Problem 4-3

4-3 Solving recurrences with a change of variables
Sometimes, a little algebraic manipulation can make an unknown recurrence similar to one you have seen before. Let's solve the recurrence

$$T(n) = 2T\left(\sqrt{n}\right) + \Theta(\lg n) \quad (4.25)$$

by using the change-of-variables method.

a. Define $m = \lg n$ and $S(m) = T(2^m)$. Rewrite recurrence (4.25) in terms of m and $S(m)$.

b. Solve your recurrence for $S(m)$.

c. Use your solution for $S(m)$ to conclude that $T(n) = \Theta(\lg n \lg \lg n)$.

d. Sketch the recursion tree for recurrence (4.25), and use it to explain intuitively why the solution is $T(n) = \Theta(\lg n \lg \lg n)$.

4.a, b, c $T(n) = 2T(\sqrt{n}) + \Theta(\lg n) =$

$T(n) = 2T(n^{1/2}) + \Theta(\lg(n)) =$ say $n = 2^m$

$T(2^m) = 2T((2^m)^{1/2}) + \Theta(\lg(2^m))$

$T(2^m) = 2T((2^m)^{1/2}) + \Theta(m \lg(2))$

$T(2^m) = 2T(2^{m/2}) + \Theta(m)$

So now, $T(2^m) = S(m)$ implies that

$T(2^{m/2}) = S(m/2)$ and then

a. $S(m) = 2S(m/2) + \Theta(m)$

b.
Now, this takes the form of an Extended Master's Theorm solution....

Ext. Master's Theorm: $T(n) = aT(n/b) + \Theta(n^k \log^p m)$

In this case: $a = 2$, $b = 2$, $k = 1$, & $p = 0$

Using the extended Masters Theorm we have:

$$S(m) = \Theta\left(m^{\lg(b)\,a}\lg^{p+1}(n)\right) \quad \text{so,}$$

$$= \Theta\left(m^{\lg_2 2}\lg^{0+1}(m)\right)$$

$$= \Theta\left(m^1 \lg^1(m)\right)$$

b. $$\boxed{= \Theta(m\lg(m))}$$

c.

Now plugging in $S(m)$ we have...

$$S(m) = T(2^m) = T(n) \quad \&$$

$$T(n) = \Theta(m\lg(m))$$

Finally, plug in n for $n = 2^m$ as stated.

$$\lg(n) = \lg(2^m)$$

$$\lg(n) = m\lg(2)$$

$$\lg(n) = m$$

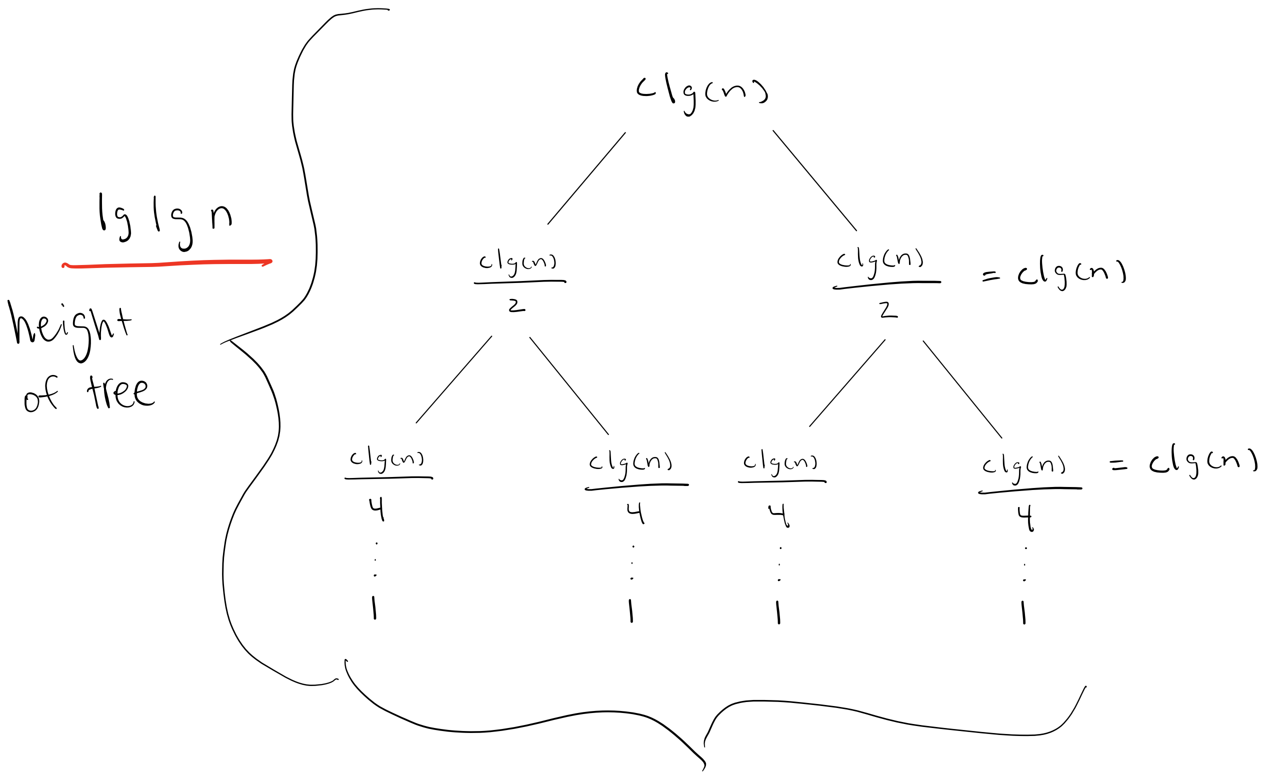
Now knowing $m = \lg(n)$ ......

