Stats 215A, Fall 2020
Homework 3

1 EM Algorithm

1. Our observed data is the X vector the latent variable is the cluster assignment which we'll denote Z with $z_i \sim \text{Ber}(\pi)$. The joint likelihood is

$$L(X, Z; \theta) = \prod_{i=1}^{n} (\pi f_{\mu_0}(x_i))^{z_i} \cdot ((1-\pi) f_{\mu_1}(x_i))^{1-z_i}$$

The vector of model parameters is $\theta = (\pi, \mu_1, \mu_2)$ It is also interesting to note that using bayes theorem

$$\begin{aligned}\mathbb{P}(z_i = 1|X, \theta) &= \frac{\pi f_{\mu_0}(x_i) \cdot \mathbb{P}(X_{(-i)})}{\pi f_{\mu_0}(x_i) \cdot \mathbb{P}(X_{(-i)}) + (1-\pi) f_{\mu_1}(x_i) \cdot \mathbb{P}(X_{(-i)})} \\ &= \frac{\pi f_{\mu_0}(x_i)}{\pi f_{\mu_0}(x_i) + (1-\pi) f_{\mu_1}(x_i)}\end{aligned}$$

Which defines the conditional distribution of Z given X and θ

2. The E step we take the conditional expectation of the log-likelihood given our current estimate of the parameters over Z

$$\begin{aligned}\mathbb{E}_{Z|X,\theta^{(t)}} \log(L(X, Z; \theta)) &= \mathbb{E}_{Z|X,\theta^{(t)}} \sum_{i=1}^{n} z_i \log(pi f_{\mu_0}(x_i)) + (1-z_i)\log((1-pi) f_{\mu_1}(x_i)) \\ &= \sum_{i=1}^{n} t_i^{(t)} \log(\pi f_{\mu_0}(x_i)) + (1-t_i^{(t)}) \log((1-\pi) f_{\mu_1}(x_i)) \\ &= \sum_{i=1}^{n} t_i^{(t)} (\log(\pi) + \log(f_{\mu_0}(x_i))) + (1-t_i^{(t)})(\log((1-\pi)) + \log(f_{\mu_1}(x_i)))\end{aligned}$$

With $t_i^{(t)} = \mathbb{P}(z_i = 1|X, \theta^{(t)})$.
What we actually do in the E step is update $t_i^{(t-1)}$ to $t_i^{(t)}$, it'll be clear why after we derive the M step formula.

3. On the M step we maximize the expected likelihood with respect to the parameters

$$\frac{\partial}{\partial \pi} \mathbb{E}_{Z|X,\theta^{(t)}} \log(L(X, Z; \theta)) = \sum_{i=1}^{n} t_i^{(t)} (\frac{1}{\pi}) - (1-t_i^{(t)})(\frac{1}{1-\pi})$$

Finding the roots of the equation we have

$$\pi^{(t+1)} = \frac{\sum_{i=1}^{n} t_i^{(t)}}{n}$$

Next we move to the estimation of μ_0, μ_1

$$\frac{\partial}{\partial \mu_0} \mathbb{E}_{Z|X,\theta^{(t)}} \log(L(X, Z; \theta)) = \sum_{i=1}^{n} t_i^{(t)} \frac{x_i}{\mu_0} - n$$

