CSE 127: Introduction to Security

Lecture 16: Public-Key Cryptography

Deian Stefan

UCSD

Fall 2020

Material from Nadia Heninger

"We stand today on the brink of a revolution in cryptography."

— Diffie and Hellman, 1976

Lecture Outline

- Key Exchange
- Public Key Encryption
- Digital Signatures

Asymmetric cryptography/public-key cryptography

Main insight: Separate keys for different operations.

Keys come in pairs, and are related to each other by the specific algorithm:

- Public key: known to everyone, used to encrypt or verify signatures
- Private key: used to decrypt and sign

Public-key encryption

- Encryption: (public key, plaintext) $\rightarrow$ ciphertext

$$\mathsf{Enc}_{pk}(m) = c$$

- Decryption: (secret key, ciphertext) $\rightarrow$ plaintext

$$\mathsf{Dec}_{sk}(c) = m$$

Properties:

- Encryption and decryption are inverse operations:

Public-key encryption

- Encryption: (public key, plaintext) $\rightarrow$ ciphertext

$$\mathrm{Enc}_{pk}(m) = c$$

- Decryption: (secret key, ciphertext) $\rightarrow$ plaintext

$$\mathrm{Dec}_{sk}(c) = m$$

Properties:

- Encryption and decryption are inverse operations:

$$\mathrm{Dec}_{sk}(\mathrm{Enc}_{pk}(m)) = m$$

- Secrecy: ciphertext reveals nothing about plaintext
  - Computationally hard to decrypt without secret key
- The point:
  - Anybody with your public key can send you a secret message! Solves key distribution problem.

Modular Arithmetic Review

Division: Let n, d, q, r be integers.

$$\begin{aligned} \lfloor n/d \rfloor &= q \\ n &= qd + r \qquad 0 \le r < d \\ n &\equiv r \bmod d \end{aligned}$$

Facts about remainders/modular arithmetic:

$$\begin{aligned} \text{Add: } & (a \bmod d) + (b \bmod d) \equiv (a + b) \bmod d \\ \text{Subtract: } & (a \bmod d) - (b \bmod d) \equiv (a - b) \bmod d \\ \text{Multiply: } & (a \bmod d) \cdot (b \bmod d) \equiv (a \cdot b) \bmod d \end{aligned}$$

Modular Inverse: "Division" for modular arithmetic

If $a \cdot b \mod d = c \bmod d$ we would like $c/b \bmod d = a \bmod d$.

Let's try this: let $a = 3$, $b = 2$, and $d = 4$

Modular Inverse: "Division" for modular arithmetic

If $a \cdot b \mod d = c \mod d$ we would like $c/b \mod d = a \mod d$.

Let's try this: let $a = 3$, $b = 2$, and $d = 4$

This doesn't quite work, it says $3 = 1 \mod 4$!

Fix: For rationals, $\frac{a}{b} = a \cdot \frac{1}{b}$ $\qquad b \cdot \frac{1}{b} = 1$.

Define modular inverse: $\frac{1}{b}$ means $b^{-1} \mod d$.

- $b^{-1} \mod d$ is a value such that $b \cdot b^{-1} \equiv 1 \mod d$.
- Example: $3 \cdot (3^{-1} \mod 5) \equiv 3 \cdot 2 \equiv 1 \mod 5$.
- If $\gcd(a, d) = 1$ then a^{-1} is well defined.
- Efficient to compute.

Modular exponentiation and discrete log

Modular exponentiation

- Over the integers, $g^a = g \cdot g \cdot g \ldots g$
- $g^a \bmod d$ it's the same:
  $g^a \bmod d = (((g \bmod d) \cdot g \bmod d) \ldots g \bmod d) \bmod d$
- Efficient to compute using the binary representation of a.

Modular exponentiation and discrete log

Modular exponentiation

- Over the integers, $g^a = g \cdot g \cdot g \ldots g$
- $g^a \bmod d$ it's the same:
  $g^a \bmod d = (((g \bmod d) \cdot g \bmod d) \ldots g \bmod d) \bmod d$
- Efficient to compute using the binary representation of a.

"Inverse" of modular exponentiation: Discrete log

- Over the reals, if $b^a = y$ then $\log_b y = a$.

