Introduction to

Algorithm Design and Analysis

[09] Hashing

Jingwei Xu
https://cs.nju.edu.cn/ics/people/jingweixu/index.html
Institute of Computer Software
Nanjing University

In the last class …

- **The searching problem**
  - “Architecture” of data
- **logn search**
  - Binary search
    - In a more general sense
  - Red-black tree: balanced BST
    - Definition
      - Black height constraint for balance
      - Color constraint for low maintenance cost
    - Operation
      - Insertion, deletion

Hashing

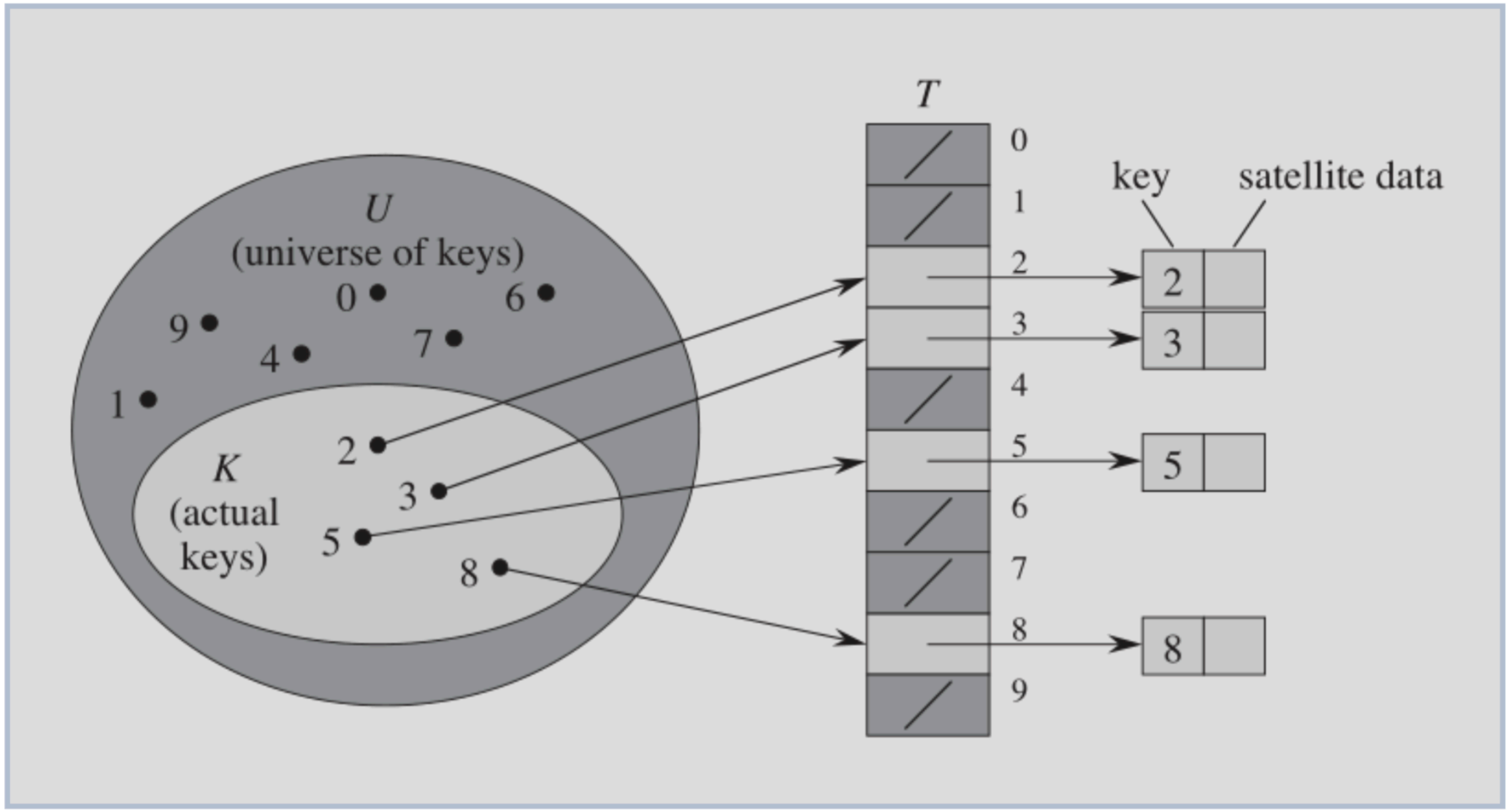
- **The searching problem**
  - The ambition of hashing
- **Hashing**
  - Brute force table: direct addressing
  - Basic idea of hashing
- **Collision Handling for Hashing**
  - Closed address hashing
  - Open address hashing
- **Amortized Analysis**
  - Array doubling

