CSE 127: Introduction to Security

Public-Key Cryptography

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UCSD

Today

- Key Exchange
- Public Key Encryption
- Digital Signatures

Asymmetric cryptography/public-key cryptography

Main insight: Separate keys for different operations.

Keys come in pairs, and are related to each other by the specific algorithm

- Public key: used to encrypt or verify signatures
- Private key: used to decrypt and sign

Public-key encryption

- Encryption: (public key, plaintext) $\rightarrow$ ciphertext

$$\mathrm{Enc}_{pk}(m) = c$$

- Decryption: (secret key, ciphertext) $\rightarrow$ plaintext

$$\mathrm{Dec}_{sk}(c) = m$$

Properties:

- Encryption and decryption are inverse operations:

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Properties:

- Encryption and decryption are inverse operations:

$$\mathrm{Dec}_{sk}(\mathrm{Enc}_{pk}(m)) = m$$

- Secrecy: ciphertext reveals nothing about plaintext
  - Computationally hard to decrypt without secret key
- The point:
  - Anybody with your public key can send you a secret message! Solves key distribution problem.

Modular Arithmetic Review

Division: Let n, d, q, r be integers.

$$\begin{aligned} \lfloor n/d \rfloor &= q \\ n &= qd + r \qquad 0 \leq r < d \\ n &\equiv r \bmod d \end{aligned}$$

Facts about remainders/modular arithmetic:

$$\begin{aligned} \text{Add: } & (a \bmod d) + (b \bmod d) \equiv (a + b) \bmod d \\ \text{Subtract: } & (a \bmod d) - (b \bmod d) \equiv (a - b) \bmod d \\ \text{Multiply: } & (a \bmod d) \cdot (b \bmod d) \equiv (a \cdot b) \bmod d \end{aligned}$$

Modular Inverse: "Division" for modular arithmetic

If $a \cdot b \mod d = c \bmod d$ we would like $c/b \bmod d = a \bmod d$.

Let's try this: let $a = 3$, $b = 2$, and $d = 4$

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This doesn't quite work, it says $3 = 1 \bmod 4$!

Fix: For rationals, $\frac{a}{b} = a \cdot \frac{1}{b}$ $\qquad b \cdot \frac{1}{b} = 1$.

Define modular inverse: $\frac{1}{b}$ means $b^{-1} \bmod d$.

- $b^{-1} \bmod d$ is a value such that $b \cdot b^{-1} \equiv 1 \bmod d$.
- Example: $3 \cdot (3^{-1} \bmod 5) \equiv$

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- $b^{-1} \bmod d$ is a value such that $b \cdot b^{-1} \equiv 1 \bmod d$.
- Example: $3 \cdot (3^{-1} \bmod 5) \equiv 3 \cdot 2 \equiv 1 \bmod 5$.
- If $\gcd(a, d) = 1$ then a^{-1} is well defined.
- Efficient to compute.

Modular exponentiation and discrete log

Modular exponentiation

- Over the integers, $g^a = g \cdot g \cdot g \ldots g$
- $g^a \bmod d$ it's the same:
  $g^a \bmod d = (((g \bmod d) \cdot g \bmod d) \ldots g \bmod d) \bmod d$
- Efficient to compute using the binary representation of a.

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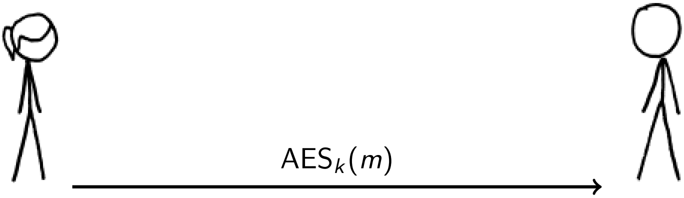
- Define discrete log similarly:
  Input b, d, y, discrete log is a such that $b^a \equiv y \bmod d$.
- No known polynomial-time algorithm to compute this.

