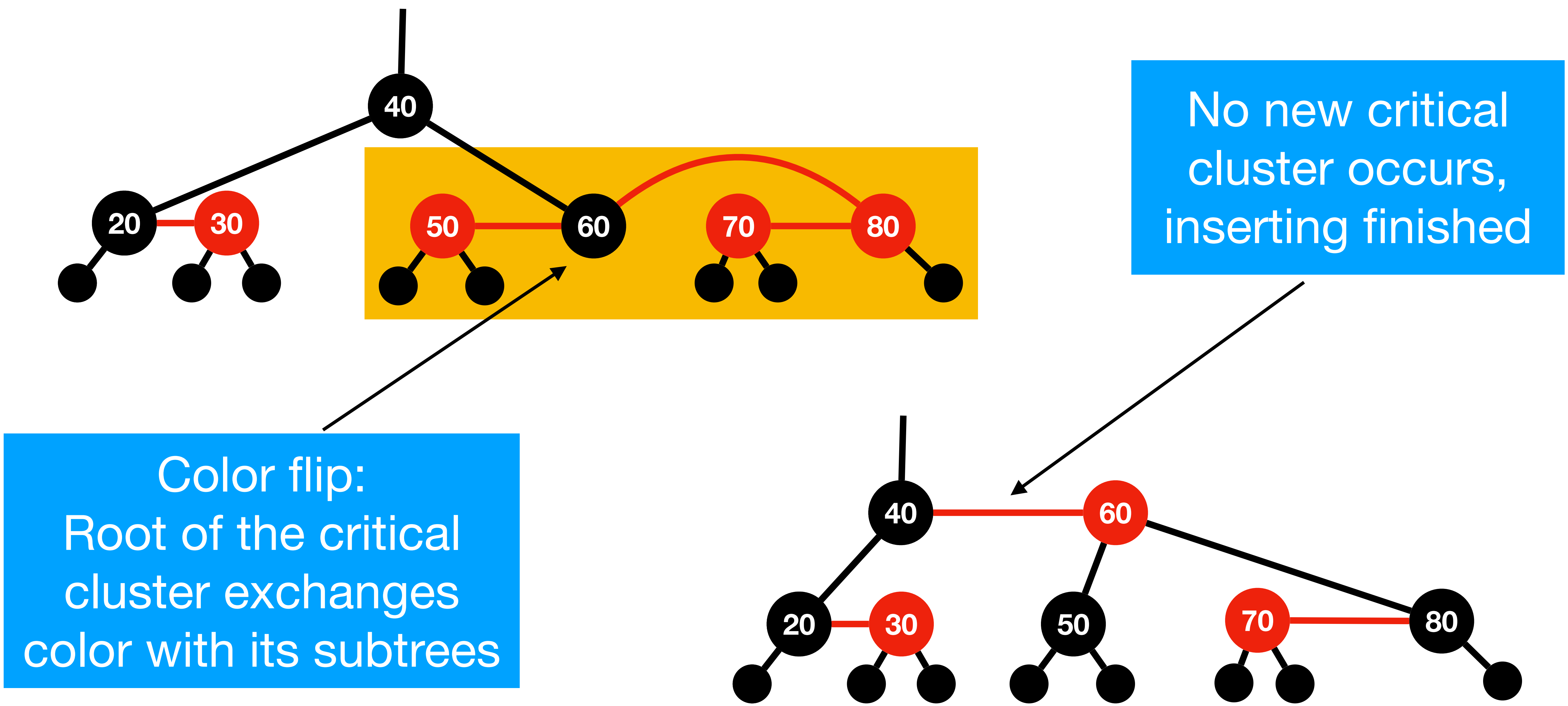
Critical clusters (external nodes excluded), which originated by color violation, with 3 or 4 nodes