Cost for Searching

- **Brute force**
  - $O(n)$
- **Balanced BST**
  - $O(\log n)$
- **Hashing - almost constant time**
  - $O(1+\alpha)$
- **"Mission impossible"**
  - $O(1)$

Searching - A Brute Force Approach

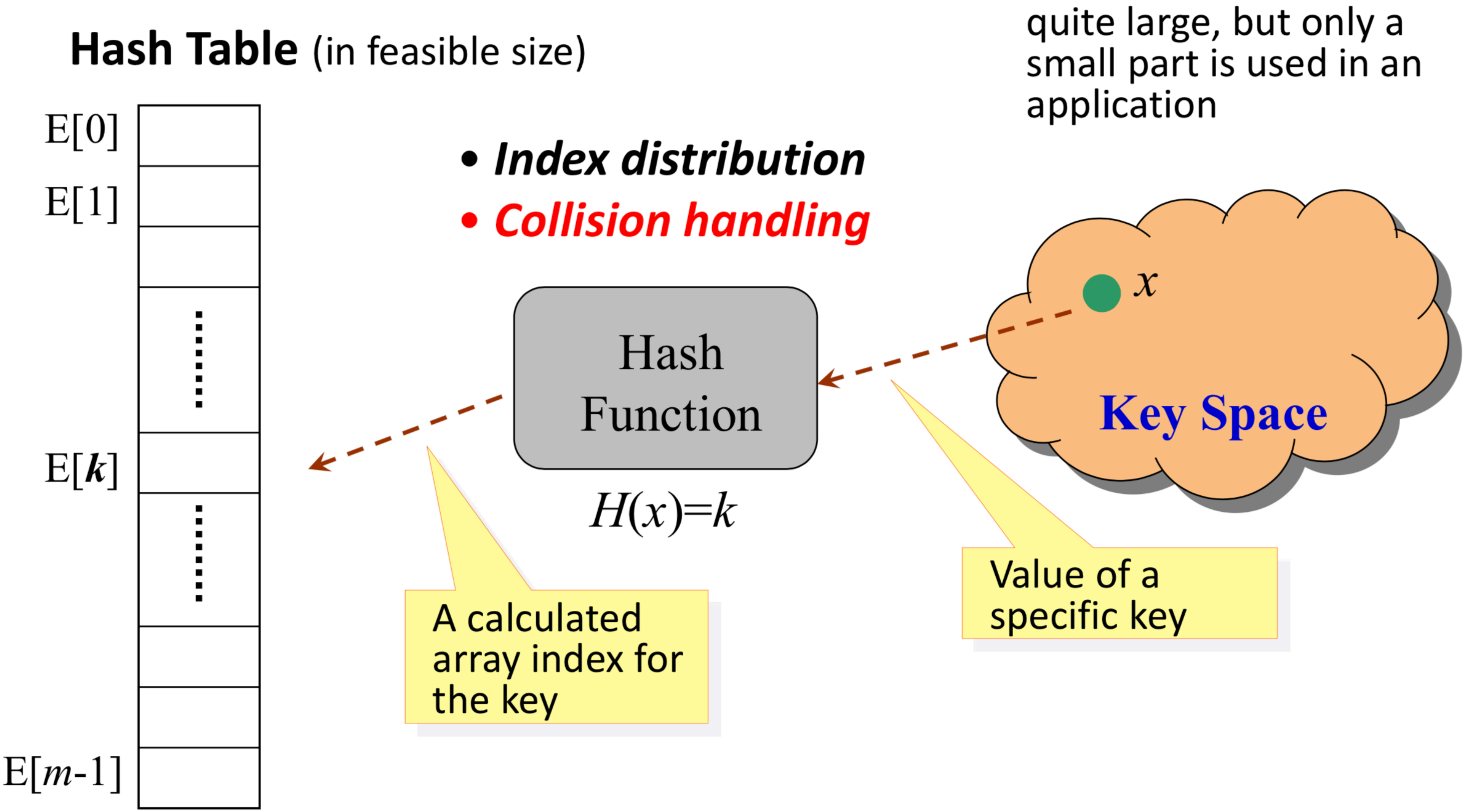
- **Direct-address table**
  - Take into account the whole universe of keys



