Introduction to

Algorithm Design and Analysis

[14] Minimum Spanning Tree

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In the last class…

- **Undirected and Symmetric Digraph**
    - DFS skeleton
- **Biconnected Components**
    - Articulation point
    - Bridge
- **Other undirected graph problems**
    - Orientation for undirected graphs
    - MST based on graph traversal

Greedy Strategy

- **Optimization Problem**
- **Greedy Strategy**

- **MST Problem**
  - Prim's Algorithm
  - Kruskal's Algorithm
- **Single-Source Shortest Path Problem**
  - Dijkstra's Algorithm

Greedy Strategy for Optimization Problems

- **Coin change Problem**
  - [candidates] A finite set of coins, of 1, 5, 10 and 25 units, with enough number for each value
  - [constraints] Pay an exact amount by a selected set of coins
  - [optimization] a smallest possible number of coins in the selected set
- **Solution by greedy strategy**
  - For each selection, choose the highest-valued coin as possible

Greedy Fails Sometimes

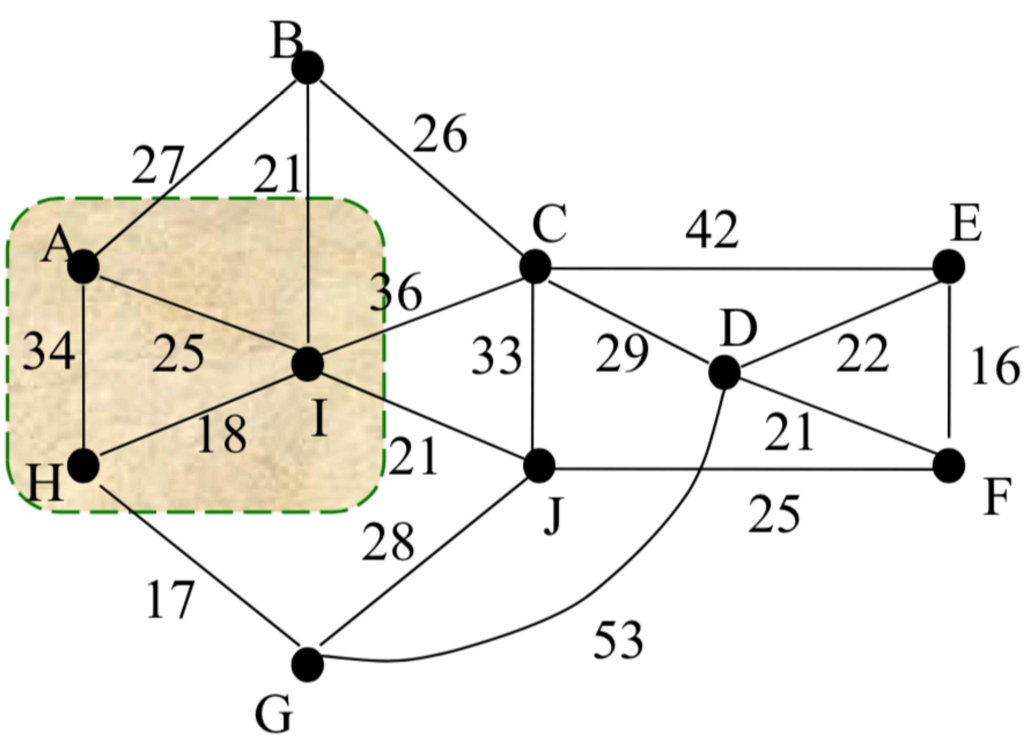
- **We have to pay 15 in total**
- **If the available types of coins are {1,5,12}**
    - The greedy choice is {12,1,1,1}
    - But the smallest set of coins is {5,5,5}
- **If the available types of coins are {1,5,10,25}**
    - The greedy choice is always correct

Greedy Strategy

- **Expanding the partial solution step by step**
- **In each step, a selection is made from a set of candidates. The choice made must be:**
  - [Feasible] it has to satisfy the problem's constraints
  - [Locally optimal] it has to be the best local choice among all feasible choices on the step
  - [Irrevocable] the choice cannot be revoked in subsequent steps

```
set greedy(set candidate)
    set S=Ø;
    while not solution(S) and   candidate≠Ø
        select locally optimizing x from candidate;
        candidate=candidate-{x};
        if feasible(x) then S=S∪{x};
    if solution(S) then return S
        else return ("no solution")
```

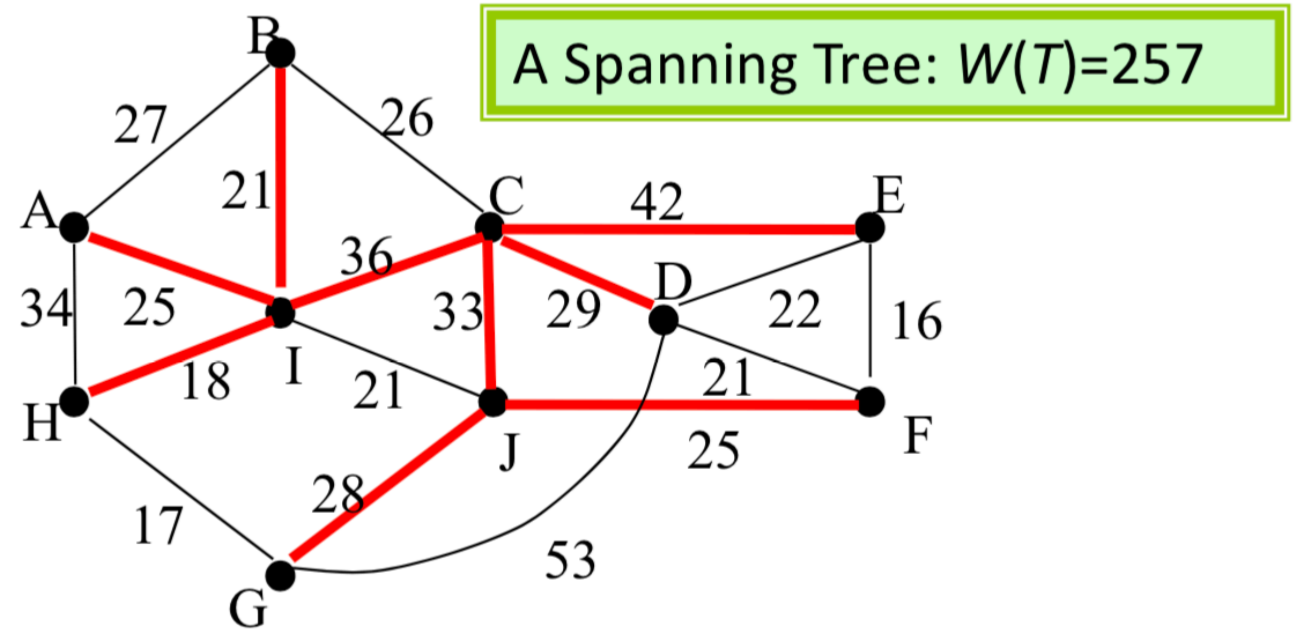
Weighted Graph and MST

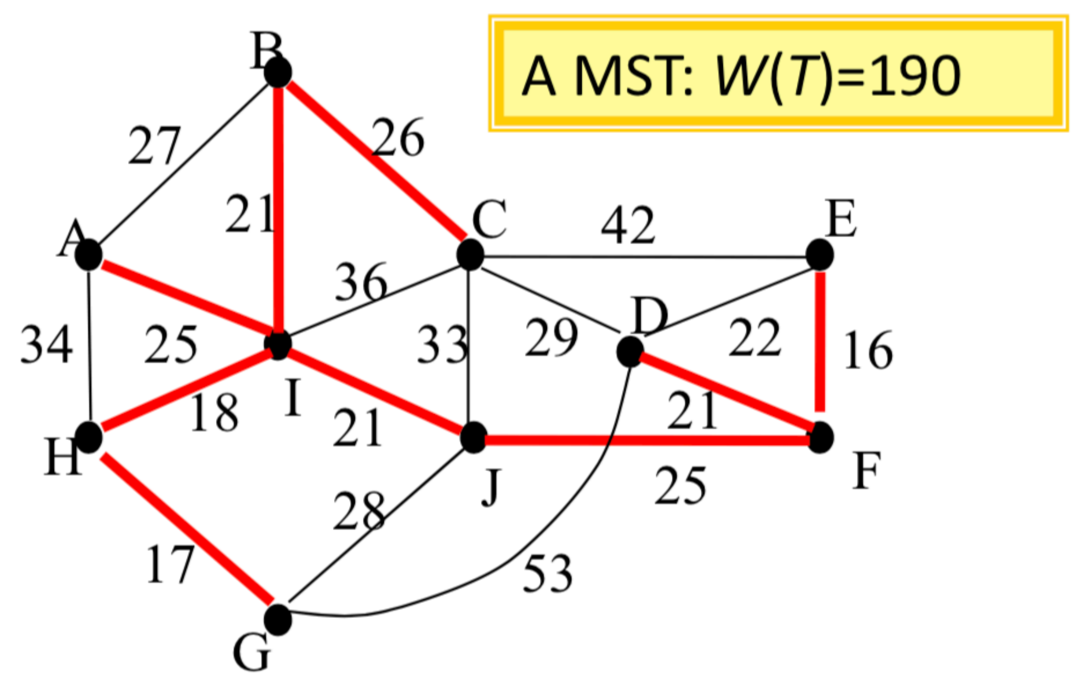


A weighted graph

The nearest neighbor of vertex ***I*** is ***H***

The nearest neighbor of shaded subset of vertex is ***G***

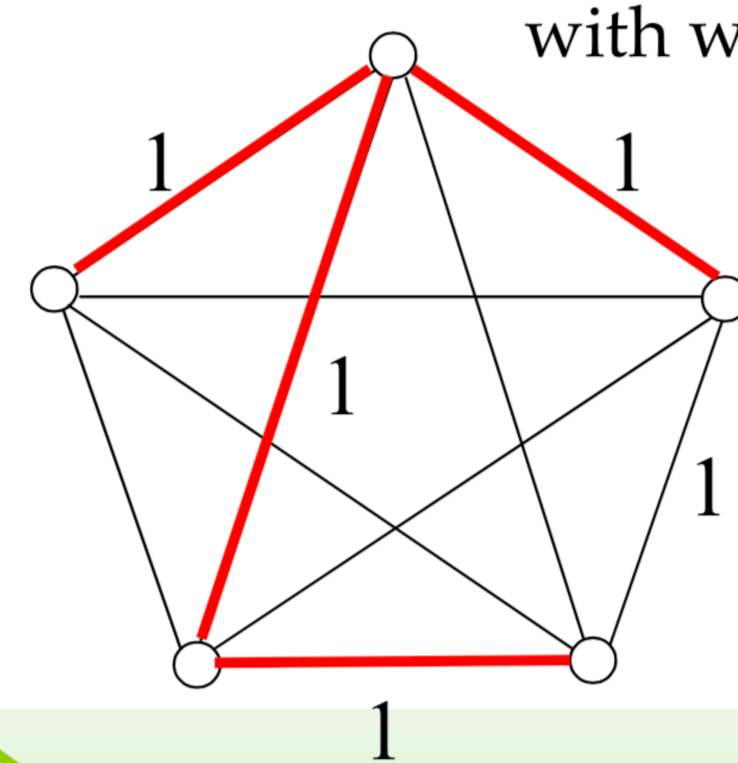




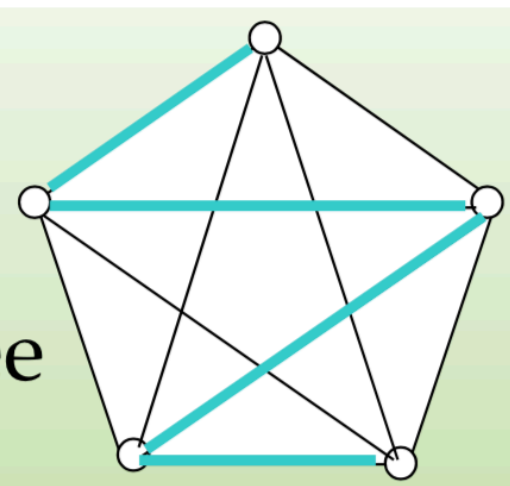
Graph Traversal and MST

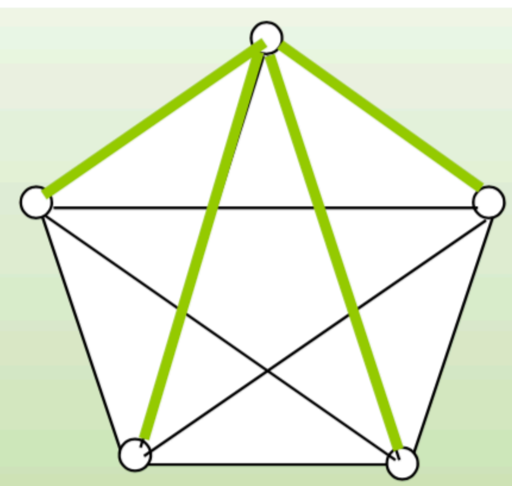
There are cases that graph traversal tree **cannot** be minimum spanning tree, with the vertices explored in any order.

All other edges with weight 5



DFS tree





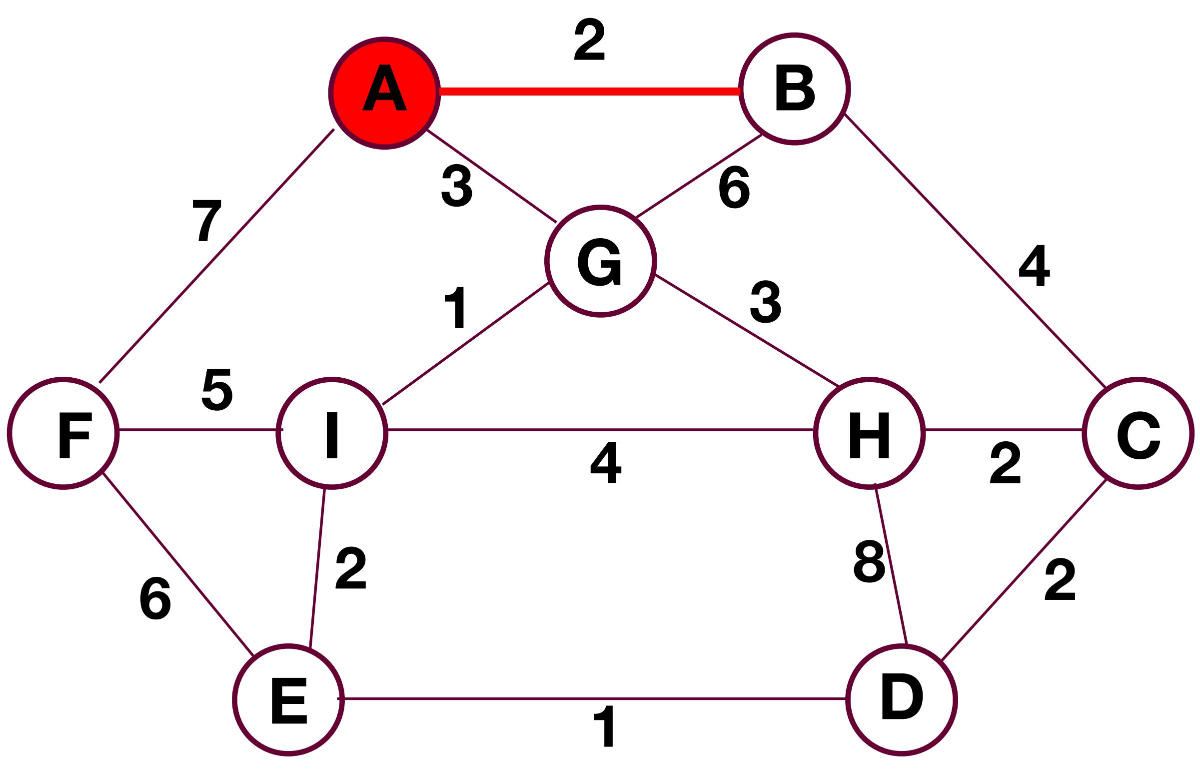
BFS tree

in any ordering of vertex

Greedy Algorithms for MST

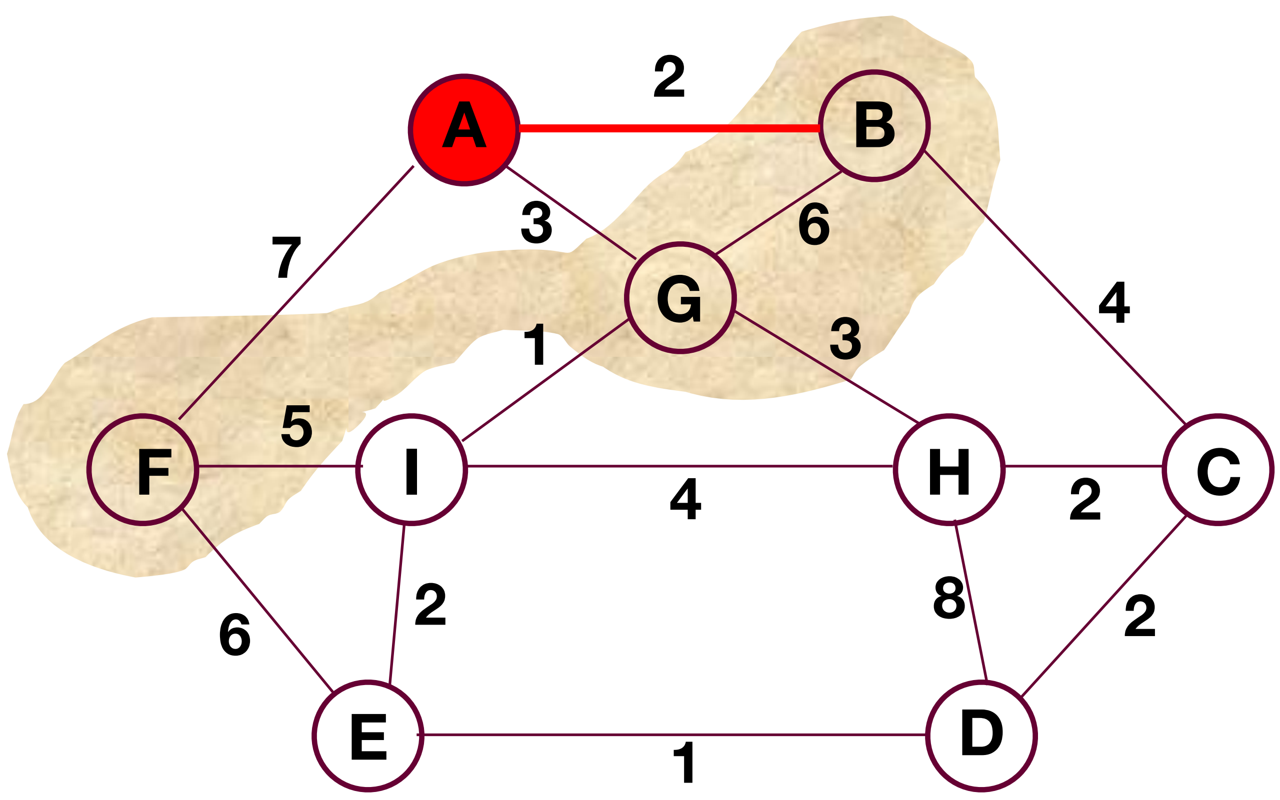
- **Prim's algorithm:**
  - Difficult selecting: "best local optimization means no cycle and small weight under limitation"
  - Easy checking: doing nothing
- **Kruskal's algorithm:**
  - Easy selecting: smallest in primitive meaning
  - Difficult checking: no cycle

Prim’s Algorithm



edges included in the MST

Prim's Algorithm



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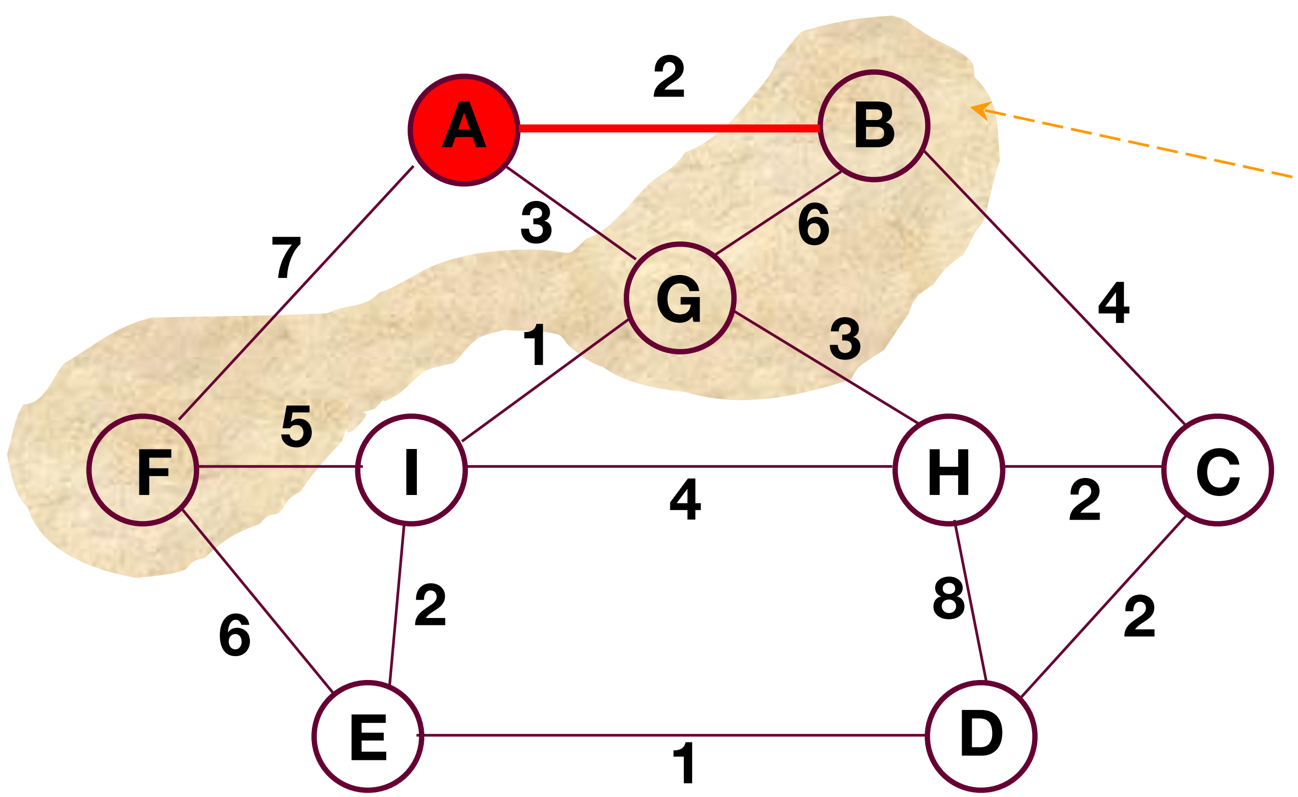
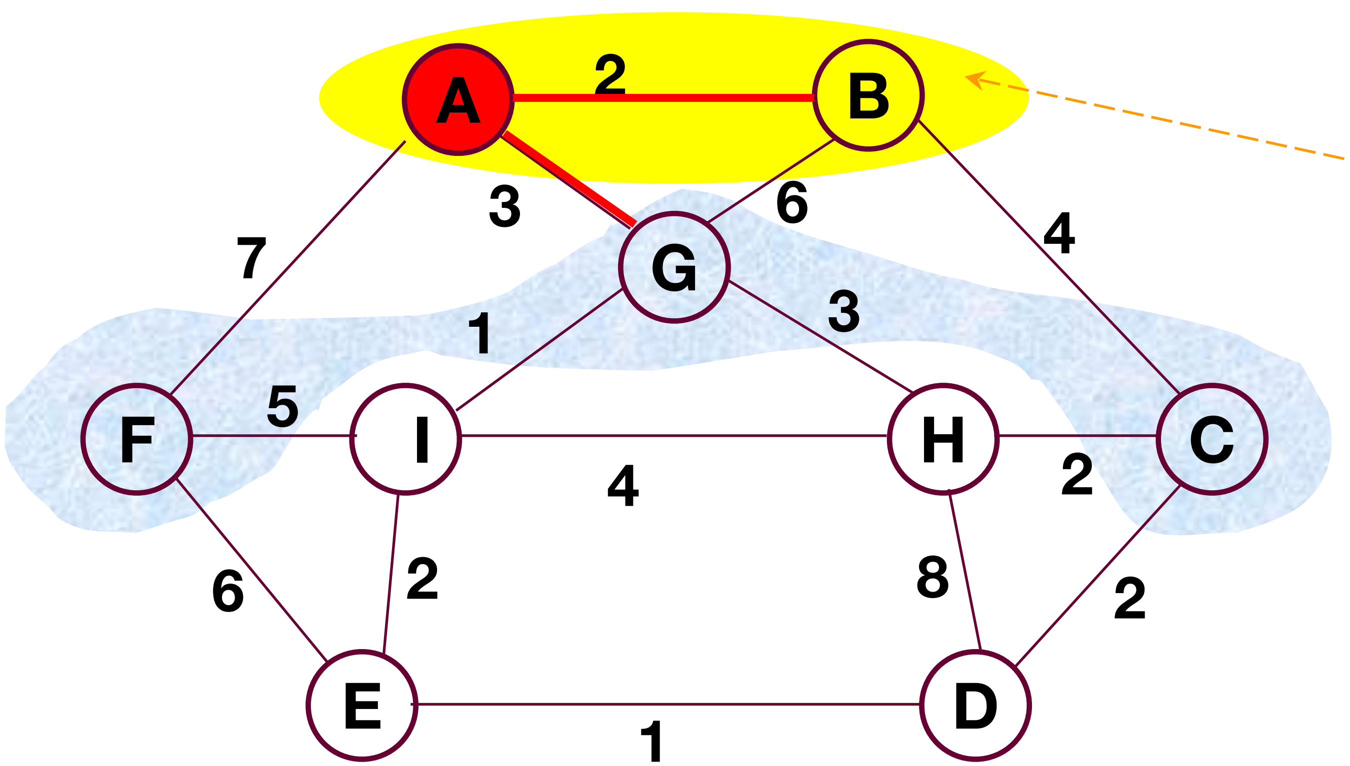


Greedy strategy:
For each set of fringe vertex, select the edge with the minimal weight, that is, local optimal.

edges included in the MST

Prim’s Algorithm



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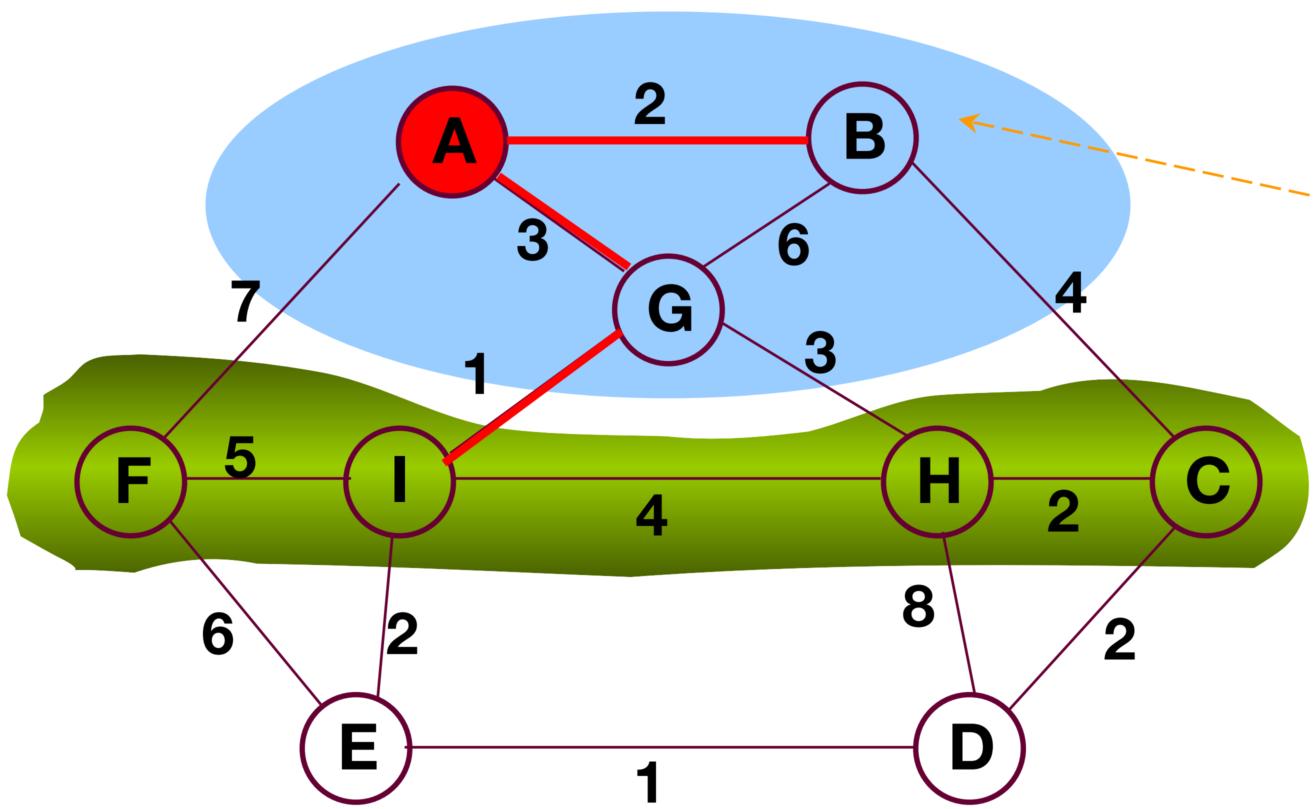

