CS 5413
Chapter 5

Probabilistic Analysis and Randomized Algorithms

The primary goals

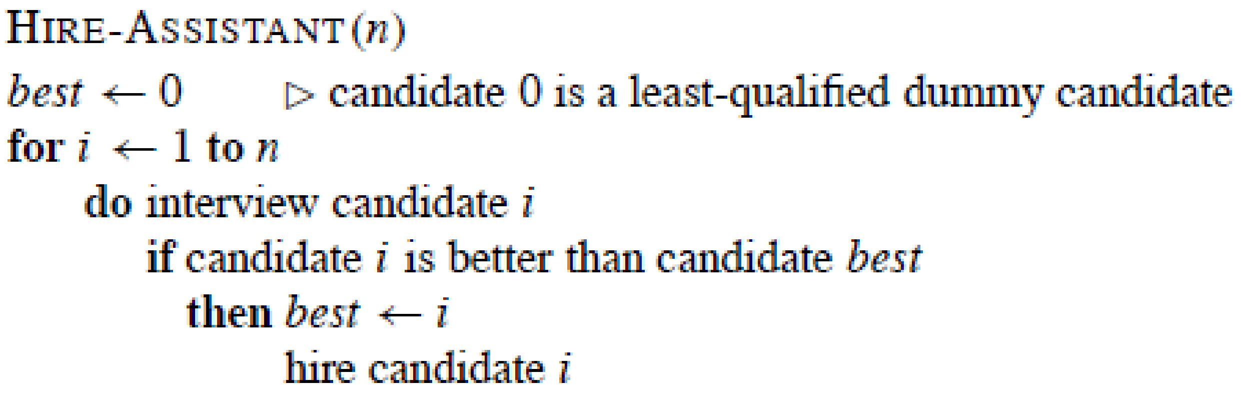
• explain the difference between probabilistic analysis and randomized algorithms
• present the technique of indicator random variables
• give another example of the analysis of a randomized algorithm (permuting an array in place).

The hiring problem Scenario:

• You are using an employment agency to hire a new office assistant.
• The agency sends you one candidate each day.
• You interview the candidate and must immediately decide whether or not to hire that person. But if you hire, you must also fire your current office assistant - even if it's someone you have recently hired.
• Cost to interview is c_i per candidate (interview fee paid to agency).
• Cost to hire is c_h per candidate (includes cost to fire current office assistant + hiring fee paid to agency).
• Assume that $c_h >> ci$.
• You are committed to having hired, at all times, the best candidate seen so far. Meaning that whenever you interview a candidate who is better than your current office assistant, you must fire the current office assistant and hire the candidate. Since you must have someone hired at all times, you will always hire the first candidate that you interview.

Goal: Determine what the price of this strategy will be.

Pseudocode to model this scenario: Assumes that the candidates are numbered l to *n* and that after interviewing each candidate, we can determine if it's better than the current ofice assistant. Uses a dummy candidate 0 that is worse than all others, so that the irst candidate is always hired.



```
HIRE-ASSISTANT($n$)
$best \leftarrow 0$        ▷ candidate 0 is a least-qualified dummy candidate
for $i \leftarrow 1$ to $n$
    do interview candidate $i$
       if candidate $i$ is better than candidate $best$
         then $best \leftarrow i$
              hire candidate $i$
```

Cost: If *n* candidates, and we hire *m* of them, the cost is $O(nc_i + mc_h)$.

- Have to pay nc_i to interview, no matter how many we hire.
- So we focus on analyzing the hiring cost *mch* .

• mc_h varies with each run - it depends on the order in which we interview the candidates.

• This is a model of a common paradigm: we need to find the maximum or minimum in a sequence by examining each element and maintaining a current "winner." The variable *m* denotes how many times we change our notion of which element is currently winning.

Worst-case analysis

In the worst case, we hire all *n* candidates.

This happens if each one is better than all who came before. In other words, if the candidates appear in increasing order of quality.

If we hire all *n*, then the cost is

$$O(nc_i + nc_h) = O(nc_h) \text{ (since } c_h > c_i\text{)}.$$

Probabilistic analysis

In general, we have no control over the order in which candidates appear.

We could assume that they come in a random order:

- Assign a rank to each candidate: $rank(i)$ is a unique integer in the range 1 to n.
- The ordered list $\langle rank(1), rank(2), \ldots, rank(n) \rangle$ is a permutation of the candidate numbers $\langle 1, 2, \ldots, n \rangle$.
- The list of ranks is equally likely to be any one of the $n!$ permutations.
- Equivalently, the ranks form a ***uniform random permutation***: each of the possible $n!$ permutations appears with equal probability.

Essential idea of probabilistic analysis: We must use knowledge of, or make assumptions about, the distribution of inputs.

- The expectation is over this distribution.
- This technique requires that we can make a reasonable characterization of the input distribution.

Randomized algorithms

We might not know the distribution of inputs, or we might not be able to model it computationally.

Instead, we use randomization within the algorithm in order to impose a distribution on the inputs.