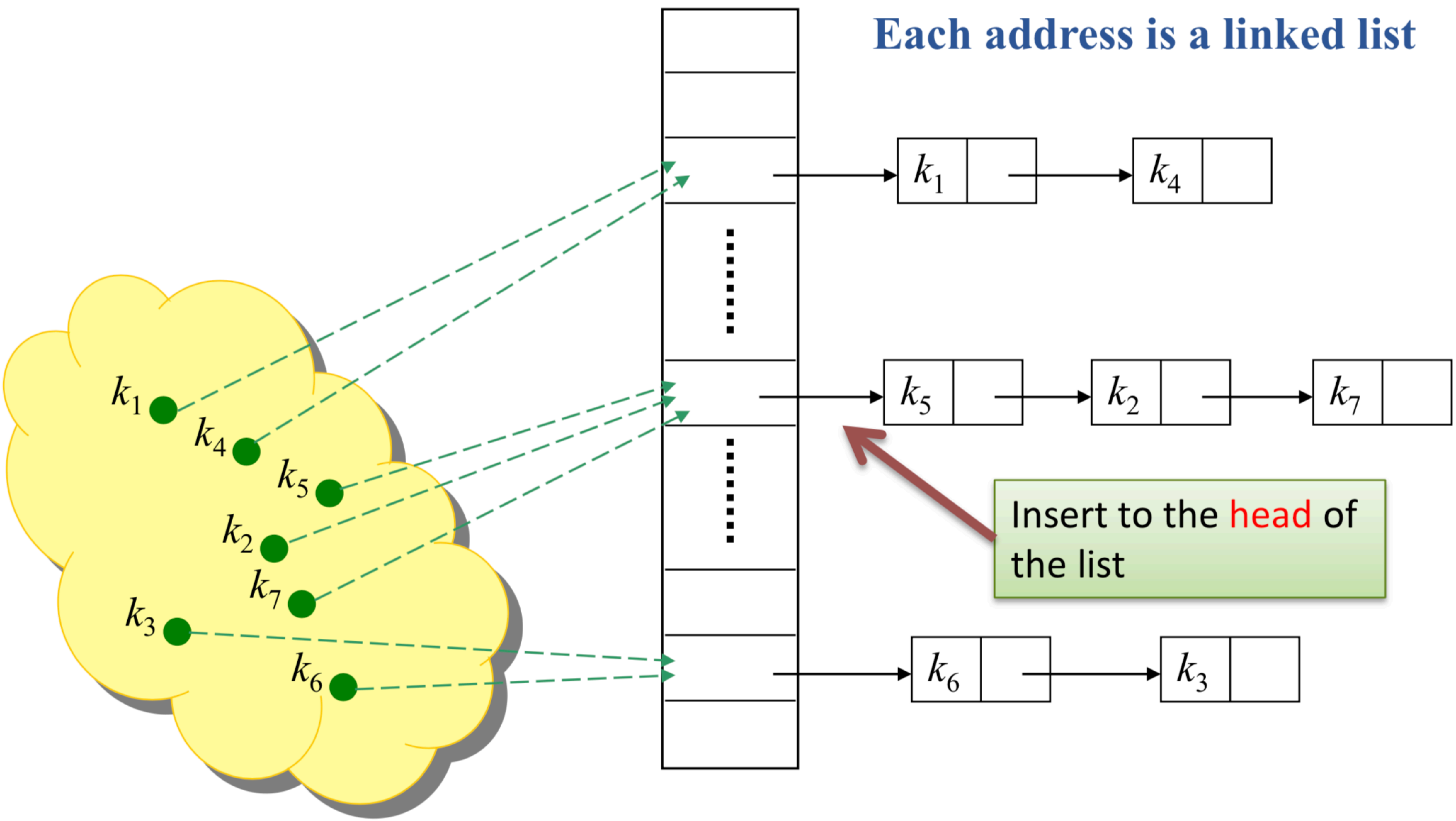
Direct-address Table

DIRECT-ADDRESS-SEARCH(T, k)
    return T[k]
DIRECT-ADDRESS-INSERT(T, x)
    T[key[x]] := x
DIRECT-ADDRESS-DELETE(T, x)
    T[key[x]] := NIL