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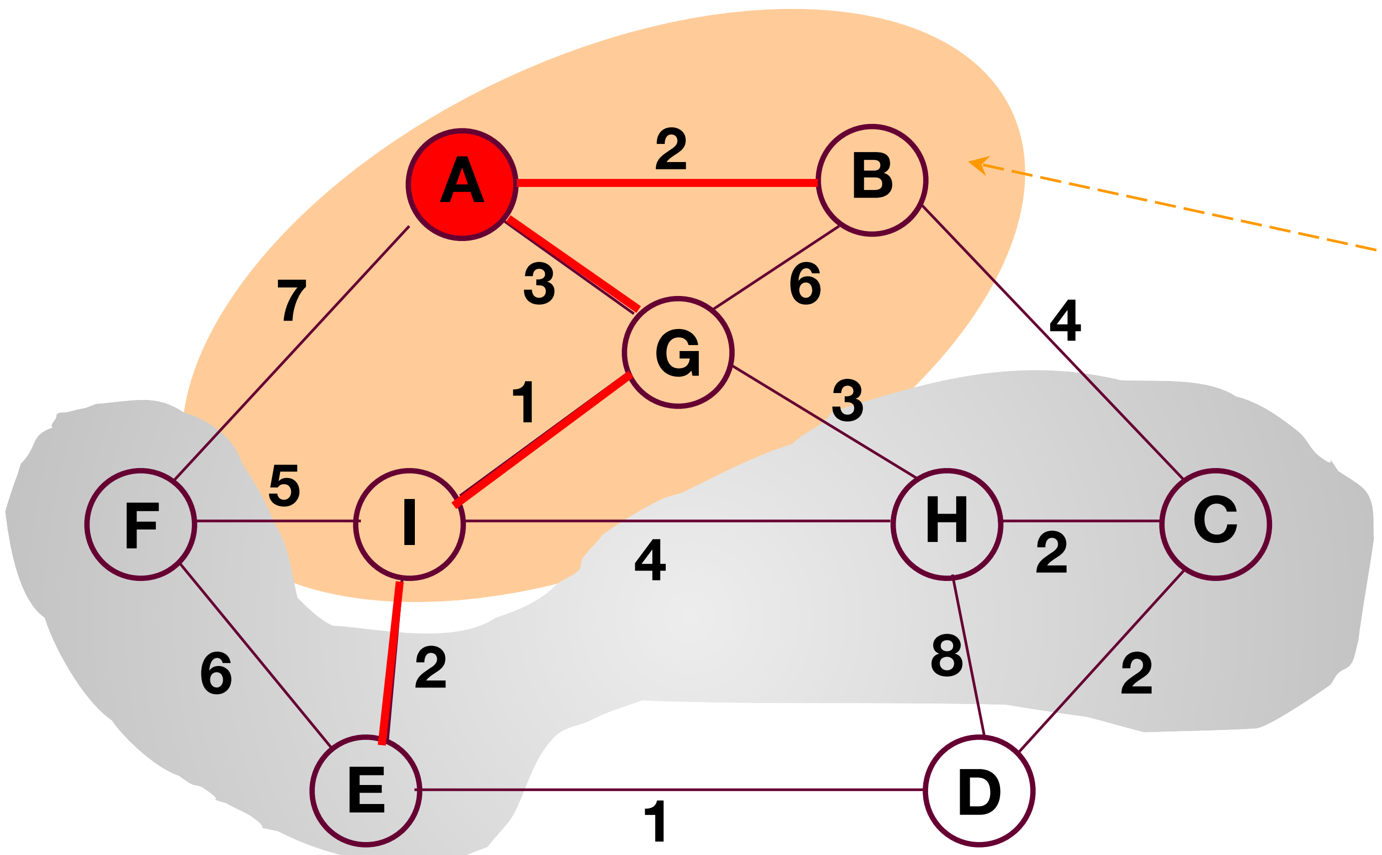

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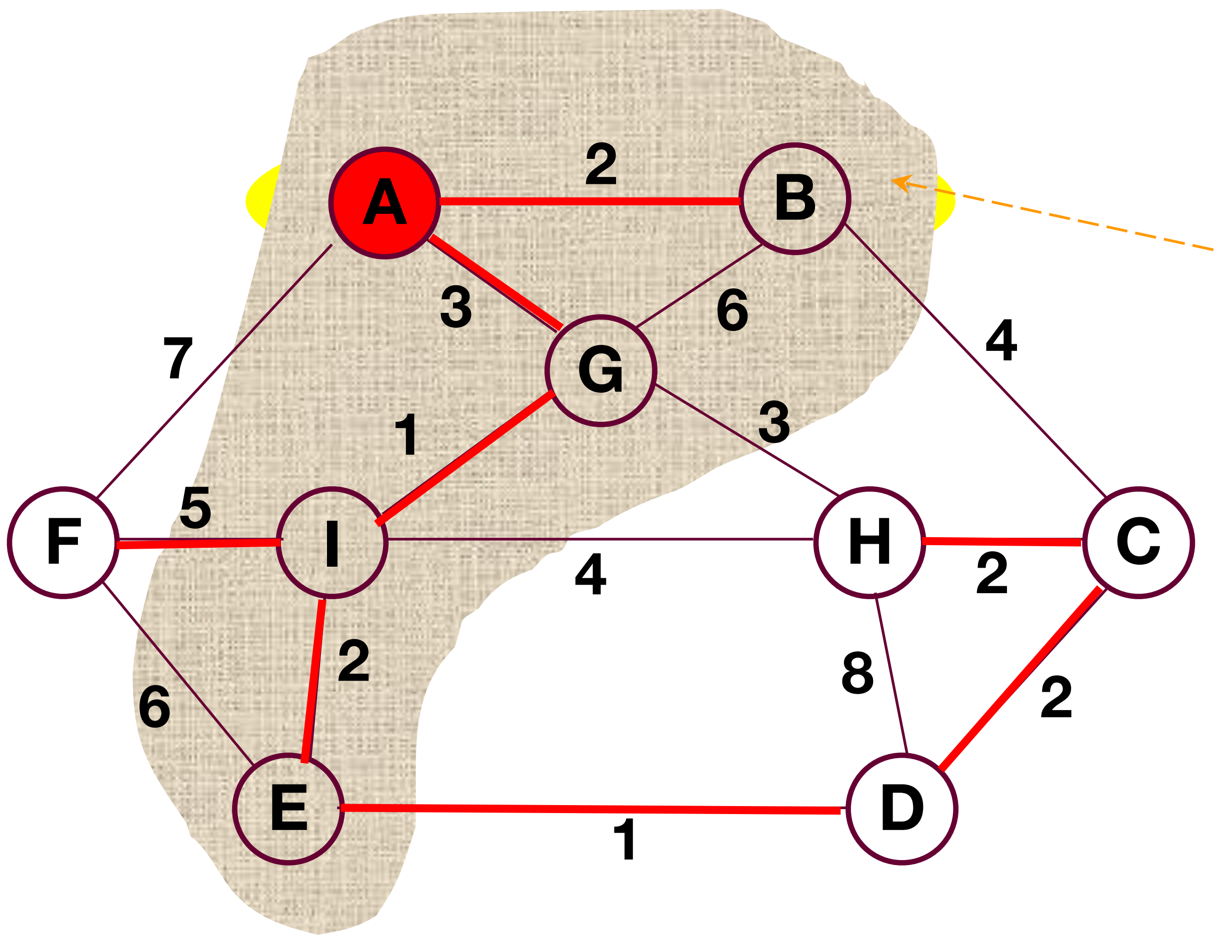

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edges included in the MST

Correctness: How to Prove

of nodes selected

$T^{(1)} \quad T^{(2)} \quad \ldots \quad T^{(k\text{-}1)} \quad T^{(k)} \quad T^{(k+1)} \quad \ldots \quad T^{(n)}$

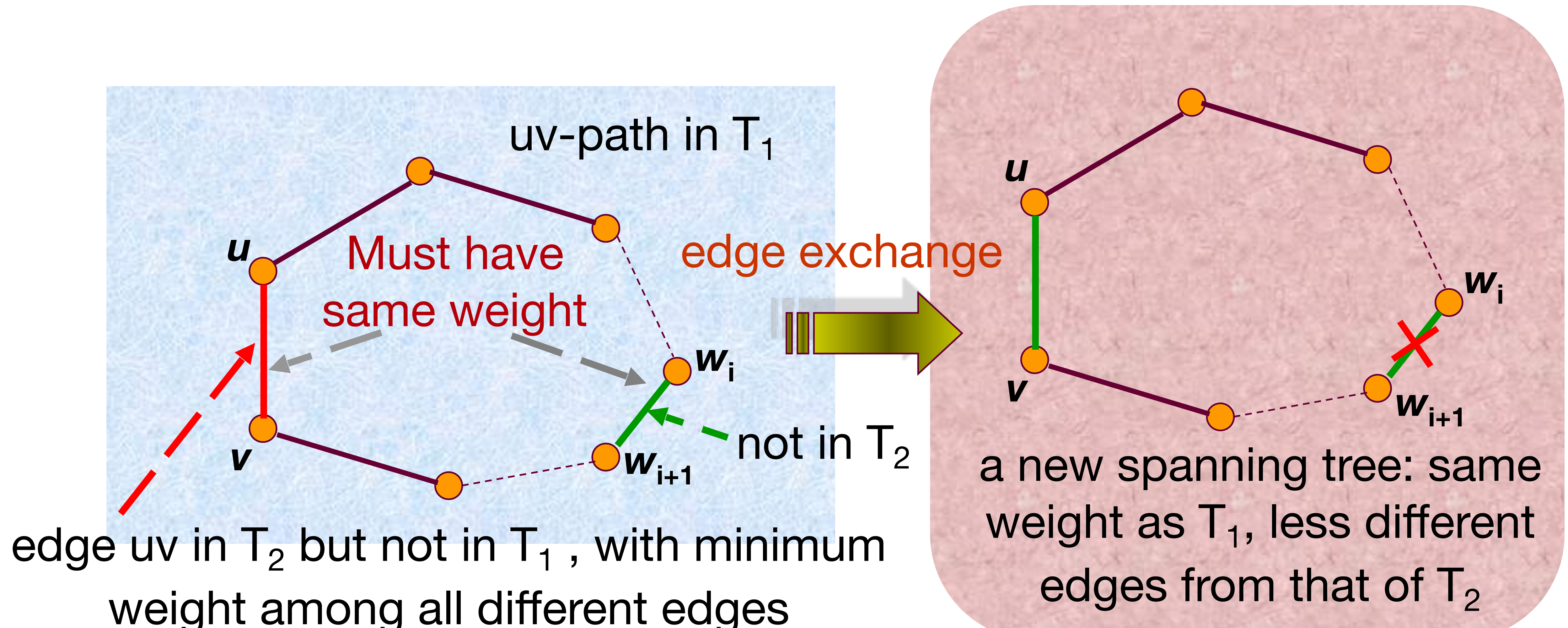
Invariance: MST

- Spanning tree
- Min weight

Definition transformation

Minimum Spanning Tree Property

- A spanning tree T of a connected, weighted graph has MST property if and only if for any non-tree edge uv, T ∪ {uv} contain a cycle in which uv is one of the maximum-weight edge.
- All the spanning trees having MST property have the same weight.



MST Property and Minimum Spanning Tree

- **In a connected, weighted graph G={V,E,W}, a tree T is a minimum spanning tree if and only if T has the MST property.**

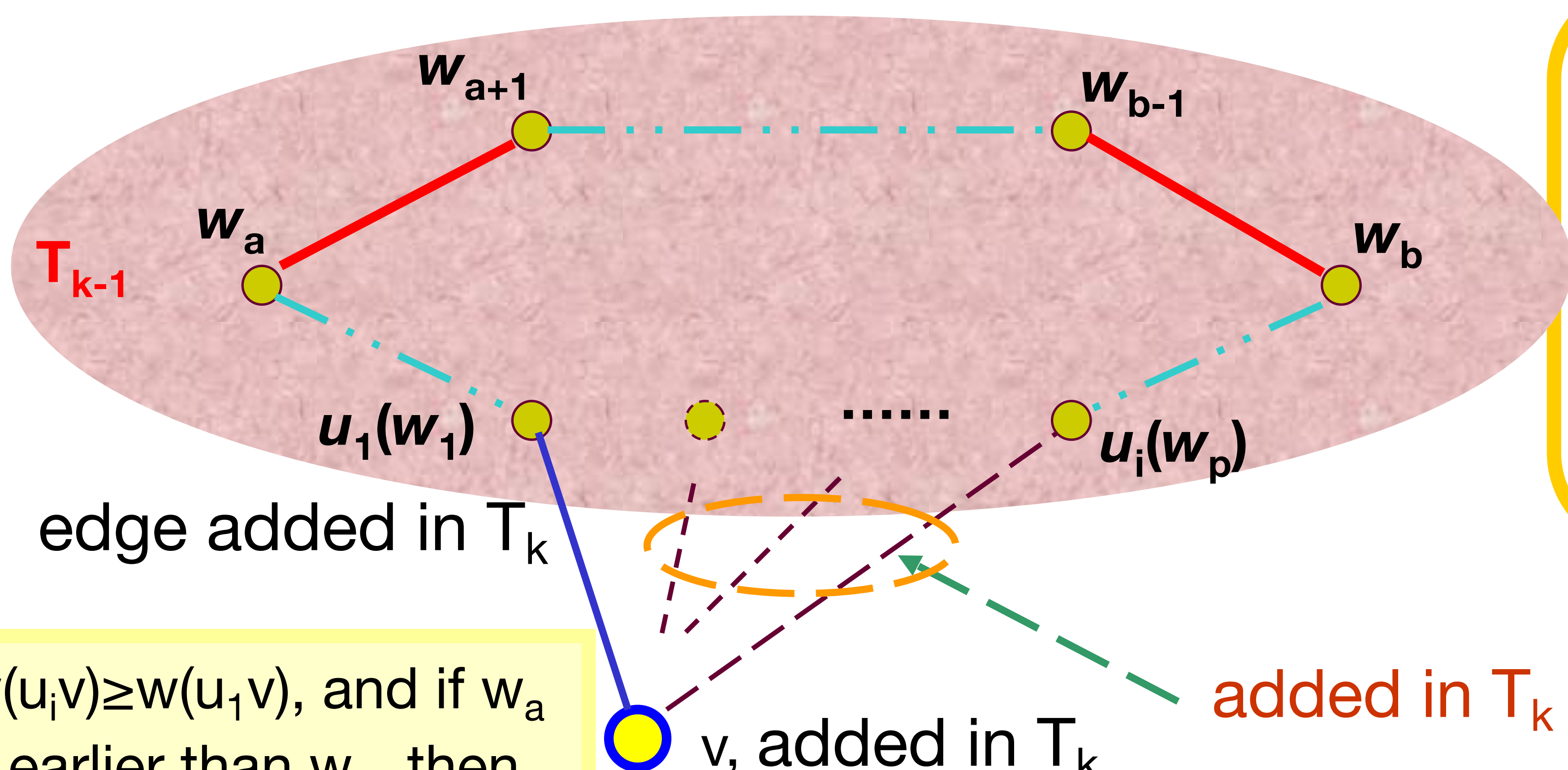
- **Proof**

    - => For a minimum spanning tree T, if it doesn't has MST property. So, there is a non-tree edge uv, and T ∪ {uv} contain an edge xy with weight larger than that of uv. Substituting uv for xy results a spanning tree with less weight than T. Contradiction.

    - <= As claimed above, any minimum spanning tree has the MST property. Since T has MST property, it has the same weight as any minimum spanning tree, i.e. T is a minimum spanning tree as well.

Correctness of Prim's Algorithm

- Let T_k be the tree constructed after the k^{th} step of Prim's algorithm is executed. Then T_k has the MST property in G_k, the subgraph of G induced by vertices of T_k.



assumed first and last edges with larger weight than $w(u_iv)$, resulting contradictions.

added in T_k to form a cycle, only these need be considered

Note: $w(u_iv) \geq w(u_1v)$, and if w_a added earlier than w_b, then w_aw_{a+1} and $w_{b-1}w_b$ added later than any edges in u_1w_a-path, and v as well

Key Issue in Implementation

- **Maintaining the set of fringe vertices**
  - Create the set and update it after each vertex is "selected" (deleting the vertex having been selected and inserting new fringe vertices)
  - Easy to decide the vertex with "highest priority"
  - Changing the priority of the vertices (decreasing key)
- **The choice: priority queue**

Implementing Prim's Algorithm

Main Procedure

```
primMST(G,n)
  Initialize the priority queue pq as empty;
  Select vertex s to start the tree;
  Set its candidate edge to (-1,s,0);
  insert(pq,s,0);
  while (pq is not empty)
     v=getMin(pq); deleteMin(pq);
     add the candidate edge of v to the tree;
     updateFringe(pq,G,v);
  return
```

getMin(pq) always be the vertex with the smallest key in the fringe set.

ADT operation executions:

insert, getMin, deleteMin: n times

decreaseKey: m times

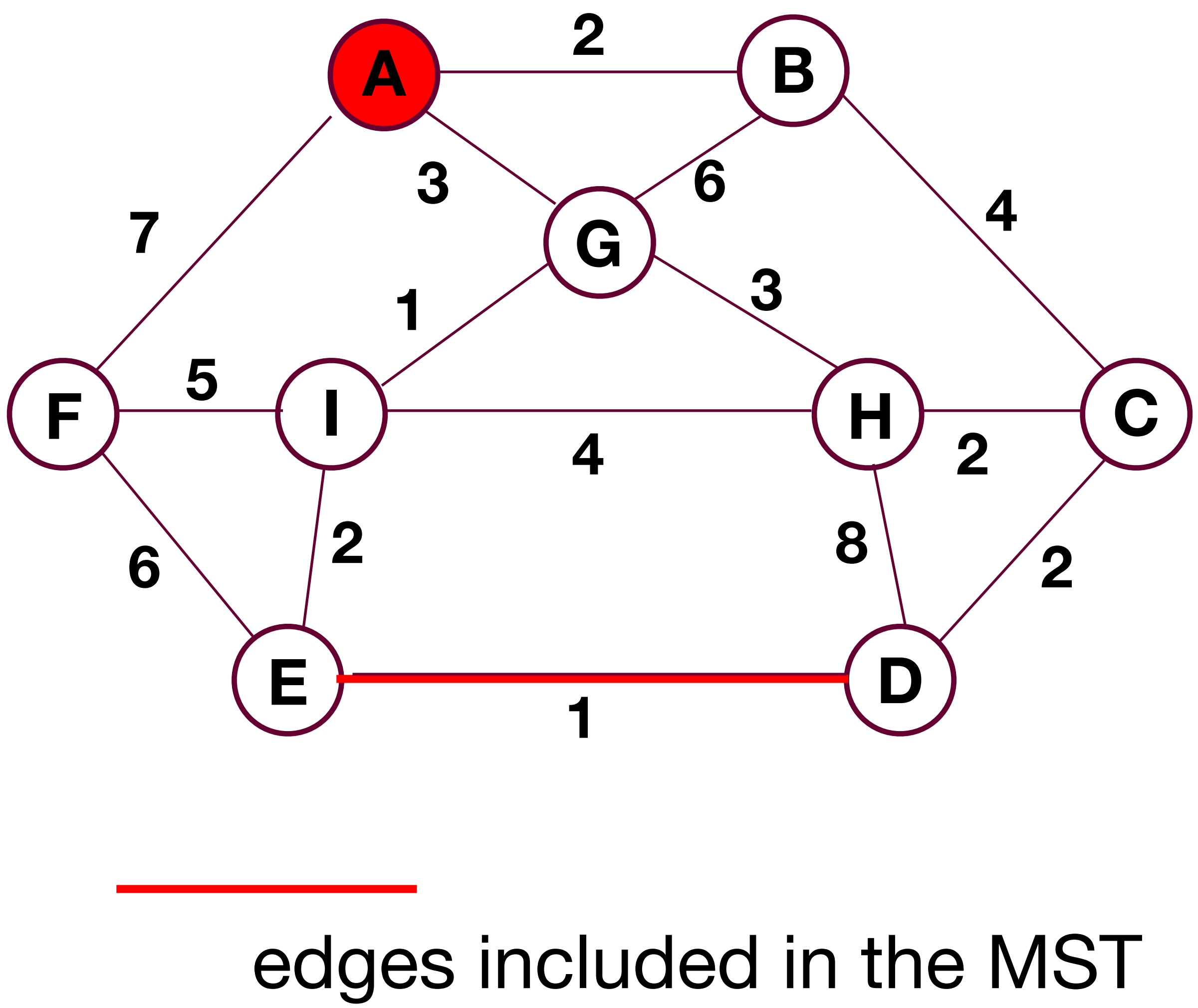
Updating the Queue

```
updateFringe(pq,G,v)
  For all vertices w adjcent to v //2m loops
     newWgt=w(v,w);
     if w.status is unseen then
        Set its candidate edge to (v,w,newWgt);
        insert(pq,w,newWgt)
     else
        if newWgt<getPriorty(pq,w)
           Revise its candidate edge to (v,w,newWgt);
           decreaseKey(pq,w,newWgt)
  return
```

Complexity

- **Operations on ADT priority queue: (for a graph with vertices and m edges)**
  - insert: n;   getMin: n;   deleteMin: n;
  - decreaseKey: m (appears in 2m loops, but execute at most m)
- **So,**
  - T(n,m)=O(nT(getMin)+nT(deleteMin+insert)+mT(decreaseKey))
- **Implementing priority queue using array, we can get $\Theta(n^2+m)$**

Kruskal's Algorithm



Also Greedy strategy:

From the set of edges not yet included in the partially built MST, select the edge with the minimal weight, that is, local optimal, in another sense.

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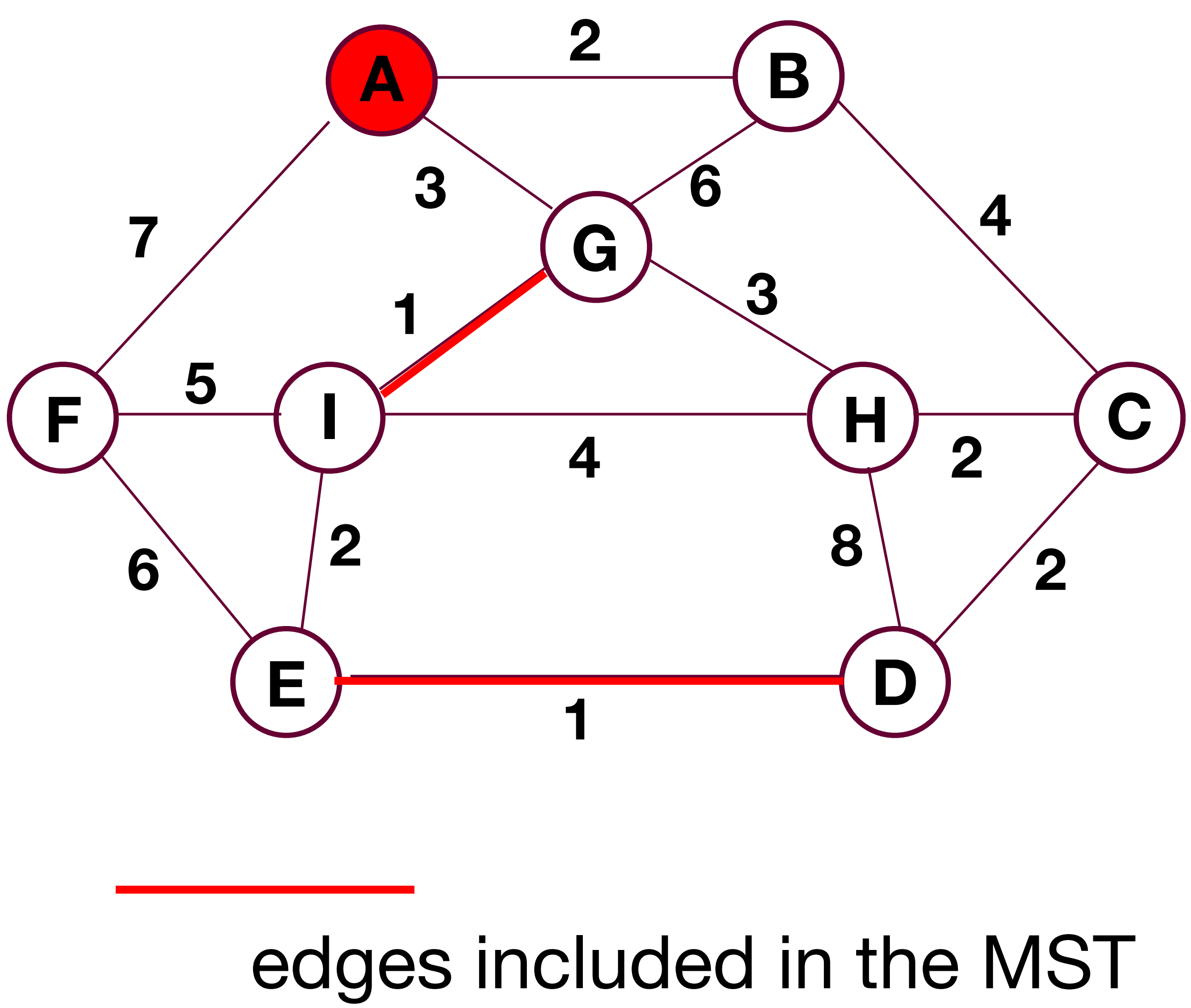

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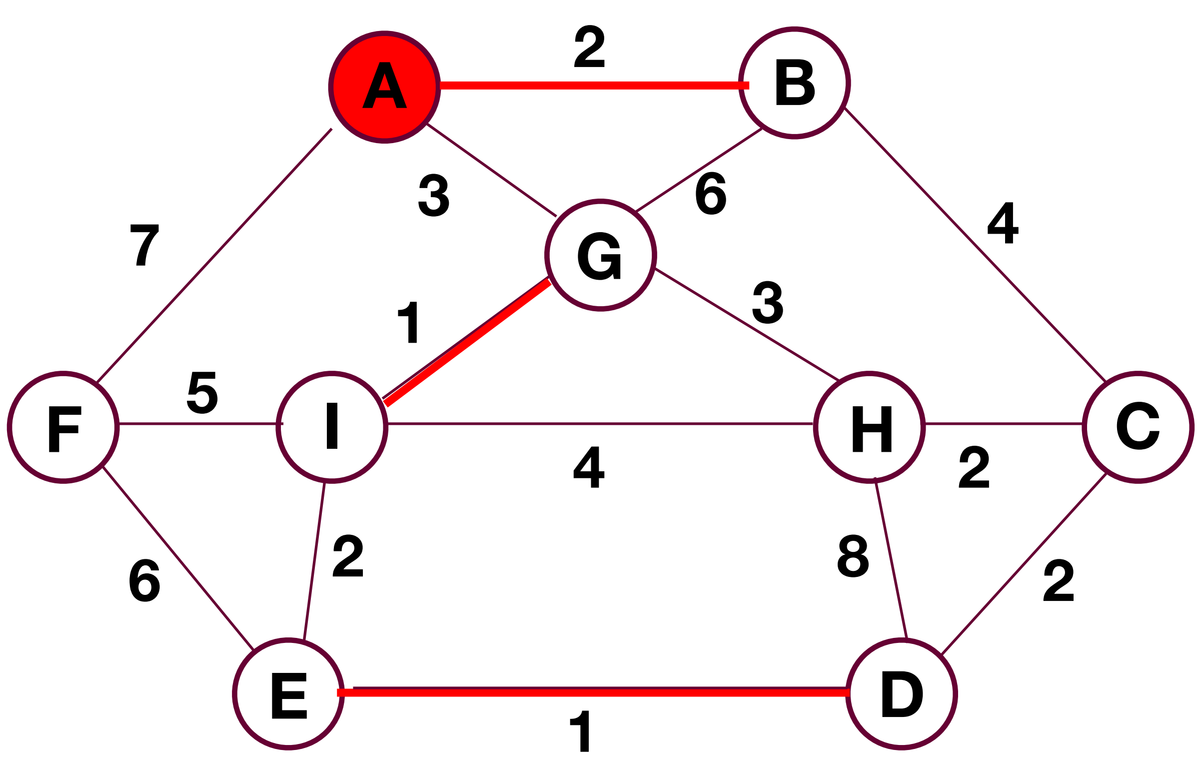