Modular exponentiation and discrete log

Modular exponentiation

- Over the integers, $g^a = g \cdot g \cdot g \ldots g$
- $g^a \bmod d$ it's the same:
  $g^a \bmod d = (((g \bmod d) \cdot g \bmod d) \ldots g \bmod d) \bmod d$
- Efficient to compute using the binary representation of a.

"Inverse" of modular exponentiation: Discrete log

- Over the reals, if $b^a = y$ then $\log_b y = a$.
- Define discrete log similarly:
  Input b, d, y, discrete log is a such that $b^a \equiv y \bmod d$.

Modular exponentiation and discrete log

Modular exponentiation

- Over the integers, $g^a = g \cdot g \cdot g \dots g$
- $g^a \bmod d$ it's the same:
  $g^a \bmod d = (((g \bmod d) \cdot g \bmod d) \dots g \bmod d) \bmod d$
- Efficient to compute using the binary representation of a.

"Inverse" of modular exponentiation: Discrete log

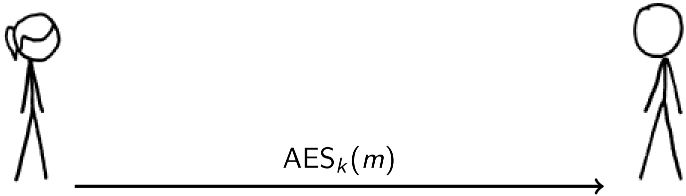
- Over the reals, if $b^a = y$ then $\log_b y = a$.
- Define discrete log similarly:
  Input b, d, y, discrete log is a such that $b^a \equiv y \bmod d$.
- No known polynomial-time algorithm to compute this.

New Directions in Cryptography

Invited Paper

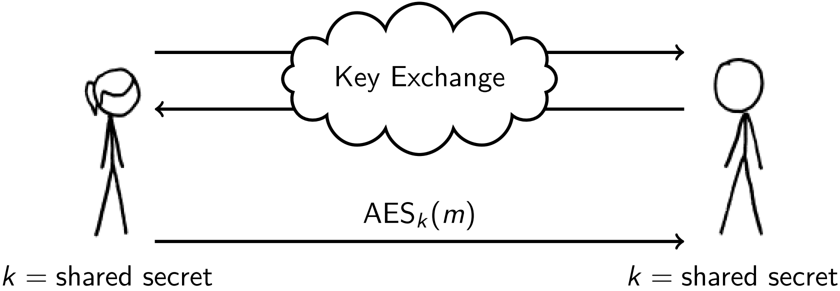
WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

Symmetric cryptography



Public key crypto idea # 1: Key exchange

Solving key distribution without trusted third parties



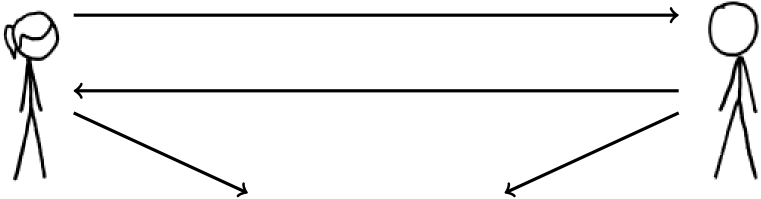
Textbook Diffie-Hellman Key Exchange

Public Parameters

p a prime

g an integer $\bmod p$

Key Exchange



Note: $(g^a)^b \bmod p = g^{ab} \bmod p = g^{ba} \bmod p = (g^b)^a \bmod p$.