Hashing: the Idea



Collision Handling: Closed Address



Closed Address - Analysis

- **Assumption - simple uniform hashing**
  - For j = 0, 1, 2, …, m-1, the average length of the list at E[j] is n/m.
- **The average cost for an unsuccessful search**
  - Any key that is not in the table is equally likely to hash to any of the m address.
  - Total cost $\Theta(1+n/m)$
    - The average cost to determine that the key is not in the list e[h(k)] is the cost to search to the end of the list, which is n/m.

Closed Address - Analysis

- **For successful search** (assuming that x_i is the ith element inserted into the table, $i = 1, 2, \ldots, n$)
  - For each i, the probability of that x_i is searched is 1/n.
  - For a specific x_i, the number of elements examined in a successful search is t+1, where t is the number of elements inserted into the same list as x_i, after x_i has been inserted

$$\frac{1}{n}\sum_{i=1}^{n}(1+t)$$

- **How to compute t?**
  - Consider the construction process of the hash table.

Closed Address - Analysis

- **For successful search:** (assuming that x_i is the i^{th} element inserted into the table, i = 1, 2, …, n)

  - For each i, the probability of that x_i is searched is 1/n.

  - For a specific x_i, the number of elements examined in a successful search is t+1, where t is the number of elements inserted into the same list as x_i, after x_i has been inserted. And for any j, the probability of that x_j is inserted into the same list of x_i is 1/m. So, the cost is:

Cost for computing hashing

$$1 + \frac{1}{n}\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n}\frac{1}{m}\right)$$

Expected number of elements in front of the searched one in the same linked list.

Closed Address: Analysis

- **The average cost of a successful search:**
  - Define α=n/m as load factor,
    - The average cost of a successful search is:

$$\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right) = 1 + \frac{1}{nm}\sum_{i=1}^{n}(n-i) = 1 + \frac{1}{nm}\sum_{i=1}^{n-1} i$$

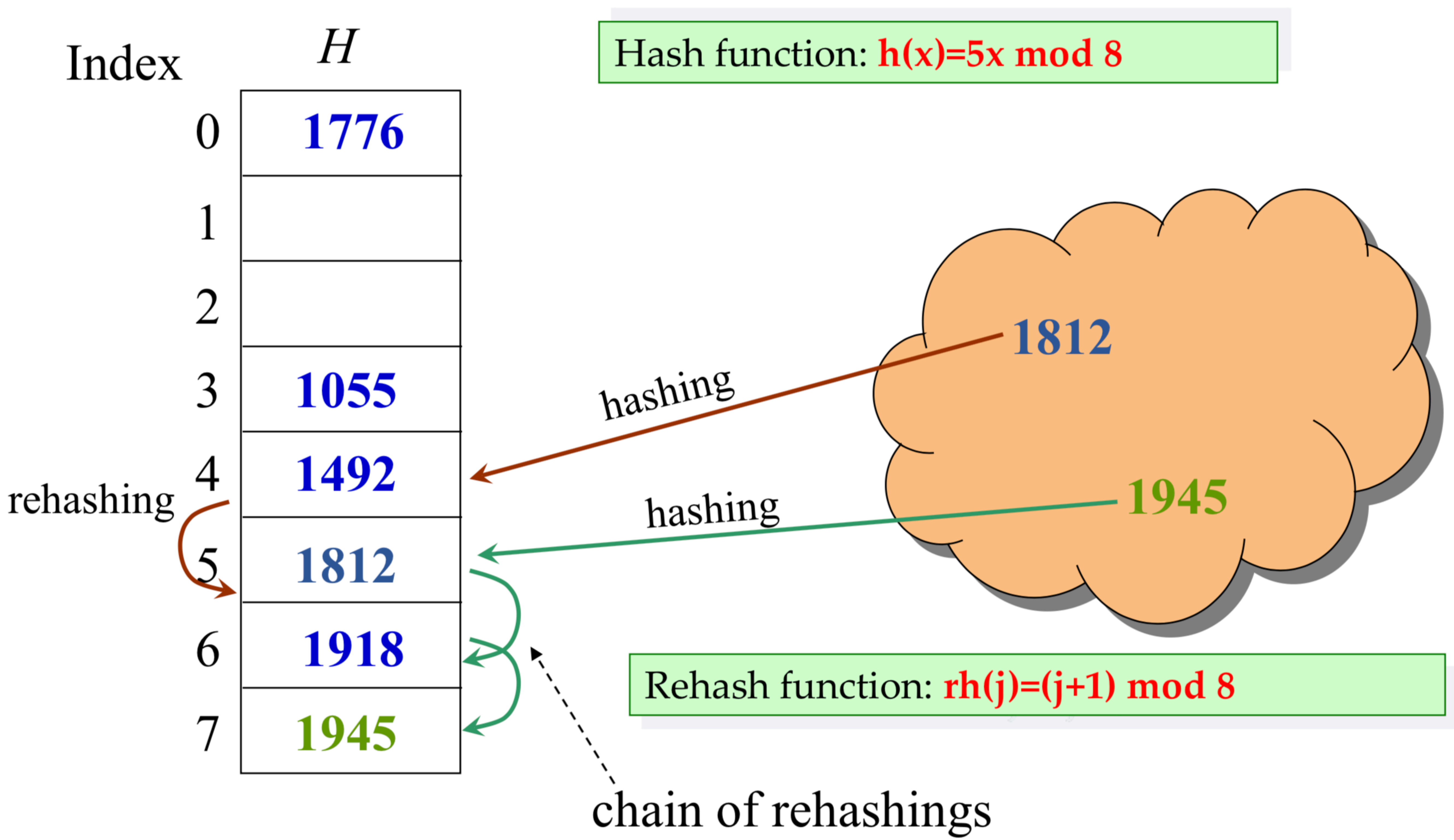
$$= 1 + \frac{n-1}{2m} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2n} = \Theta(1+\alpha)$$