New Directions in Cryptography

Invited Paper

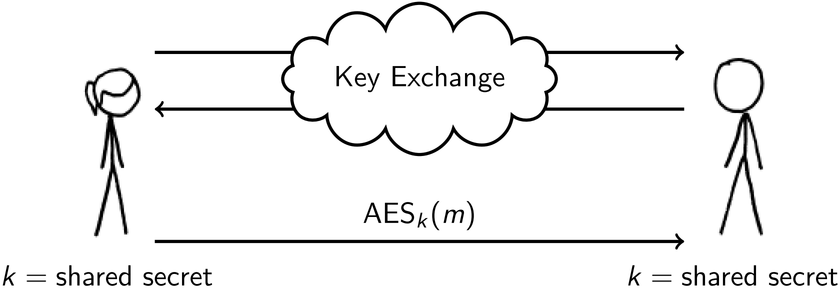
WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

Symmetric cryptography



Public key crypto idea # 1: Key exchange

Solving key distribution without trusted third parties



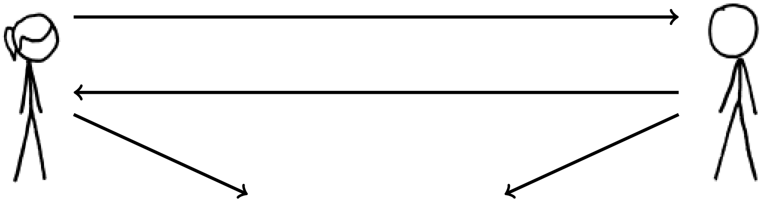
Textbook Diffie-Hellman Key Exchange

Public Parameters

p a prime

g an integer mod p

Key Exchange



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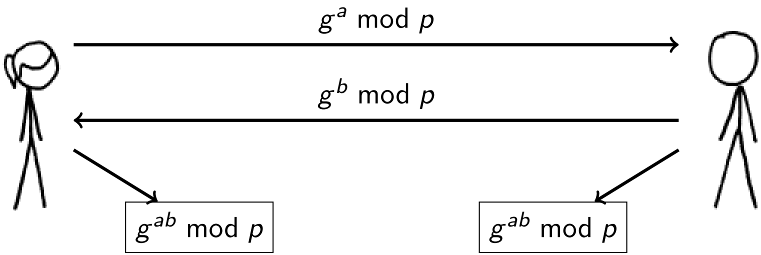
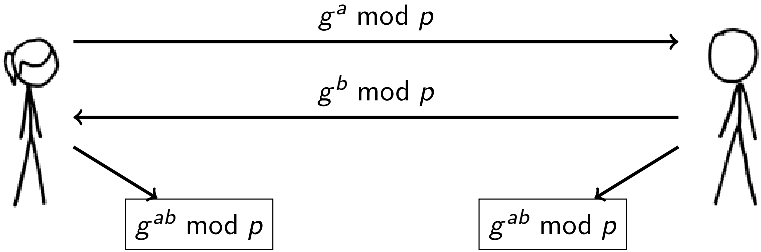


Note: $(g^a)^b \bmod p = g^{ab} \bmod p = g^{ba} \bmod p (g^b)^a \bmod p$.

Diffie-Hellman Security



- Most efficient algorithm for passive eavesdropper to break: Compute discrete log of public values $g^a \bmod p$ or $g^b \bmod p$.

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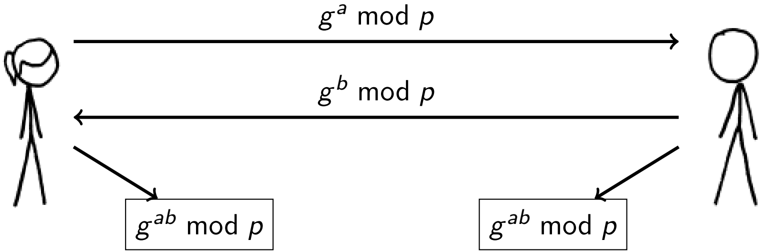

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- Parameter selection: p should be ≥ 2048 bits.