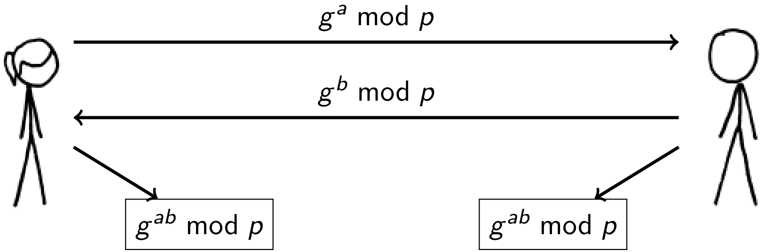
Textbook Diffie-Hellman Key Exchange

Public Parameters

p a prime

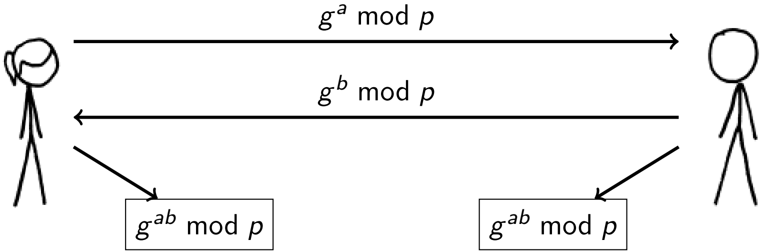
g an integer mod p

Key Exchange



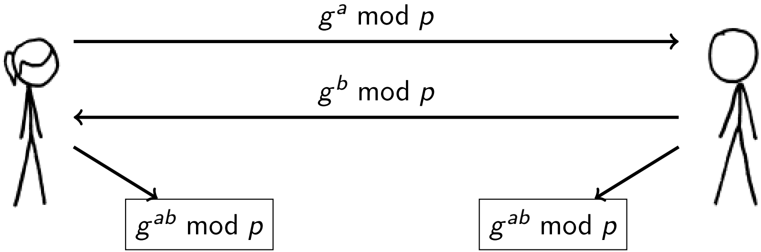
Note: $(g^a)^b \bmod p = g^{ab} \bmod p = g^{ba} \bmod p (g^b)^a \bmod p$.

Diffie-Hellman Security



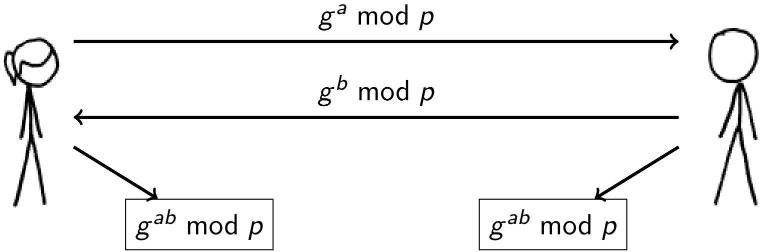
- Most efficient algorithm for passive eavesdropper to break: Compute discrete log of public values $g^a \bmod p$ or $g^b \bmod p$.

Diffie-Hellman Security



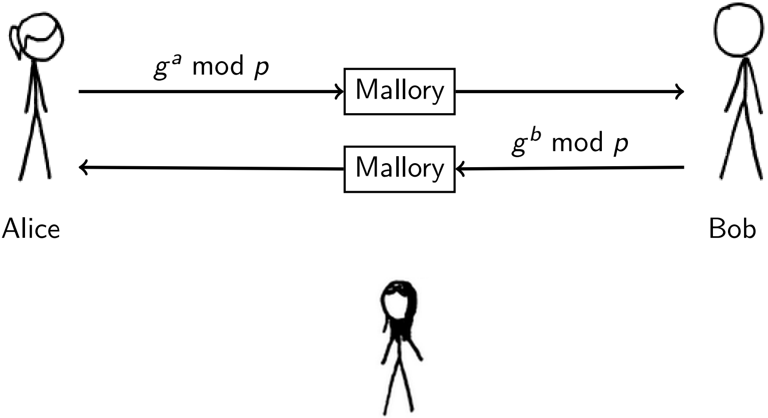
- Most efficient algorithm for passive eavesdropper to break: Compute discrete log of public values $g^a \bmod p$ or $g^b \bmod p$.
- Parameter selection: p should be ≥ 2048 bits.

Diffie-Hellman Security



- Most efficient algorithm for passive eavesdropper to break: Compute discrete log of public values $g^a \bmod p$ or $g^b \bmod p$.
- Parameter selection: p should be ≥ 2048 bits.
- Do <u>not</u> implement this yourself ever: discrete log is only hard for certain choices of p and g.
- Best current choice: Use elliptic curve Diffie-Hellman. (Similar idea, more complicated math.)