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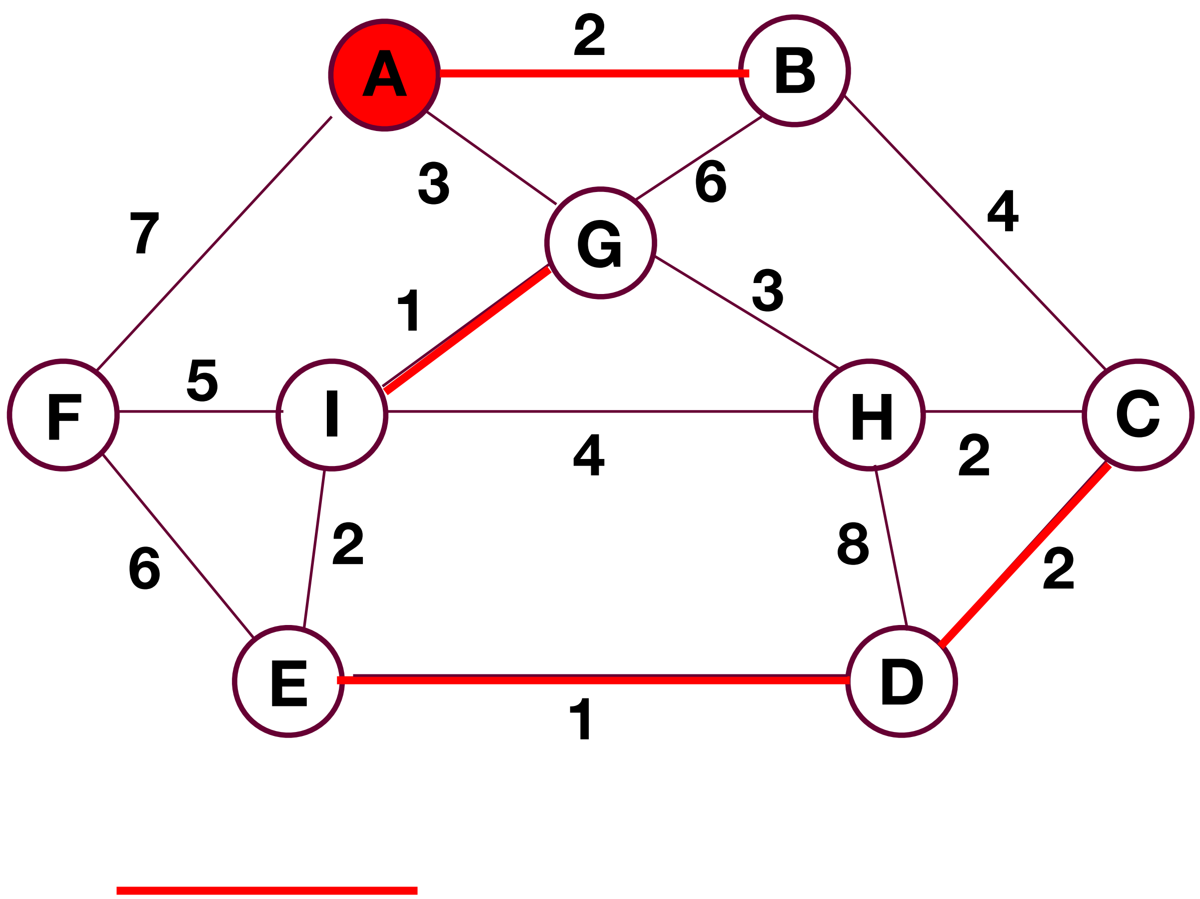

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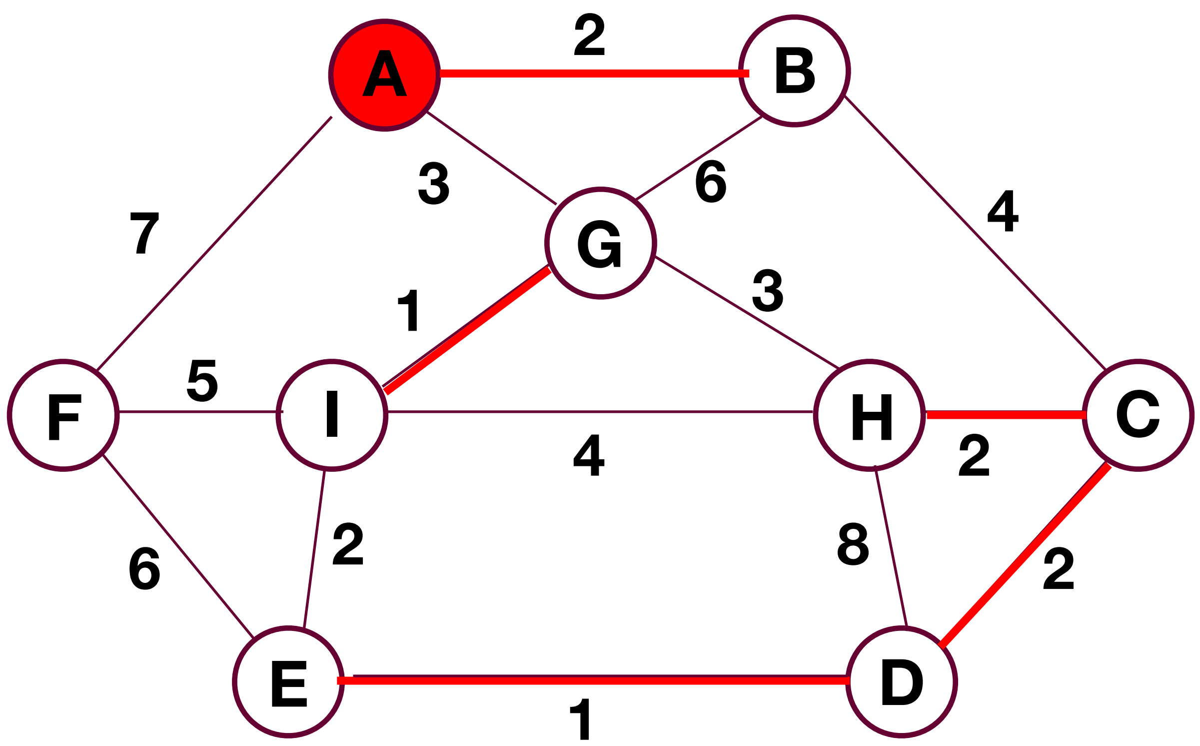

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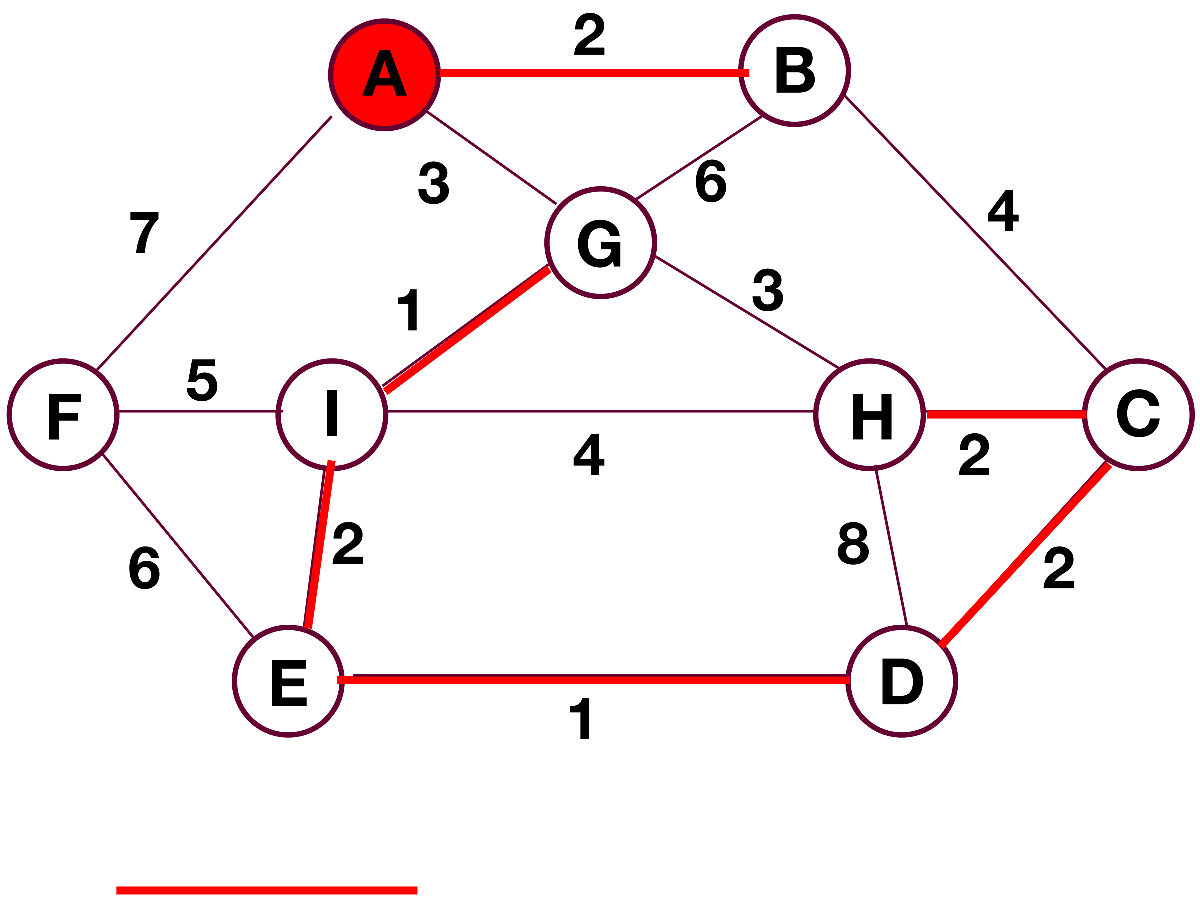

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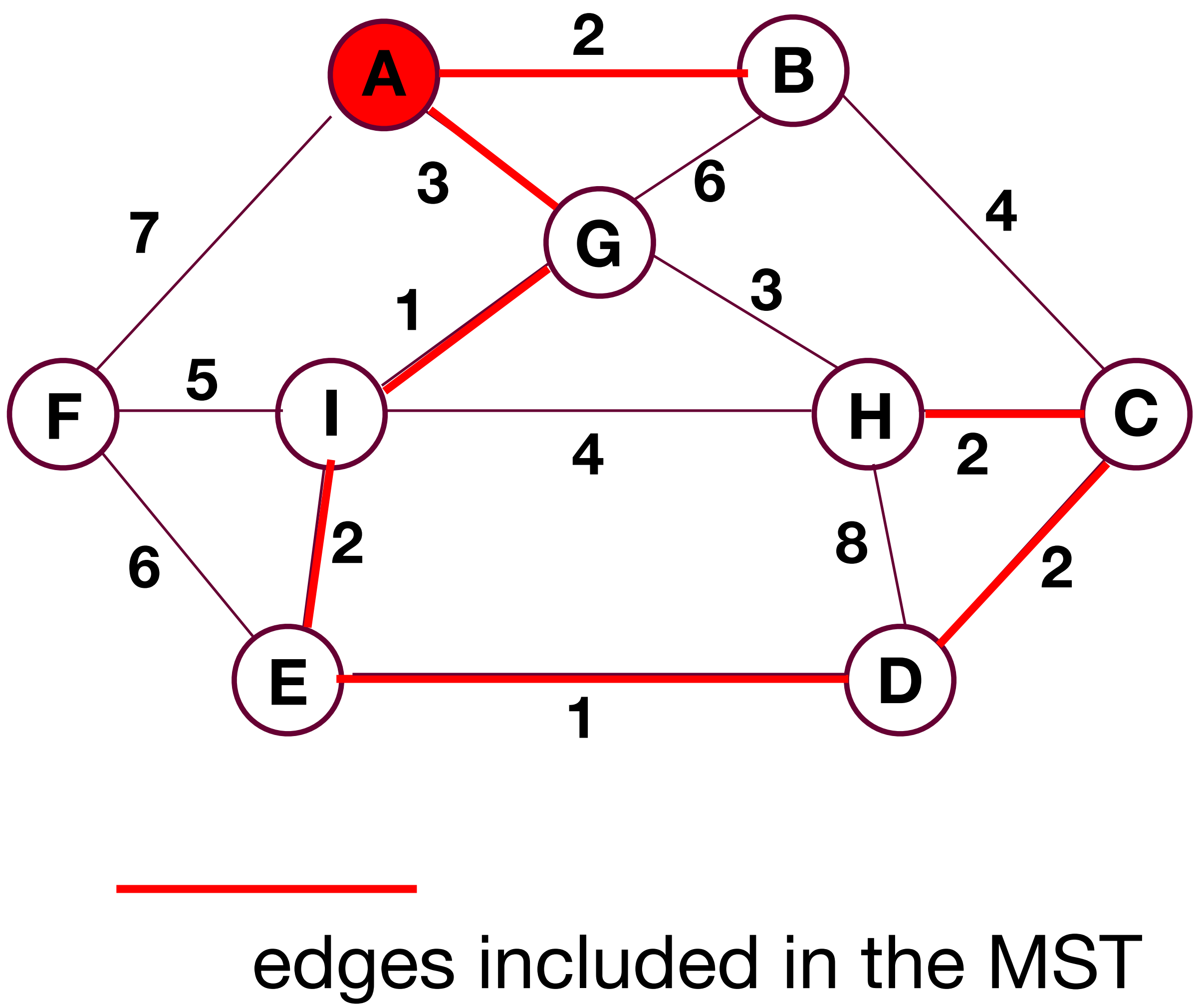

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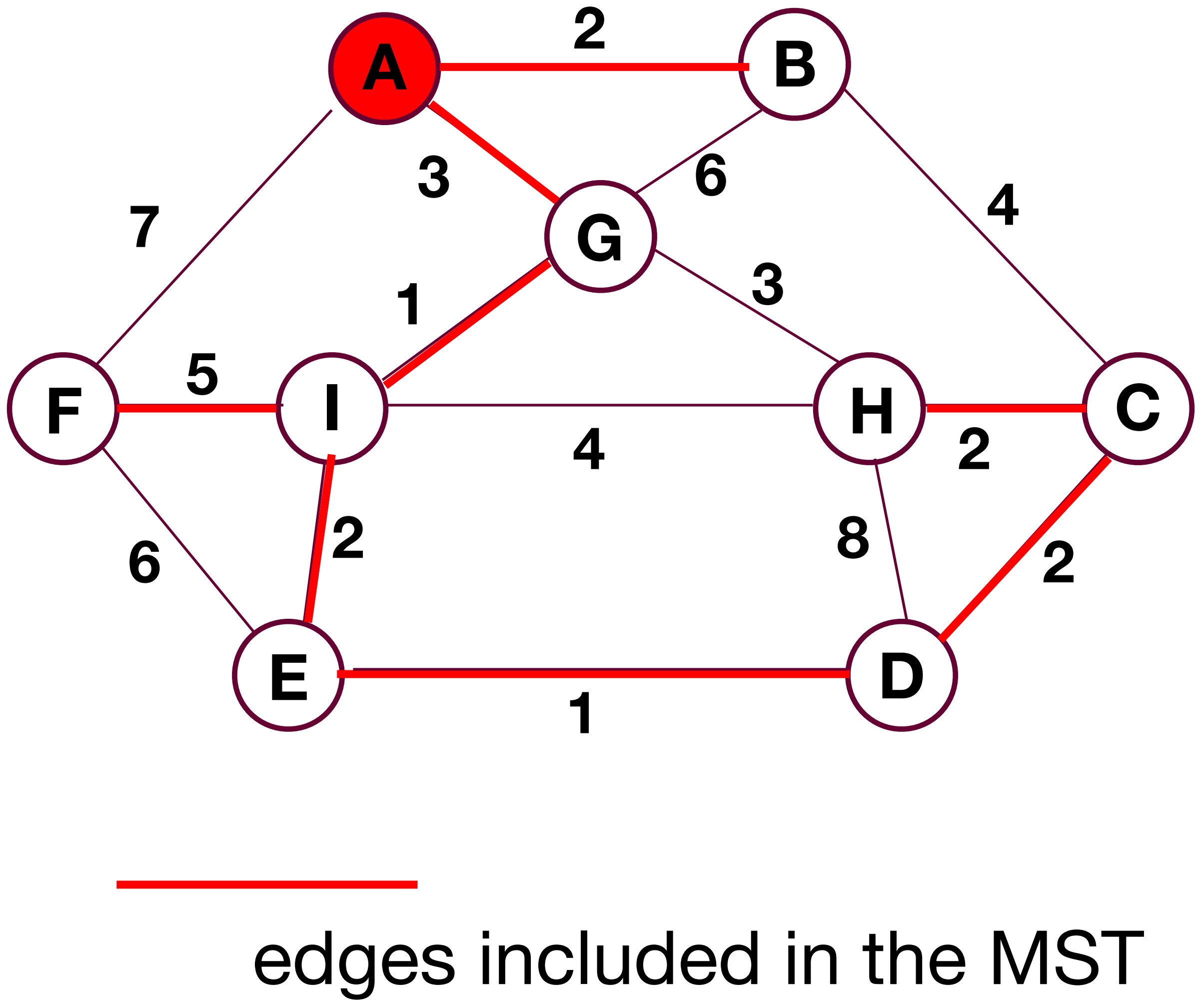

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Key Issue in Implementation

- How to know an insertion of edge will result in a cycle efficiently?
- For correctness: the two endpoints of the selected edge cannot be in the same connected components.
- For the efficiency: connected components are implemented as dynamic equivalence classes using union-find.

Kruskal's Algorithm: the Procedure

- kruskalMST(G,n,F) //outline
- **int** count;
- Build a minimizing priority queue, pq, of edges of G, prioritized by weight.
- Initialize a Union-Find structure, sets, in which each vertex of G is in its own set.
-
- F=ϕ;
- **while** (isEmpty(pq) == **false**)
- vwEdge = getMin(pq);
- deleteMin(pq);
- **int** vSet = find(sets, vwEdge.from);
- **int** wSet = find(sets, vwEdge.to);
- **if** (vSet ≠ wSet)
- Add vwEdge to F;
- union(sets, vSet, wSet)
- **return**

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Simply sorting, the cost will be Θ(mlogm)

Prim vs. Kruskal

- **Lower bound for MST**
  - For a correct MST, each edge I the graph should be examined at least once.
  - So, the lower bound is $\Omega(m)$.
- **$\Theta(n^2+m)$ and $\Theta(m\log m)$, which is better?**
  - Generally speaking, depends on the density of edge of the graph.

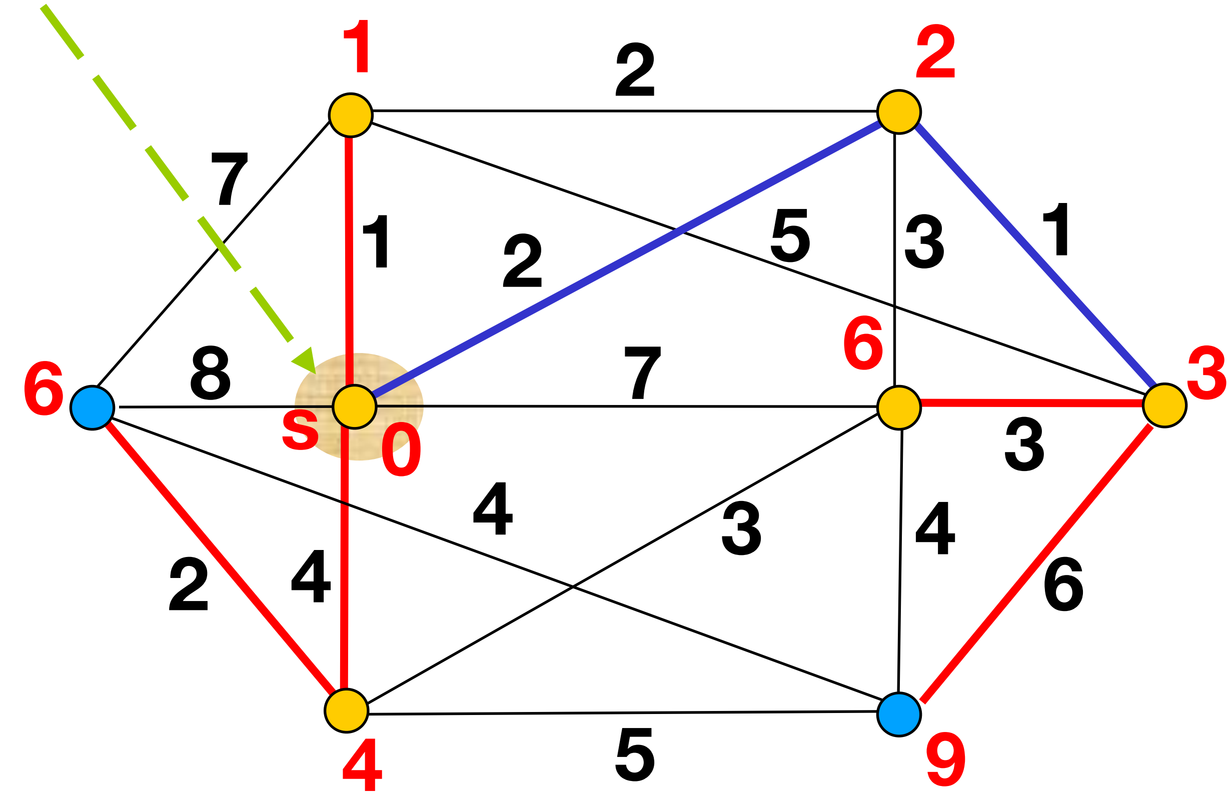
Single Source Shortest Paths

Red labels on each vertex is the length of the shortest path from s to the vertex.

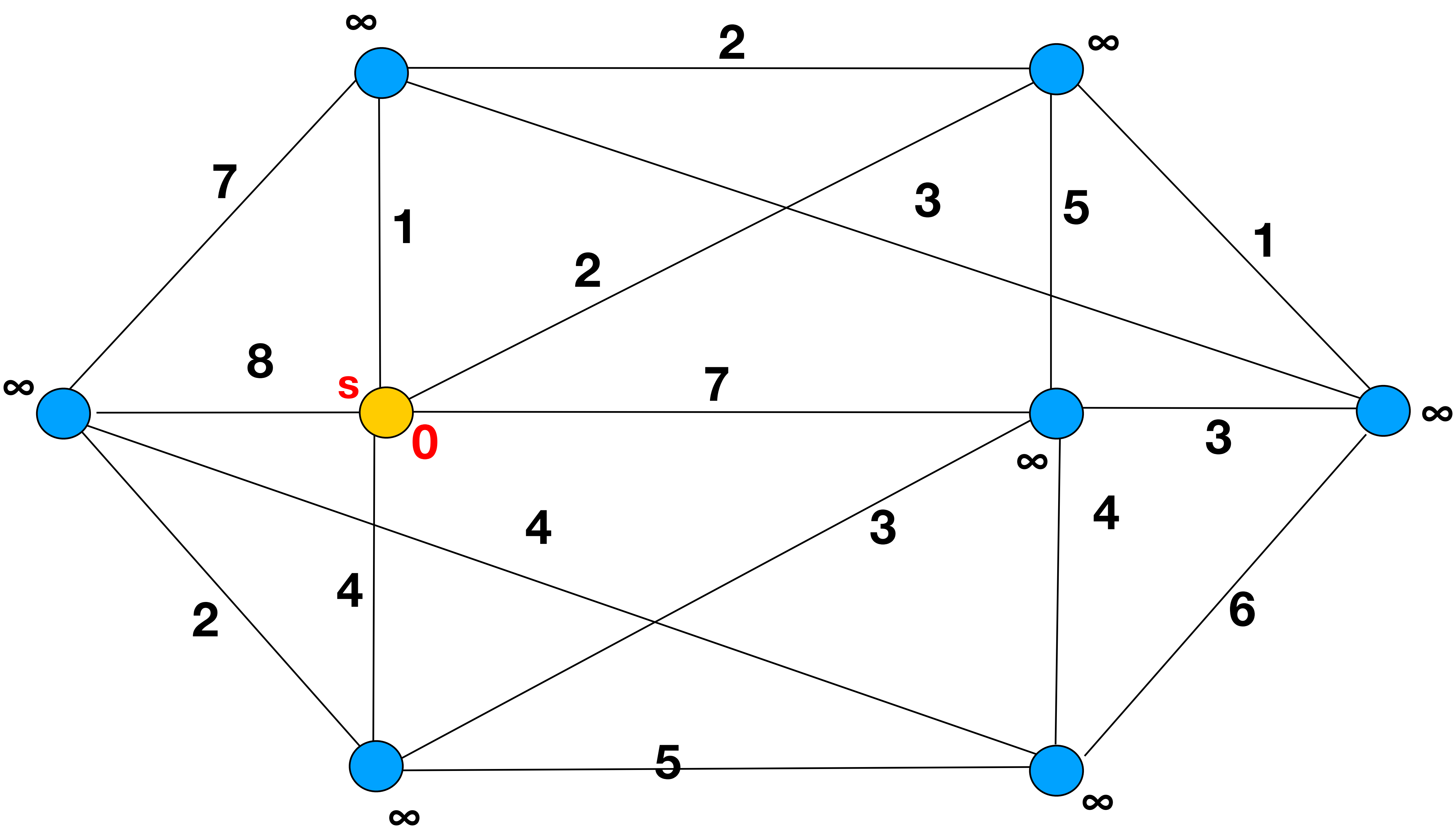
The single source

Note:

The shortest [0, 3]-path doesn't contain the shortest edge leaving s, the edge [0,1]



Dijstra's Algorithm: an Example



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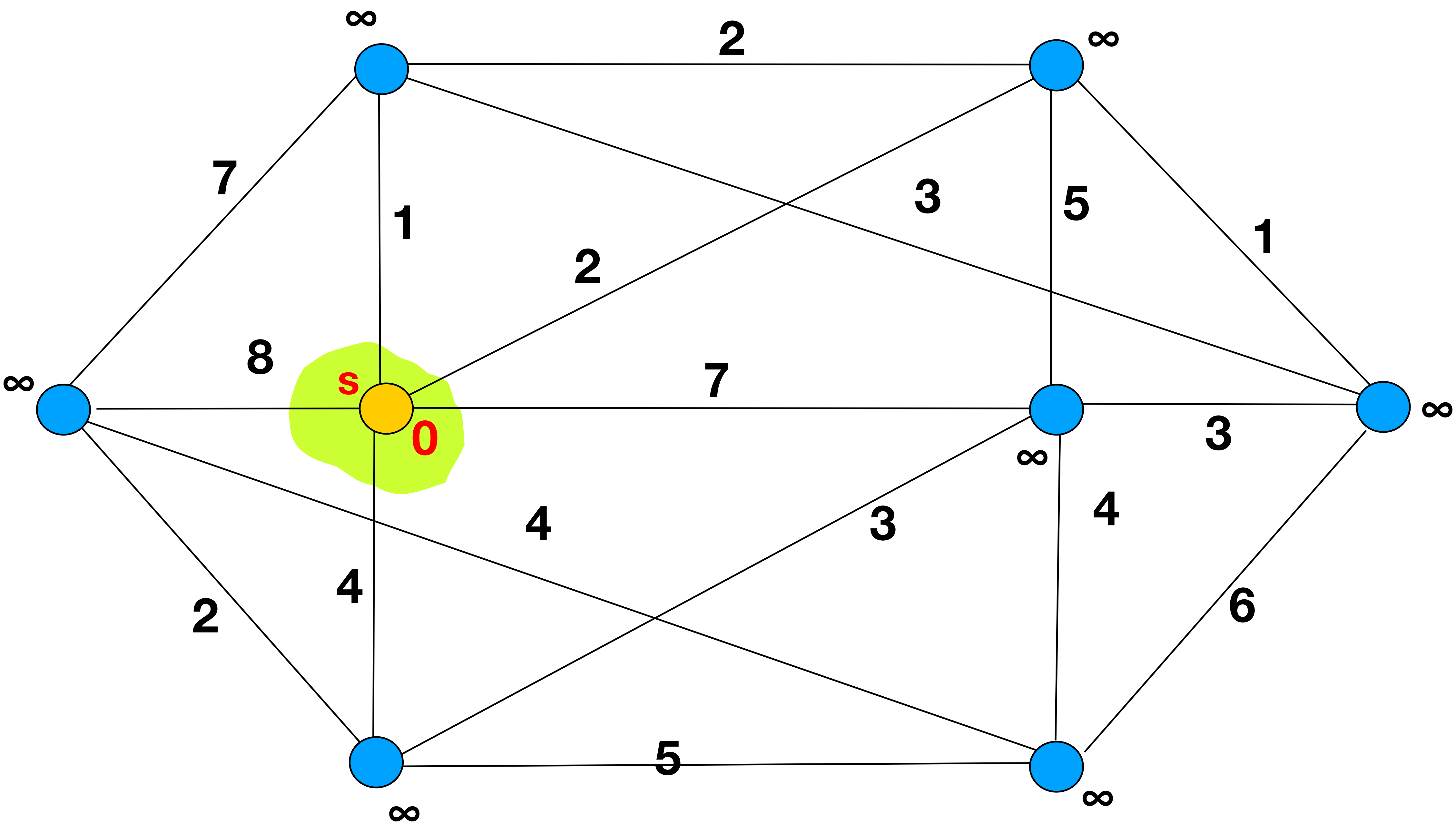

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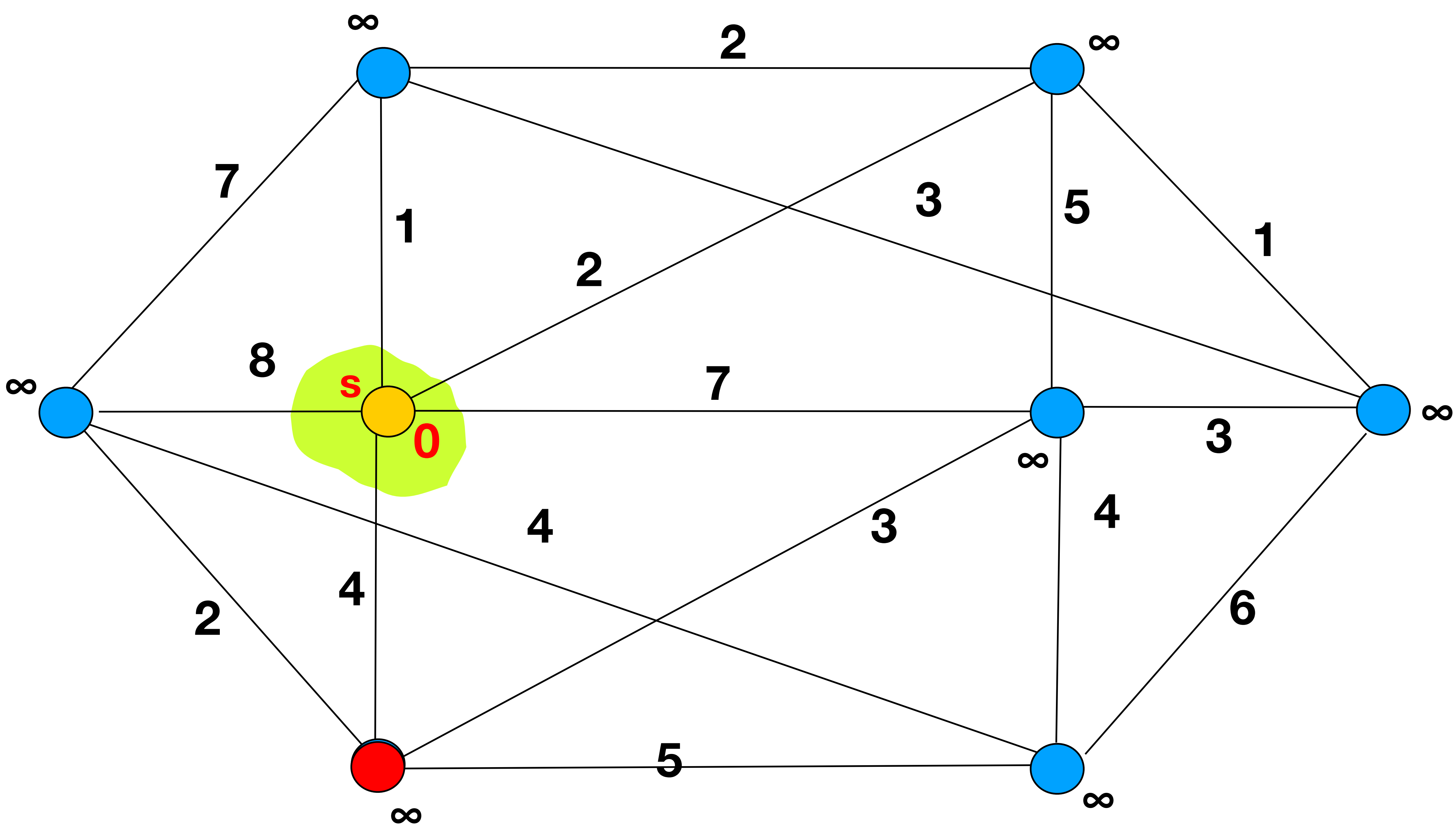