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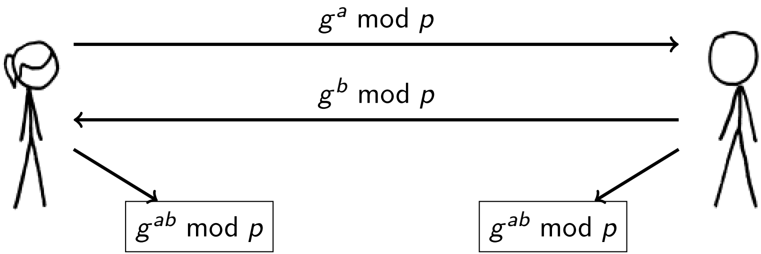
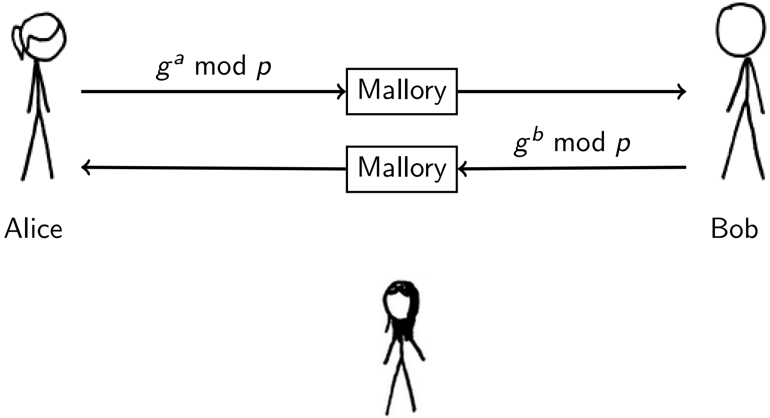


- Most efficient algorithm for passive eavesdropper to break: Compute discrete log of public values g^a mod p or g^b mod p.
- Parameter selection: p should be ≥ 2048 bits.
- Do <u>not</u> implement this yourself ever: discrete log is only hard for certain choices of p and g.
- Best current choice: Use elliptic curve Diffie-Hellman. (Similar idea, more complicated math.)

Diffie-Hellman insecure against man-in-the-middle



Active adversary can modify Diffie-Hellman messages in transit and learn both shared secrets.

Allows transparent MITM attack against later encryption.

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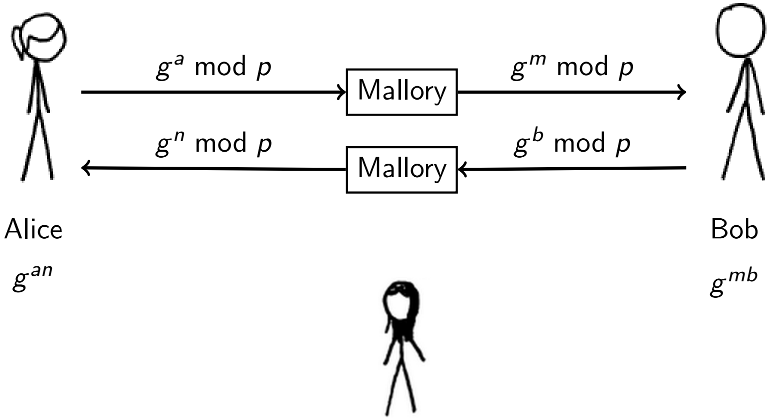

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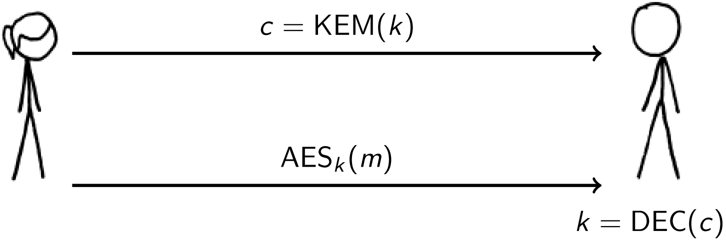
Fix: Need to authenticate messages.