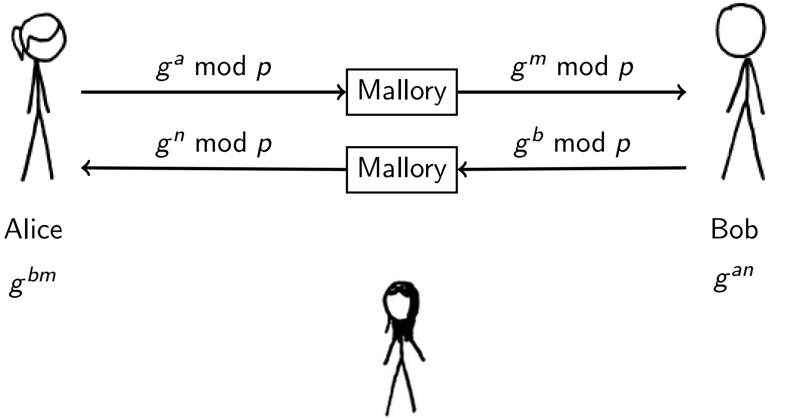
Diffie-Hellman insecure against man-in-the-middle



Active adversary can modify Diffie-Hellman messages in transit and learn both shared secrets.

Allows transparent MITM attack against later encryption.

Diffie-Hellman insecure against man-in-the-middle



Active adversary can modify Diffie-Hellman messages in transit and learn both shared secrets.

Allows transparent MITM attack against later encryption.

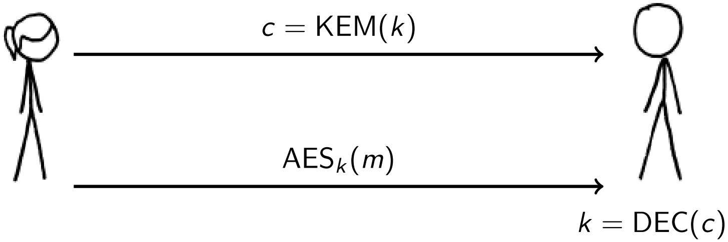
Fix: Need to authenticate messages.

Computational complexity for integer problems

- Integer multiplication is efficient to compute.
- There is no known polynomial-time algorithm for general-purpose factoring.
- Efficient factoring algorithms for many types of integers. Easy to find small factors of random integers.
- Modular exponentiation is efficient to compute.
- Modular inverses are efficient to compute.

Idea # 2: Key encapsulation/public-key encryption

Solving key distribution without trusted third parties



A Method for Obtaining Digital Signatures and Public-Key Cryptosystems

R.L. Rivest, A. Shamir, and L. Adleman*

Textbook RSA Encryption

[Rivest Shamir Adleman 1977]

Public Key pk

$N = pq$ modulus

e encryption exponent

Secret Key sk

p, q primes

d decryption exponent

$(d = e^{-1} \bmod (p-1)(q-1) = e^{-1} \bmod \phi(N))$

$pk = (N, e)$

$c = \mathrm{Enc}_{pk}(m) =$

$\boxed{m = \mathrm{Dec}_{sk}(c) =}$

$\mathrm{Dec}(\mathrm{Enc}(m)) =$

Textbook RSA Encryption

[Rivest Shamir Adleman 1977]

Public Key pk

$N = pq$ modulus

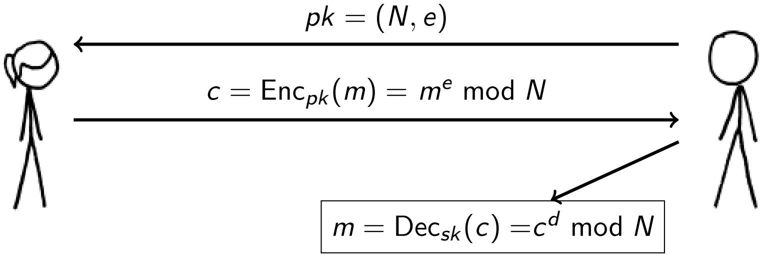
e encryption exponent

Secret Key sk

p, q primes

d decryption exponent

$(d = e^{-1} \bmod (p-1)(q-1) = e^{-1} \bmod \phi(N))$



$\mathrm{Dec}(\mathrm{Enc}(m)) = m^{ed} \bmod N \equiv m^{1+k\phi(N)} \equiv m \bmod N$ by Euler's theorem.

RSA Security

- Best algorithm to break RSA: Factor N and compute d.
- Factoring is not efficient in general.
- Current key size recommendations: N should be ≥ 2048 bits.
- Do not ever implement this yourself. Factoring is only hard for some integers, and textbook RSA is insecure.