In terms of m & $S(m)$

$$T(n) = \Theta(m\lg(m)) \rightarrow$$

c. $$\boxed{T(n) = \Theta\left(\underbrace{\lg(n)}_{m}\lg(\underbrace{\lg(n)}_{m})\right)}$$

d. Recursion Tree $T(n) = 2T(n^{1/2}) + \Theta(\lg n)$



c lg(n) operations each level

Intuitively, we can easily see the height of the tree is lglgn. A clgn is getting square rooted each time & halved.

$$\underset{\text{height}}{\lg\lg n} \times \underset{\text{width}}{\lg n} = \lg n \lg\lg n$$

$\boxed{T(n) = \Theta(\lg n \lg\lg n) \text{ as seen by the width \& height of the tree.}}$

e. solve using change of variables

$$T(n) = 2T(\sqrt{n}) + \Theta(1)$$
$$T(n) = 2T(n^{1/2}) + \Theta(1)$$

define a new recurrence R $\longrightarrow R(k) = T(2^k)$

$$= 2T(\sqrt{2^k}) + 1$$
$$= 2T(2^{k/2}) + 1$$
$$= 2R(k/2) + 1 \quad \ldots \text{ in form of } aT(n/b) + f(n)$$

Now,

$$R(k) = \Theta(k) \Longrightarrow T(n) \simeq R(\lg(n))$$

& $R(\lg n) = T(2^{\lg n}) = T(n)$

$$\boxed{T(n) = R(\lg n) = \Theta(\lg n)}$$

n must be ≥ 2 to divide.

$$\sqrt[2^k]{n} = 0$$
$$n = 2^{2^k}$$
$$k = \lg\lg n$$
$$2(2^{\lg\lg n}) + 1 = 2\lg n - 1$$

$\boxed{\text{Therefore, } T(n) = \Theta(\lg n) \text{ as shown above}}$

f.

solve using change of variables

$$T(n) = 3T(\sqrt[3]{n}) + \Theta(n)$$

$$T(n) = 3T(n^{1/3}) + \Theta(n)$$

$n = 3^m$ $\quad n^{1/3} = 3^{m/3}$ $\quad n = m$

$$\underbrace{T(3^m)}_{S(m)} = 3\underbrace{T(3^{m/3})}_{S(m/3)} + m$$

This looks very similar to merge sort

$$S(m) = 3S(m/3) + m$$

$$S(m) = \Theta(m \lg m) \Longrightarrow T(3^m) = \Theta(m \lg m)$$

$$T(n) = \Theta(m \lg m)$$

$$\boxed{T(n) = \Theta(n \lg n)}$$

Problem 4-4 b, f, g, h, i, j

b. $T(n) = 3T(n/3) + n/\lg n$.

$$t(n) = 3t(n/3) + \frac{n}{\lg n}$$

Rate of Increase For Subproblems : 3

Rate of Decrease in subproblem size : 3

Hence, at depth $i = 0, 1, 2, \ldots \lg n$ of the tree, there are 3^i nodes each with a cost of $n/3^i$

$$t(n) = 3t(n/3) + \frac{n}{\lg(n)}$$

Master's Theorm

$$t(n) = at\left(\frac{n}{b}\right) + f(n)$$

where

$a = 3$, $b = 3$ so

$$n^{\log_b a} = n^{\log_3(3)} = n^1 = n$$

Similarily

$n-1 < n < n+1$

$n > f(n) = \frac{n}{\lg(n)}$ So this falls under case 1 of Master Theorm

Therefore, $\boxed{O(n^{\log_3 3}) = \Theta(n).}$

$T(n) = n+1 = O(n)$

$T(n) = n-1 = \Omega(n)$

f. $T(n) = T(n/2) + T(n/4) + T(n/8) + n.$

$T(n) = T(n/2) + T(n/4) + T(n/8) + n$

$T(n) = T(n4/8) + T(n2/8) + T(n1/8) + n$

$= T(7n/8) + n$

$= \sum_{i=1}^{\lg(n)} (7/8)^i n + n \quad \leq \sum_{i=1}^{\infty} \left(\frac{7}{8}\right)^i n + n = \frac{n}{1-\frac{7}{8}} + n = 9n$

$9n = \Theta(n) \longrightarrow T(n) = \Theta(n)$

$T(n) = O(n+1) \longrightarrow O(n)$

$T(n) = \Omega(n-1) \longrightarrow \Omega(n)$

g. $T(n) = T(n-1) + 1/n$.