Computational complexity for integer problems

- Integer multiplication is efficient to compute.
- There is no known polynomial-time algorithm for general-purpose factoring.
- Efficient factoring algorithms for many types of integers. Easy to find small factors of random integers.
- Modular exponentiation is efficient to compute.
- Modular inverses are efficient to compute.

Idea # 2: Key encapsulation/public-key encryption

Solving key distribution without trusted third parties



A Method for Obtaining Digital Signatures and Public-Key Cryptosystems

R.L. Rivest, A. Shamir, and L. Adleman*

Textbook RSA Encryption

[Rivest Shamir Adleman 1977]

Public Key pk

$N = pq$ modulus

e encryption exponent

Secret Key sk

p, q primes

d decryption exponent

$(d = e^{-1} \bmod (p-1)(q-1) = e^{-1} \bmod \phi(N))$

$pk = (N, e)$

$c = \mathrm{Enc}_{pk}(m) = m^e \bmod N$

$$\boxed{m = \mathrm{Dec}_{sk}(c) = c^d \bmod N}$$

$\mathrm{Dec}(\mathrm{Enc}(m)) = m^{ed} \bmod N \equiv m^{1+k\phi(N)} \equiv m \bmod N$ by Euler's theorem ($m^{\phi(N)} \equiv 1 \bmod N$).

RSA Security

- Best algorithm to break RSA: Factor N and compute d.
- Factoring is not efficient in general.
- Current key size recommendations: N should be ≥ 2048 bits.
- Do not ever implement this yourself. Factoring is only hard for some integers, and textbook RSA is insecure.

Textbook RSA is super insecure

Unpadded RSA encryption is homomorphic under multiplication.

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Attack: Malleability

Given a ciphertext $c = \text{Enc}(m) = m^e \bmod N$, attacker can forge ciphertext $\text{Enc}(ma) = ca^e \bmod N$ for any a.

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Attack: Chosen ciphertext attack

Given a ciphertext $c = \mathrm{Enc}(m)$ for unknown m, attacker asks for $\mathrm{Dec}(ca^e \bmod N) = d$ and computes $m = da^{-1} \bmod N$.

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Fix: always use padding on messages.

RSA PKCS #1 v1.5 padding

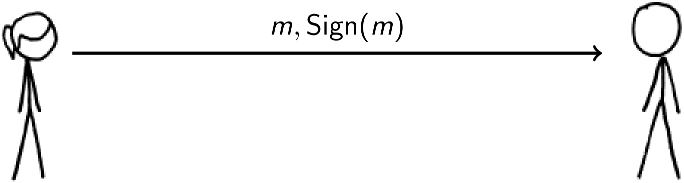
Most common implementation choice even though it is insecure

```
pad(m) = 00 02 [random padding string] 00 [m]
```

- Encrypter pads message, then encrypts padded message using RSA public key: $\text{Enc}_{pk}(m) = pad(m)^e \bmod N$
- Decrypter decrypts using RSA private key, strips off padding to recover original data: $\text{Dec}_{sk}(c) = c^d \bmod N = pad(m)$

PKCS#1v1.5 padding is vulnerable to a number of padding attacks. It is still commonly used in practice.

Idea #3: Digital Signatures



Verify $\mathrm{Sign}(m)$

Bob wants to verify Alice's signature using only a public key.

- Signature verifies that Alice was the only one who could have sent this message.
- Signature also verifies that the message hasn't been modified in transit.

Digital Signatures

- Signing: (secret key, message) $\rightarrow$ signature

$$\mathrm{Sign}_{sk}(m) = s$$

- Verification: (public key, message, signature) $\rightarrow$ bool

$$\mathrm{Verify}_{pk}(m, s) = \mathrm{true} \mid \mathrm{false}$$

Signature properties:

- Verification of signed message succeeds:

Digital Signatures

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Signature properties:

- Verification of signed message succeeds:
  - $\text{Verify}_{pk}(m, Sign_{sk}(m)) = \text{true}$
- Unforgeability: Can't compute signature for message m that verifies with public key without corresponding secret key.
- The point:
  - Anybody with your public key can verify that you signed something!