Textbook RSA is super insecure

Unpadded RSA encryption is homomorphic under multiplication.
Let's have some fun!

Textbook RSA is super insecure

Unpadded RSA encryption is homomorphic under multiplication. Let's have some fun!

Attack: Malleability

Given a ciphertext $c = \mathrm{Enc}(m) = m^e \bmod N$, attacker can forge ciphertext $\mathrm{Enc}(ma) = ca^e \bmod N$ for any a.

Textbook RSA is super insecure

Unpadded RSA encryption is homomorphic under multiplication.
Let's have some fun!

Attack: Malleability

Given a ciphertext $c = \mathrm{Enc}(m) = m^e \bmod N$, attacker can forge ciphertext $\mathrm{Enc}(ma) = ca^e \bmod N$ for any a.

Attack: Chosen ciphertext attack

Given a ciphertext $c = \mathrm{Enc}(m)$ for unknown m, attacker asks for $\mathrm{Dec}(ca^e \bmod N) = d$ and computes $m = da^{-1} \bmod N$.

Textbook RSA is super insecure

Unpadded RSA encryption is homomorphic under multiplication.
Let's have some fun!

Attack: Malleability

Given a ciphertext $c = \mathrm{Enc}(m) = m^e \bmod N$, attacker can forge ciphertext $\mathrm{Enc}(ma) = ca^e \bmod N$ for any a.

Attack: Chosen ciphertext attack

Given a ciphertext $c = \mathrm{Enc}(m)$ for unknown m, attacker asks for $\mathrm{Dec}(ca^e \bmod N) = d$ and computes $m = da^{-1} \bmod N$.

Fix: always use padding on messages.

RSA PKCS #1 v1.5 padding

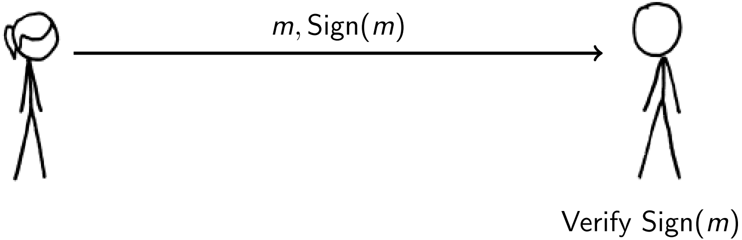
Most common implementation choice even though it is insecure

```
pad(m) = 00 02 [random padding string] 00 [m]
```

- Encrypter pads message, then encrypts padded message using RSA public key: $\text{Enc}_{pk}(m) = pad(m)^e \bmod N$
- Decrypter decrypts using RSA private key, strips off padding to recover original data: $\text{Dec}_{sk}(c) = c^d \bmod N = pad(m)$

PKCS#1v1.5 padding is vulnerable to a number of padding attacks. It is still commonly used in practice.

Idea #3: Digital Signatures



Bob wants to verify Alice's signature using only a public key.

- Signature verifies that Alice was the only one who could have sent this message.
- Signature also verifies that the message hasn't been modified in transit.

Digital Signatures

- Signing: (secret key, message) $\rightarrow$ signature

$$\text{Sign}_{sk}(m) = s$$

- Verification: (public key, message, signature) $\rightarrow$ bool

$$\text{Verify}_{pk}(m, s) = \text{true} \mid \text{false}$$

Signature properties:

- Verification of signed message succeeds:

Digital Signatures

- Signing: (secret key, message) $\rightarrow$ signature

$$\text{Sign}_{sk}(m) = s$$

- Verification: (public key, message, signature) $\rightarrow$ bool

$$\text{Verify}_{pk}(m, s) = \text{true} \mid \text{false}$$

Signature properties:

- Verification of signed message succeeds:
  - $\text{Verify}_{pk}(m, Sign_{sk}(m)) = \text{true}$
- Unforgeability: Can't compute signature for message m that verifies with public key without corresponding secret key.
- The point:
  - Anybody with your public key can verify that you signed something!

