Homework 2 - Omer Ronen
Stats 215A, Fall 2020

1 Principal Components Analysis and SVD

1. First we define the Lagrangian

$$L(v, \lambda) = v^T X^T X v + \lambda(v^T v - 1)$$

Next we take the derivatives

$$\frac{\partial L}{\partial v} = 2v^T X^T X + 2\lambda v^T$$
$$\frac{\partial L}{\partial \lambda} = v^T v - 1$$

Solving we first equation we have that

$$v^T X^T X = \lambda v^T$$

Which mean that v is an eigen vector for the matrix $X^T X$ using the SVD of X we note that $X^T X = VDU^T UDV^T = VD^2 V^T$. We see that the columns of V are the eigen vectors and D^2 is the diagonal matrix of the eigen values.
As each eigen vector is a stationary point for the problem, we note that for every eigen vector $v^\star$

$$(v^\star)^T X^T X v^\star = d^\star$$

where $d^\star$ is the corresponding eigen value in D^2 matrix. Since we can assume that all values in D are positive (if not we set $u_i' = -u_i$) this means that $v = v_1$.

2. We know from the previous section that $X^T X = VD^2 V^T$, now let v be any vector satisfying the conditions in the question then

$$v^T X^T X v = v^T V D^2 V^T v$$

It's easy to to see that

$$h_j = (v^T V D)_i = v^T v_i$$

And this vector is equal to zero for $i < j$. To conclude we simply note that

$$v^T X^T X v = hh^T = \sum_{k=j}^{p} h_k h_k^T = \sum_{k=j}^{p} v^T (d_k^2 v_k v_k^T) v$$

3. It is easy too see that when we plug in $X = X_{(j)}$ and repeat the same calculations as in (1) we would reach the same eigen value problem where we would select the eigen vector corresponding to the largest eigen value of $X_{(j)}^T X_{(j)}$
We prove the equivalence by induction, (1) is the induction basis, assume that $\hat{v_1}, \dots, \hat{v_{j-1}}$ are the eigen vectors corresponding to the $j-1$ largest eigen values of the matrix $X^T X$ and therefore solution to PC1-PCj-1 problems.
Let v any feasible solution for the PCj problem, and note that the objective can be written as

$$v^T X^T X v = \sum_{h=j}^{p} \sum_{k=j}^{p} v^T (v_h d_h d_k v_k^T) v$$

Any feasible solution must be orthogonal to $\{v_1, \dots, v_{j-1}\}$ by the induction assumption, so $v^T \in \text{Span}(\{v_j, \dots, v_p\})$ and we can write $v = \sum_{k=j}^{p} \alpha_k v_k$; $\sum_{k=j}^{p} \alpha_k = 1$, and see that

$$\sum_{h=j}^{p} \sum_{k=j}^{p} v^T (v_h d_h d_k v_k^T) v = \sum_{h=j}^{p} \sum_{k=j}^{p} \sum_{n=j}^{p} \alpha_n \cdot \left(v_n^T (v_h d_h d_k v_k^T) v_n \right)$$

By orthogonality this sum reduces to

$$\sum_{k=j}^{p} \alpha_k \cdot \left(v_k^T (v_k d_k d_k v_k^T) v_k \right) = \sum_{k=j}^{p} \alpha_k d_k^2$$

.

We note that this is once again an eigen value problem $\square$.

4. Follows immediately from (3)

2 Ordinary Least Squares

1. Let define the objective function as:

$$\|y - X\beta\|_2^2 = (f \circ g)(x); \quad f(g) = g^T g, \; g(X) = y - X\beta$$

Using the chain rule we take derivative

$$\frac{df}{d\beta} = \frac{df}{dg} \cdot \frac{dg}{dX} = 2g(X)^T \cdot (-X) = -2(y - \beta X)^T X$$

Equating to zero and dividing the -2 we have that

$$y^T X = \beta^T X^T X \Rightarrow b^T = y^T X (X^T X)^{-1}$$

The inversion of $X^T X$ requires that X is of full rank

2.
$$X \hat{\beta}_{\text{OLS}} = \underbrace{X (X^T X)^{-1} X^T}_{H} y$$

Indeed

$$H^2 = X(X^T X)^{-1} \underbrace{X^T X (X^T X)^{-1}}_{I} X^T = X(X^T X)^{-1} X^T = H$$

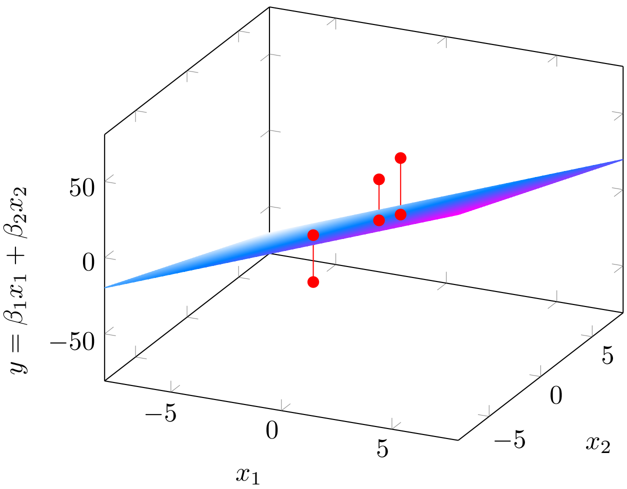
3. We can write the residual as

$$\hat{r} = y - \hat{y} = (I - H) y$$

Noticing that H is a symmetric matrix we conclude that

$$\hat{r}^T \hat{y} = y^T \underbrace{(I - H)(H)}_{0} y = 0$$

Where $H(I - H) = H - H^2 = H - H = 0$ by (2) $\square$


$y = \beta_1 x_1 + \beta_2 x_2$
50
0
−50
−5
0
5
x_1
−5
0
5
x_2