For the hiring problem: Change the scenario:

- The employment agency sends us a list of all n candidates in advance.
- On each day, we randomly choose a candidate from the list to interview (but considering only those we have not yet interviewed).
- Instead of relying on the candidates being presented to us in a random order, we take control of the process and enforce a random order.

What makes an algorithm randomized: An algorithm is *randomized* if its behavior is determined in part by values produced by a *random-number generator*.

- RANDOM(a, b) returns an integer r, where $a \le r \le b$ and each of the $b - a + 1$ possible values of r is equally likely.
- In practice, RANDOM is implemented by a *pseudorandom-number generator*, which is a deterministic method returning numbers that "look" random and pass statistical tests.

Indicator random variables

A simple yet powerful technique for computing the expected value of a random variable.

Helpful in situations in which there may be dependence.

Given a sample space and an event A, we define the *indicator random variable*

$$\mathrm{I}\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs} , \\ 0 & \text{if } A \text{ does not occur} . \end{cases}$$

Lemma

For an event A, let $X_A = \mathrm{I}\{A\}$. Then $\mathrm{E}[X_A] = \Pr\{A\}$.

Proof Letting $\overline{A}$ be the complement of A, we have

$$\begin{aligned} \mathrm{E}[X_A] &= \mathrm{E}[\mathrm{I}\{A\}] \\ &= 1 \cdot \Pr\{A\} + 0 \cdot \Pr\{\overline{A}\} \quad \text{(definition of expected value)} \\ &= \Pr\{A\} . \end{aligned}$$

■ (lemma)

Simple example: Determine the expected number of heads when we flip a fair coin one time.

- Sample space is $\{H, T\}$.
- $\Pr\{H\} = \Pr\{T\} = 1/2$.
- Define indicator random variable $X_H = \mathrm{I}\{H\}$. X_H counts the number of heads in one flip.
- Since $\Pr\{H\} = 1/2$, lemma says that $\mathrm{E}[X_H] = 1/2$.

Slightly more complicated example: Determine the expected number of heads in n coin flips.

- Let X be a random variable for the number of heads in n flips.
- Could compute $\mathrm{E}[X] = \sum_{k=0}^{n} k \cdot \Pr\{X = k\}$. In fact, this is what the book does in equation
- Instead, we'll use indicator random variables.
- For $i = 1, 2, \ldots, n$, define $X_i = \mathrm{I}\{\text{the } i\text{th flip results in event } H\}$.
- Then $X = \sum_{i=1}^{n} X_i$.
- Lemma says that $\mathrm{E}[X_i] = \Pr\{H\} = 1/2$ for $i = 1, 2, \ldots, n$.
- Expected number of heads is $\mathrm{E}[X] = \mathrm{E}[\sum_{i=1}^{n} X_i]$.

- ***Problem:*** We want $\mathrm{E}\left[\sum_{i=1}^{n} X_i\right]$. We have only the individual expectations $\mathrm{E}[X_1], \mathrm{E}[X_2], \ldots, \mathrm{E}[X_n]$.
- *Solution:* Linearity of expectation says that the expectation of the sum equals the sum of the expectations. Thus,

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} \mathrm{E}[X_i] \\
&= \sum_{i=1}^{n} 1/2 \\
&= n/2 .
\end{aligned}$$

- Linearity of expectation applies even when there is dependence among the random variables.

Analysis of the hiring problem

Assume that the candidates arrive in a random order.

Let X be a random variable that equals the number of times we hire a new office assistant.

Define indicator random variables $X_1, X_2, \ldots, X_n$, where

$X_i = \mathrm{I}\{\text{candidate } i \text{ is hired}\}$.

Useful properties:

- $X = X_1 + X_2 + \cdots + X_n$.
- Lemma $\Rightarrow \mathrm{E}[X_i] = \Pr\{\text{candidate } i \text{ is hired}\}$.

We need to compute $\Pr\{\text{candidate } i \text{ is hired}\}$.

- Candidate i is hired if and only if candidate i is better than each of candidates $1, 2, \ldots, i-1$.
- Assumption that the candidates arrive in random order $\Rightarrow$ candidates $1, 2, \ldots, i$ arrive in random order $\Rightarrow$ any one of these first i candidates is equally likely to be the best one so far.
- Thus, $\Pr\{\text{candidate } i \text{ is the best so far}\} = 1/i$.
- Which implies $\mathrm{E}[X_i] = 1/i$.

Now compute $\mathrm{E}[X]$:

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} \mathrm{E}[X_i] \\
&= \sum_{i=1}^{n} 1/i \\
&= \ln n + O(1)
\end{aligned}$$

the sum is a harmonic series

Thus, the expected hiring cost is $O(c_h \ln n)$, which is much better than the worst-case cost of $O(nc_h)$.

Randomized algorithms

Instead of assuming a distribution of the inputs, we impose a distribution.