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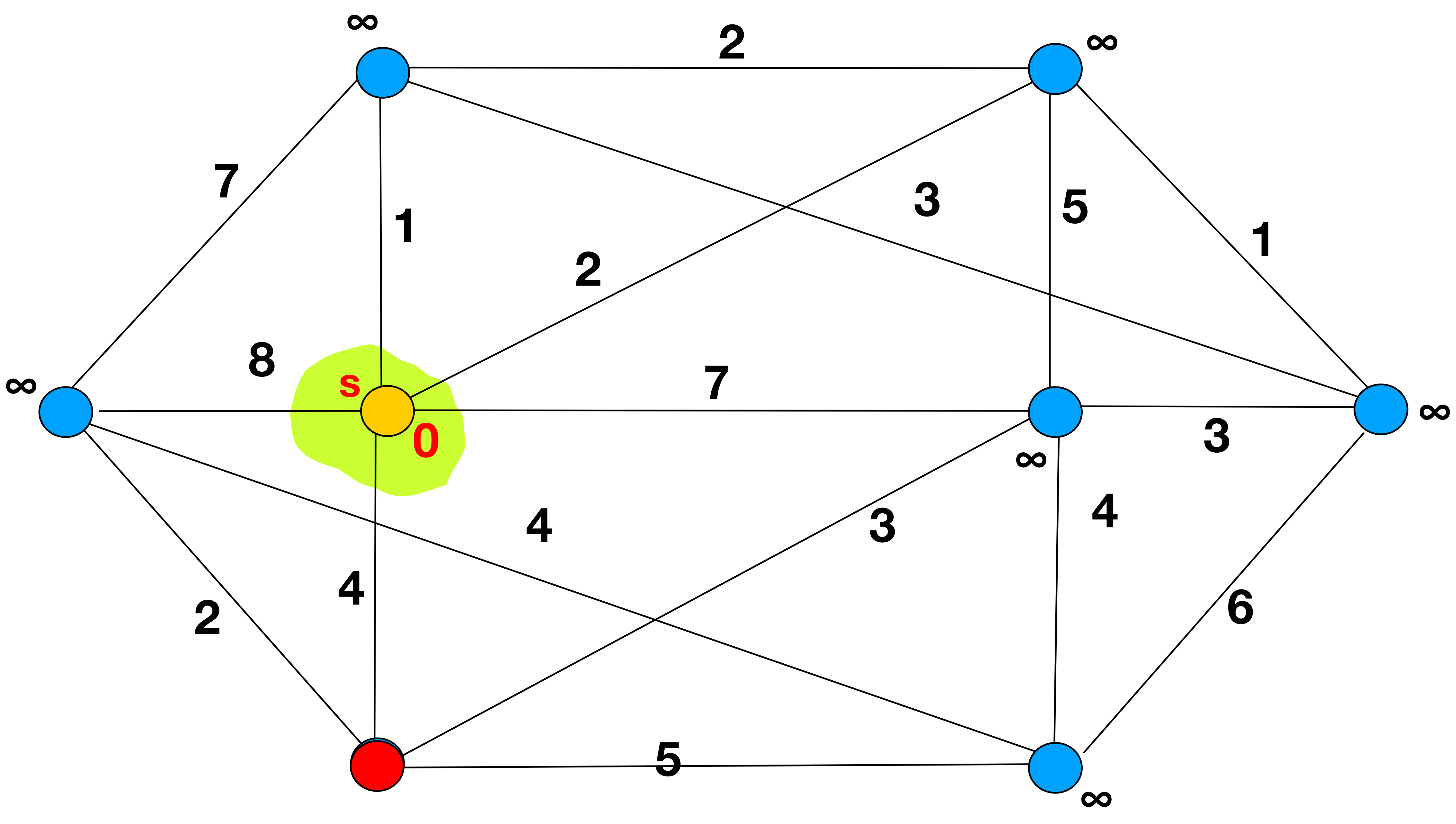

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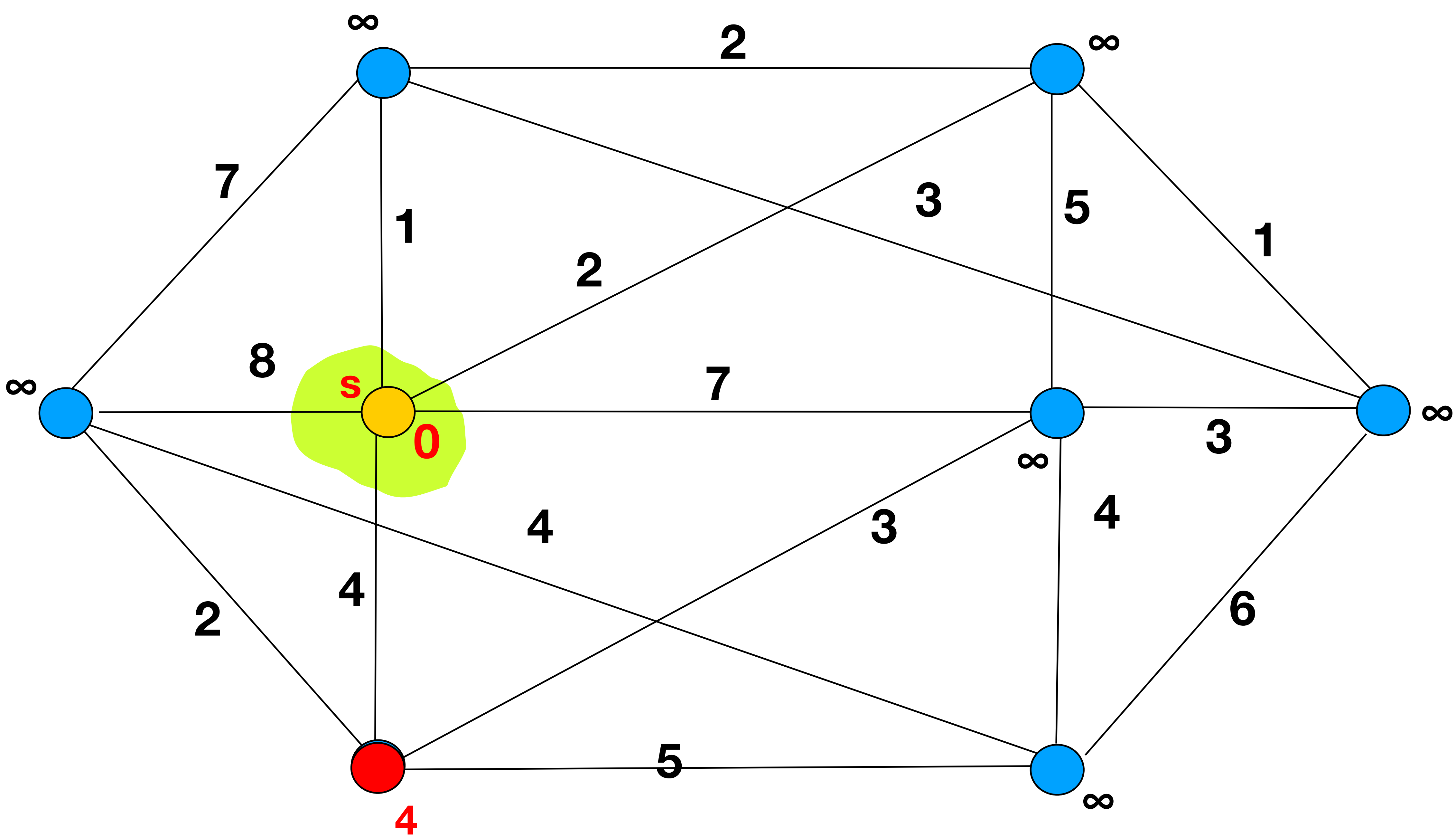

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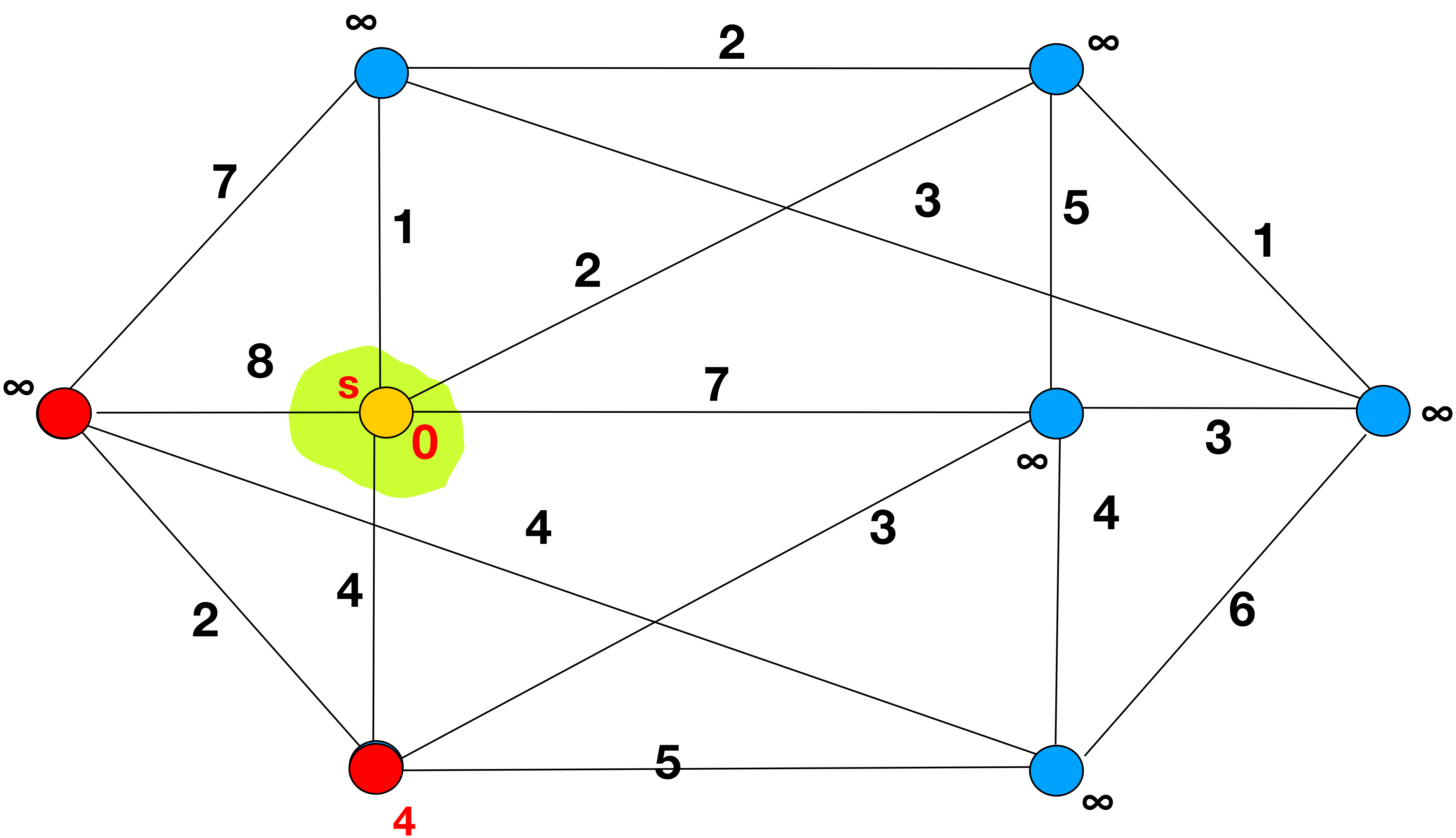

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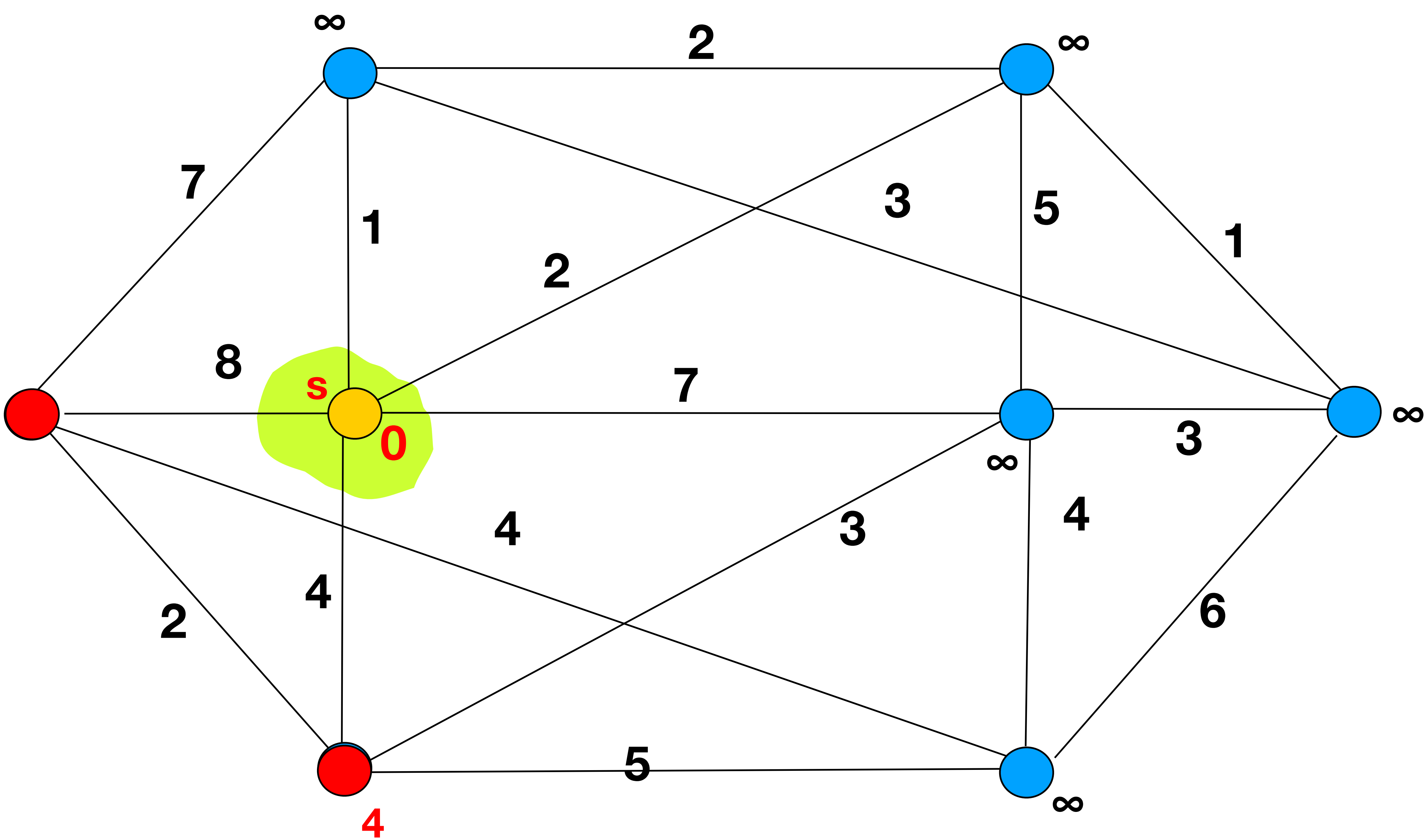

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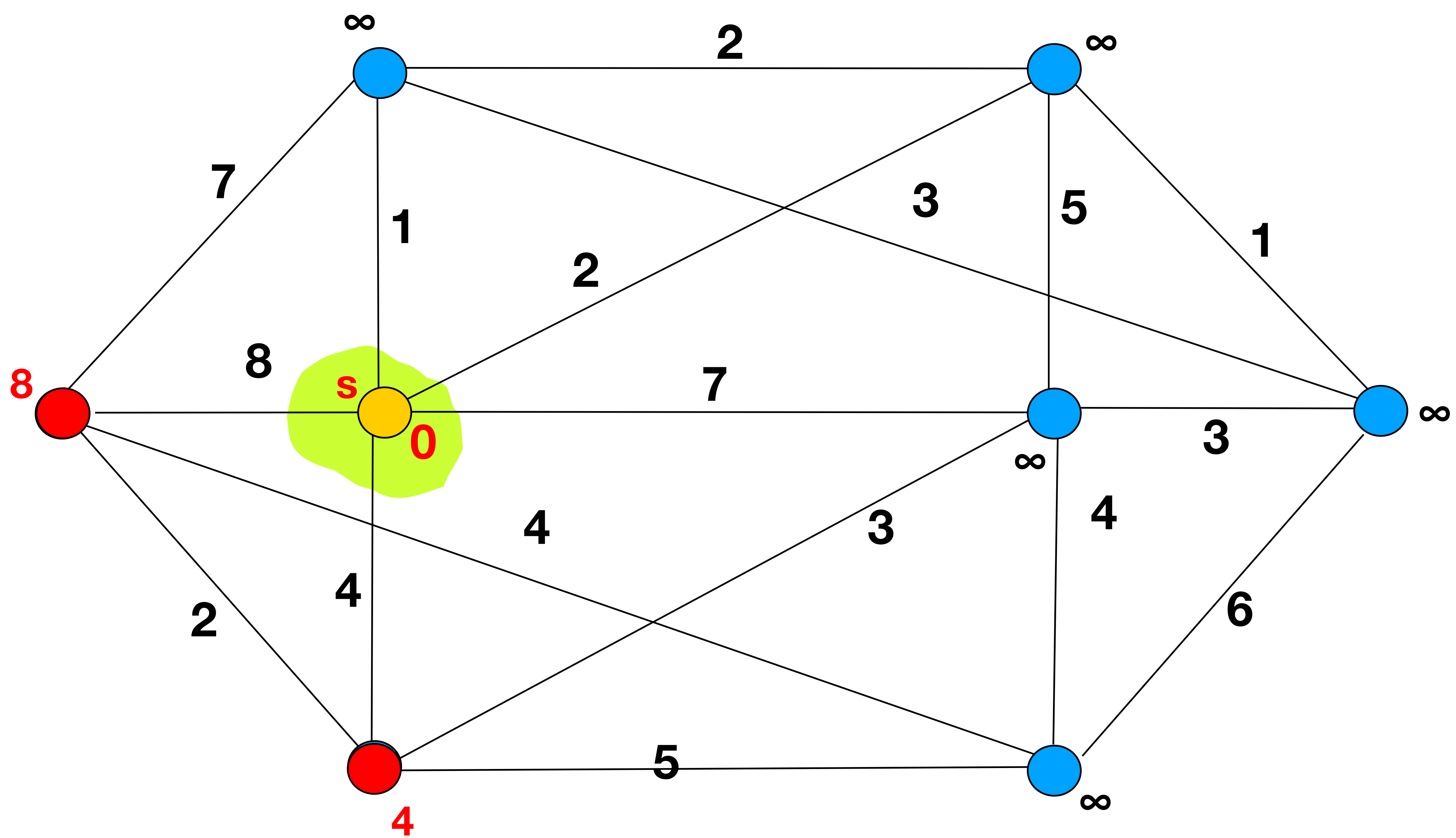

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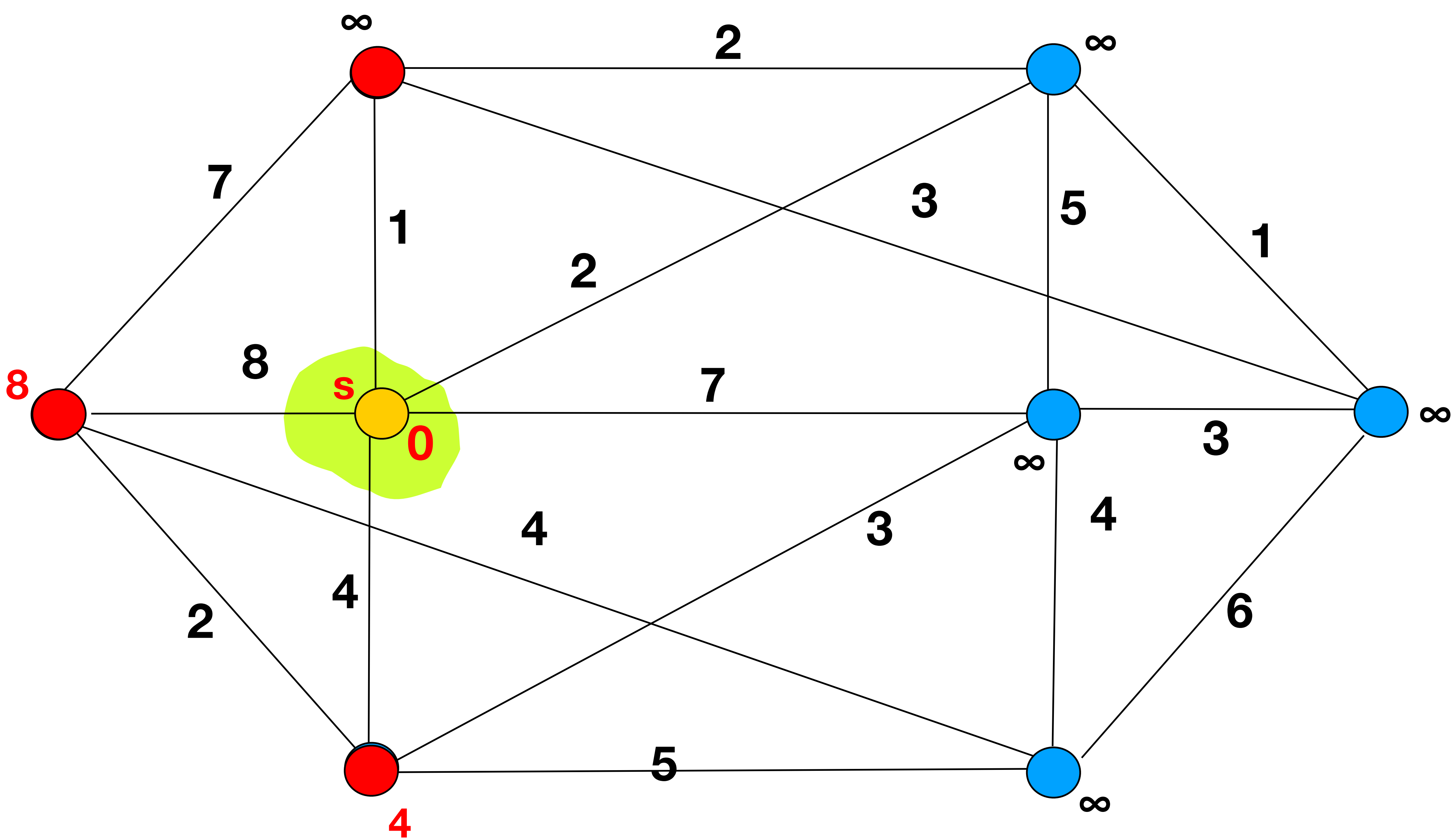

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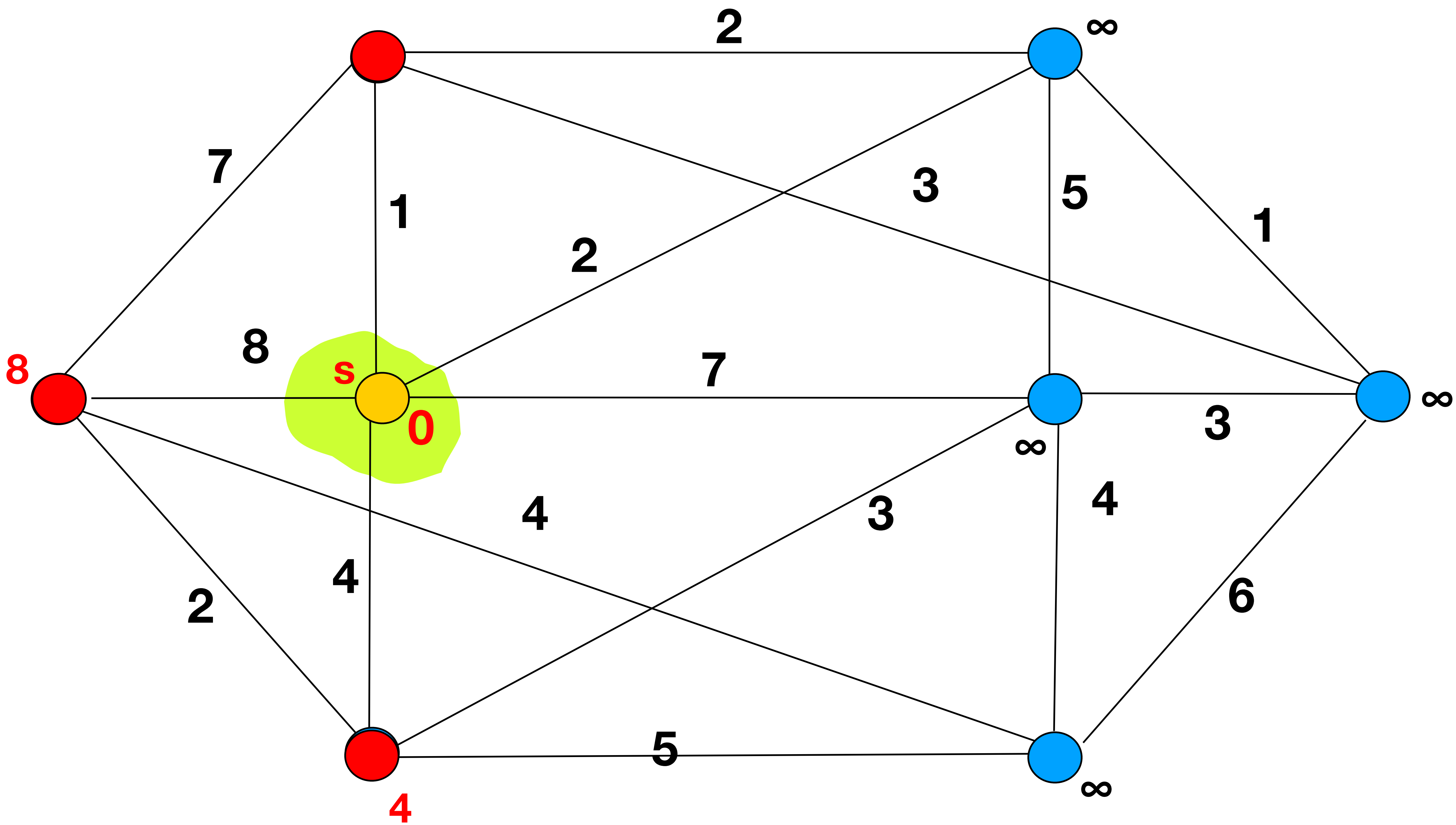

