

Introduction to Algorithm Design and Analysis

[10] Union-Find

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In the last class ...

- Hashing
 - Basic idea
- Collision handling for hashing
 - Closed address
 - Open address
- Amortized analysis
 - Array doubling
 - Stack operations
 - Binary counter

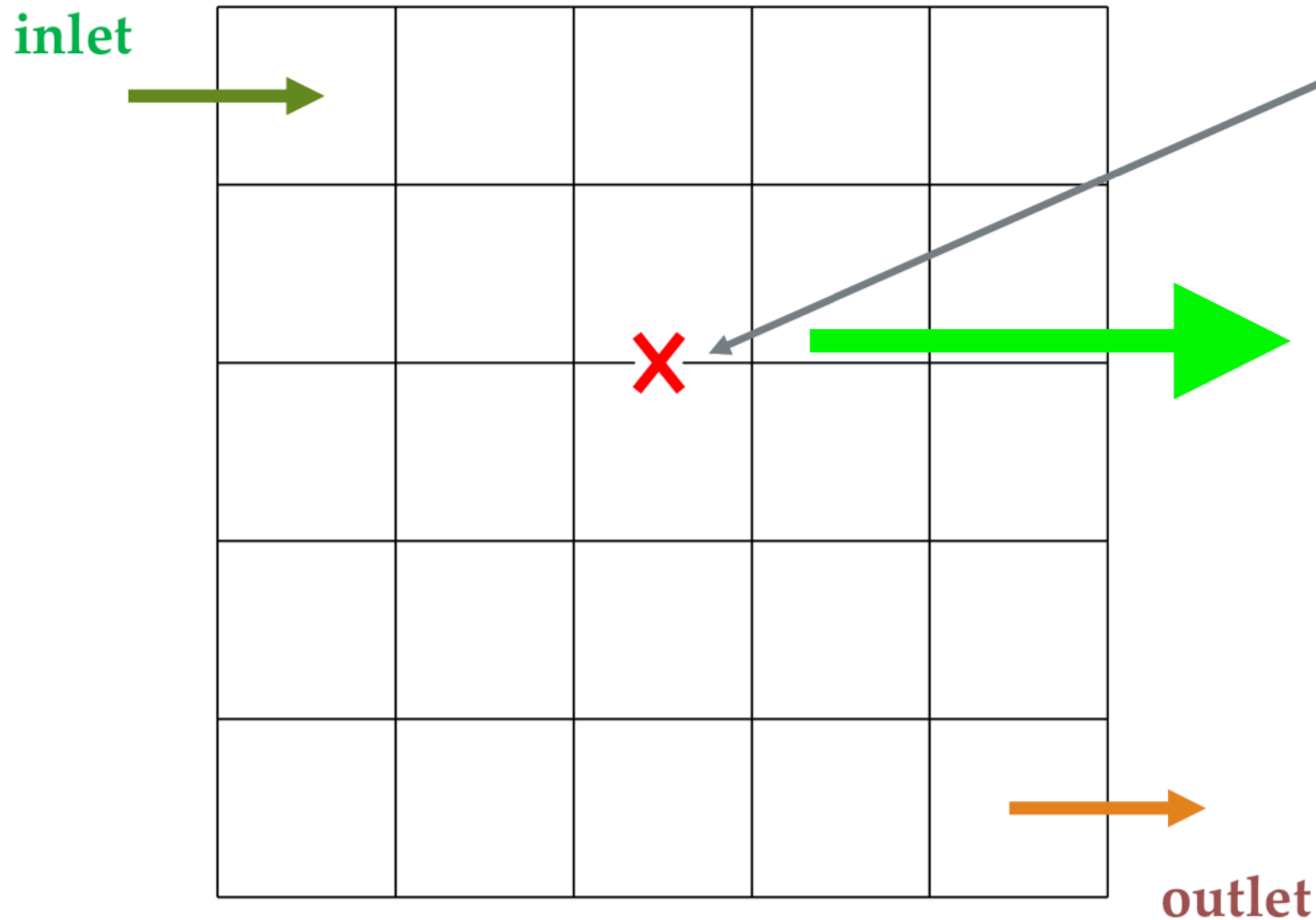
Union-Find

- **Dynamic Equivalence Relation**
 - Examples
 - Definitions
 - Brute force implementations
- **Disjoint Set**
 - Straightforward Union-Find
 - Weighted Union + Straightforward Find
 - Weighted Union + Path-compressing Find

Minimum Spanning Tree

- Kruskal's algorithm, greedy strategy:
 - Select one edge
 - With the minimum weight
 - Not in the tree
 - Evaluate this edge
 - This edge will **NOT** result in a cycle
- Critical issue:
 - How to know **“NO CYCLE”**?