Number of elements in front of the searched one in the same linked list

Collision Handling: Open Address

- **All elements are stored in the hash table**
  - No linked list is used
  - The load factor α cannot be larger than 1
- **Collision is settled by "rehashing"**
  - A function is used to get a new hashing address for each collided address
    - The hash table slots are probed successively, until a valid location is found.
- **The probe sequence can be seen as a permutation of (0,1,2,…,m-1)**

Linear Probing: An Example



Commonly Used Probing

- **Linear probing:**
  - Given an ordinary hash function h', which is called an auxiliary hash function, the hash function is: (clustering may occur)

$$h(k,i) = (h'(k) + i) \mod m (i = 0,1,\ldots,m-1)$$

- **Quadratic probing:**
  - Given auxiliary function h' and nonzero auxiliary constant c_1 and c_2, the hash function is: (secondary clustering may occur)

$$h(k,i) = (h'(k) + c_1 i + c_2 i^2) \mod m (i = 0,1,\ldots,m-1)$$

- **Double hashing:**
  - Given auxiliary functions h_1 and h_2, the hash function is:

$$h(k,i) = (h_1(k) + i h_2(k)) \mod m (i = 0,1,\ldots,m-1)$$

Equally Likely Permutations

- **Assumption**
  - Each key is equally likely to have any of the m! Permutations of (1,2,…,m) as its probe sequence
- **Note**
  - Both linear and quadratic probing have only m distinct probe sequence, as determined by the first probe

Analysis for Open Address hashing

- **The average number of probes in an unsuccessful search is at most $1/(1-\alpha)$ ($\alpha=n/m<1$)**
  - Assuming uniform hashing

The probability of the first probed position being occupied is $\frac{n}{m}$, and that of the $j^{th}(j>1)$ position occupied is $\frac{n-j+1}{m-j+1}$. So the probability of the number of probes no less than i will be:

$$\frac{n}{m}\cdot\frac{n-1}{m-1}\cdot\frac{n-2}{m-2}\cdot\cdots\cdot\frac{n-i+2}{m-i+2}\le\left(\frac{n}{m}\right)^{i-1}=\alpha^{i-1}$$

The the average number of probe is: $\sum\limits_{i=1}^{\infty}\alpha^{i-1}=\sum\limits_{i=0}^{\infty}\alpha^{i}=\frac{1}{1-\alpha}$