Inserting 70

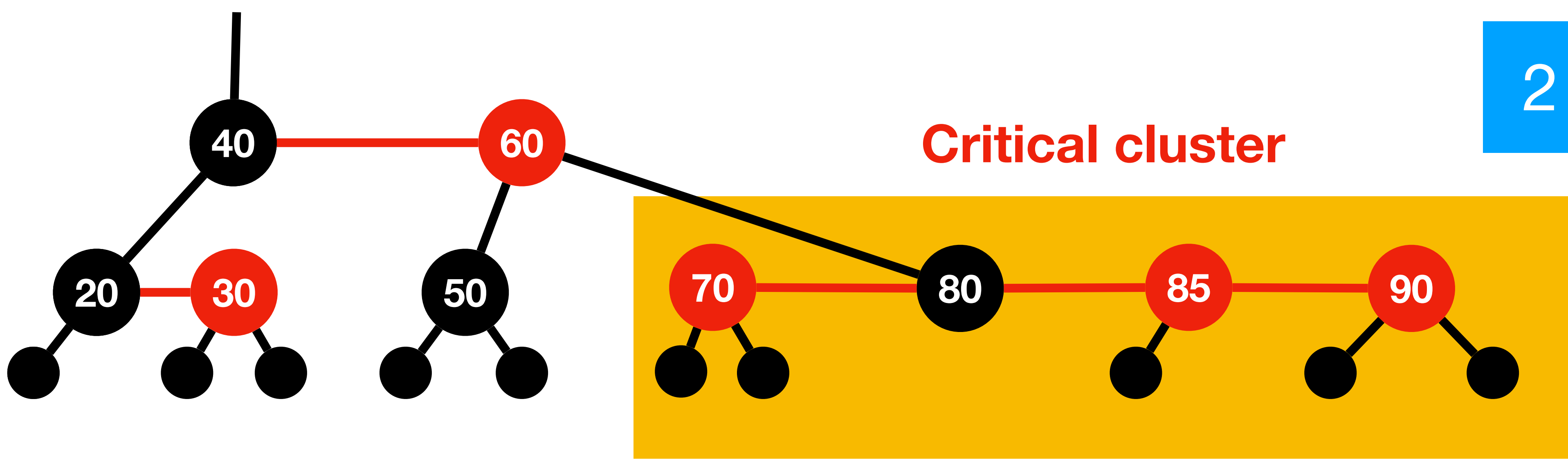


Repairing 4-node Critical Cluster

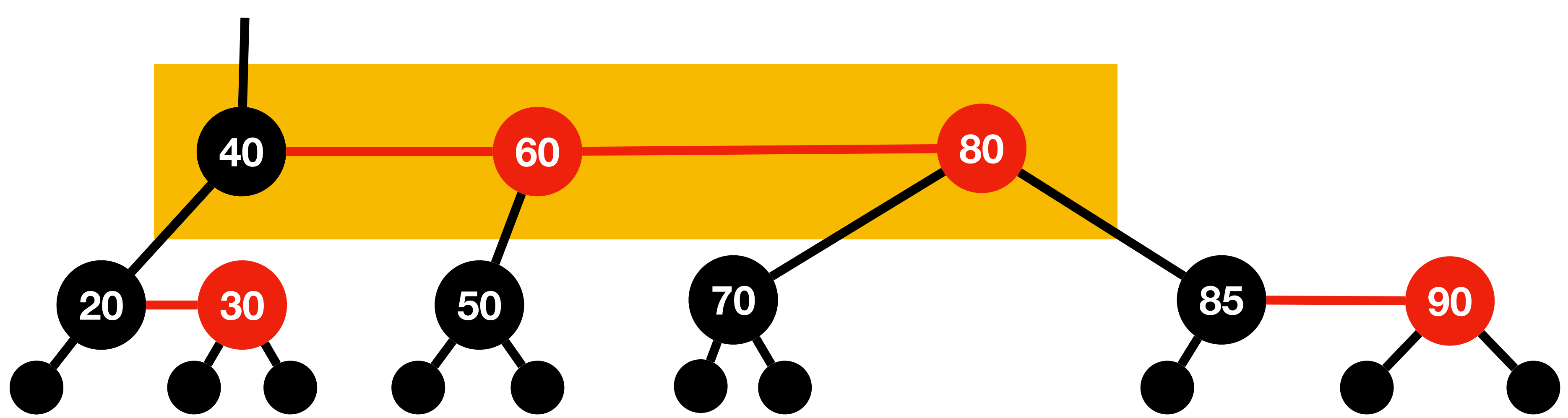


Repairing 4-node Critical Cluster

2 more insertions

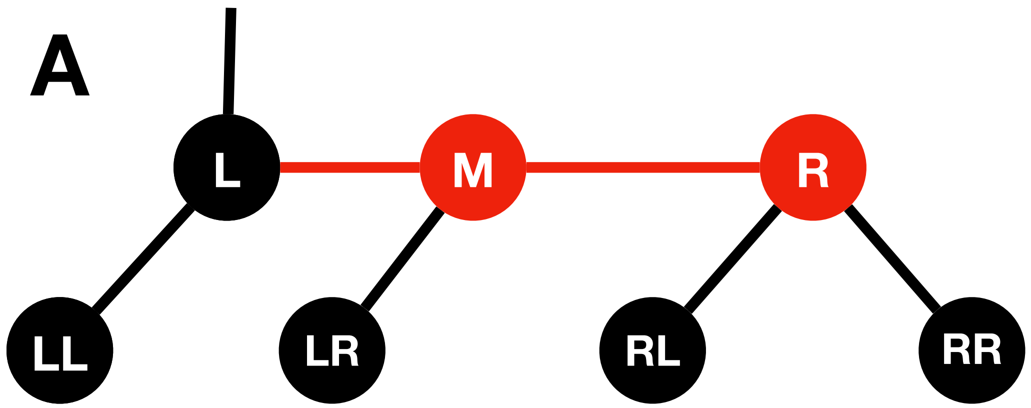


New critical cluster with 3 nodes. Color flip doesn't work, why?