Maze Generation



Select a wall to pull down randomly

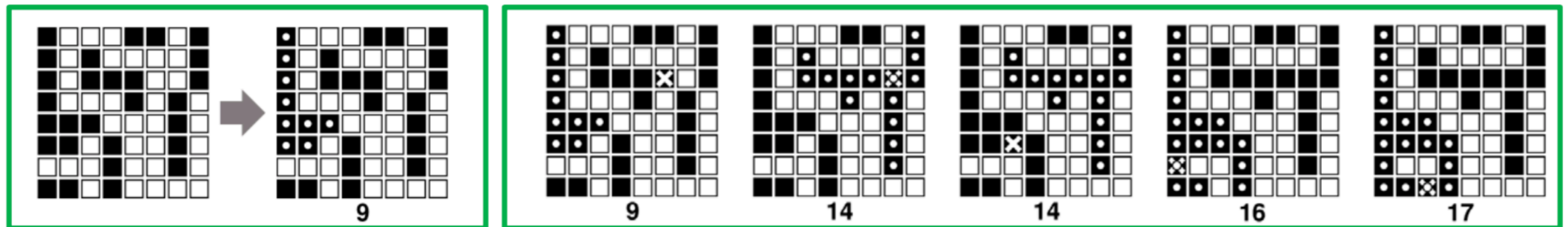
If i and j are in same *equivalence class*, then select another wall to pull down.

Otherwise, joint the two classes into one.

The maze is complete when the inlet and outlet are in one equivalence class.

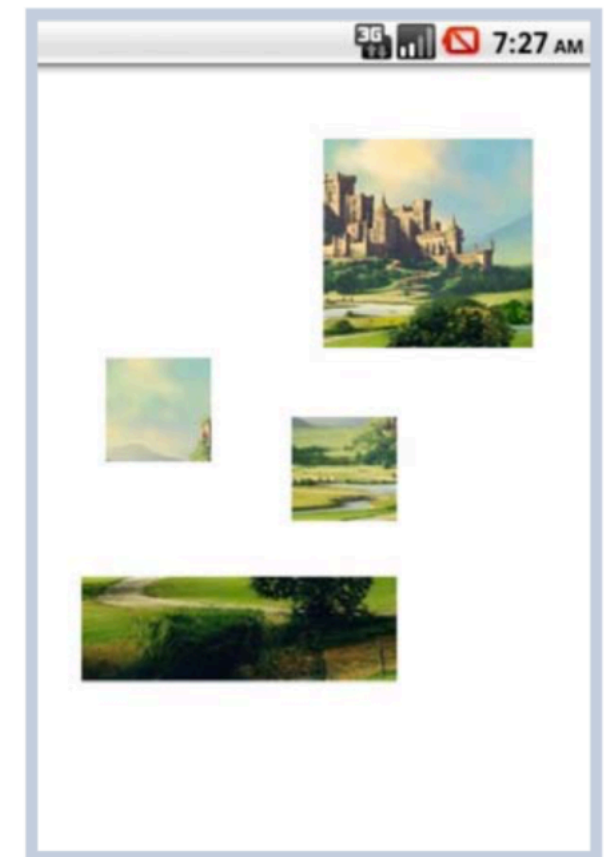
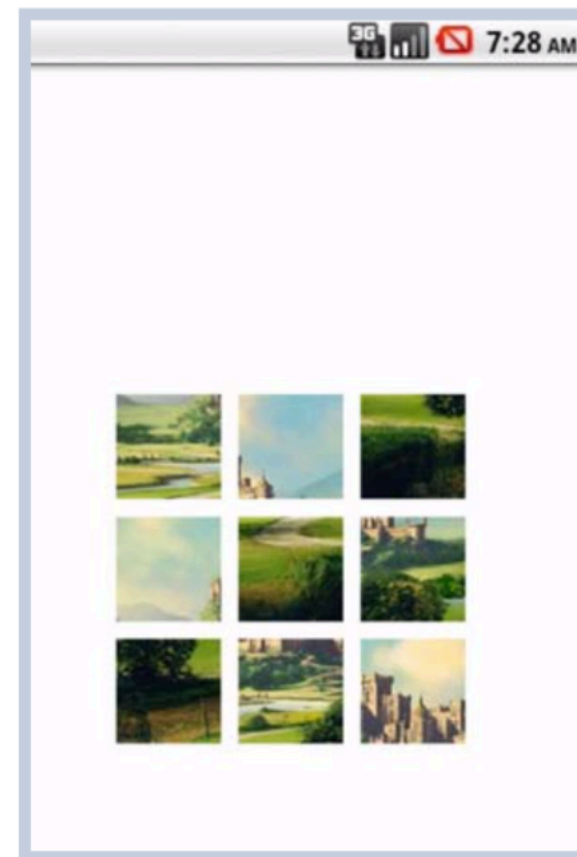
Black Pixels

- Maximum black pixel component
 - Let α be the size of the component
- Color one pixel black
 - How α changes?
 - How to choose the pixel, to accelerate the change in α



Jigsaw Puzzle

- Multiple pieces may be glued together
- From “one player” to “two players”
- Each group can only be moved in mutual exclusive way
- How to decide the relation of “in the same group”



Dynamic Equivalence Relations

- **Equivalence**
 - Reflexive, symmetric, transitive
 - Equivalent classes forming a **partition**
- **Dynamic equivalence relation**
 - Changing in the process of computation
 - **IS** instruction: yes or no (in the same equivalence class)
 - **MAKE** instruction: combining two equivalent classes, by relating two unrelated elements, and influencing the results of subsequent IS instructions.
 - Starting as equality relation

Implementation:

How to Measure

- The number of basic operations for processing a sequence of **m** **MAKE** and/or **IS** instructions on a set **S** with **n** elements.
- An example: $S = \{1, 2, 3, 4, 5\}$
 - 0. [create] $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$
 - 1. **IS** $2 \equiv 4?$ **NO**
 - 2. **IS** $3 \equiv 5?$ **NO**
 - 3. **MAKE** $3 \equiv 5.$ $\{\{1\}, \{2\}, \{3, 5\}, \{4\}\}$
 - 4. **MAKE** $2 \equiv 5.$ $\{\{1\}, \{2, 3, 5\}, \{4\}\}$
 - 5. **IS** $2 \equiv 3?$ **YES**
 - 6. **MAKE** $4 \equiv 1.$ $\{\{1, 4\}, \{2, 3, 5\}\}$
 - 7. **IS** $2 \equiv 4?$ **NO**

Union-Find based Implementation

- The maze problem
 - Randomly delete a wall and **union** two cells
 - Loop until you **find** the inlet and outlet are in one equivalent class
- The Kruskal algorithm
 - **Find** whether u and v are in the same equivalent class
 - If not, add the edge and **union** the two nodes
- The black pixels problem
 - **Find** two black pixels not in the same group
 - How the **union** will increase α

Implementation: Choices

- Matrix (**relation matrix**)
 - Space in $\Theta(n^2)$, and worst-case cost in $\Omega(mn)$ (mainly for row copying for MAKE)
- Array (**for equivalence class ID**)
 - Space in $\Theta(n)$, and worst-case cost in $\Omega(mn)$ (mainly for search and change for MAKE)
- Forest of rooted trees
 - A collection of disjoint sets, supporting Union and Find operations
 - Not necessary to traverse all the elements in one set

Union-Find ADT

- **Constructor:** `Union-Find create(int n)`
 - `sets = create(n)` refers to a newly created group of sets $\{1\}, \{2\}, \dots, \{n\}$ (n singletons)
- **Access Function:** `int find(UnionFind sets, e)`
 - `find(sets, e) = <e>`
- **Manipulation Procedures**
 - `void makeSet(UnionFind sets, int e)`
 - `void union(UnionFind sets, int s, int t)`

Using Rooted Tree

- **IS** $s_i \equiv s_j$:

- $t = \text{find}(s_i);$
- $u = \text{find}(s_j);$
- $(t == u)?$

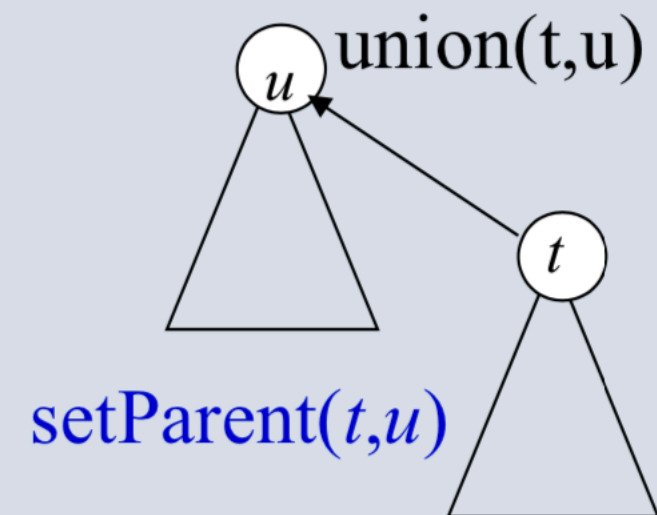
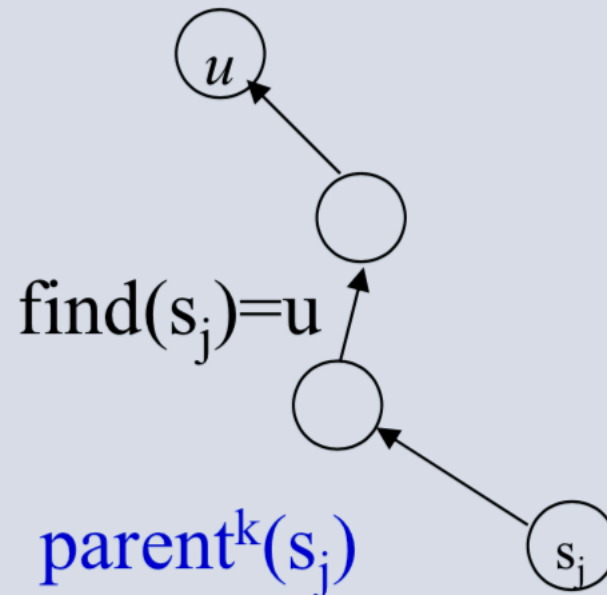
implementation by inTree

create(n): sequence of makeNode



- **MAKE** $s_i \equiv s_j$:

- $t = \text{find}(s_i);$
- $u = \text{find}(s_j);$
- $\text{union}(t, u);$



Union-Find Program

- A **union-find program of length m**
 - is (a `create(n)` operation followed by) a sequence of **m** union and/or find operations in any order
- A union-find program is considered an input
 - The object on which the analysis is conducted
- The measure: number of accesses to the **parent**
 - **assignments**: for union operations
 - **lookups**: for find operations

link operation

Worst-case Analysis for Union-Find Program

- Assuming each lookup/assignment take $O(1)$
- Each makeSet or union does one assignment, and each find does $d+1$ lookups, where d is the depth of the node.