$T(n) = T(n-1) + 1/n$

Substitutions

$T(n) = T(n-1) + 1/n$

$T(n) = T(n-2) + 2/n$

$T(n-1) = T(n-2) + 1/n$

$T(n) = T(n-3) + 3/n$

$T(n-2) = T(n-3) + 1/n$

$\vdots$

continue for k times

$T(n) = T(n-k) + \frac{k}{n}$

$T(1) = 1$

$T(2) = T(1) + 1/2$

$T(3) = T(2) + 1/3$

Now, assume $n-k=0$

then, $n = k$

$T(3) = T(1) + 1/2 + 1/3$

$= 1 + 1/2 + 1/3$

$T(n) = T(n-n) + \frac{n}{n}$

$T(4) = 1 + 1/2 + 1/3 + 1/4$

$T(n) = T(0) + 1$

$T(n) = 1/n + 1$

$T(n)\ 1 + \frac{1}{\ } \dots 1/n$

So, $T(n) = \Theta(\lg n)$

$\sum_{i=1}^{n} \frac{1}{i} = \lg(n)$

$T(n) = O(\lg n)$ iff $T(n) = \lg n\, c$ where $c \geq 1$

$T(n) = \Omega(\lg n)$ iff $T(n) = \lg n\, c$ where $c < 0$

h. $T(n) = T(n-1) + \lg n.$

$T(1) = 1$

$T(2) = T(1) + \lg 2$

$T(3) = T(2) + \lg 3$

$T(4) = 1 + \lg 2 + \lg 3 + \lg 4$

$$T(n) = 1 + \sum_{i=1}^{n} \lg(i)$$

In other words, $T(n) = T(n-1) + \lg n$

$= T(n-2) + \lg(n-1) + \lg n$

$= T(n-3) + \lg(n-2) + \lg(n-1) + \lg n$

$\vdots$

$= T(0) + \lg 1 + \lg 2 + \dots \lg(n-1) + \lg n$

$= T(0) + \lg n!$

So $T(n) = \Theta(\lg n!)$

Note $\Theta(\lg n!) = \Theta(n \lg n)$ according to Stirling's Approximation for factorials

Worst case $O(\lg n!)$

Best case $\Omega(\lg n! - 1) = \Omega(\lg n!)$

i. $T(n) = T(n-2) + 1/\lg n.$

$T(n) = T(n-2) + 1/\lg n$

$T(n-1) = T(n-3) + \frac{1}{\lg(n-1)}$

$T(n-2) = T(n-4) + \frac{1}{\lg(n-2)}$

Suppose repeats k times

$T(n) = T(n-k-1) + \frac{1}{\lg(n)}$ $n = k+1$

$T(n-n) + \frac{1}{\lg(n)(n-1)(n-2)\dots(1)}$

worst

$T(0) + \lg(n!)$ $T(n) = \lg(n!+1) = O(\lg n!)$

Best

$T(n) = \lg(n!-1) = \Omega(\lg n!)$

$T(n) = \Theta(\lg(n!))$

or

$\Theta(n \lg n)$ if using Stirling's Approximation for Factorials

$\lg n! - 1 < \lg n! < \lg n! + 1$

j. $T(n) = \sqrt{n}\,T(\sqrt{n}) + n$.

$$T(n) = n^{1/2}\,T(n^{1/2}) + n$$

$$= n^{1/2}\left(n^{1/2^2}\right)T\left(n^{1/2^2}\right) + n^{1/2} + n$$

$$= n^{3/2^2}\,T\left(n^{1/2^2}\right) + 2n$$

$$= n^{3/2^2}\, n^{1/2^3}\, T\left(n^{1/2^3}\right) + n^{1/2^3} + 2n$$

$$\vdots$$

assume $n^{1/2^k} = 2$

$$1/2^k = \lg n = 1$$

$$\lg n = 2k$$

$$\lg \lg n = k$$

$$= n^{\frac{1-1}{2}^k}\, T\left(n^{1/2^k}\right) + kn$$

$$= \frac{n}{n}\left(1/2^k\right) T(2) + kn$$

$$= n/2 * 2 + n \lg \lg n$$

$$= n + n \lg\lg n$$

$n\lg\lg n - 1 < n\lg\lg n < n\lg\lg n + n$

Best Case

$T(n) = n\lg\lg n - 1 = \Omega(n\lg\lg n)$

Worst case

& $T(n) = n + n\lg\lg n = O(n\lg\lg n)$

$$\boxed{T(n) = \Theta(n \lg\lg n)}$$