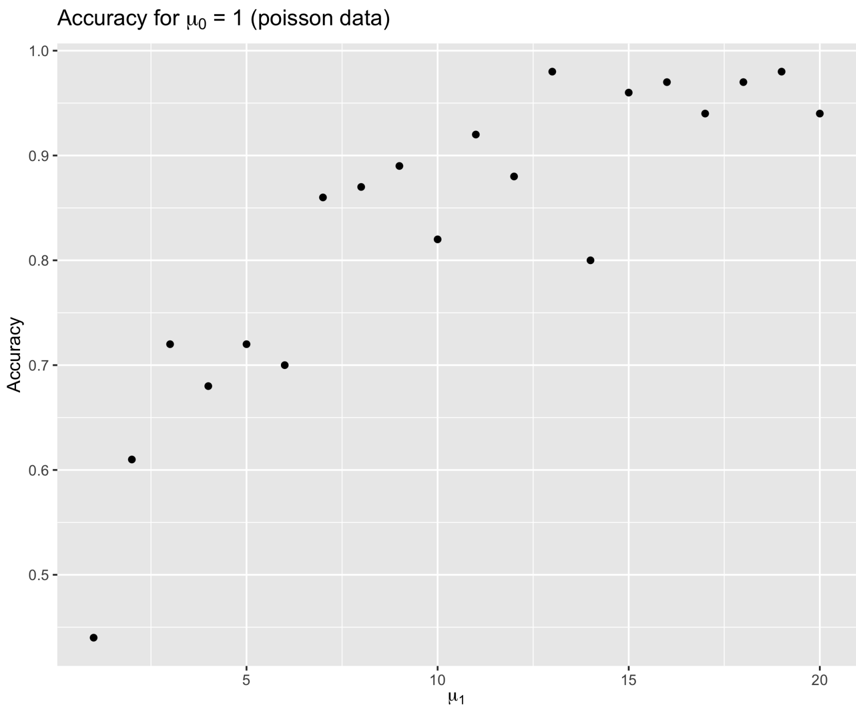
Finding the roots of the equation we have

$$\mu_0^{(t)} = \frac{\sum_{i=1}^{n} t_i^{(t)} x_i}{n}$$

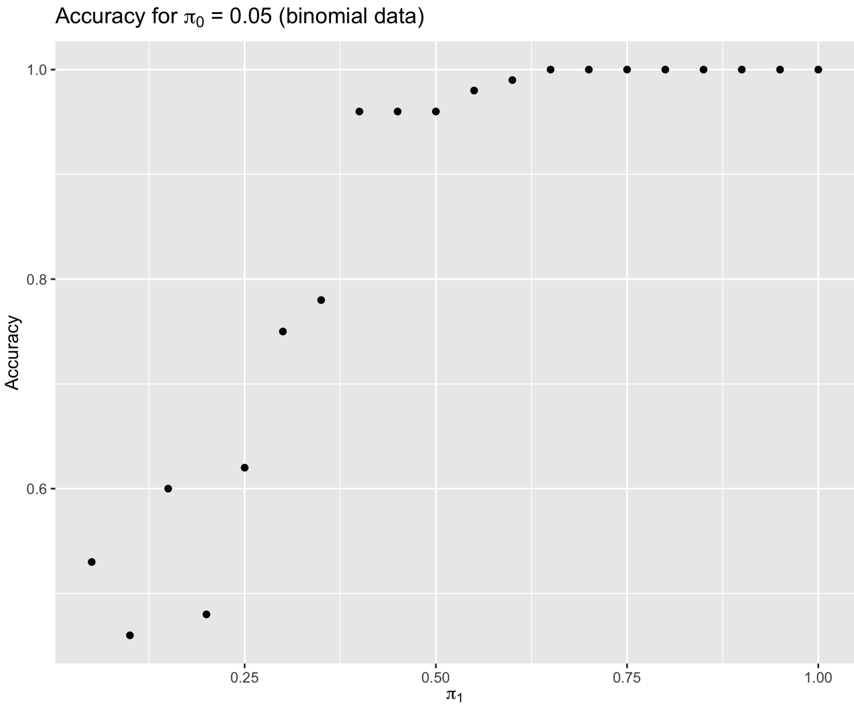
and using identical calculations we have

$$\mu_1^{(t)} = \frac{\sum_{i=1}^{n} (1 - t_i^{(t)}) x_i}{n}$$

4. Poisson data



5. Binomial data


Accuracy for π0 = 0.05 (binomial data)
1.0
0.8
0.6
Accuracy
0.25
0.50
0.75
1.00
π1