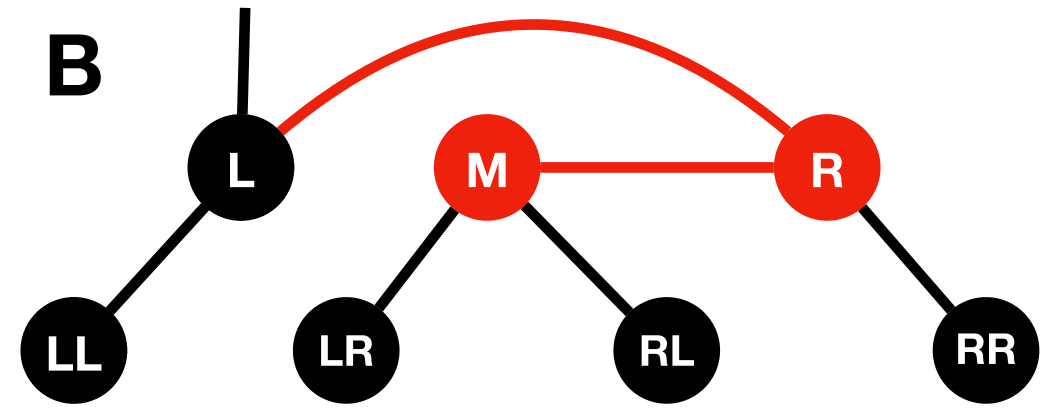


Patterns of 3-node Critical Cluster

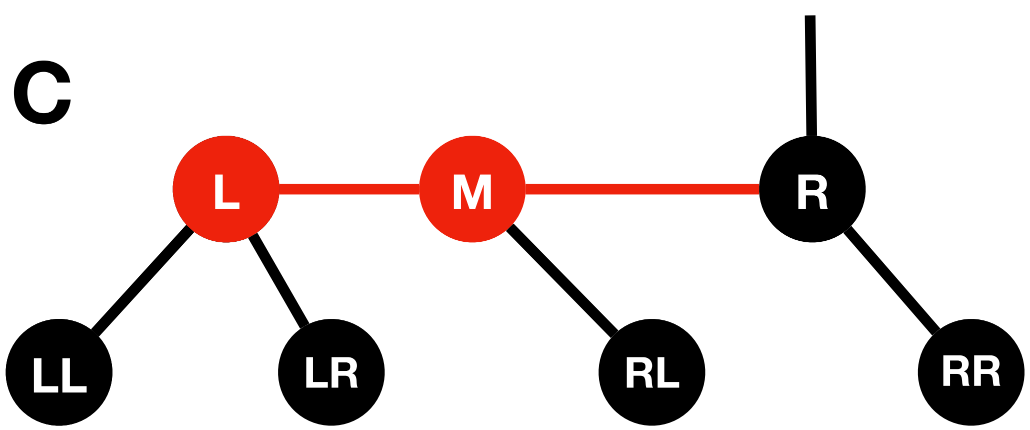
A



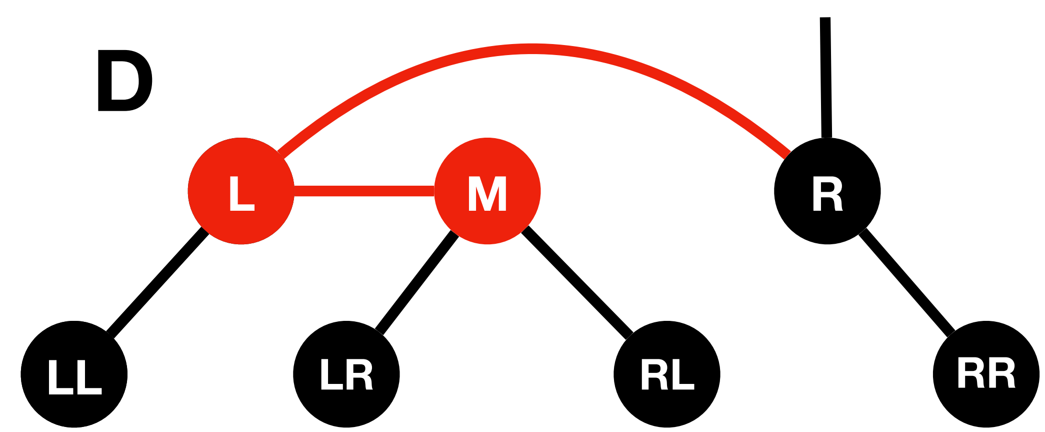
B



C

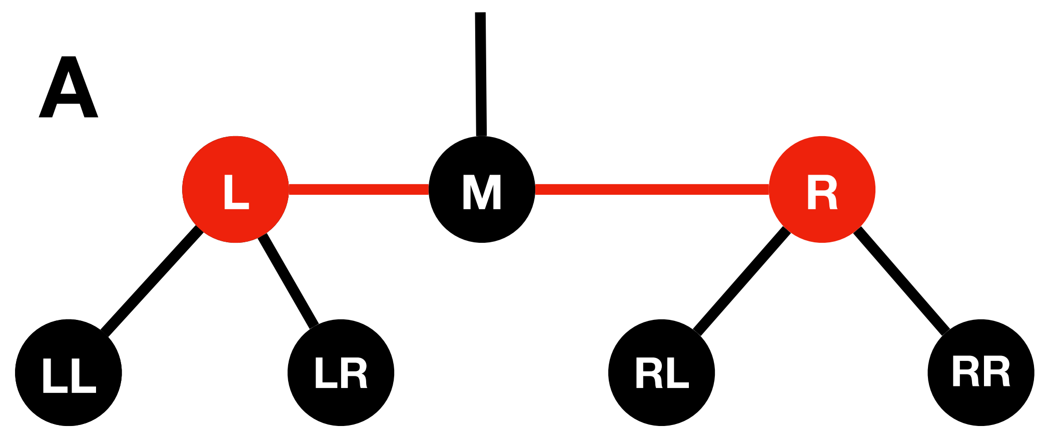


D

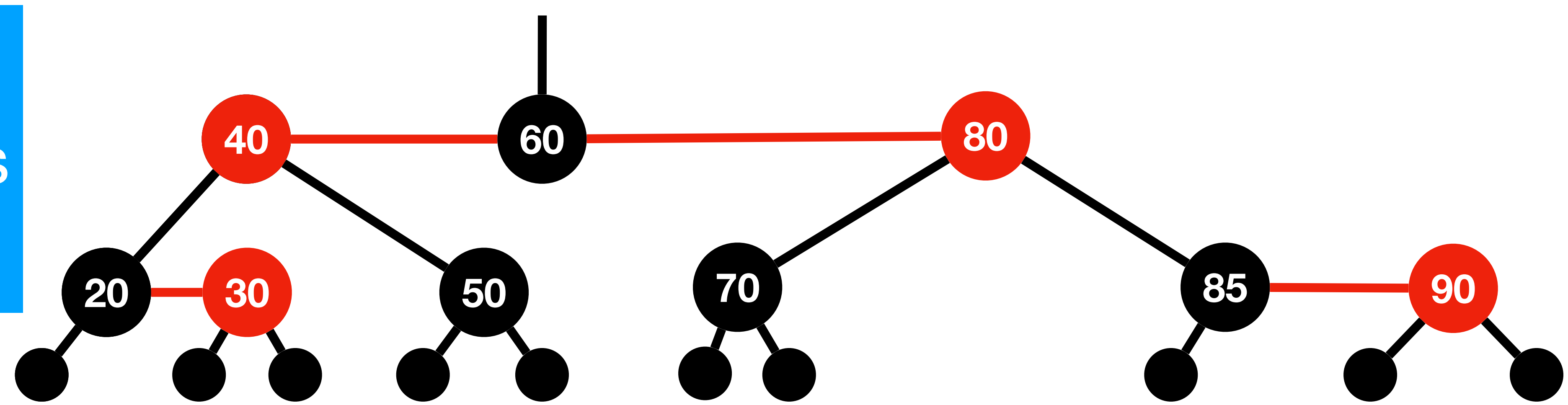


Repairing 3-node Critical Cluster

Root of the critical cluster is changed to M, and the parent ship is adjusted accordingly



The incurred critical cluster is of pattern A



Implementing Insertion: Class

```
class RBTree
    Element root;
    RBTree leftSubtree;
    RBTree rightSubtree;
    int color; /*red, black*/;

    static class InsReturn
        public RBTree newTree;
        public int status /* ok, rbr, brb, rrb, brr */
```

Implementing Insertion: Procedure

```
RBTree rbtInsert(RBtree oldRBtree, Element newNode)
    InsReturn ans = rbtIns(oldREtree, newNode);
    if(ans.newTree.color != black)
        ans.newTree.color = black;
    return ans.newTree;
```

Implementing Insertion: Procedure

InsReturn **rbtIns**(RBtree oldRBtree, Element newNode)

InsReturn ans, ansLeft, ansRight;

if (oldRBtree = nil) **then** <*Inserting simply*>;

else

if (newNode.key < oldRBtree.root.key)

ansLeft = **rbtIns**(oldRBtree.leftSubtree, newNode);

ans = **repairLeft**(oldRBtree, ansLeft);

else

ansRight = **rbtIns**(oldRBtree.rightSubtree, newNode);