Textbook RSA Signatures

[Rivest Shamir Adleman 1977]

Public Key pk

$N = pq$ modulus

e public exponent

Secret Key sk

p, q primes

d private exponent

$(d = e^{-1} \bmod (p-1)(q-1))$

$pk = (N, e)$

m, s $= \mathrm{Sign}(m) = m^d \bmod N$

$\boxed{\mathrm{Verify}(m, s):\ m = s^e \bmod N}$

Works for the same reason RSA encryption does.

Textbook RSA signatures are super insecure

Attack: Signature forgery

1. Attacker wants $\mathrm{Sign}(x)$.
2. Attacker computes $z = xy^e \bmod N$ for some y.
3. Attacker asks signer for $s = \mathrm{Sign}(z) = z^d \bmod N$.
4. Attacker computes $\mathrm{Sign}(x) = sy^{-1} \bmod N$.

Countermeasures:

- **Always use padding with RSA.**
- **Sign hash of m and not raw message m.**

Positive viewpoint:

- Blind signatures: Lots of neat crypto applications.

RSA PKCS #1 v1.5 signature padding

Most widely used padding scheme in practice

```
pad(m) = 00 01 [FF FF FF ...  FF FF] 00 [data H(m)]
```

- Signer hashes and pads message, then signs padded message using RSA private key.
- Verifier verifies using RSA public key, strips off padding to recover hash of message.

Q: What happens if a decrypter doesn't correctly check padding length?

A: **Bleichenbacher low exponent signature forgery attack.**

Bleichenbacher RSA Signature Forgery

```
pad(m) = 00 01 [FF FF FF ...  FF FF] 00 [data H(m)]
```

If victim shortcuts padding check: just looks for padding format but doesn't check length, and signature uses $e = 3$:

1. Construct a perfect cube over the integers, ignoring N, such that

   $$x = 0001FF \dots FF00[\text{hash of forged message}][\text{garbage}]$$

2. Compute s such that $s^3 = x$.
   (Easy way: set garbage to zero and take cube root, i.e., $s = \lceil x \rceil^{1/3}$.)

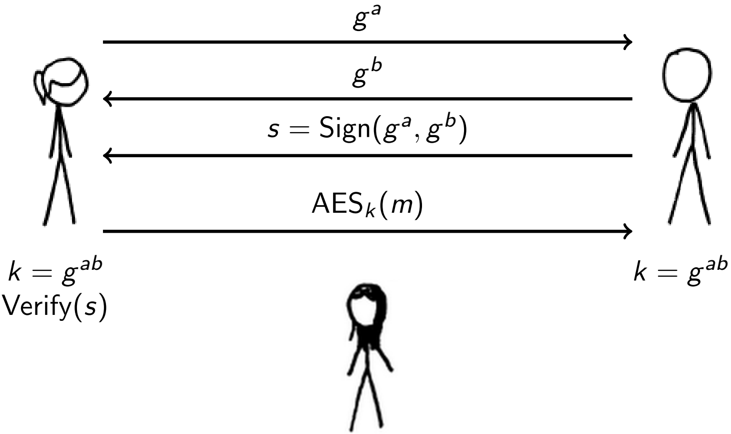
3. Lazy implementation validates bad signature!

Security for RSA signatures

- Same as RSA encryption.
- Recommendation: Use ECDSA or ed25519 instead.

Putting it all together

How public-key cryptography is used in practice



- Diffie-Hellman used to negotiate shared session key.
- Alice verifies Bob's signature to ensure that key exchange was not man-in-the-middled.
- Shared secret used to symmetrically encrypt data.

Public-key cryptography and quantum computers

Right now, all public-key cryptography used in the real world involves three "hard" problems:

- Factoring
- Discrete log mod primes
- Elliptic curve discrete log

All of these problems can be solved efficiently by a general-purpose quantum computer.

Big standardization effort now to develop replacements:

- Lattice-based cryptography
- Multivariate cryptography
- Hash-based signatures
- Supersingular isogeny Diffie-Hellman

These will likely be used more in the real world in the next few years.