See [CLRS] p.1199, C.25

Analysis for Open Address Hashing

- **The average cost of probes in an successful search is at most** $\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$ **(α=n/m<1)**
  - Assuming uniform hashing

To search for the $(i+1)^{th}$ inserted element in the table, the cost is the same as that for inserting it when there are just i elements in the table. At that time, $\alpha = \frac{i}{m}$. So the cost is $\frac{1}{1-\frac{i}{m}} = \frac{m}{m-i}$.
So the average cost for a successful search is:

For your reference:
Half full: 1.387;
90% full: 2.559

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1}\frac{m}{m-i} &= \frac{m}{n}\sum_{i=0}^{n-1}\frac{1}{m-i} = \frac{1}{\alpha}\sum_{i=m-n+1}^{m}\frac{1}{i} \\ &\leq \frac{1}{\alpha}\int_{m-n}^{m}\frac{dx}{x} = \frac{1}{\alpha}\ln\frac{m}{m-n} = \frac{1}{\alpha}\ln\frac{1}{1-\alpha}\end{aligned}$$

Hash Function

- **A good hash function satisfies the assumption of simple uniform hashing**
  - Heuristic hashing functions
    - The division method: $h(k) = k \mod m$
    - The multiplication method: $h(k) = \lfloor m(kA \mod 1) \rfloor (0 < A < 1)$
  - No single function can avoid the worst case Θ(n)
    - So “universal hashing” is proposed.
  - Rich resource about hashing function
    - Gonnet and Baeza-Yates: Handbook of Algorithms and Data Structures, Addison-Wesley, 1991.

Array Doubling

- **Cost for search in a hash table is $\Theta(1+\alpha)$**
  - If we can keep α constant, the cost will be $\Theta(1)$
- **What if the hash table is more and more loaded?**
  - Space allocation techniques such as array doubling may be needed
- **The problem of "unusually expensive" individual operation**

Looking at the Memory Allocation

```
hashingInsert(HASHTABLE H, ITEM x)
    integer size = 0, num = 0;
    if size = 0 then allocate a block of size 1; size = 1;
    if num = size then
        allocate a block of size 2size;
        move all item into new table;
        size = 2size;
    insert x into the table;
    num = num + 1;
return
```

Insertion with expansion: cost size

Elementary insertion: cost 1

Worst-case Analysis

- **For n execution of insertion operations**
  - A bad analysis: the worst case for one insertion is the case when expansion is required, up to n
  - So, the worst case cost is in $O(n^2)$
- **Note the expansion is required during the i^{th} operation only if $i=2^k$, and the cost of the i^{th} operation**

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is exactly the power of 2} \\ 1 & \textit{otherwise} \end{cases}$$

So the total cost is: $\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \log n \rfloor} 2^j < n + 2n = 3n$

Amortized Analysis - Why?

- **Unusually expensive operations**
  - E.g., Insert-with-array-doubling
- **Relation between expensive and usuallay operations**
  - Each piece of the doubling cost corresponds to some previous insert

Amortized Analysis - How?

- **Amortized equation:**
  - amortized cost = actual cost + accounting cost
- **Design goal for accounting cost**
  - In any legal sequence of operations, the sum of the accounting costs is nonnegative
  - The amortized cost of each operation is fairly regular, in spite of the wide fluctuate possible for the actual cost of individual operations