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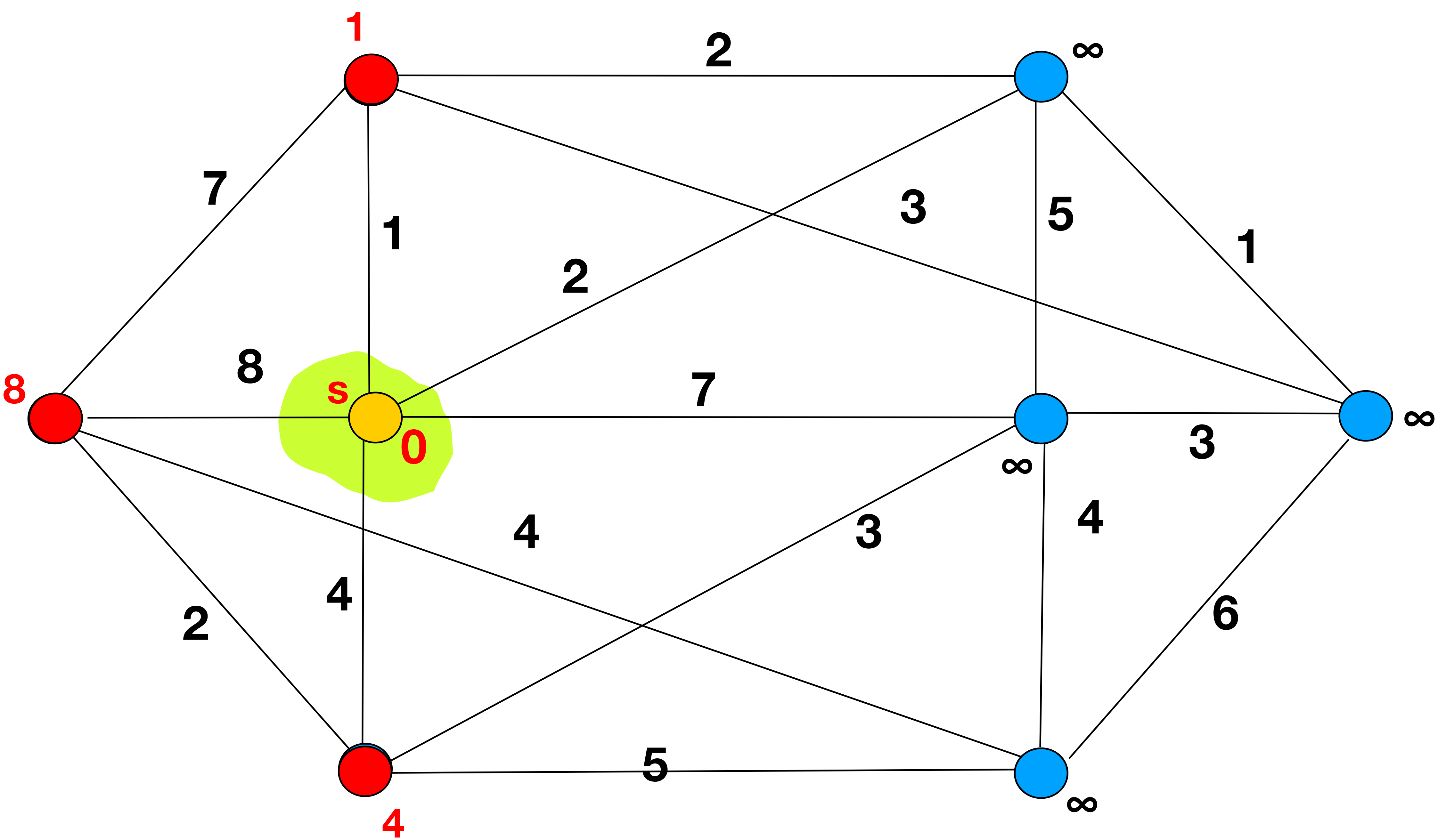

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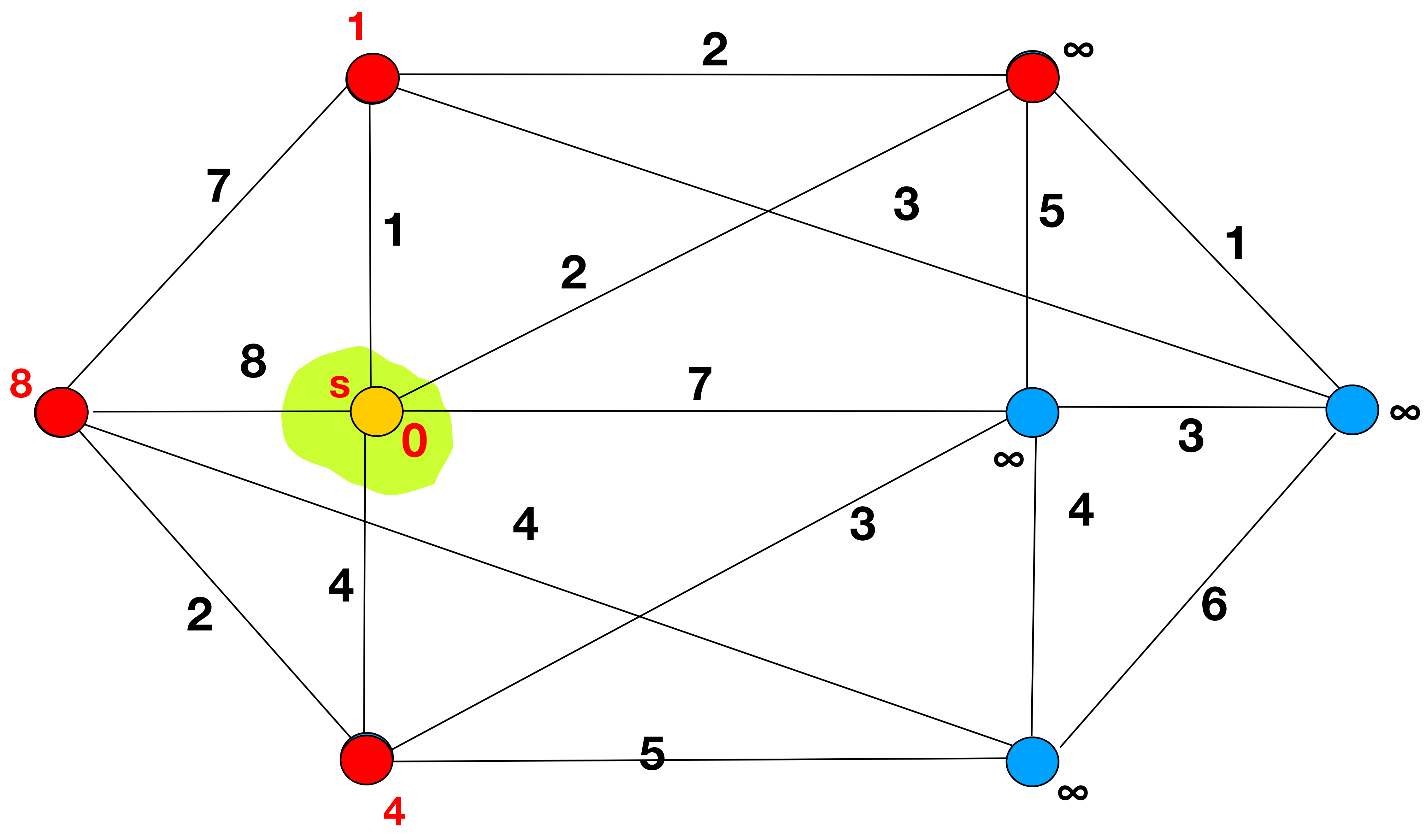

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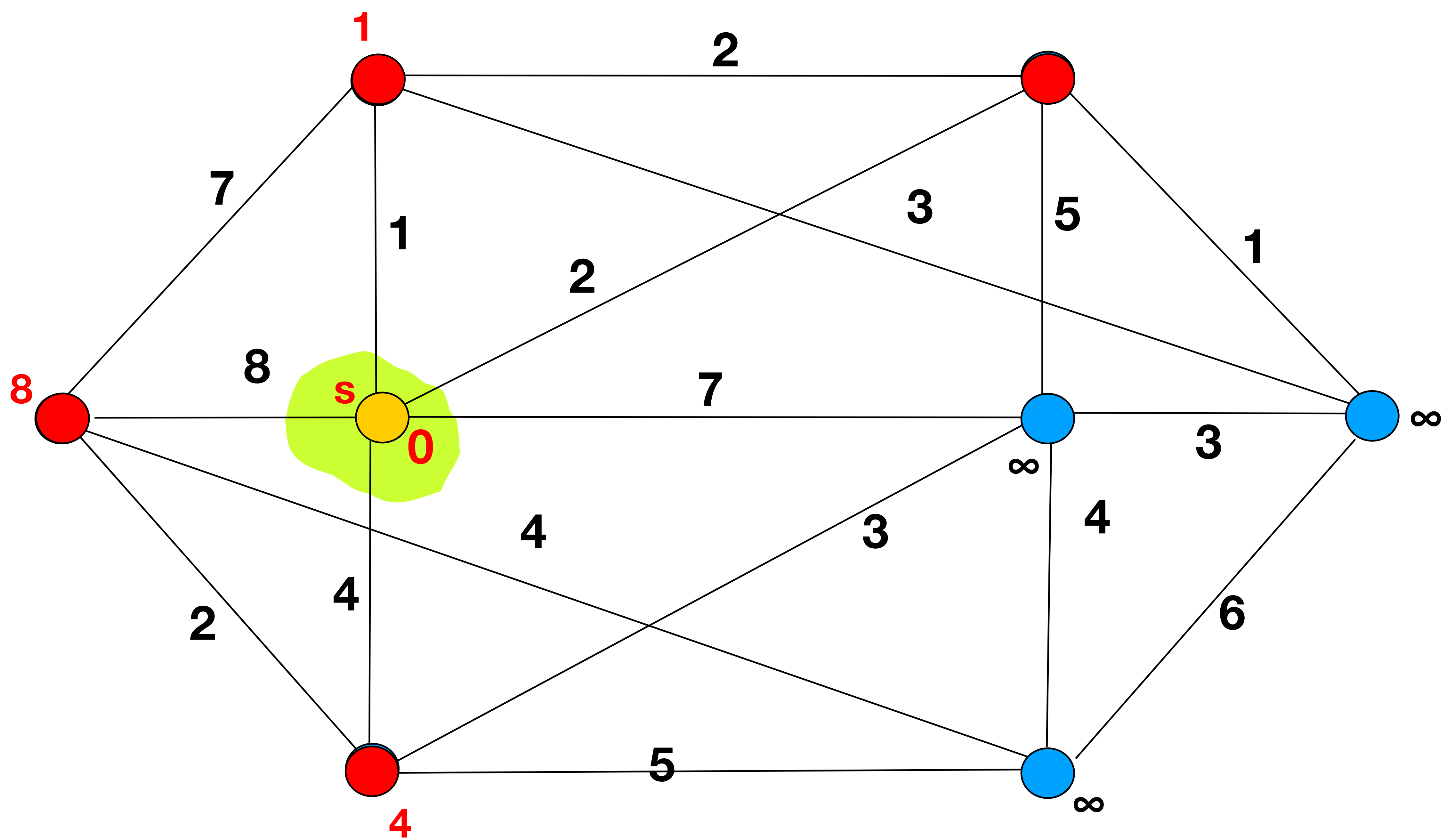

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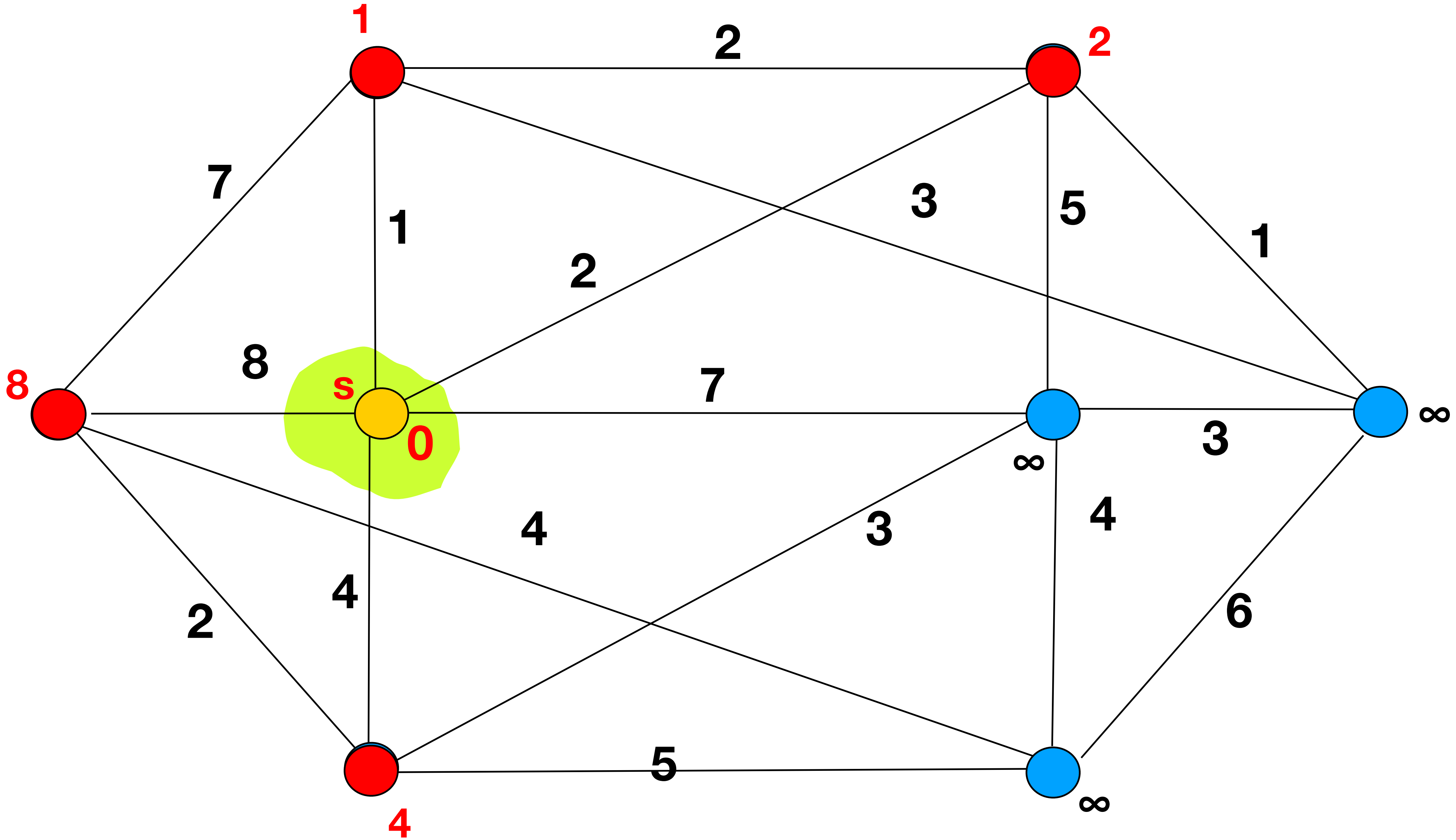

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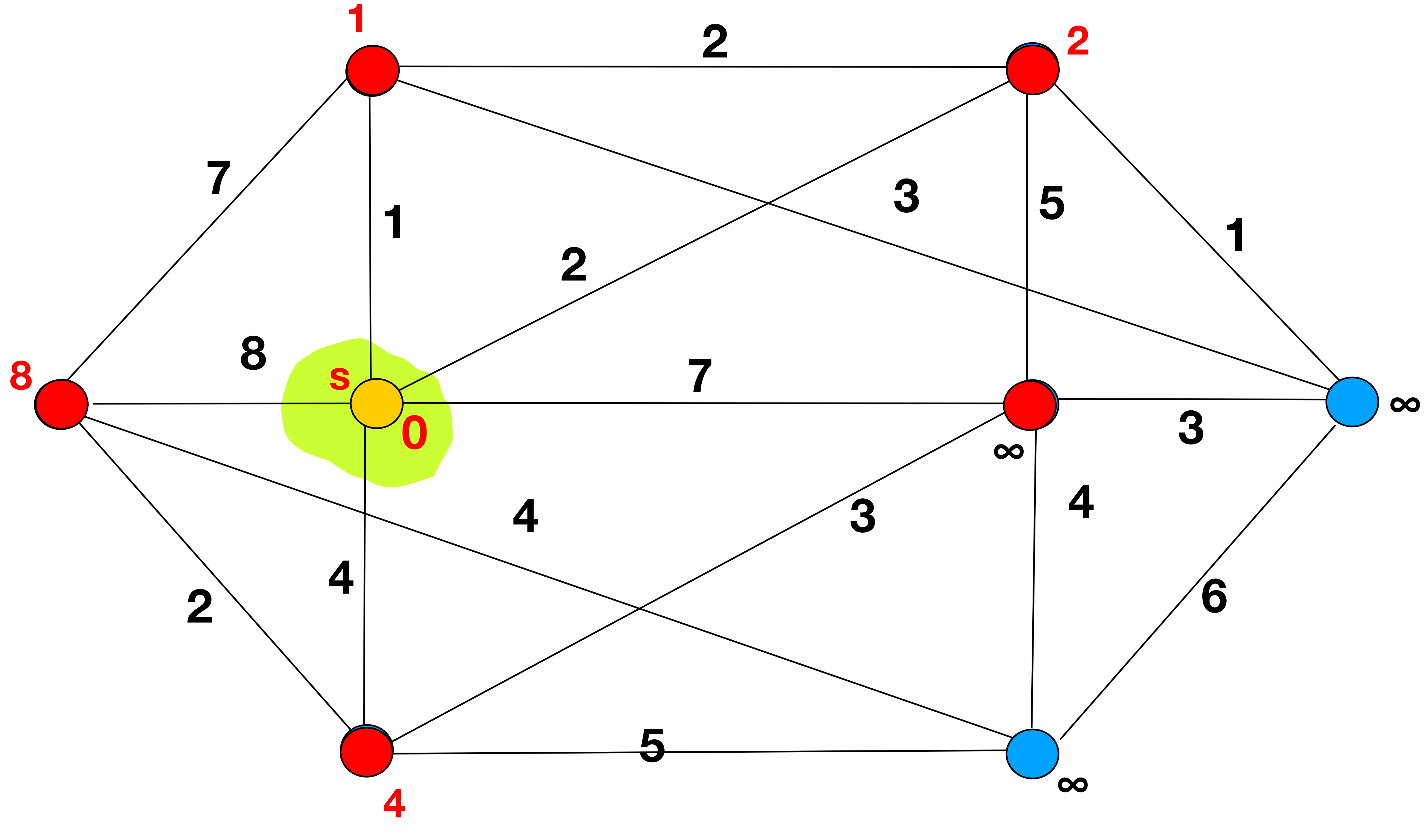

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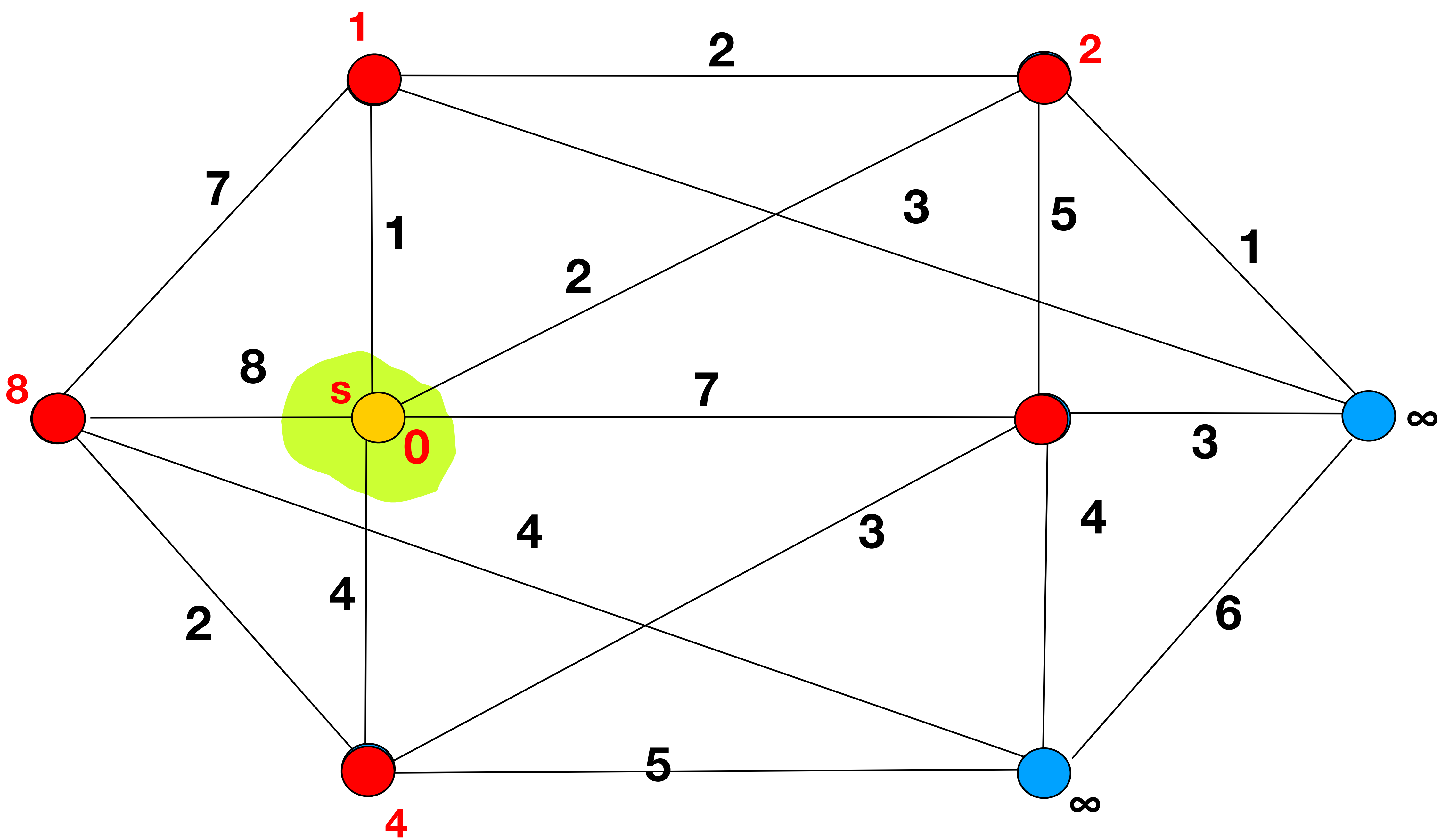

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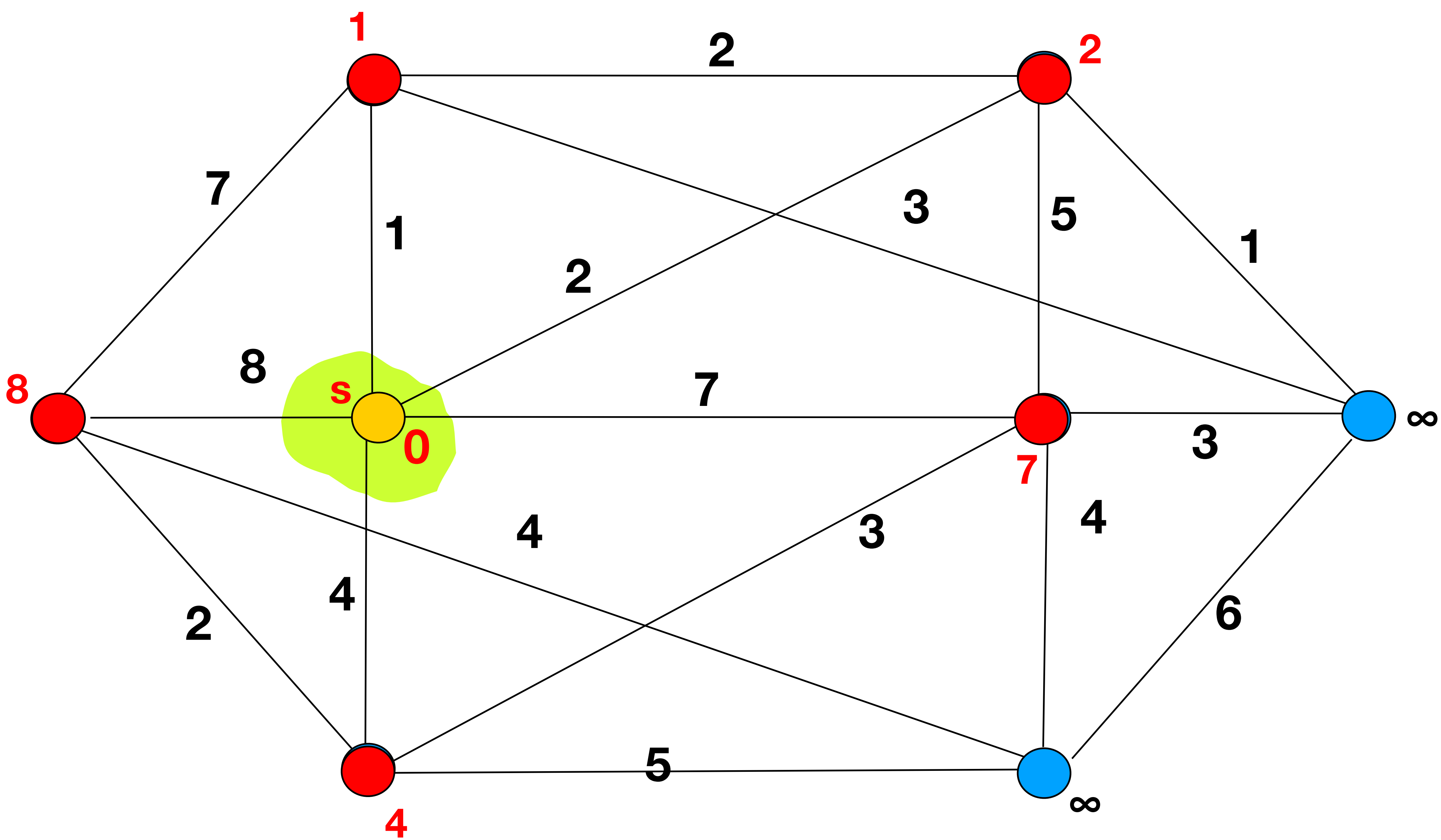
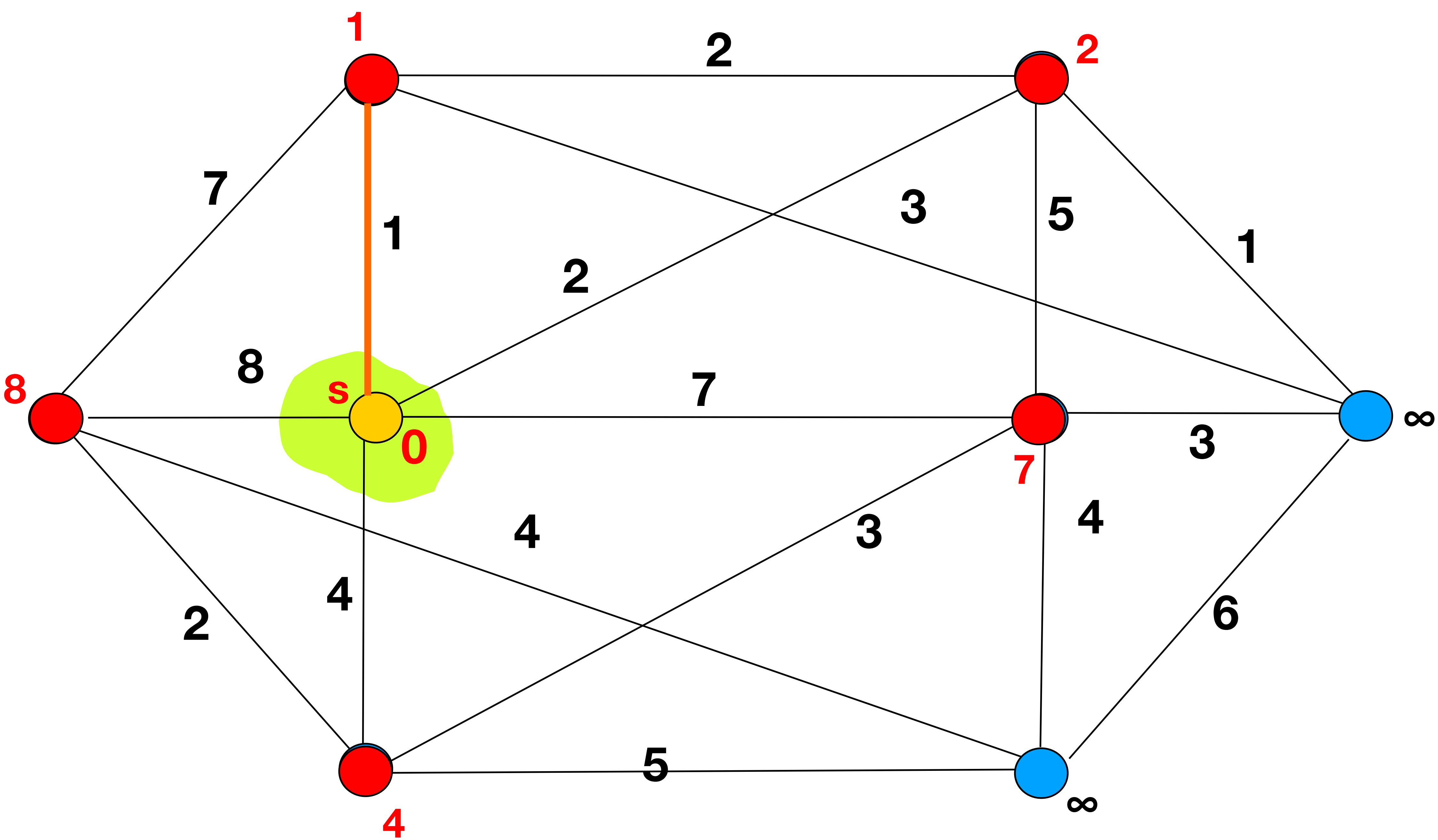
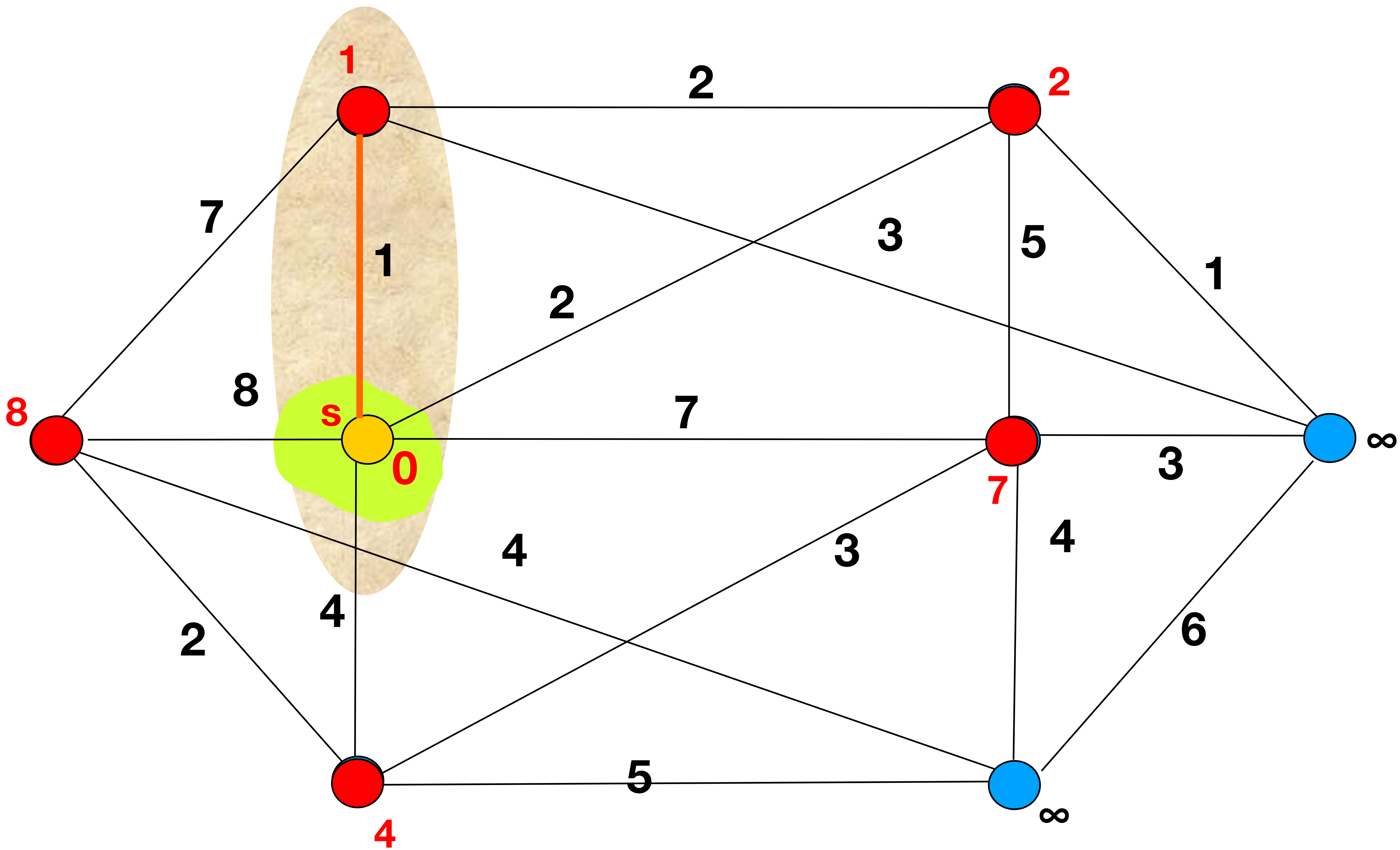