1. Union(1,2)
2. Union(2,3)
 ⋮
n-1. Union(n-1,n)
n. Find(1)
 ⋮
m. Find(1)

Example

The sequence of *Union* makes a chain of length $n-1$, which is the tree with the largest height

operations done:

$$n + (n-1) + (m-n+1)n$$

$$\Theta(mn)$$

Find(1) needs n array lookups

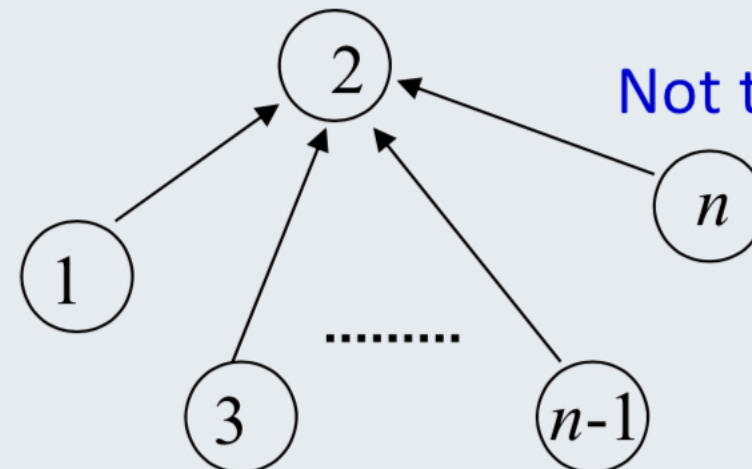
Weighted Union: for Short Trees

- Weighted union (wUnion)
 - always have the tree with **fewer nodes** as subtree

To keep the *Union* valid,
each *Union* operation is
replaced by:

```
t=find(i);  
u=find(j);  
union(t,u)
```

The order of (t,u)
satisfying the
requirement



Not the worst case!

Tree made by wUnion

Cost for the program:
 $n+3(n-1)+2(m-n+1)$

Upper Bound of Tree Height

- After any sequence of Union instructions, implemented by wUnion, any tree that has k nodes will have height at most $\lfloor \lg k \rfloor$

- Proof by induction on k :

- base case: $k=1$, the height is 0

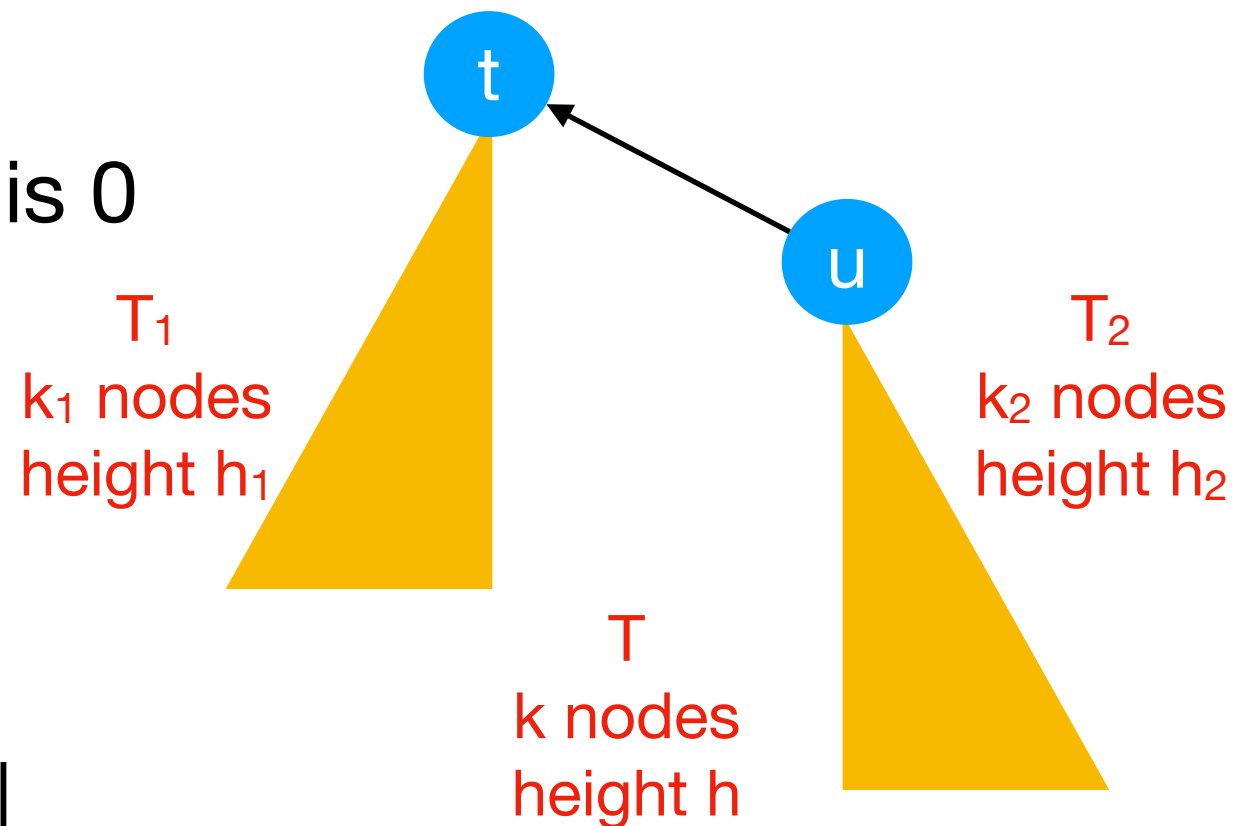
- by inductive hypothesis:

- $h_1 \leq \lfloor \lg k_1 \rfloor$, $h_2 \leq \lfloor \lg k_2 \rfloor$

- $h = \max(h_1, h_2 + 1)$ $k = k_1 + k_2$

- if $h = h_1$, $h_1 \leq \lfloor \lg k_1 \rfloor \leq \lfloor \lg k \rfloor$

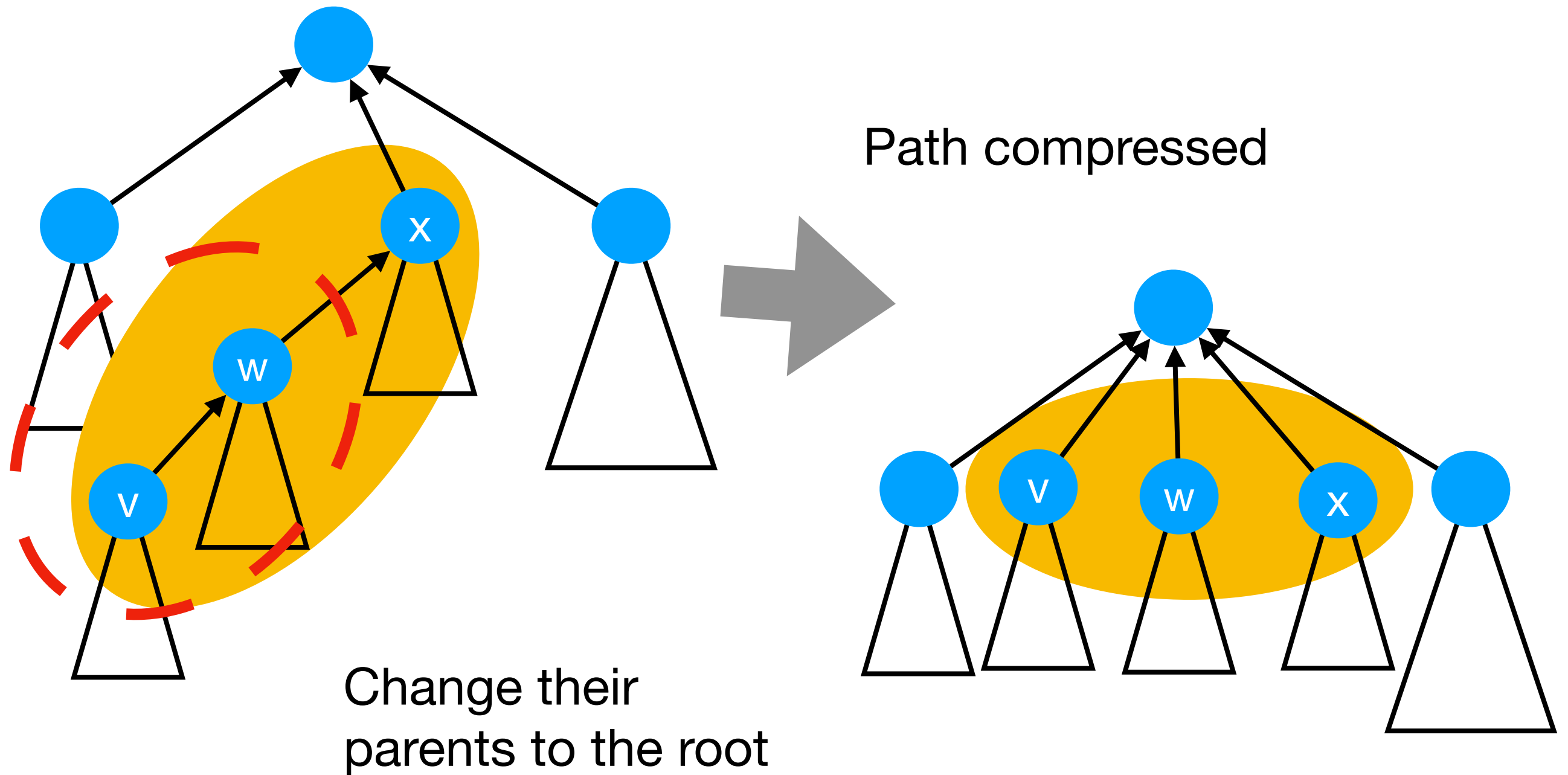
- if $h = h_2 + 1$, note: $k_2 \leq k/2$, so $h_2 + 1 \leq \lfloor \lg k_2 \rfloor + 1 \leq \lfloor \lg k \rfloor$