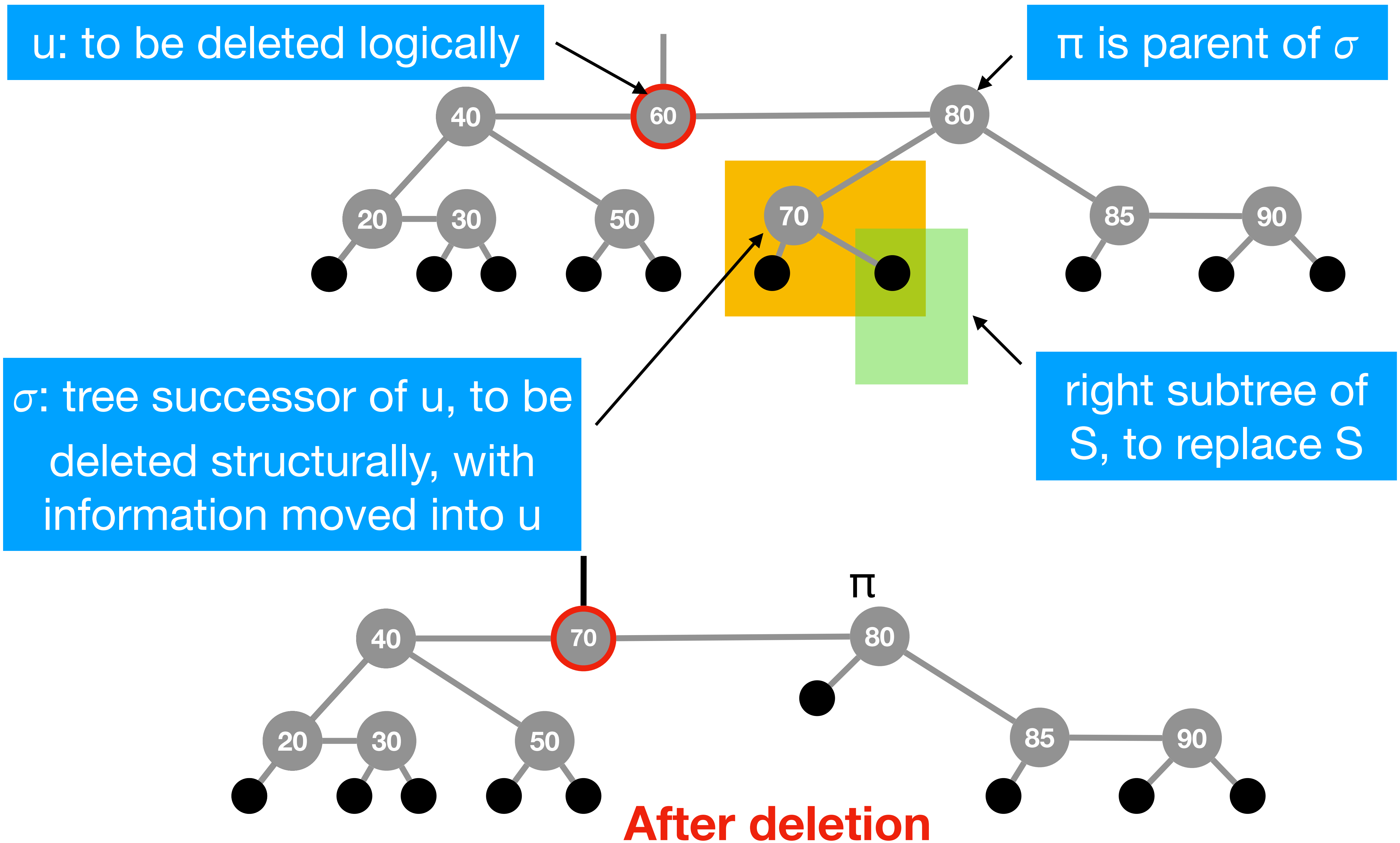
ans = **repairRight**(oldRBtree, ansRight);

return ans

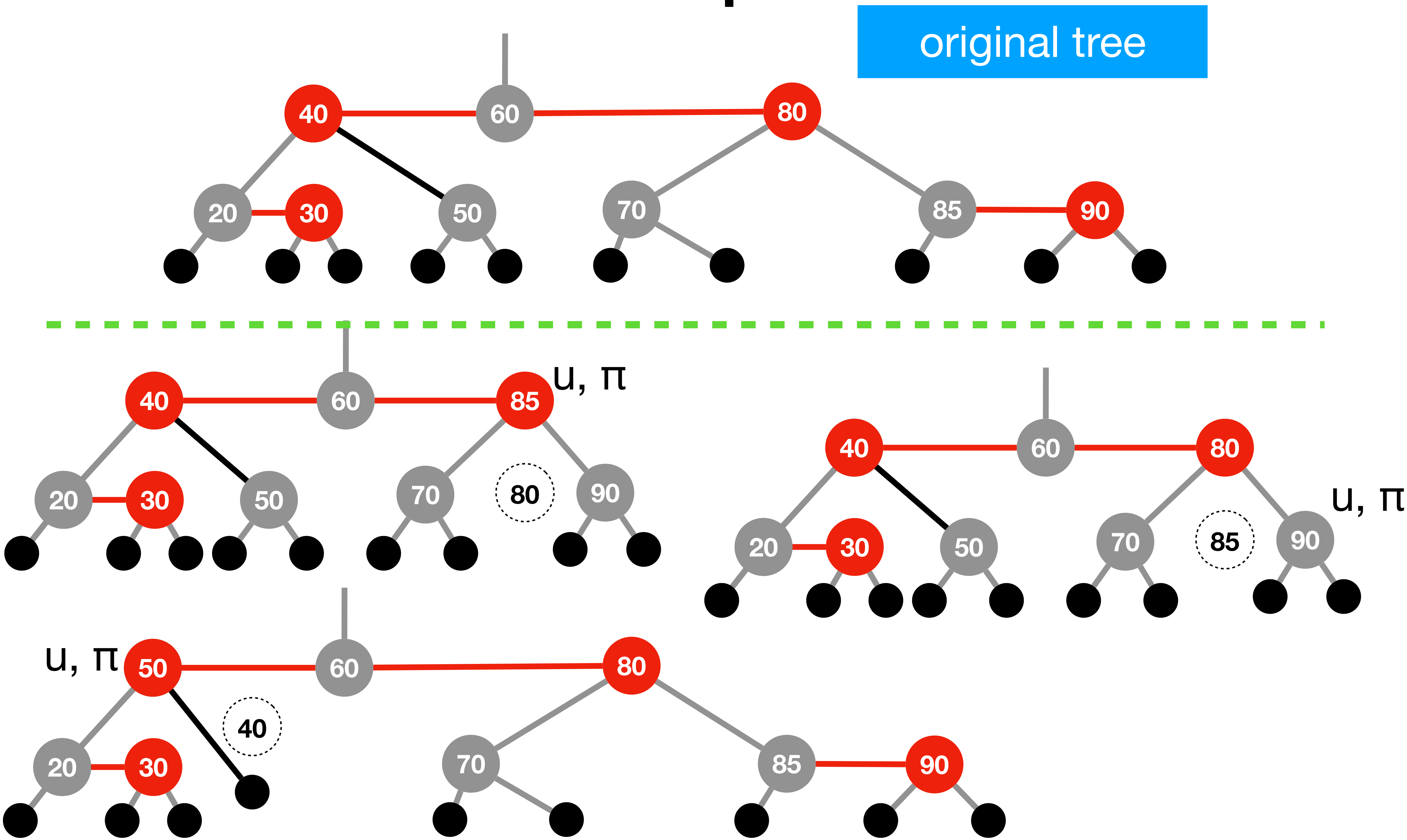
Correctness of Insertion

- **If the parameter oldRBtree of rbtIns is an RB_h tree or an ARB_{h+1} tree (which is true for the recursive calls on rbtIns), then the newTree and status fields returned are one of the following combinations:**
  - Status=ok, and newTree is an RB_h or an ARB_{h+1} tree,
  - Status=rbr, and newTree is an RB_h,
  - Status=brb, and newTree is an ARB_{h+1} tree,
  - Status=rrb, and newTree.color=red, newTree.leftSubtree is an ARB_{h+1} tree and newTree.rightSubtree is an RB_h tree,
  - Status=brr, and newTree.color=red, newTree.rightSubtree is an ARB_{h+1} tree and newTree.leftSubtree is an RB_h tree
- **For those cases with red root, the color will be changed to black, with other constraints satisfied by repairing subroutines.**