Dijstra's Algorithm: an Example



Dijstra's Algorithm: an Example



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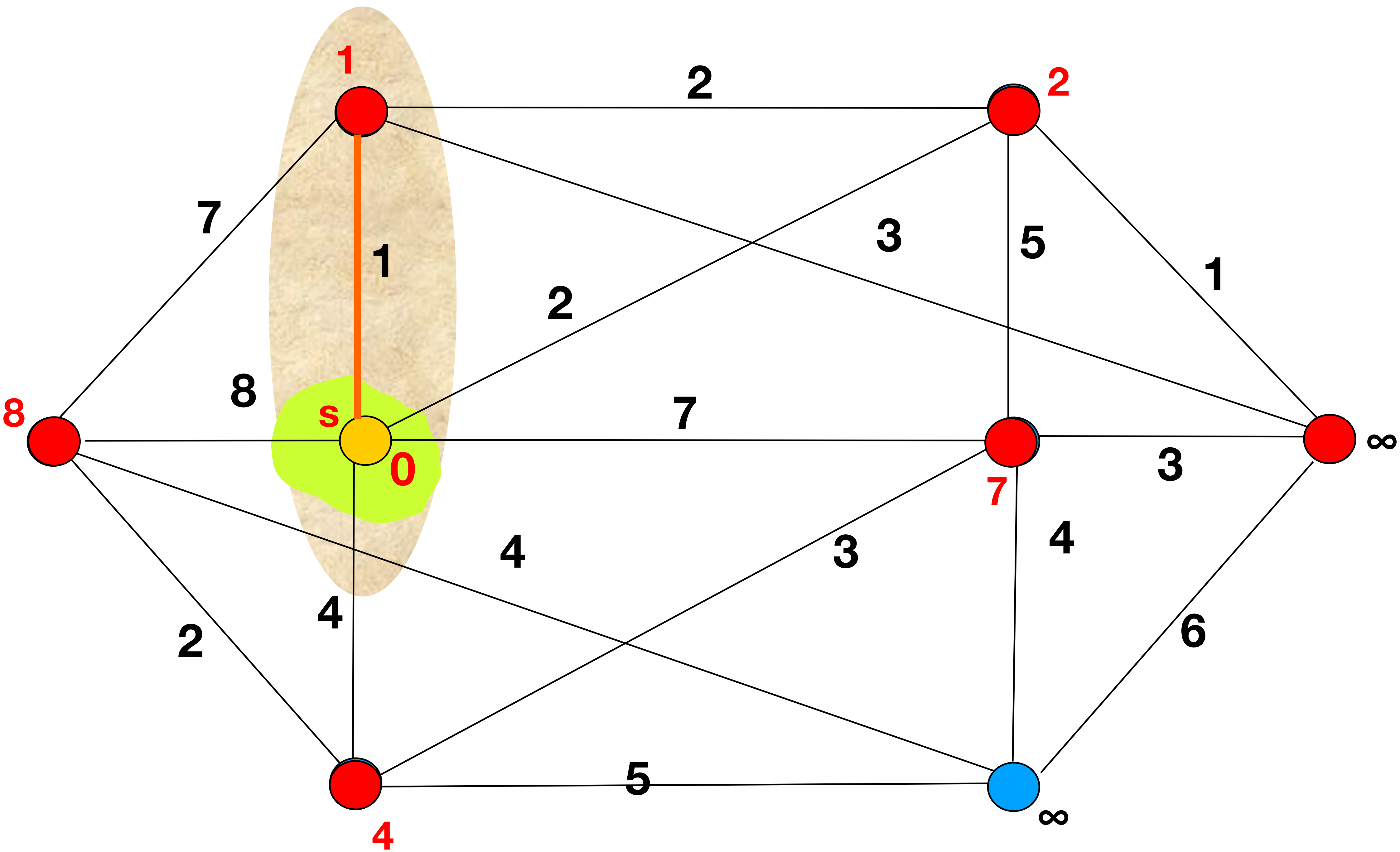

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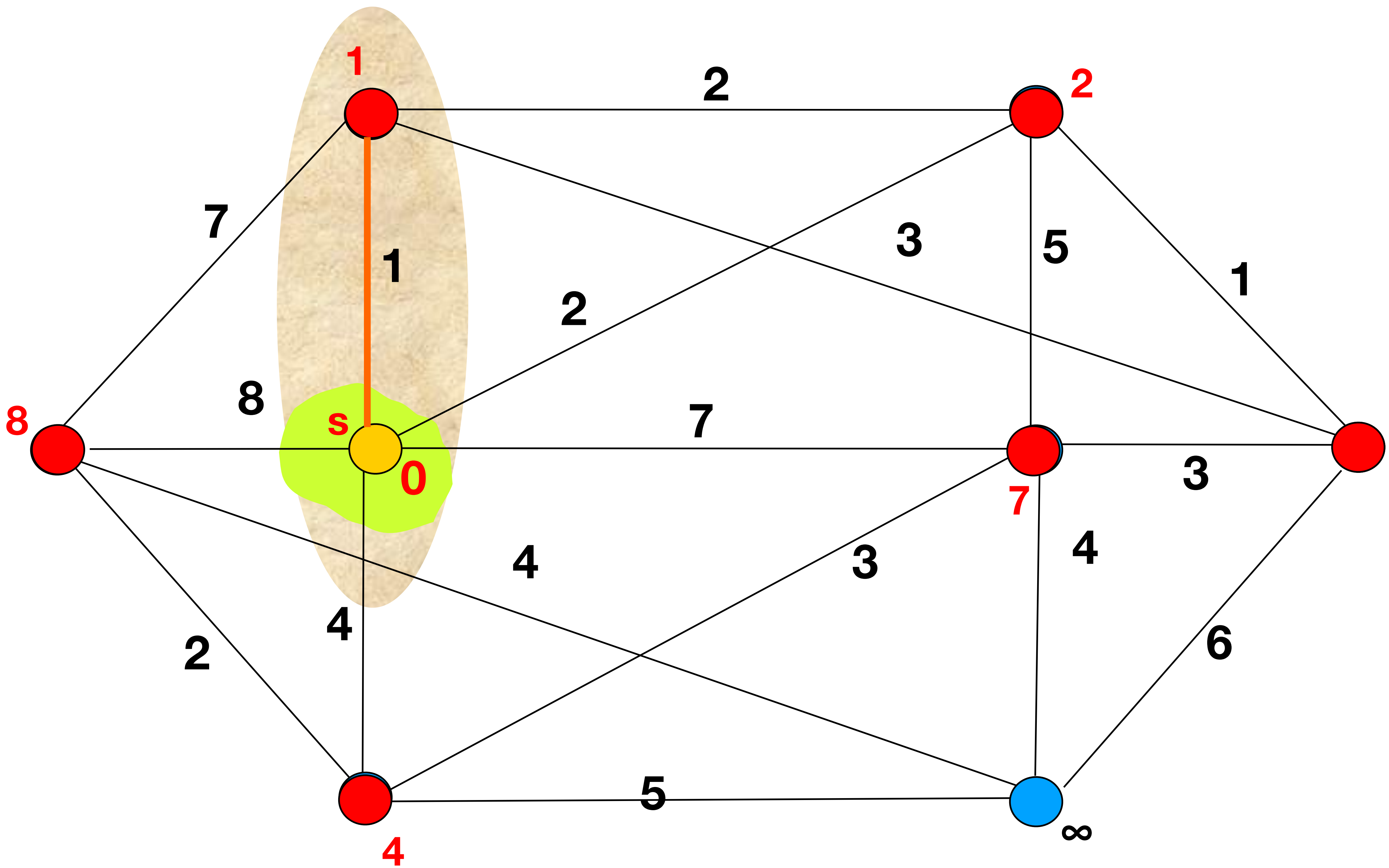

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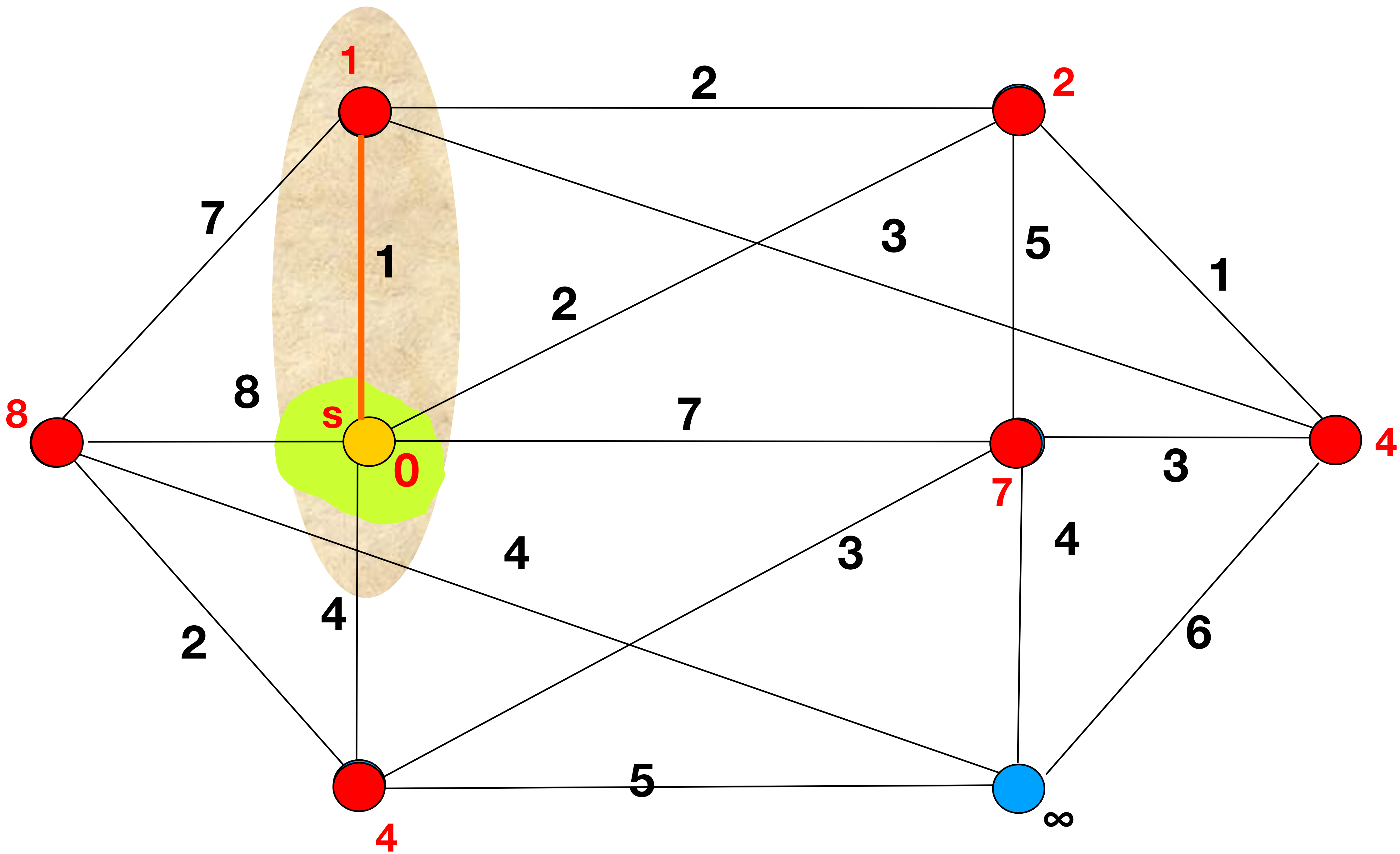

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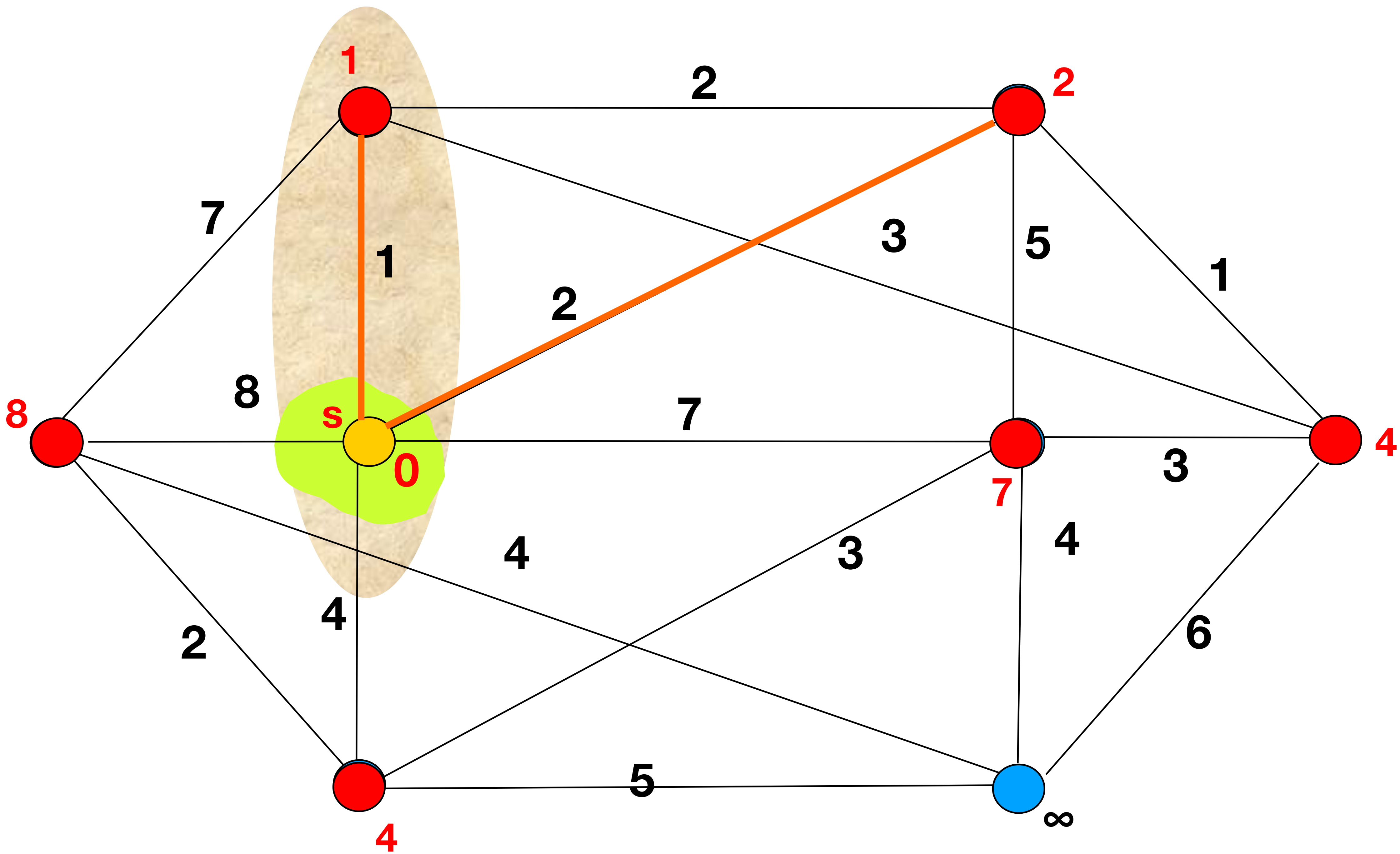

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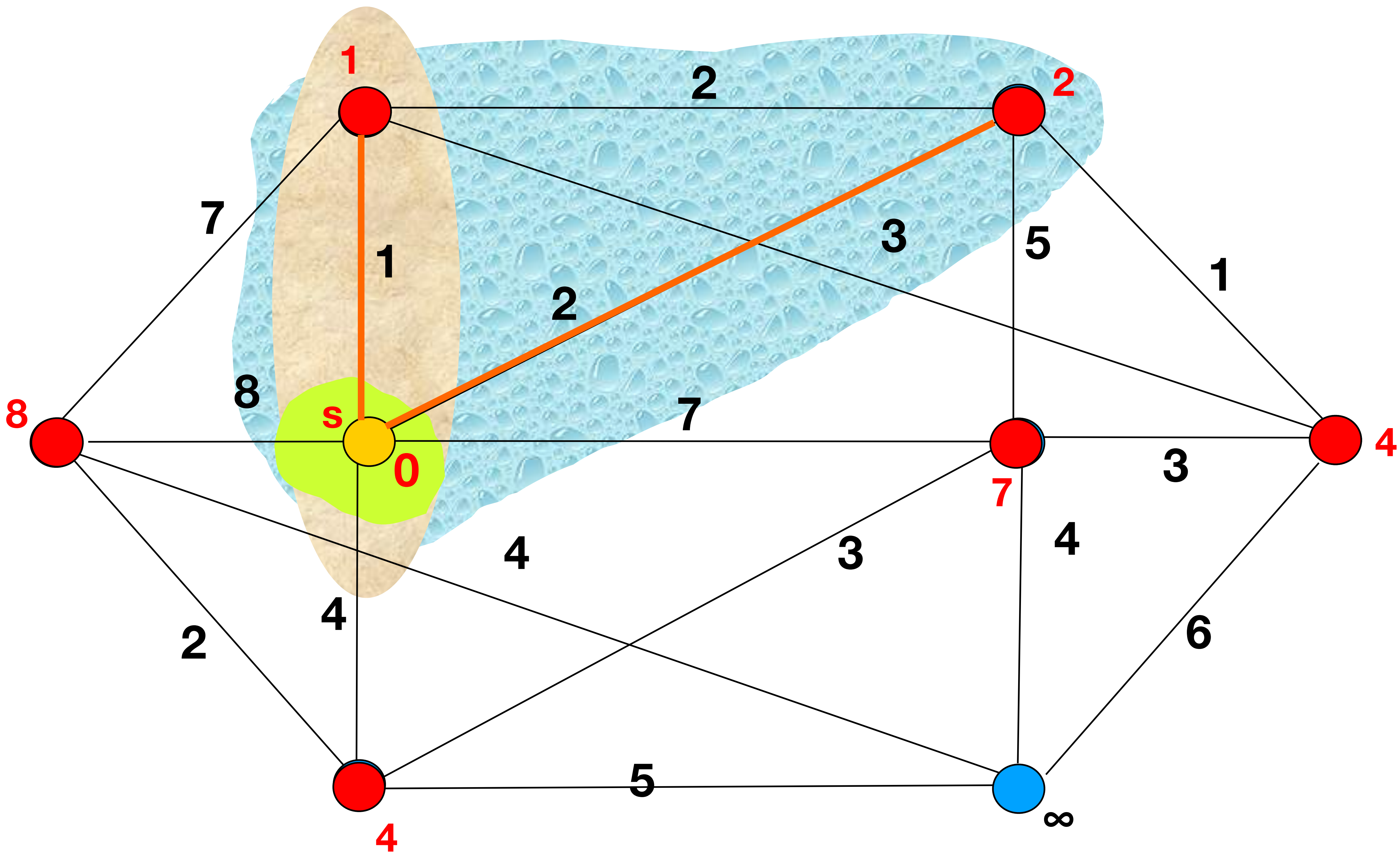

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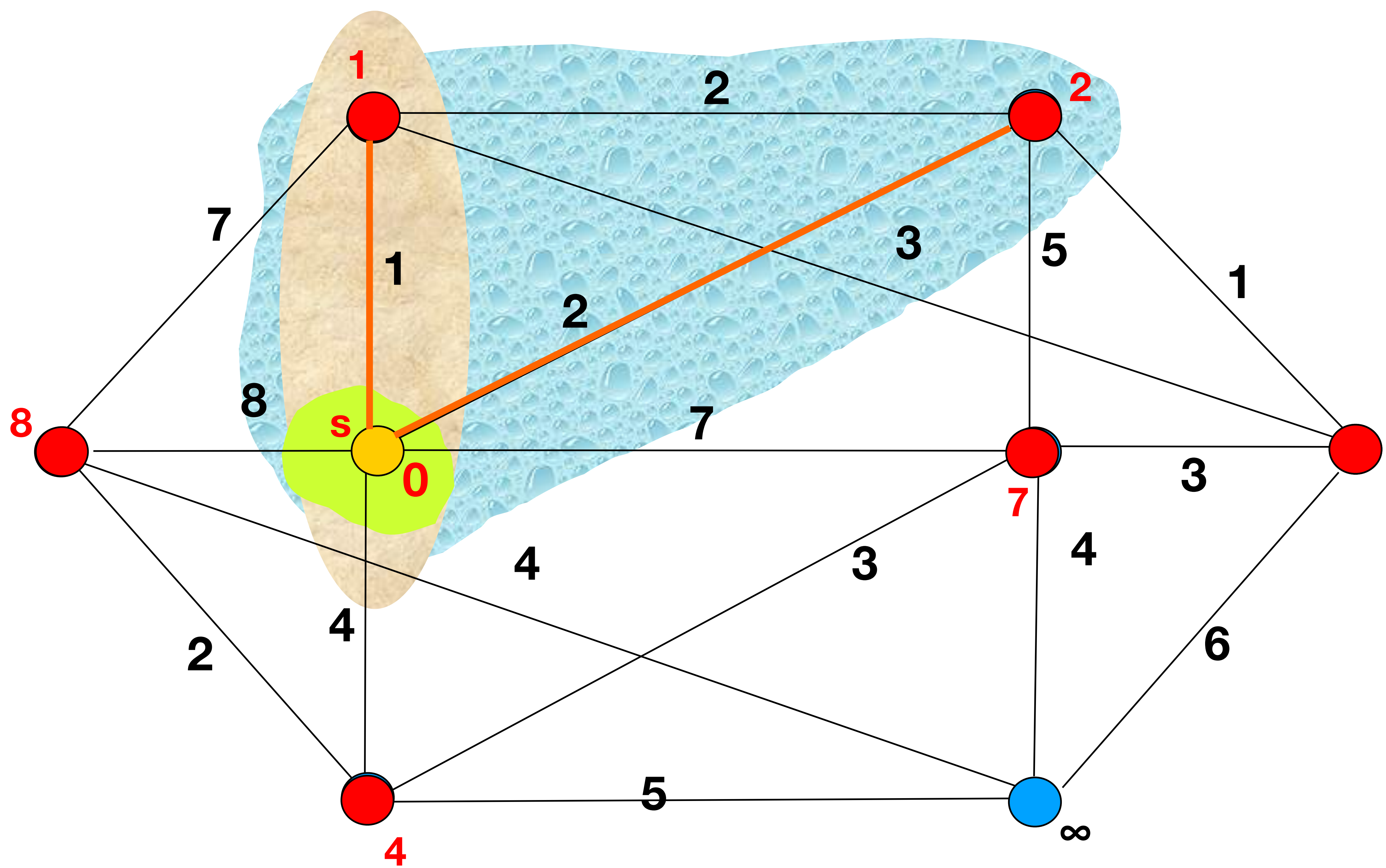

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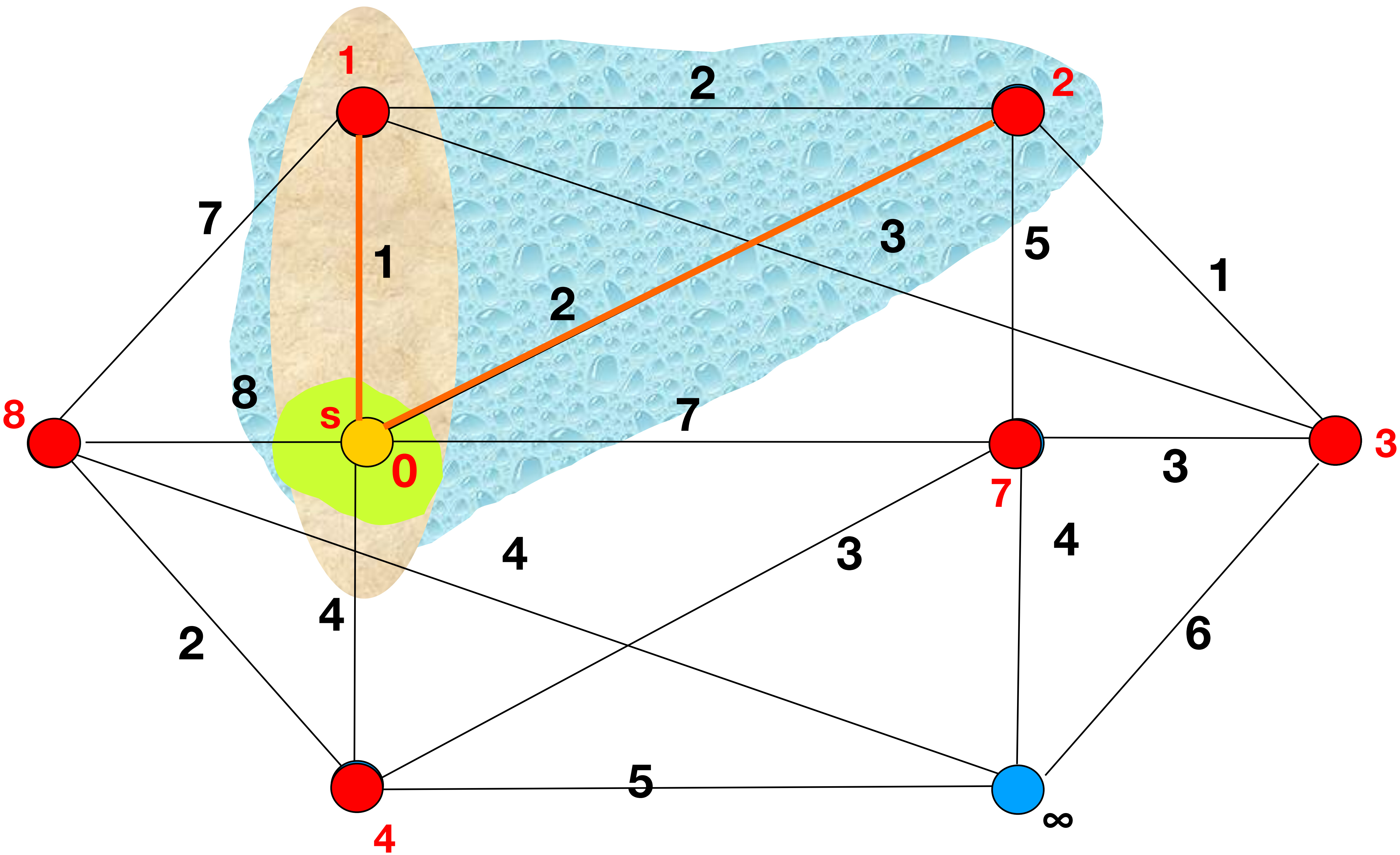