Textbook RSA Signatures

[Rivest Shamir Adleman 1977]

Public Key pk

N $= pq$ modulus

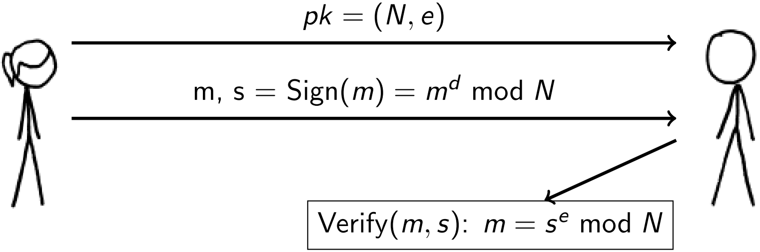
e public exponent

Secret Key sk

p, q primes

d private exponent

$(d = e^{-1} \bmod (p-1)(q-1))$



Works for the same reason RSA encryption does.

Textbook RSA signatures are super insecure

Attack: Signature forgery

1. Attacker wants $\text{Sign}(x)$.
2. Attacker computes $z = xy^e \bmod N$ for some y.
3. Attacker asks signer for $s = \text{Sign}(z) = z^d \bmod N$.
4. Attacker computes $\text{Sign}(x) = sy^{-1} \bmod N$.

Countermeasures:

- **Always use padding with RSA.**
- **Sign hash of m and not raw message m.**

Positive viewpoint:

- Blind signatures: Lots of neat crypto applications.

RSA PKCS #1 v1.5 signature padding

Most widely used padding scheme in practice

```
pad(m) = 00 01 [FF FF FF ...  FF FF] 00 [data H(m)]
```

- Signer hashes and pads message, then signs padded message using RSA private key.
- Verifier verifies using RSA public key, strips off padding to recover hash of message.

Q: What happens if a decrypter doesn't correctly check padding length?

A: **Bleichenbacher low exponent signature forgery attack.**

Bleichenbacher RSA Signature Forgery

```
pad(m) = 00 01 [FF FF FF ...  FF FF] 00 [data H(m)]
```

If victim shortcuts padding check: just looks for padding format but doesn't check length, and signature uses $e = 3$:

1. Construct a perfect cube over the integers, ignoring N, such that

$$s = 0001FF \dots FF00[\text{hash of forged message}][\text{garbage}]$$

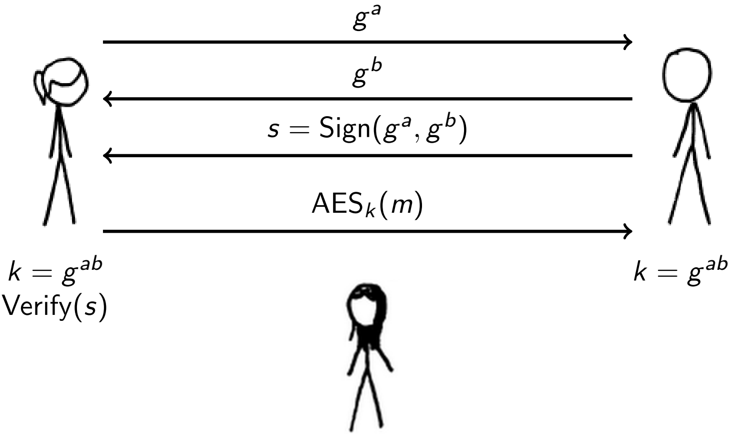
2. Compute x such that $x^3 = s$.
   (Easy way: $x = \lceil [\text{desired values}]000 \dots 0000 \rceil^{1/3}$.)
3. Lazy implementation validates bad signature!

Security for RSA signatures

- Same as RSA encryption.
- Recommendation: Use ECDSA or ed25519 instead.

Putting it all together

How public-key cryptography is used in practice



- Diffie-Hellman used to negotiate shared session key.
- Alice verifies Bob's signature to ensure that key exchange was not man-in-the-middled.
- Shared secret used to symmetrically encrypt data.

Public-key cryptography and quantum computers

Right now, all public-key cryptography used in the real world involves three "hard" problems:

- Factoring
- Discrete log mod primes
- Elliptic curve discrete log

All of these problems can be solved efficiently by a general-purpose quantum computer.

Big standardization effort now to develop replacements:

- Lattice-based cryptography
- Multivariate cryptography
- Hash-based signatures
- Supersingular isogeny Diffie-Hellman

These will likely be used more in the real world in the next few years.