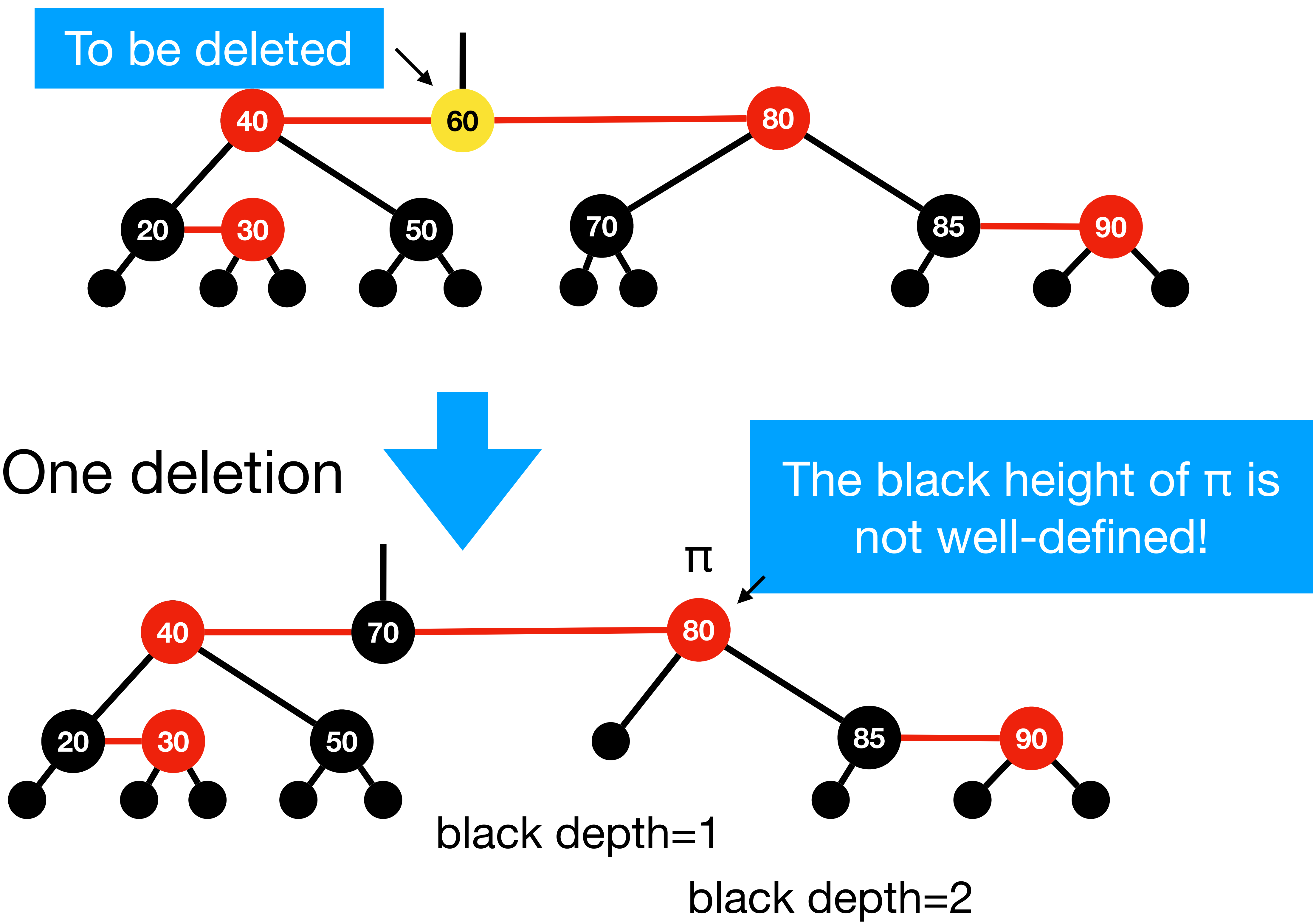
Deletion: Logical and Structural



Deletion from RBT - Examples



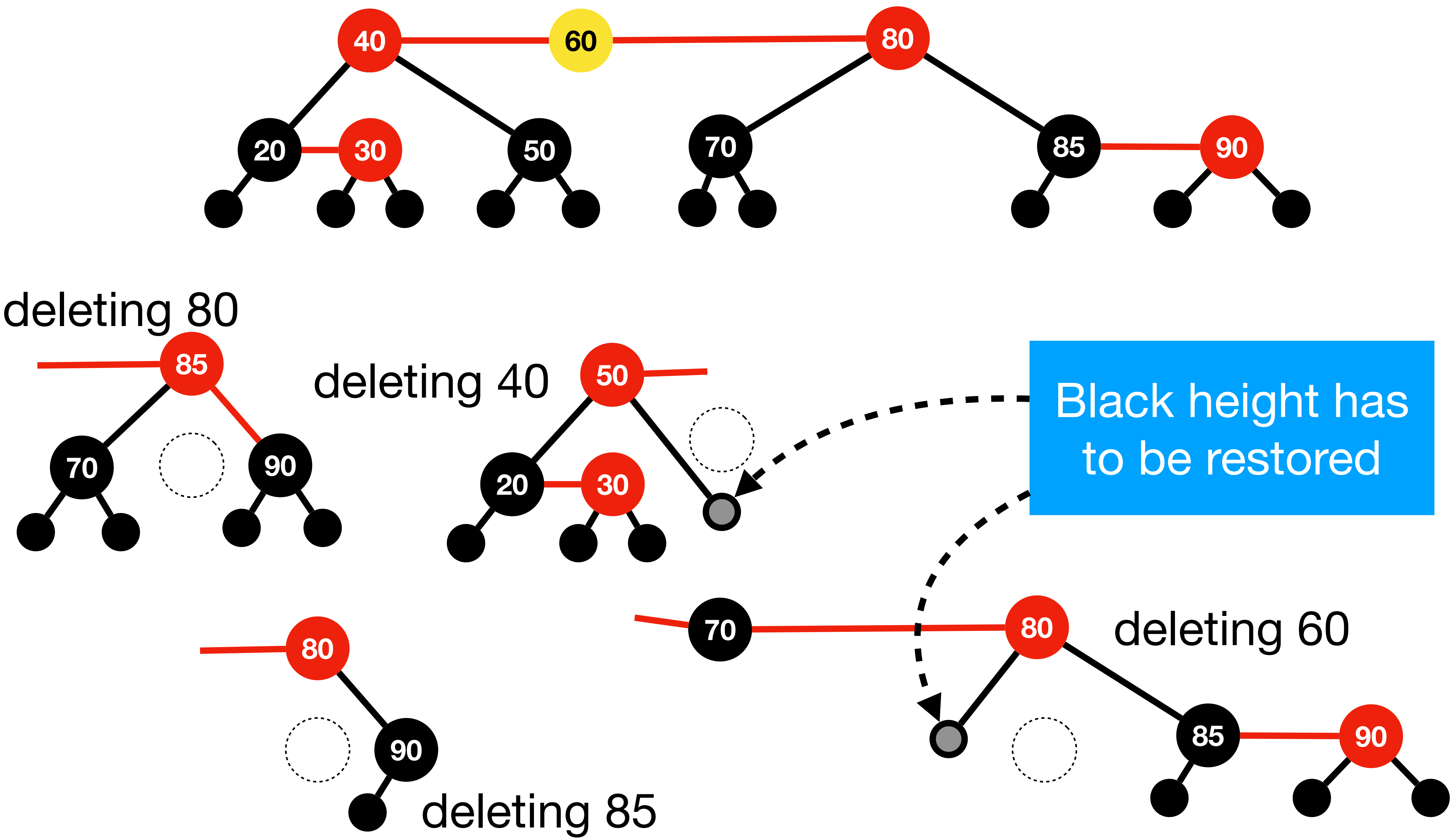
Deletion in RBT



Procedure of Red-Black Deletion

- Do a standard BST search to locate the node to be logically deleted, call it u
- If the right child of u is an external node, identify u as the node to be structurally deleted.
- If the right child of u is an internal node, find the tree successor of u, call it σ, copy the key and information from σ to u. (color of u not changed) Identify σ as the node to be deleted structurally.
- Carry out the structural deletion and repair any imbalance of black height.