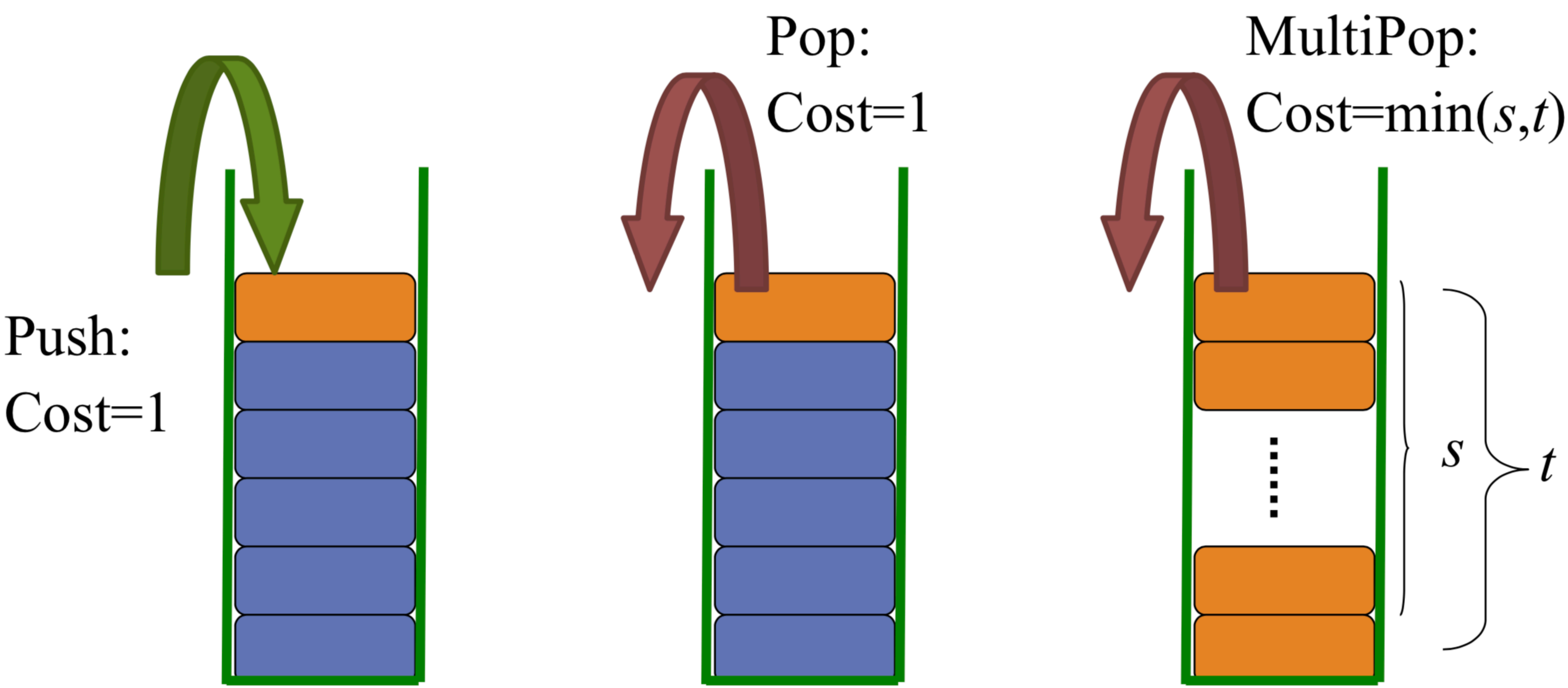
Array Doubling

- **Why non-negative accounting cost?**
  - For any possible sequence of operations?

	Amortized	Actual	Accounting
Insert (normal)	3	1	2
Insert (doubling)	3	k+1	-k+2

K is the number of elements upon doubling

Multi-pop Stack



Amortized cost: push:2; pop, multipop: 0

Multi-pop Stack

- **Why non-negative accounting cost?**
  - For any possible sequence of operations?

	Amortized	Actual	Accounting
Push	2	1	1
Multi-pop	0	k	-k

K is the number of elements upon multi-pop

Binary Counter

0	0 0 0 0 0 0 0 0	0
1	0 0 0 0 0 0 0 1	1
2	0 0 0 0 0 0 1 0	3
3	0 0 0 0 0 0 1 1	4
4	0 0 0 0 0 1 0 0	7
5	0 0 0 0 0 1 0 1	8
6	0 0 0 0 0 1 1 0	10
7	0 0 0 0 0 1 1 1	11
8	0 0 0 0 1 0 0 0	15
9	0 0 0 0 1 0 0 1	16
10	0 0 0 0 1 0 1 0	18
11	0 0 0 0 1 0 1 1	19
12	0 0 0 0 1 1 0 0	22
13	0 0 0 0 1 1 0 1	23
14	0 0 0 0 1 1 1 0	25
15	0 0 0 0 1 1 1 1	26
16	0 0 0 1 0 0 0 0	31

Cost measure: bit flip

amortized cost:

set 1: 2

set 0: 0

Binary Counter

- **Why non-negative accounting cost?**
  - For any possible sequence of operations?

	Amortized	Actual	Accounting
Set 1	2	1	1
Set 0	0	1	-1

Thank you!
Q & A