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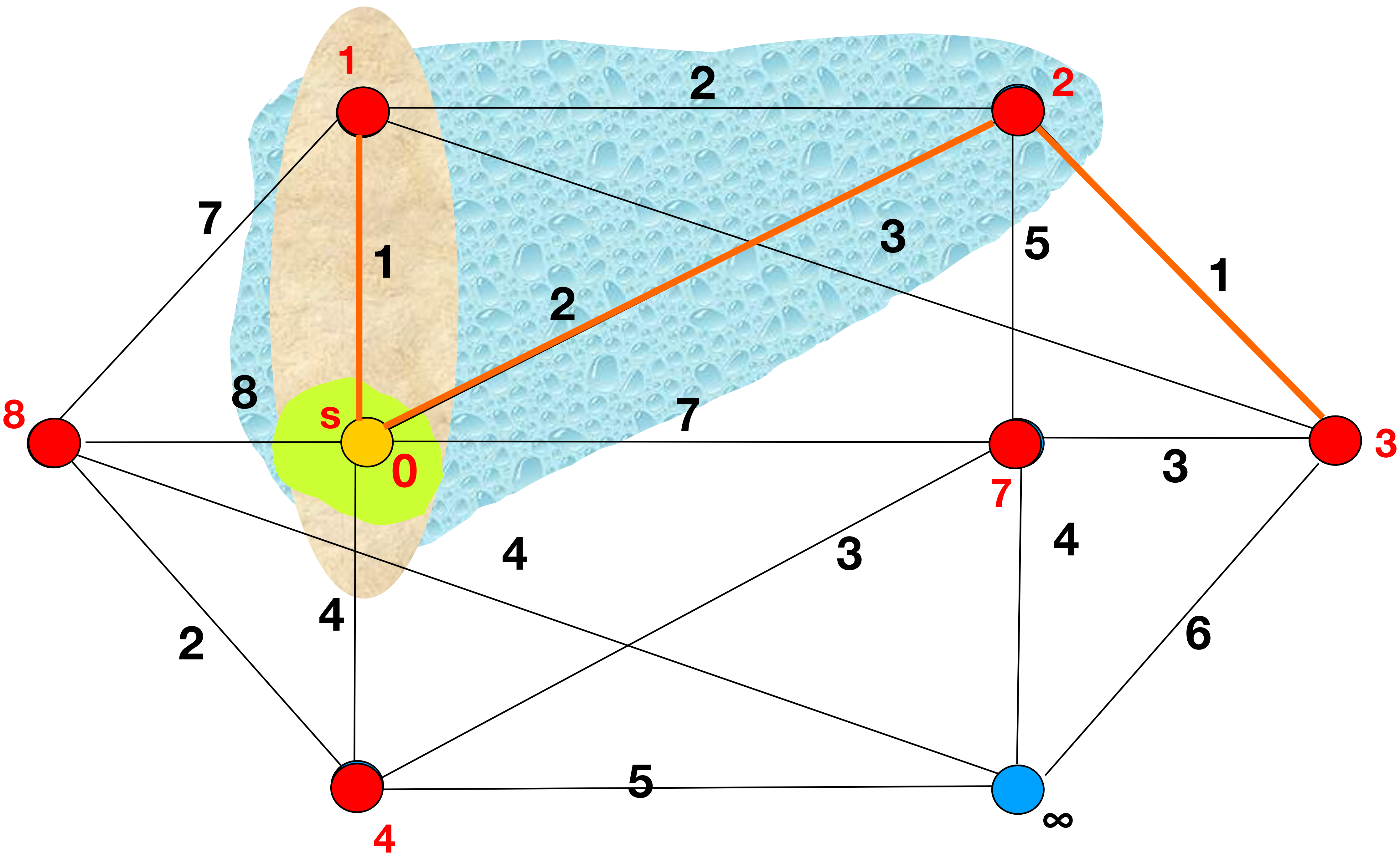

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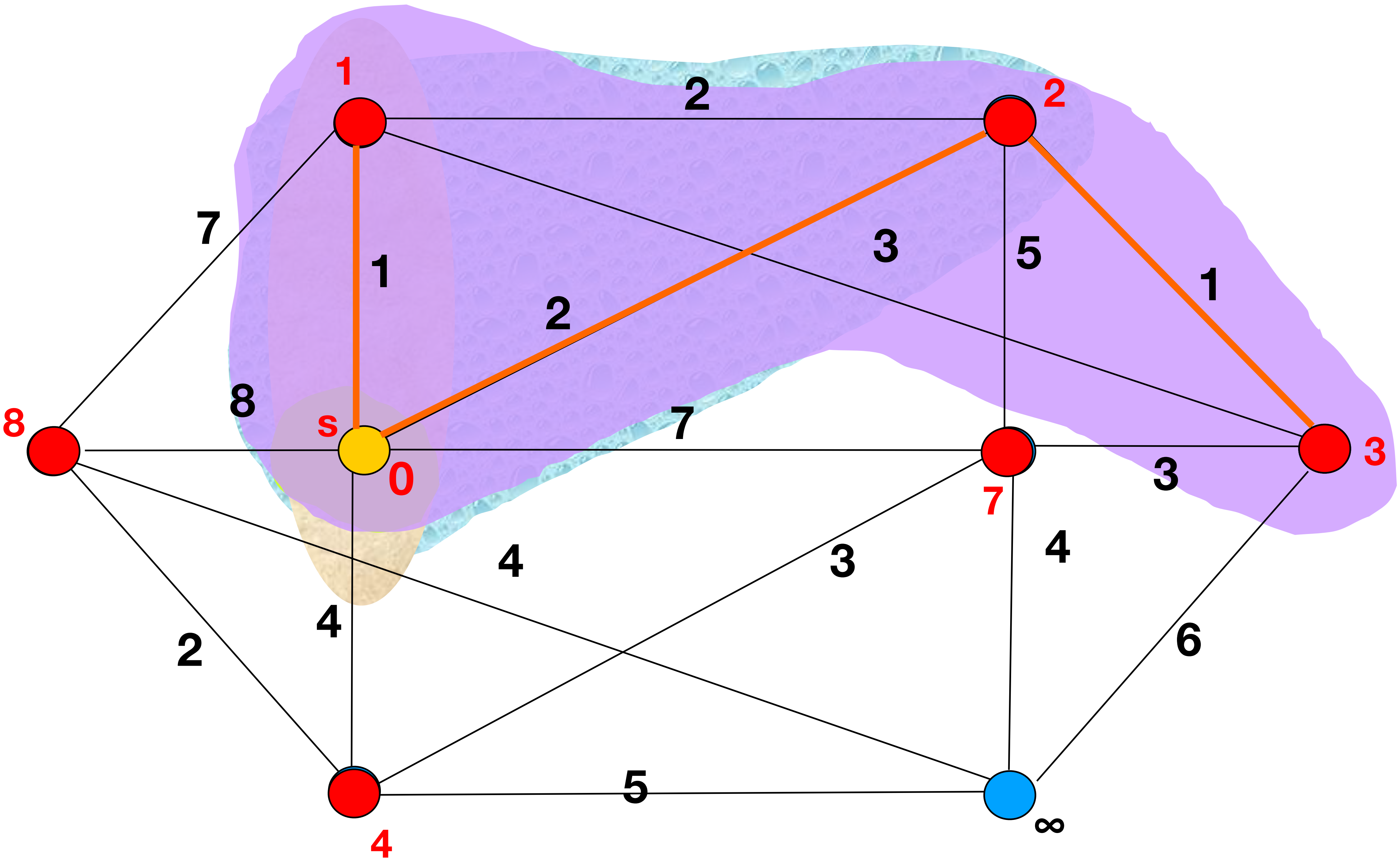

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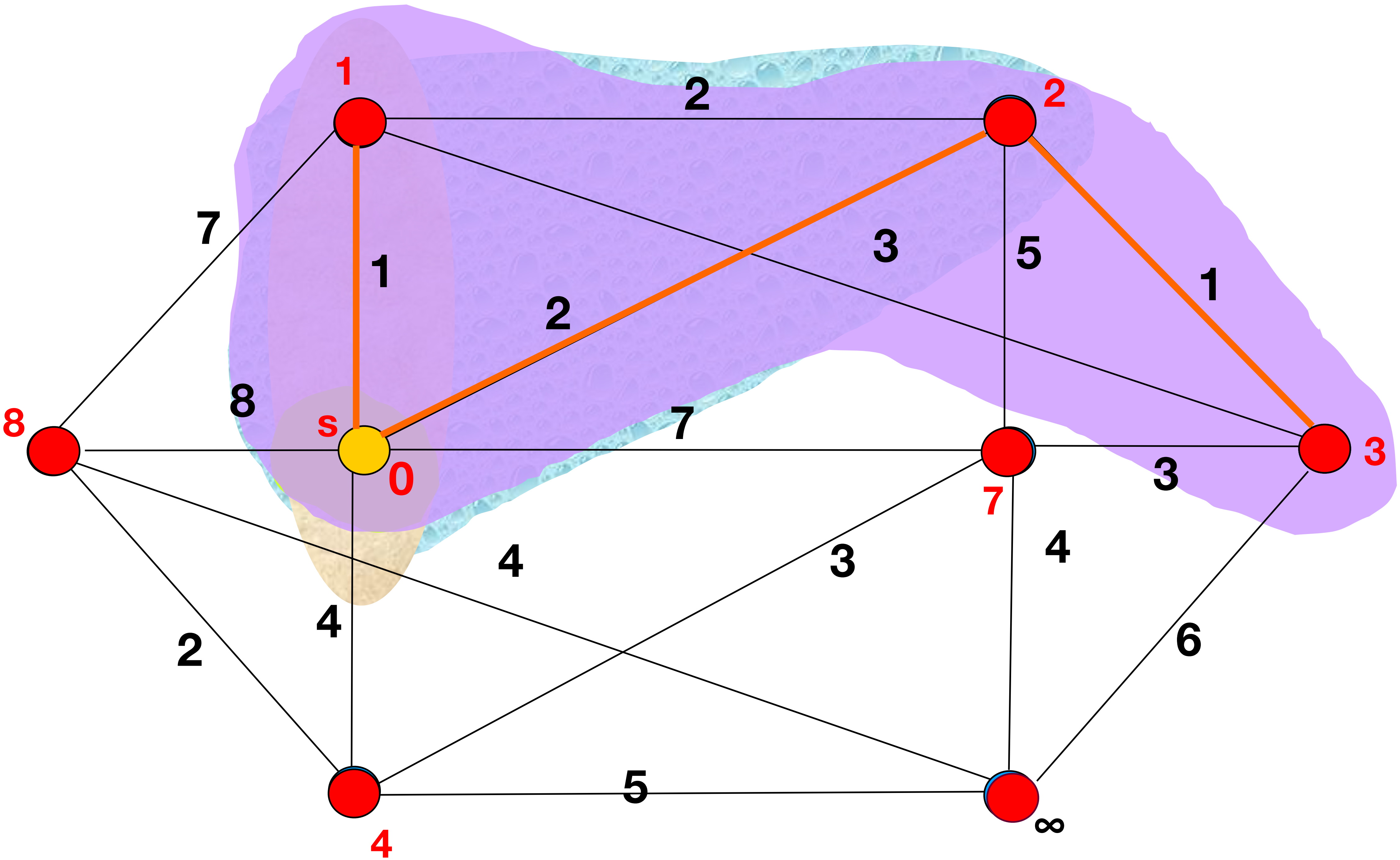

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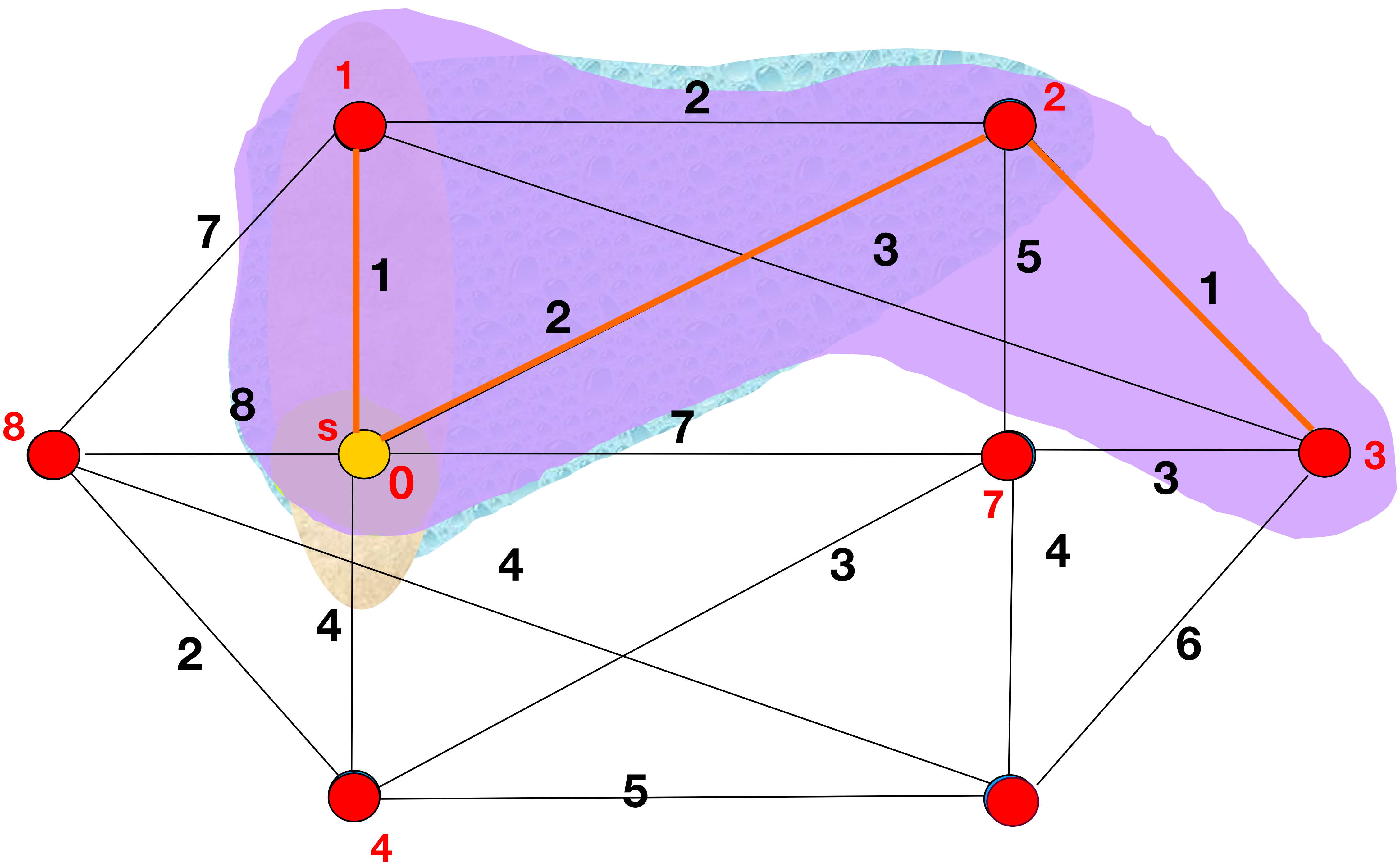

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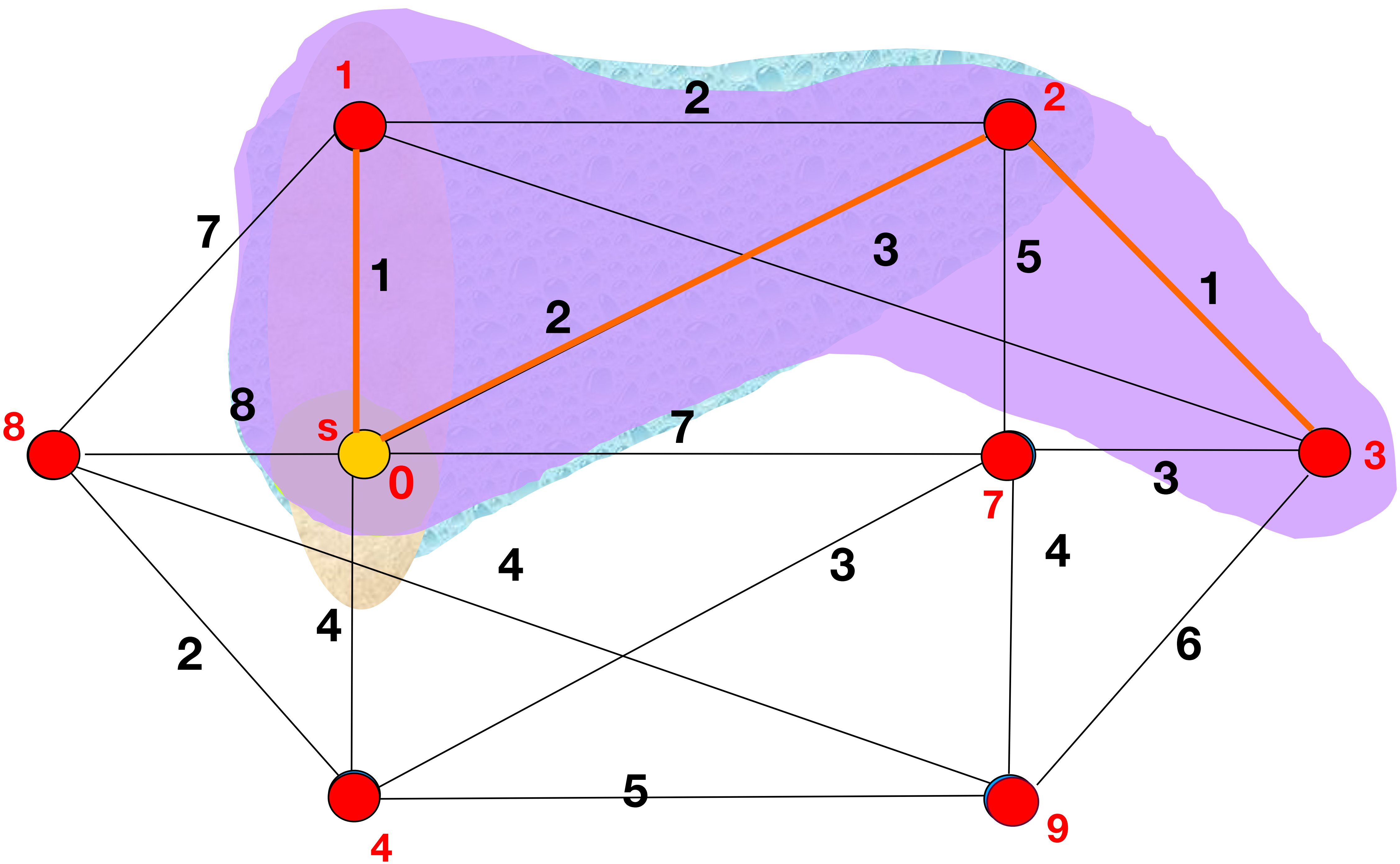

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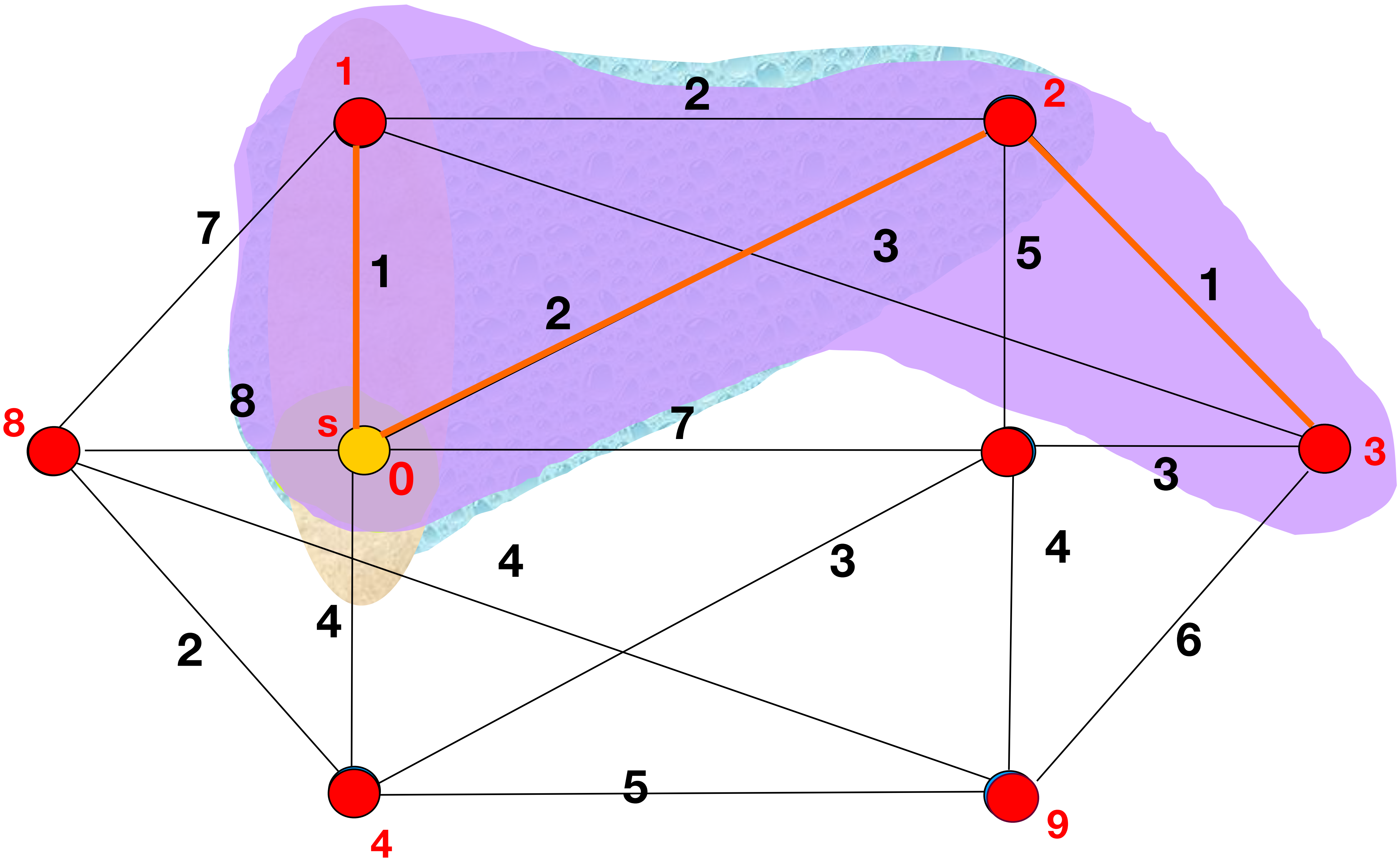

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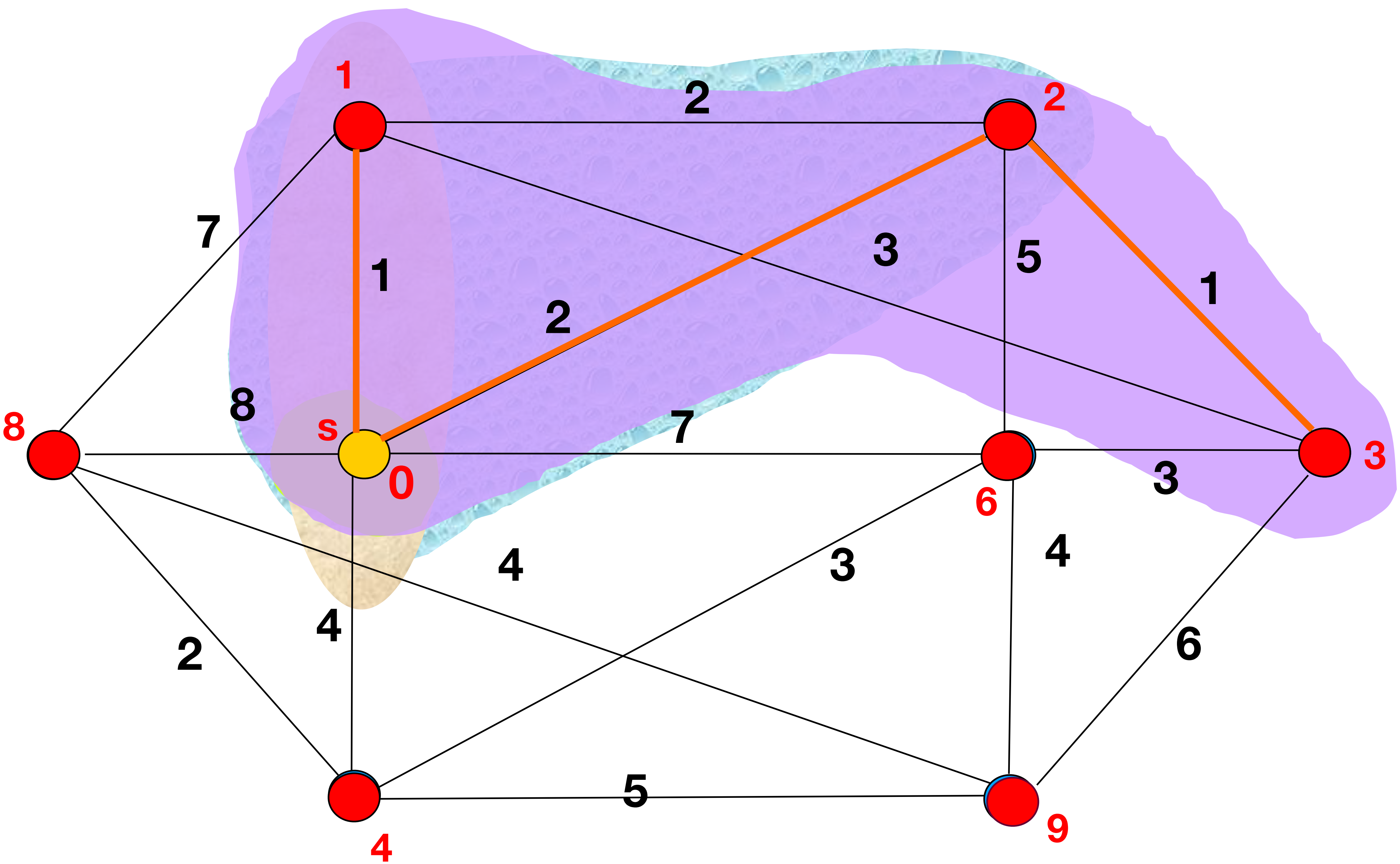

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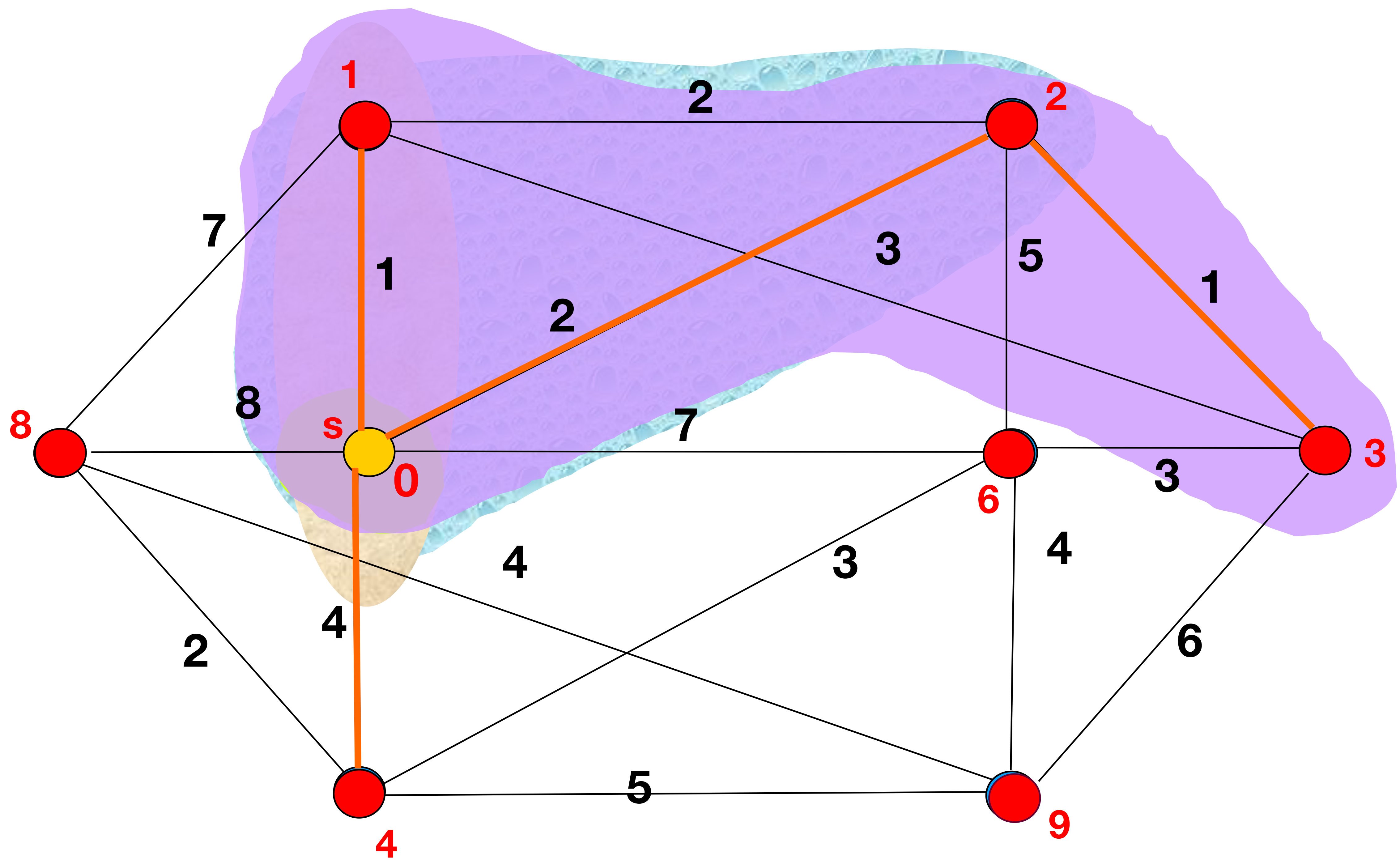