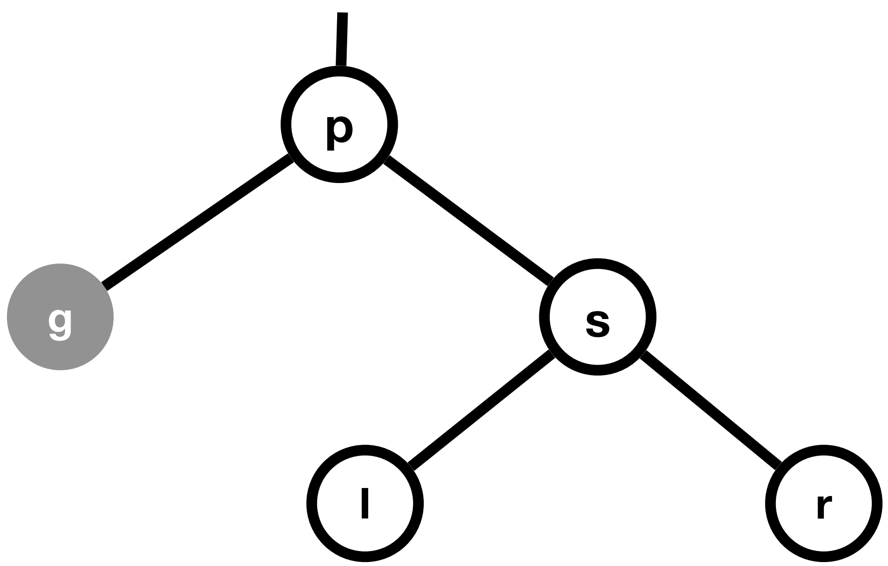
Imbalance of Black Height



Analysis of Black Imbalance

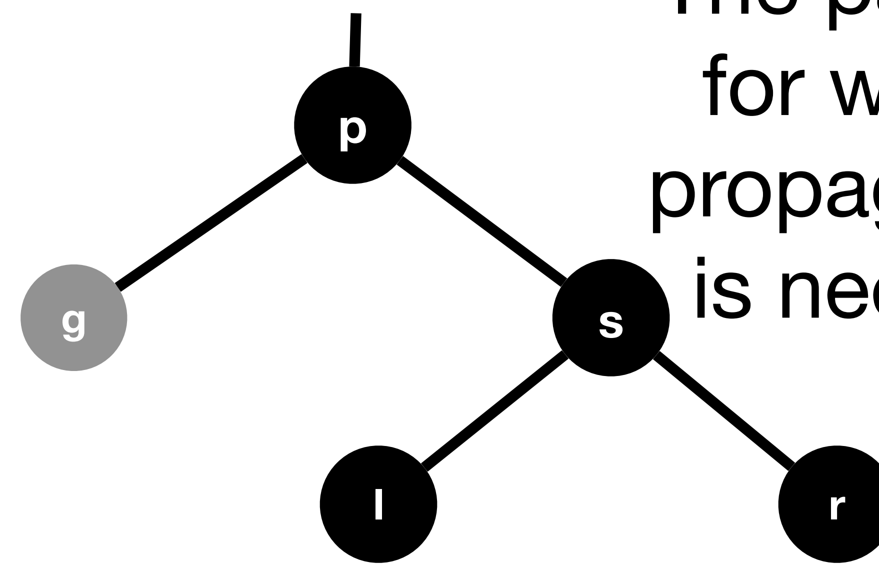
- **The imbalance occurs when:**
  - A black node is deleted structurally, and
  - Its right subtree is black (external)
- **The result is:**
  - An RB_{h-1} occupies the position of an RB_h as required by its parent, coloring it as a "gray" node.
- **Solution:**
  - Find a red node and turn it black as locally as possible.
  - The gray color might propagate up the tree.

Propagation of Gray Node



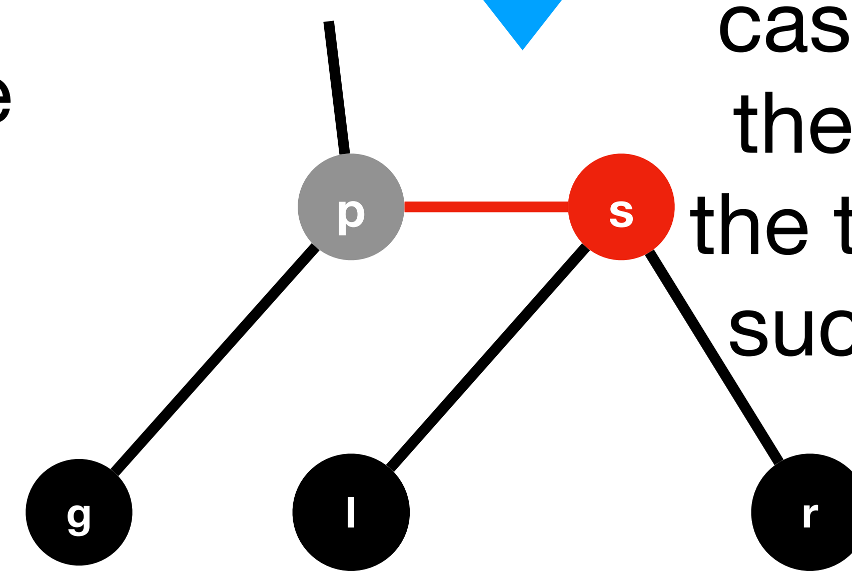
Map of the vicinity of **g**, the gray node