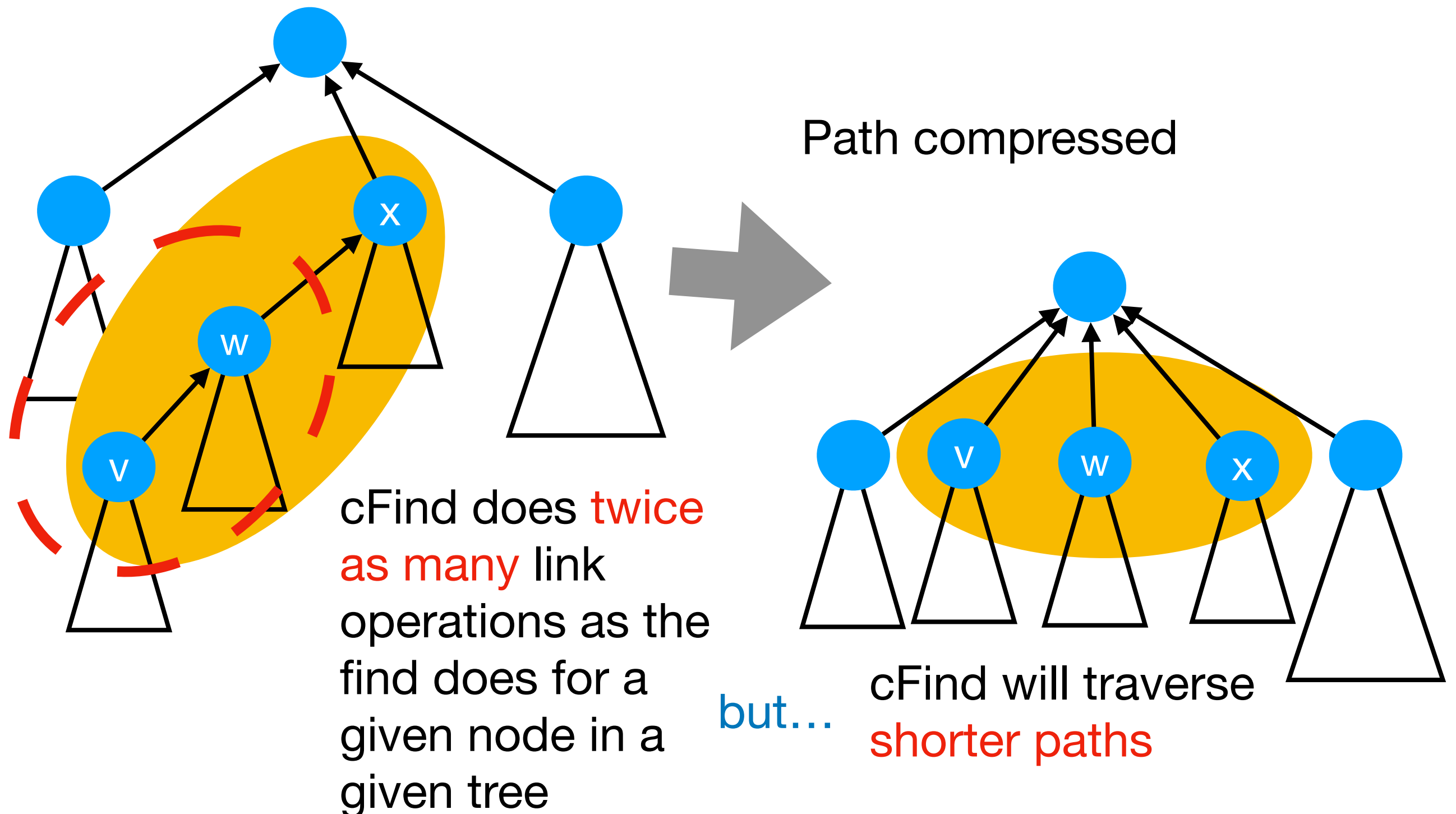
Upper Bound for Union-Find Program

- A Union-Find program of size m , on a set of n elements, performs $O(n+m\log n)$ link operations in the worst case if *wUnion* and straight *find* are used
- **Proof:**
 - At most $n-1$ *wUnion* can be done, building a tree with height at most $\lfloor \log n \rfloor$,
 - Then, each *find* costs at most $\lfloor \log n \rfloor + 1$.
 - Each *union* costs in $O(1)$, so, the upper bound on the cost of any combination of m *wUnion/find* operations is the cost of m *find* operations, that is $m(\lfloor \log n \rfloor + 1) \in O(n+m\log n)$
 - There do exist programs requiring $\Omega(n+(m-n)\log n)$ steps.

Path Compression



Challenges for the Analysis



Analysis: the Basic Idea

- cFind may be an expensive operation
 - in the case that find(i) is executed and the node i has great depth.
- However, such cFind can be executed only for limited times
 - Path compressions depends on previous unions
- So, **amortized analysis** applies

Co-Strength of wUnion and cFind

- $O((n+m)\log^*(n))$

- Link operations for a Union-Find program of length m on a set of n elements is in the worst case.
- Implemented with wUnion and cFind

- What's $\log^*(n)$?