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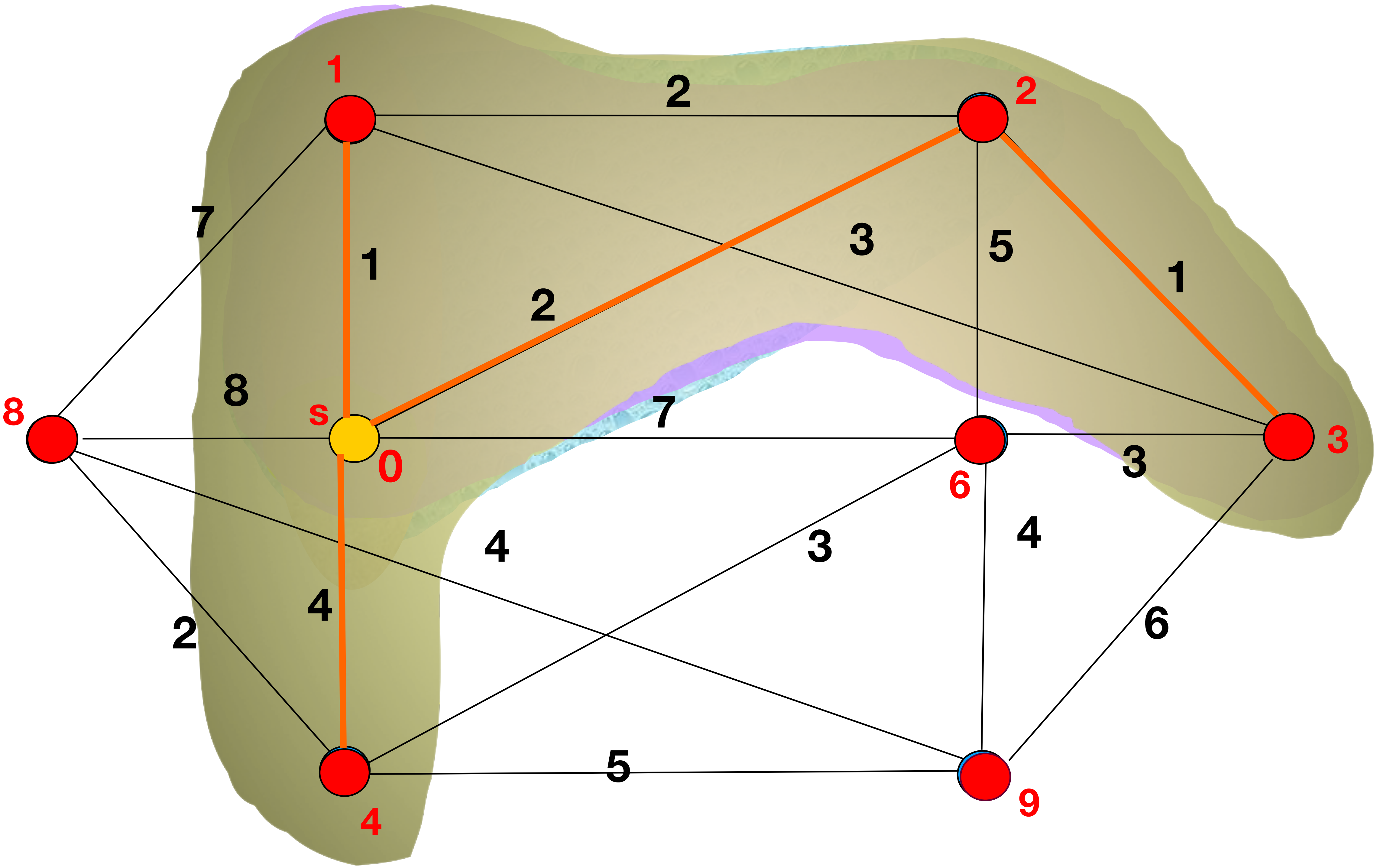

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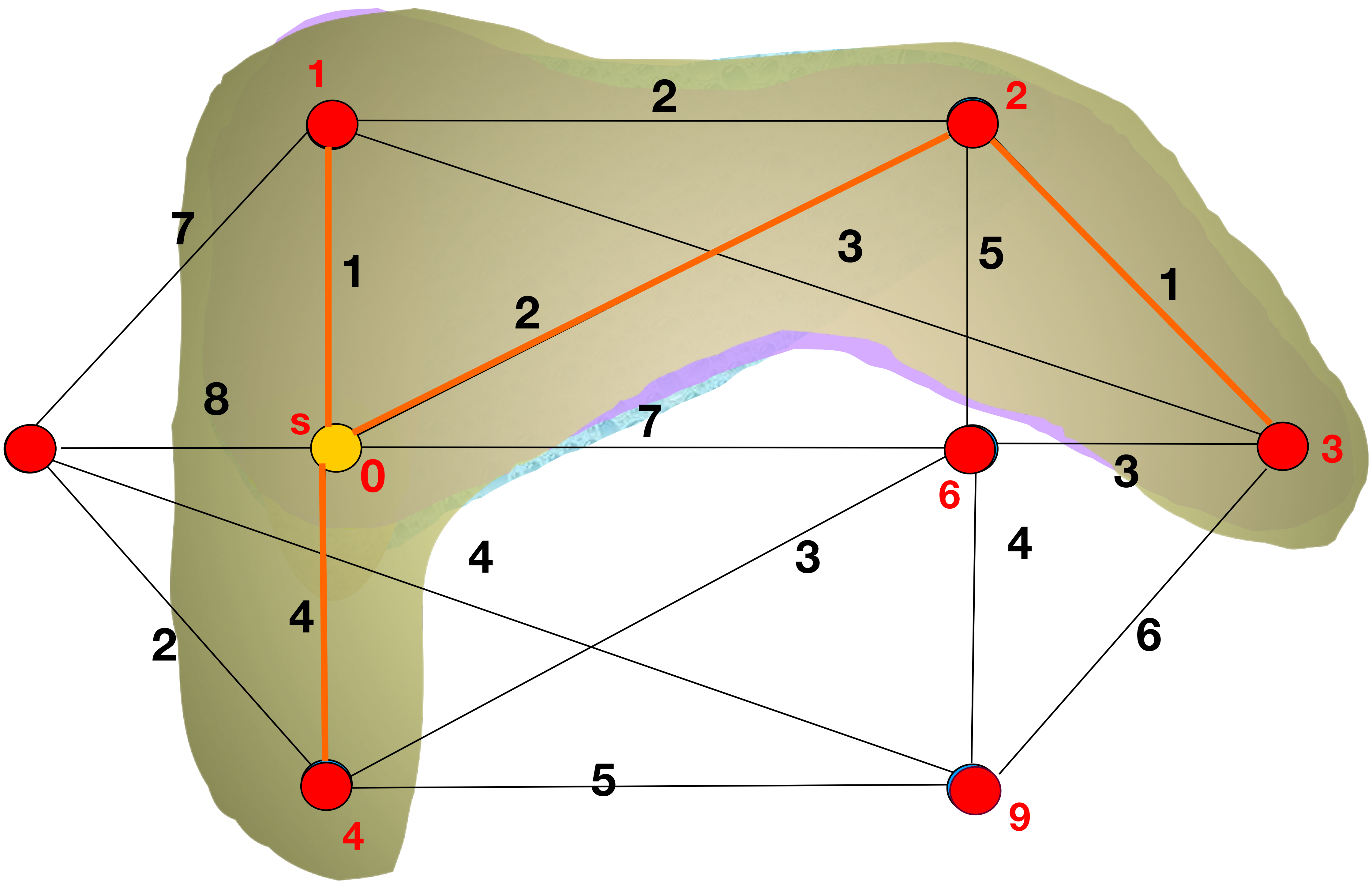

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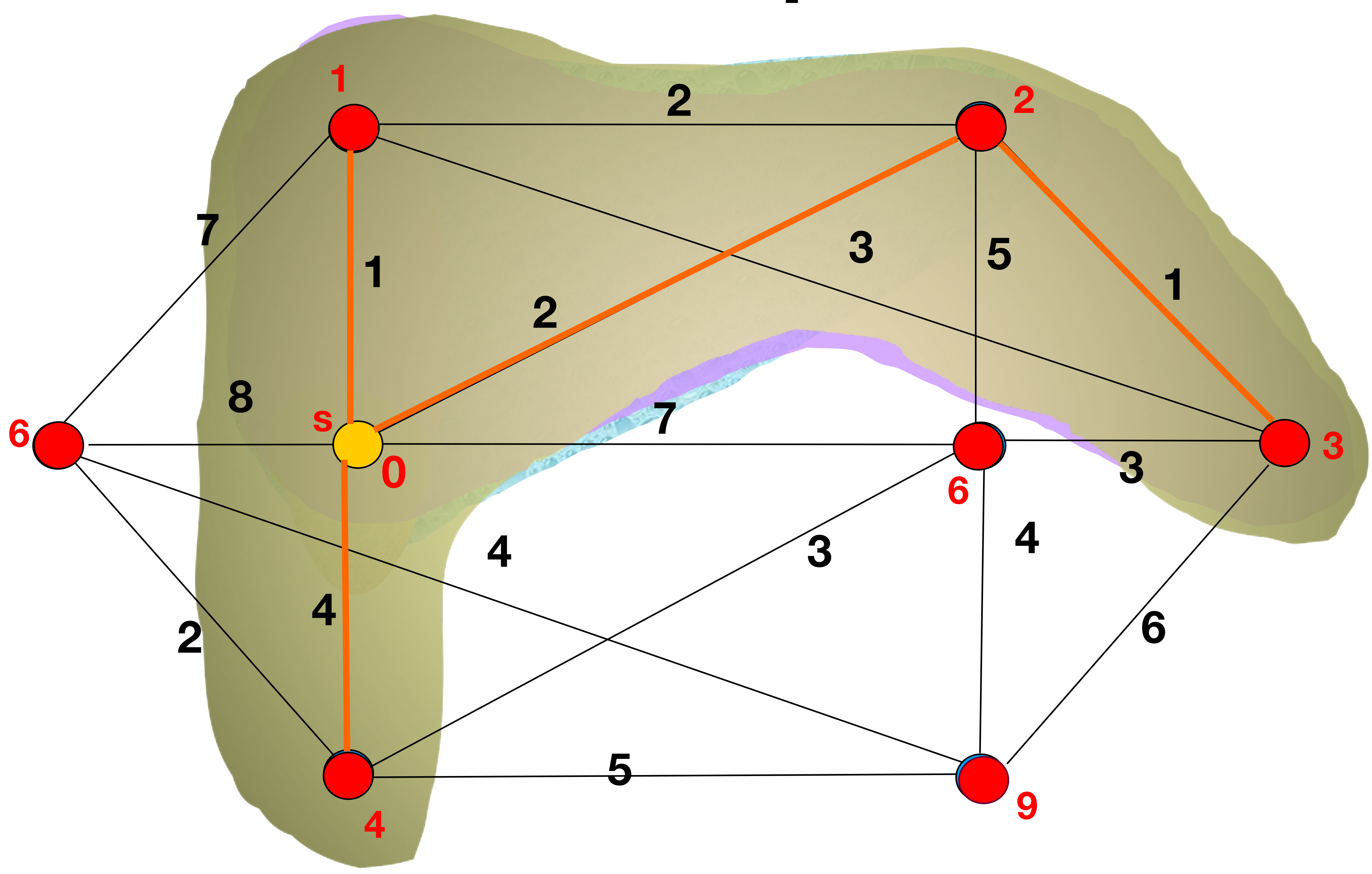

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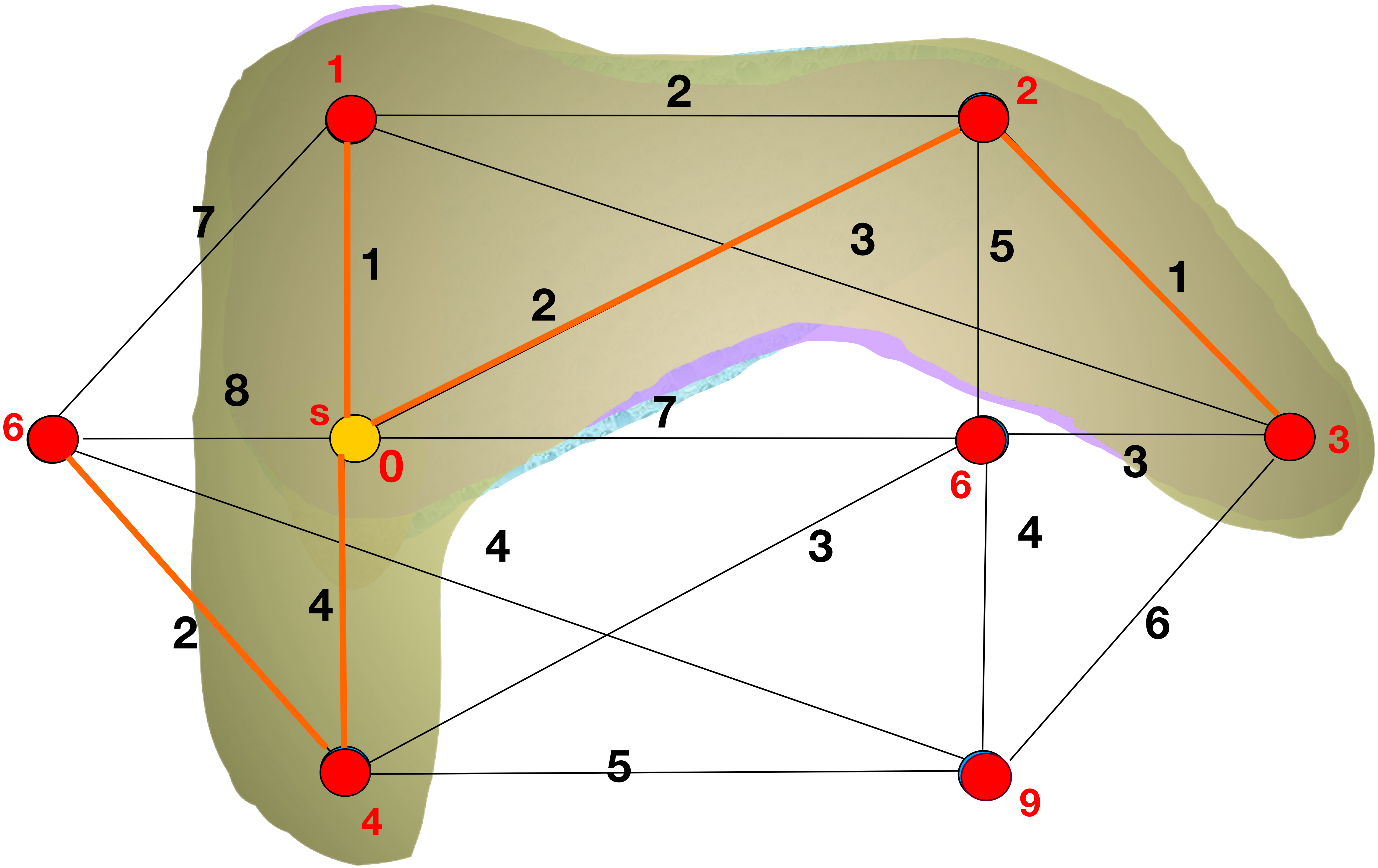

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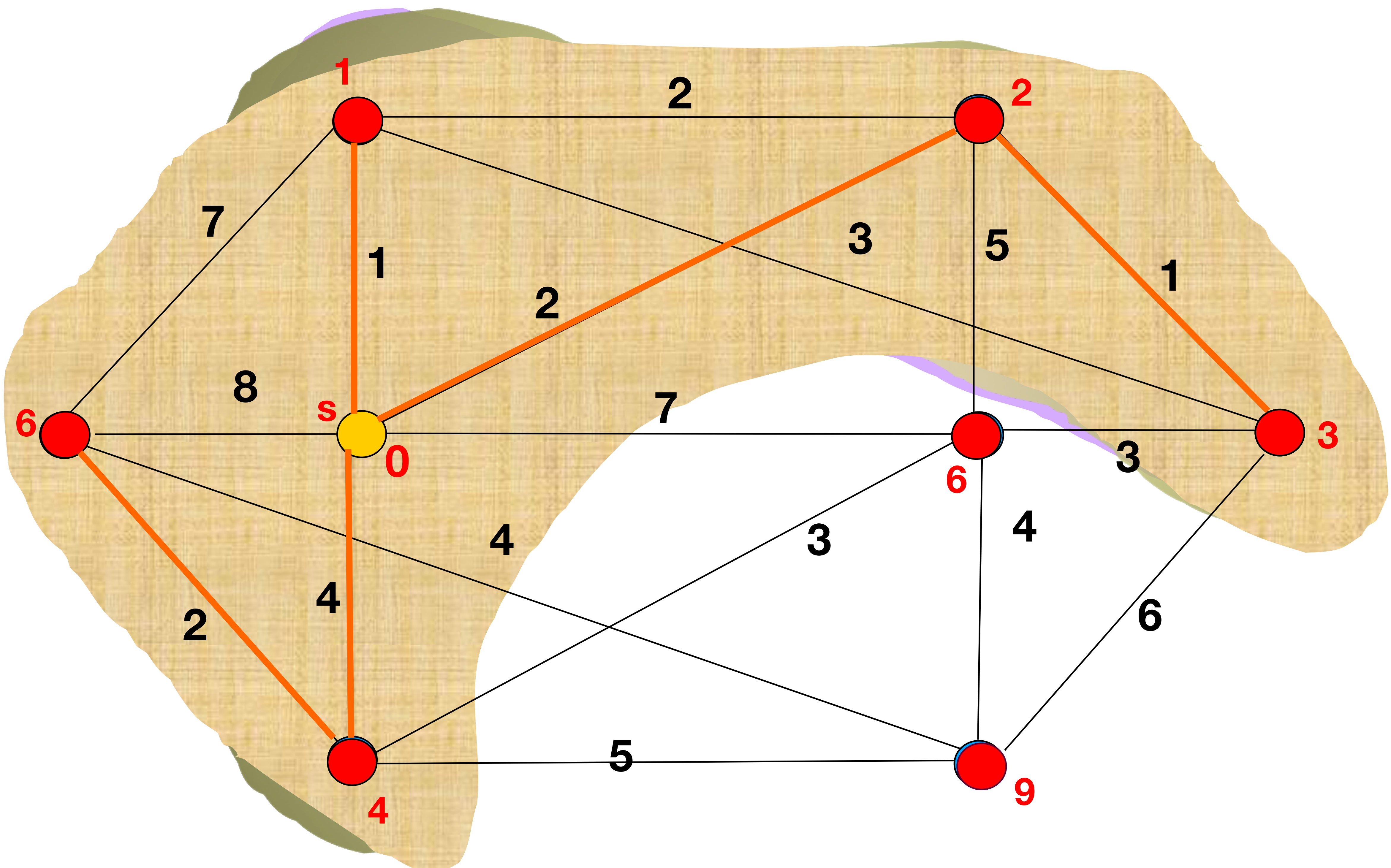

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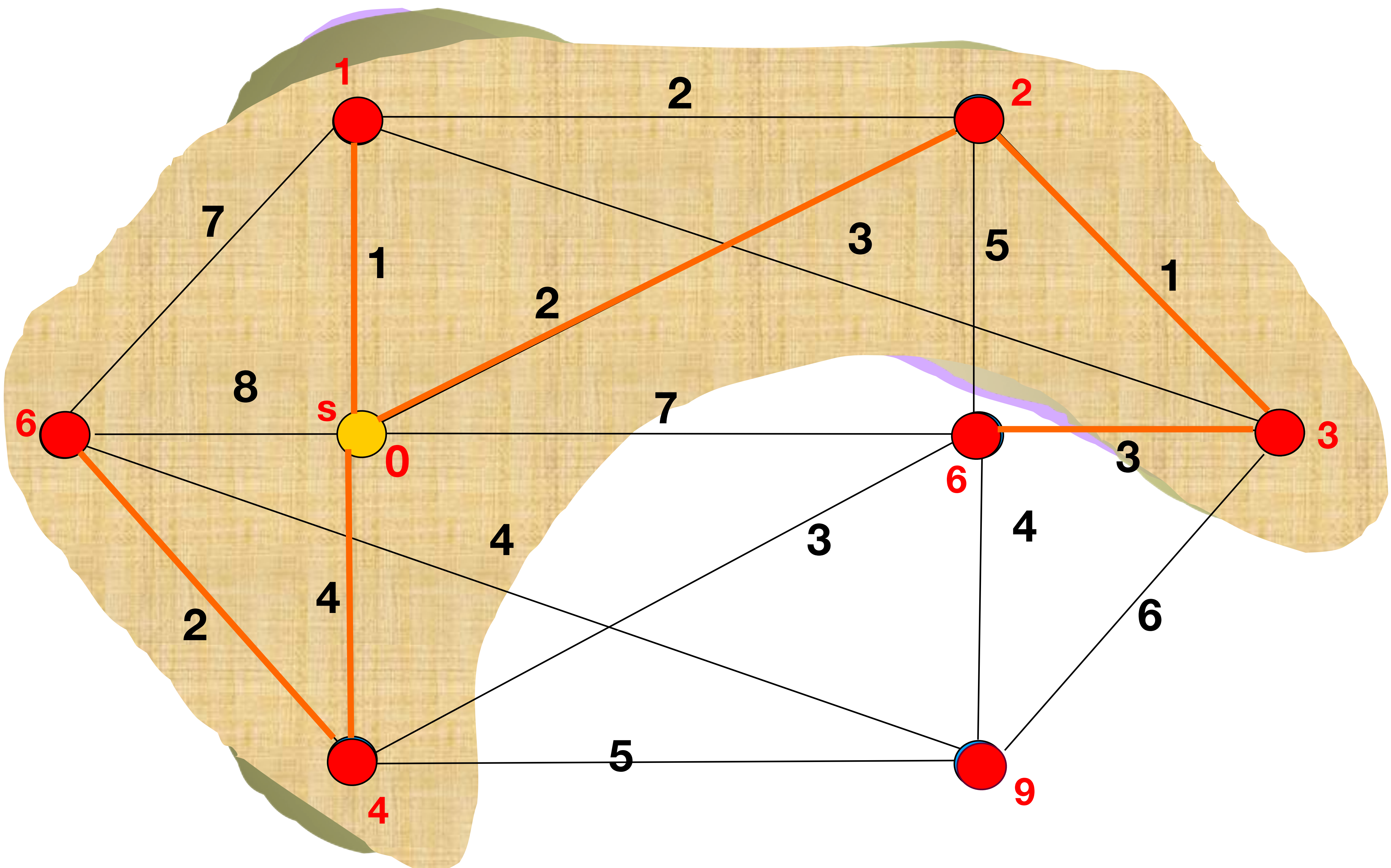

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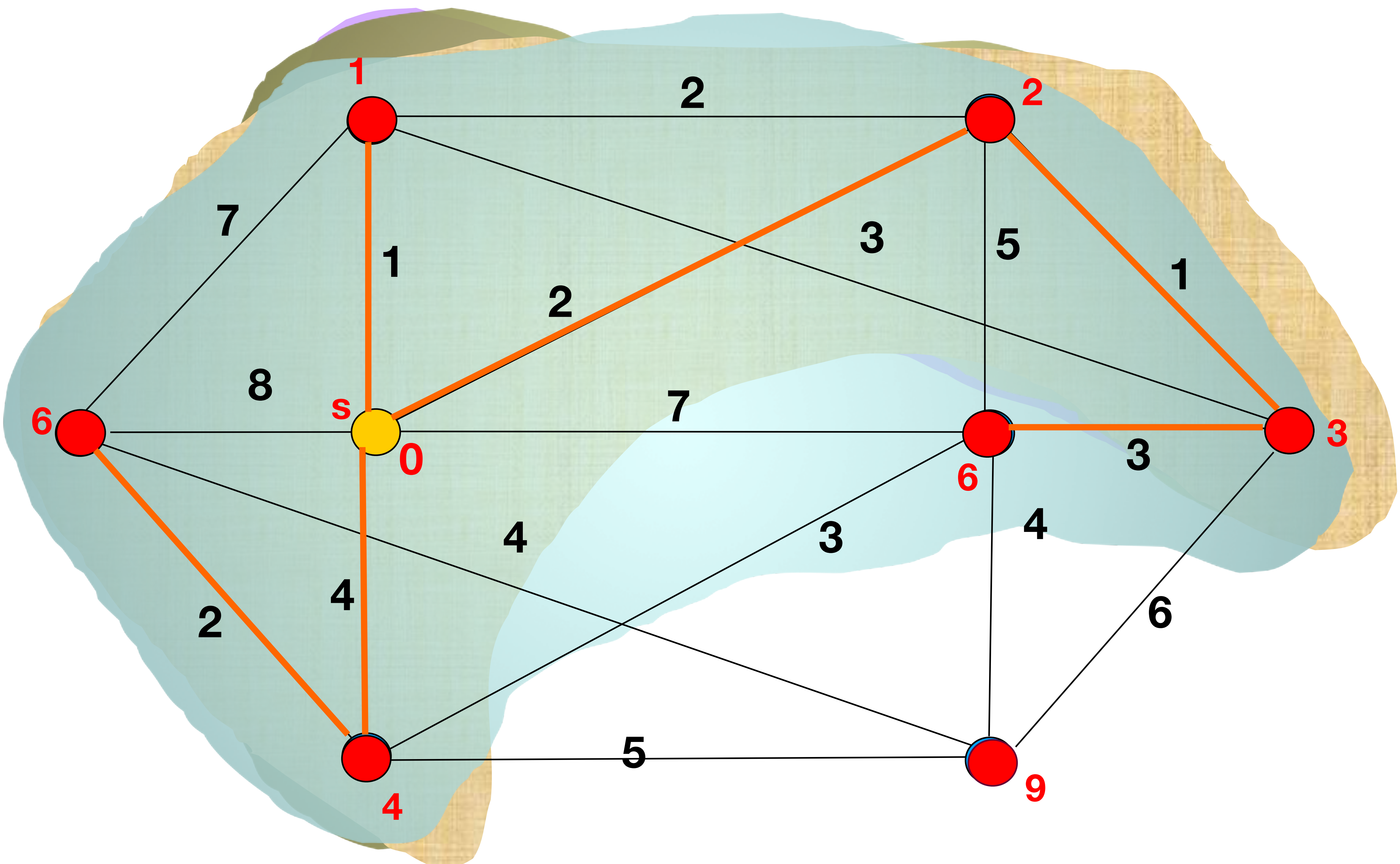

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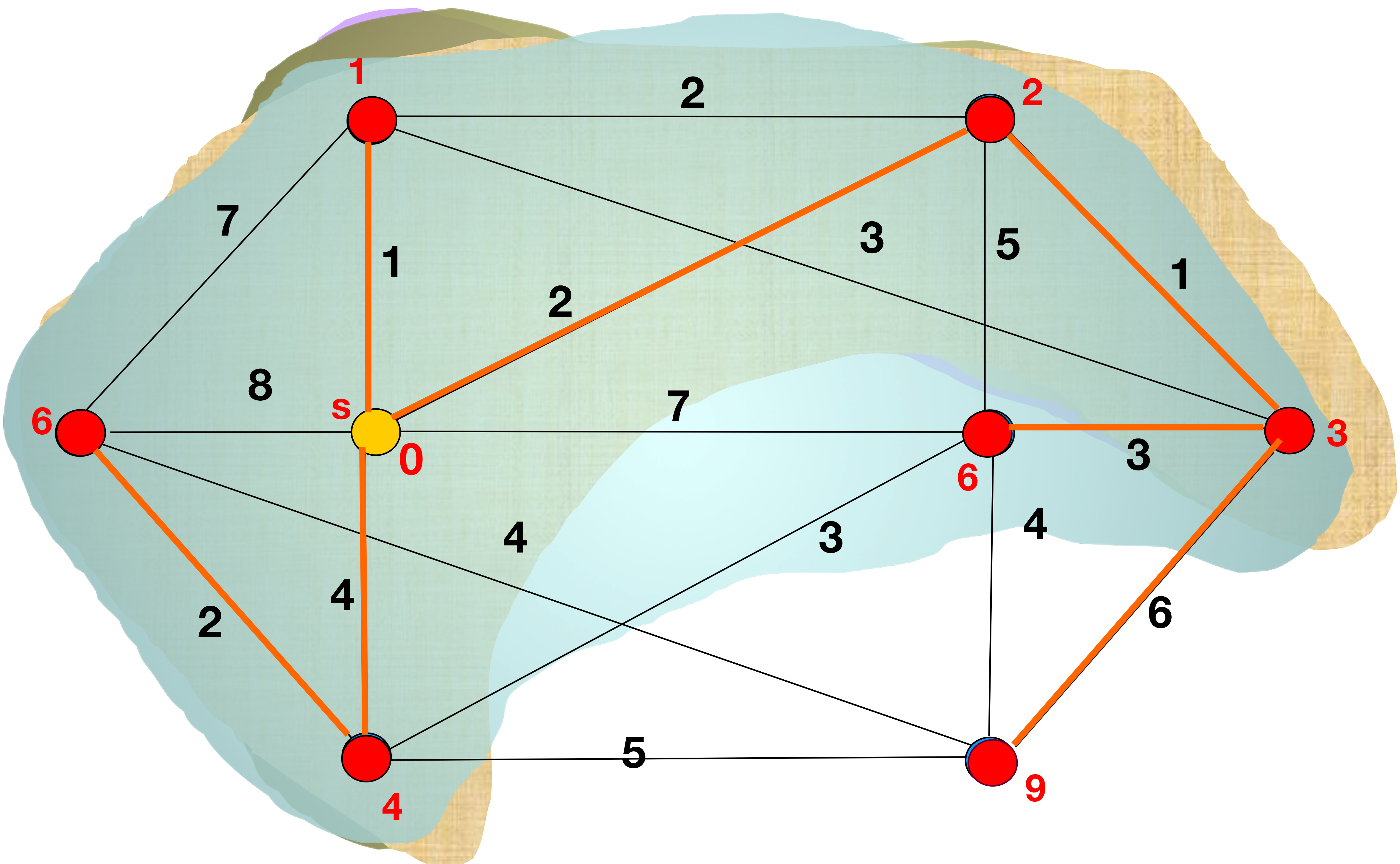

Thank you!
Q & A