The pattern for which propagation is needed



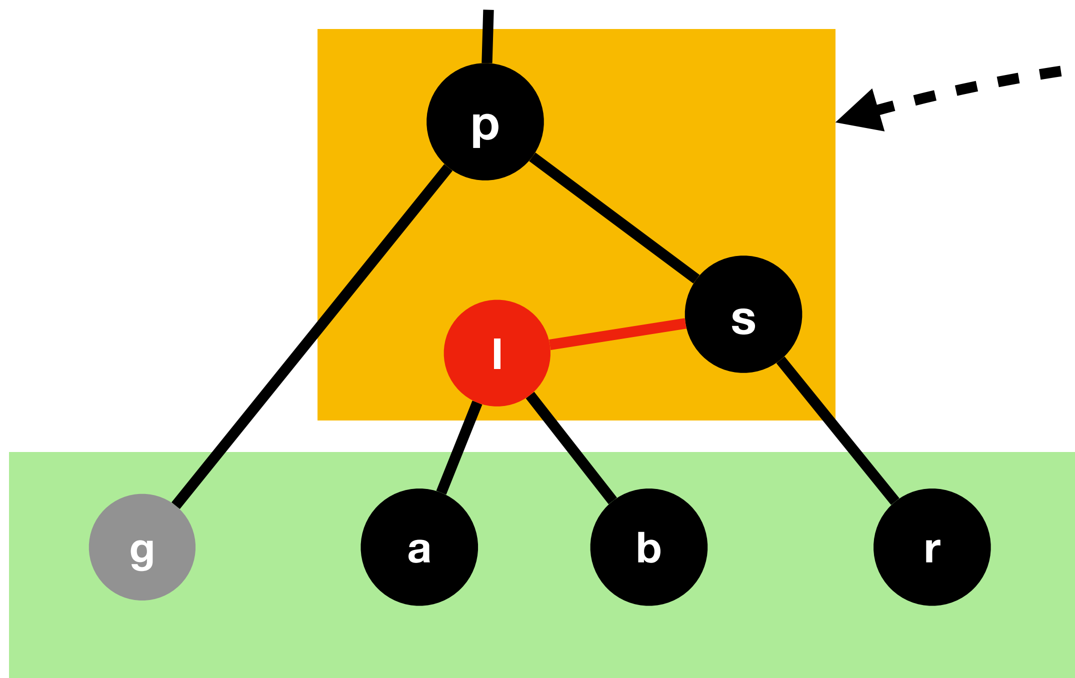
Gray up

In the worst case, up to the root of the tree, and successful

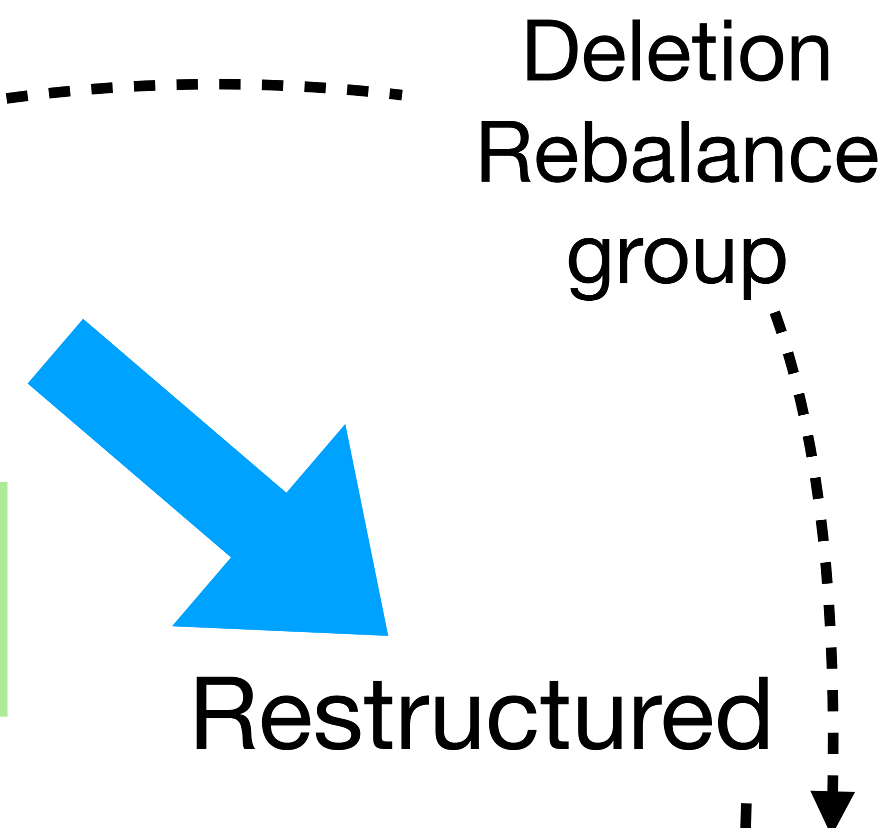


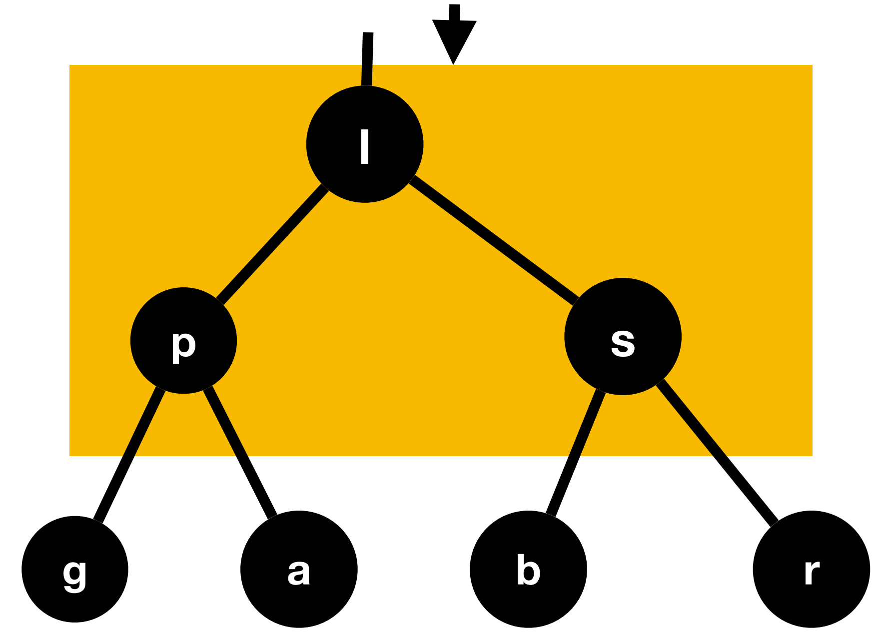
G-subtree gets well-defined black height, but that is less than that required by its parent