- Define the function H as following:

$$H(0) = 1$$

$$H(i) = 2^{H(i-1)}$$

- Then, $\log^*(j)$ for $j \geq 1$ is defined as:

$$\log^*(j) = \min\{k \mid H(k) \geq j\}$$

Definitions with a Union-Find Program P

- **Forest F:** the forest constructed by the sequence of union instructions in P, assuming:
 - wUnion is used;
 - the finds in the P are ignored
- **Height of a node v in any tree:** the height of the subtree rooted at v
- **Rank of v:** the height of v in F

Note: cFind changes the height of a node, but the rank for any node is invariable.

Constraints on Ranks in F

- The upper bound of the number of nodes with rank $r (r \geq 0)$ is $n/2^r$
 - Remember that the height of the tree built by wUnion is at most $\lfloor \log n \rfloor$, which means the subtree of height r has at least 2^r nodes.
 - The subtrees with root at rank r are disjoint.
- There are at most $\lfloor \log n \rfloor$ different ranks.
 - There are altogether n elements in S , that is, n nodes in F .

Increasing Sequence of Ranks

- The ranks of the nodes on a path from a leaf to a root of a tree in F form a strictly increasing sequence.
- When a $cFind$ operation changes the parent of a node, the new parent has higher rank than the old parent of that node.
 - Note: the new parent was an ancestor of the previous parent.

A function Growing Extremely Slowly

- **Function H:**

$$H(0) = 1$$

$$H(i) = 2^{H(i-1)}$$

$$\text{That is: } H(k) = 2^{2^{\dots^2}} \quad k \text{ 2's}$$

Note:

H grows extremely fast:

$$H(4) = 2^{16} = 65536$$

$$H(5) = 2^{65536}$$

- **Function log-star**

$\log^*(j)$ is defined as the least i such that:

$$H(i) \geq j \quad \text{for } j > 0$$

- **log-star grows extremely slowly**

$$\lim_{n \rightarrow \infty} \frac{\log^*(n)}{\log^{(p)}(n)} = 0$$

p is any fixed nonnegative constant

For any x : $2^{16} \leq x \leq 2^{65536} - 1$, $\log^*(x) = 5$

Grouping Nodes by Ranks

- Node $v \in s_i$ ($i \geq 0$) iff. $\log^*(1 + \text{rank of } v) = i$
 - which means that: if node v is in group i , then $r_v \leq H(i) - 1$, but not in group with smaller labels
- So,
 - Group 0: all nodes with rank 0
 - Group 1: all nodes with rank 1
 - Group 2: all nodes with rank 2 or 3
 - Group 3: all nodes with its rank in $[4, 15]$
 - Group 4: all nodes with its rank in $[16, 65535]$
 - Group 5: all nodes with its rank in $[65535, ???]$

Group 5 exists only when n is at least 2^{65536} . What is that?

Very Few Groups

- Node $v \in S_i$ ($i \geq 0$) iff.
 $\log^*(1 + \text{rank of } v) = i$
- Upper bound of the number of distinct node groups is $\log^*(n+1)$
 - The rank of any node in F is at most $\lfloor \log n \rfloor$, so the largest group index is $\log^*(1 + \lfloor \log n \rfloor) = \log^*(\lceil \log n + 1 \rceil) = \log^*(n+1) - 1$

If $\log^*(n+1) = k$, then

$$\underbrace{k \text{ 2's}}_{2^2} \geq n+1$$

The diagram shows a tree structure with a root node labeled 'k 2's'. Below the root, there are two branches, each labeled '2'. A dashed line is drawn below the tree, and the text $\geq n+1$ is written to the right of the tree.