The hiring problem

For the hiring problem, the algorithm is deterministic:

- For any given input, the number of times we hire a new office assistant will always be the same.
- The number of times we hire a new office assistant depends only on the input.
- In fact, it depends only on the ordering of the candidates' ranks that it is given.
- Some rank orderings will always produce a high hiring cost. Example: $\langle 1, 2, 3, 4, 5, 6 \rangle$, where each candidate is hired.
- Some will always produce a low hiring cost. Example: any ordering in which the best candidate is the first one interviewed. Then only the best candidate is hired.
- Some may be in between.

Instead of always interviewing the candidates in the order presented, what if we first randomly permuted this order?

- The randomization is now in the algorithm, not in the input distribution.
- Given a particular input, we can no longer say what its hiring cost will be. Each time we run the algorithm, we can get a different hiring cost.
- In other words, each time we run the algorithm, the execution depends on the random choices made.
- No particular input always elicits worst-case behavior.
- Bad behavior occurs only if we get "unlucky" numbers from the random-number generator.

Pseudocode for randomized hiring problem:

```
RANDOMIZED-HIRE-ASSISTANT(n)
randomly permute the list of candidates
HIRE-ASSISTANT(n)
```

Lemma

The expected hiring cost of RANDOMIZED-HIRE-ASSISTANT is $O(c_h \ln n)$.

Proof After permuting the input array, we have a situation identical to the probabilistic analysis of deterministic HIRE-ASSISTANT. ■

Randomly permuting an array

Goal: Produce a uniform random permutation (each of the $n!$ permutations is equally likely to be produced).

Non-goal: Show that for each element $A[i]$, the probability that $A[i]$ moves to position j is $1/n$.

The following procedure permutes the array $A[1\,..\,n]$ in place (i.e., no auxiliary array is required).

```
RANDOMIZE-IN-PLACE($A, n$)
for $i \leftarrow 1$ to $n$
    do swap $A[i] \leftrightarrow A[\text{RANDOM}(i, n)]$
```

Idea:

- In iteration i, choose $A[i]$ randomly from $A[i \,..\, n]$.
- Will never alter $A[i]$ after iteration i.

Time: $O(1)$ per iteration $\Rightarrow O(n)$ total.

Correctness: Given a set of n elements, a ***k-permutation*** is a sequence containing k of the n elements. There are $n!/(n-k)!$ possible k-permutations.

Lemma
RANDOMIZE-IN-PLACE computes a uniform random permutation.

Proof Use a loop invariant:

> **Loop invariant:** Just prior to the ith iteration of the **for** loop, for each possible $(i-1)$-permutation, subarray $A[1 \mathinner{.\,.} i-1]$ contains this $(i-1)$-permutation with probability $(n-i+1)!/n!$.

Initialization: Just before first iteration, $i = 1$. Loop invariant says that for each possible 0-permutation, subarray $A[1\,..\,0]$ contains this 0-permutation with probability $n!/n! = 1$. $A[1\,..\,0]$ is an empty subarray, and a 0-permutation has no elements. So, $A[1\,..\,0]$ contains any 0-permutation with probability 1.

Maintenance: Assume that just prior to the ith iteration, each possible $(i-1)$-permutation appears in $A[1 \mathinner{.\,.} i-1]$ with probability $(n-i+1)!/n!$. Will show that after the ith iteration, each possible i-permutation appears in $A[1 \mathinner{.\,.} i]$ with probability $(n-i)!/n!$. Incrementing i for the next iteration then maintains the invariant.

Consider a particular i-permutation $\pi = \langle x_1, x_2, \ldots, x_i \rangle$. It consists of an $(i-1)$-permutation $\pi' = \langle x_1, x_2, \ldots, x_{i-1} \rangle$, followed by x_i.

Let E_1 be the event that the algorithm actually puts π' into $A[1 \mathinner{.\,.} i-1]$. By the loop invariant, $\Pr\{E_1\} = (n-i+1)!/n!$.

Let E_2 be the event that the ith iteration puts x_i into $A[i]$.

We get the i-permutation π in $A[1 \mathinner{.\,.} i]$ if and only if both E_1 and E_2 occur $\Rightarrow$ the probability that the algorithm produces π in $A[1 \mathinner{.\,.} i]$ is $\Pr\{E_2 \cap E_1\}$.

Equation (C.14) $\Rightarrow \Pr\{E_2 \cap E_1\} = \Pr\{E_2 \mid E_1\} \Pr\{E_1\}$.

The algorithm chooses x_i randomly from the $n-i+1$ possibilities in $A[i \mathinner{.\,.} n]$ $\Rightarrow \Pr\{E_2 \mid E_1\} = 1/(n-i+1)$. Thus,

$$\begin{aligned}
\Pr\{E_2 \cap E_1\} &= \Pr\{E_2 \mid E_1\} \Pr\{E_1\} \\
&= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!} \\
&= \frac{(n-i)!}{n!} .
\end{aligned}$$

Termination: At termination, $i = n + 1$, so we conclude that $A[1 \,.\,.\, n]$ is a given n-permutation with probability $(n - n)!/n! = 1/n!$. ■ (lemma)