Repairing without Propagation



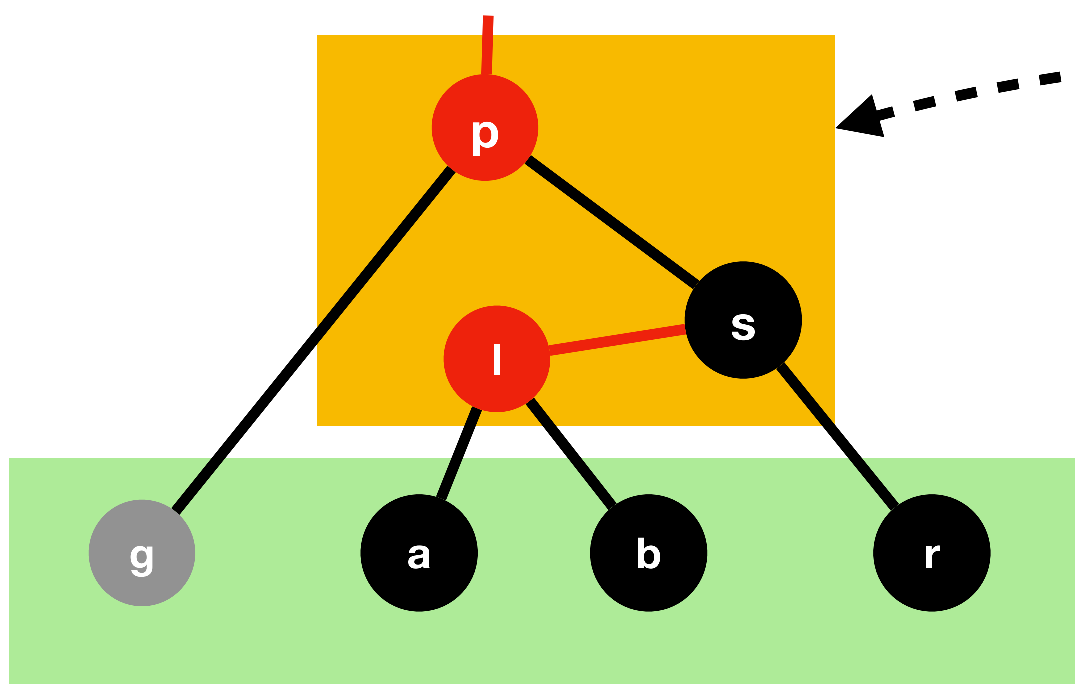
4 principal subtrees, RB_{h-1}





Restructuring the deletion rebalance group:
Red p: form an RB_1 or ARB_2 tree
Black p: form an RB_2 tree

Repairing without Propagation



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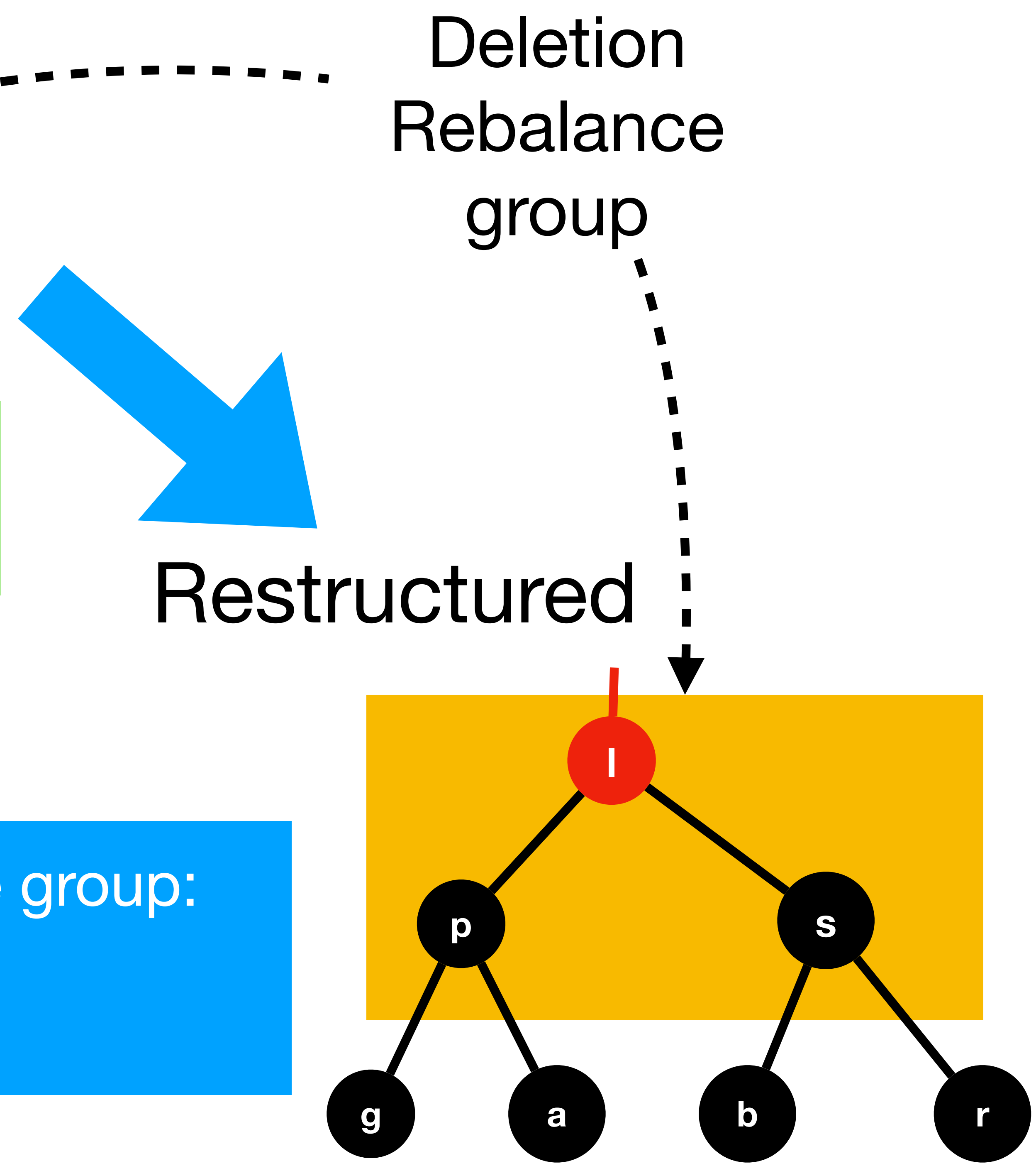

Restructuring the deletion rebalance group:
Red p: form an RB_1 or ARB_2 tree
Black p: form an RB_2 tree

Complexity of Operations on RBT

- **With reasonable implementation**
  - A new node can be inserted correctly in a red-black tree with n nodes in (logn) time in the worst case.
  - Repairs for deletion do O(1) structural changes, but may do O(logn) color changes.

Thank you!
Q & A