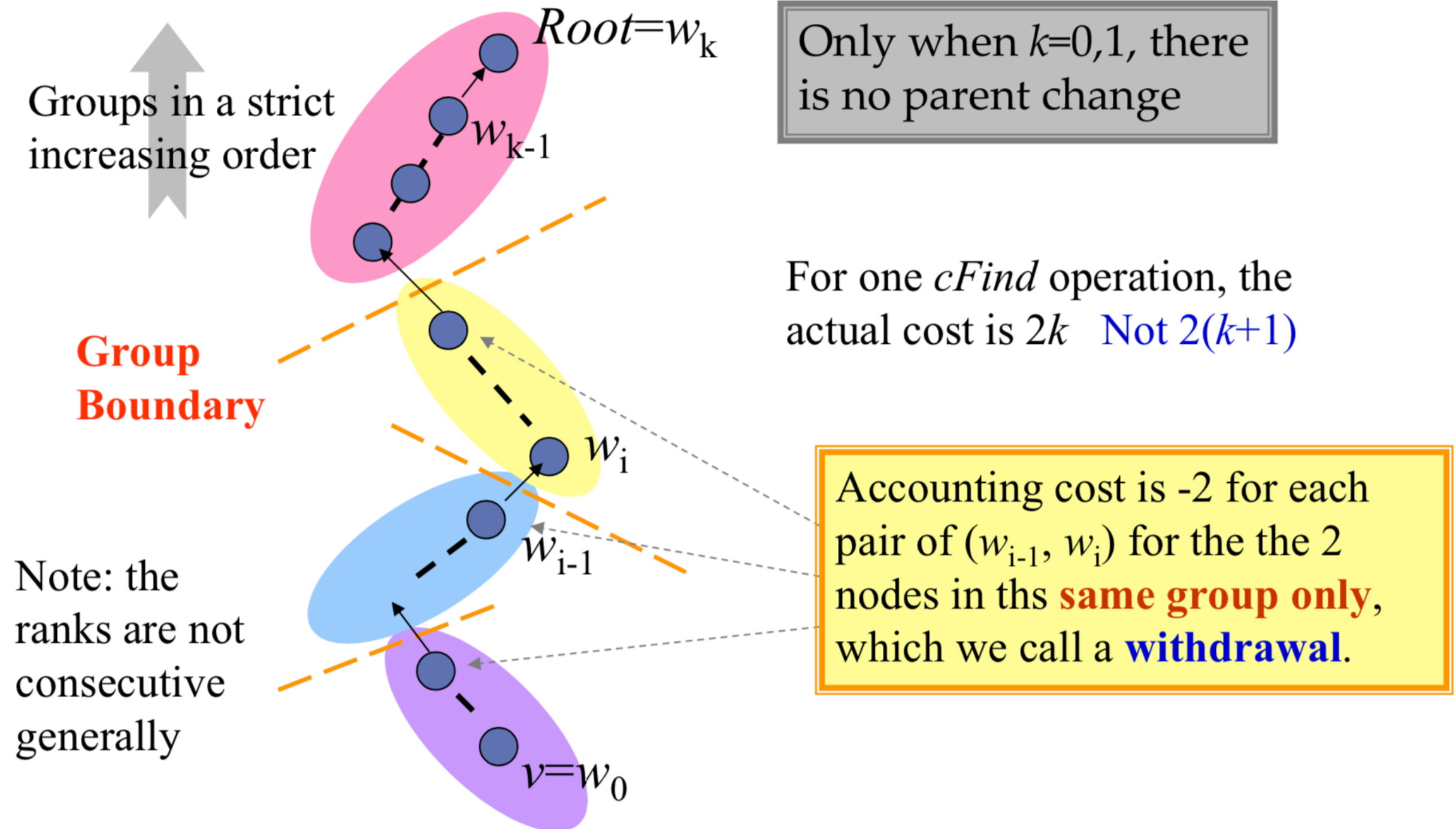
$$\underbrace{(k-1) \text{ 2's}}_{2^2} \geq \log(n+1)$$

The diagram shows a tree structure with a root node labeled '(k-1) 2's'. Below the root, there are two branches, each labeled '2'. A blue arrow points from the right side of the tree towards the right edge of the slide.

Amortized Cost of Union-Find

- Amortized Equation Recalled
 - amortized cost = actual cost + accounting cost
- The operations to be considered:
 - n makeSets
 - m union & find (with at most $n-1$ unions)

One Execution of $cFind(w_0)$



Amortizing Scheme for $wUnion-cFind$

- **makeSet**

- Accounting cost is $4\log^*(n+1)$
- So, the amortized cost is $1+4\log^*(n+1)$

- **$wUnion$**

- Accounting cost is 0
- So the amortized cost is 1

- **$cFind$**

- Accounting cost is describes as in the previous page.
- Amortized cost $\leq 2k-2((k-1)-(\log^*(n+1)-1))=2\log^*(n+1)$
(Compare with the worst case cost of $cFind$, $2\log n$)

Number of withdrawal



Validation of the Amortizing Scheme

- We must be assure that **the sum of the accounting costs is never negative.**
- The sum of the negative charges, incurred by *cFind*, does not exceed $4n\log^*(n+1)$
- We prove this by showing that at most $2n\log^*(n+1)$ withdrawals on nodes occur during all the executions of *cFind*.

Key Idea in the Derivation

- For any node, the number of withdrawal will be less than the number of different ranks in the group it belongs to
 - When a *cFind* changes the parent of a node, the new parent is always has higher rank than the old parent.
 - Once a node is assigned a new parent in a **higher group**, no more negative amortized cost will incurred for it again.
- The number of different ranks is limited within a group.

Derivation

- Bounding the number of withdrawals

The number of withdrawals from all $w \in S$ is:

a loose upper bound
of ranks in a group

$$\sum_{i=0}^{\log^*(n+1)-1} H(i) (\text{number of nodes in group } i)$$

The number of nodes in group i is at most:

$$\sum_{r=H(i-1)}^{H(i)-1} \frac{n}{2^r} \leq \frac{n}{2^{H(i-1)}} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{2n}{2^{H(i-1)}} = \frac{2n}{H(i)}$$

So,

$$\sum_{i=0}^{\log^*(n+1)-1} H(i) \frac{2n}{H(i)} = 2n \log^*(n+1)$$

Conclusion

- The number of link operations done by a Union-Find program implemented with wUnion and cFind, of length m on a set of n elements is in $O((n+m)\log^*(n))$ in the worst case.
- Note: since the sum of accounting cost is never negative, the actual cost is always not less than amortized cost. The upper bound of amortized cost is: $(n+m)(1+4\log^*(n+1))$

Thank you!

Q & A