CS 5413
Chapter 8

Sorting in Linear Time

How fast can we sort?

We will prove a lower bound, then beat it by playing a different game.

Comparison sorting

- The only operation that may be used to gain order information about a sequence is comparison of pairs of elements.
- All sorts seen so far are comparison sorts: insertion sort, selection sort, merge sort, quicksort, heapsort, treesort.

Lower bounds for sorting

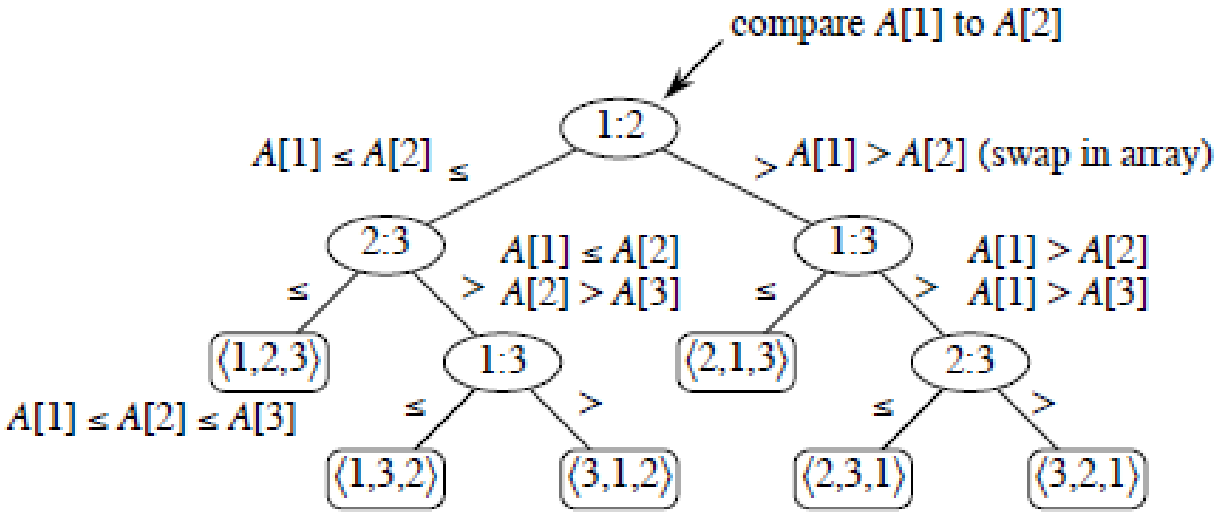
Lower bounds

- $\Omega(n)$ to examine all the input.
- All sorts seen so far are $\Omega(n \lg n)$.
- We'll show that $\Omega(n \lg n)$ is a lower bound for comparison sorts.

Decision tree

- Abstraction of any comparison sort.
- Represents comparisons made by
  - a specific sorting algorithm
  - on inputs of a given size.
- Abstracts away everything else: control and data movement.
- We're counting *only* comparisons.

For insertion sort on 3 elements:



How many leaves on the decision tree? There are $\geq n!$ leaves, because every permutation appears at least once.

For any comparison sort,

- 1 tree for each n.
- View the tree as if the algorithm splits in two at each node, based on the information it has determined up to that point.
- The tree models all possible execution traces.

What is the length of the longest path from root to leaf?

- Depends on the algorithm
- Insertion sort: $\Theta(n^2)$
- Merge sort: $\Theta(n \lg n)$

Lemma

Any binary tree of height h has $\leq 2^h$ leaves.

In other words:

- $l =$ # of leaves,
- $h =$ height,
- Then $l \leq 2^h$.

Theorem

Any decision tree that sorts n elements has height $\Omega(n \lg n)$.

Proof

- $l \geq n!$
- By lemma, $n! \leq l \leq 2^h$ or $2^h \geq n!$
- Take logs: $h \geq \lg(n!)$
- Use Stirling's approximation: $n! > (n/e)^n$ (by equation (3.16))

$$\begin{aligned} h &\geq \lg(n/e)^n \\ &= n \lg(n/e) \\ &= n \lg n - n \lg e \\ &= \Omega(n \lg n) \ . \end{aligned}$$

■ (theorem)

Proof By induction on h.

Basis: $h = 0$. Tree is just one node, which is a leaf. $2^h = 1$.

Inductive step: Assume true for height $= h - 1$. Extend tree of height $h - 1$ by making as many new leaves as possible. Each leaf becomes parent to two new leaves.

$$\begin{aligned} \#\text{ of leaves for height } h &= 2 \cdot (\#\text{ of leaves for height } h-1) \\ &= 2 \cdot 2^{h-1} && \text{(ind. hypothesis)} \\ &= 2^h . && \blacksquare \text{ (lemma)} \end{aligned}$$

Corollary

Heapsort and merge sort are asymptotically optimal comparison sorts.

Sorting in linear time

Non-comparison sorts.

Counting sort

Depends on a *key assumption*: numbers to be sorted are integers in $\{0, 1, \ldots, k\}$.

Input: $A[1..n]$, where $A[j] \in \{0, 1, \ldots, k\}$ for $j = 1, 2, \ldots, n$. Array A and values n and k are given as parameters.

Output: $B[1..n]$, sorted. B is assumed to be already allocated and is given as a parameter.

Auxiliary storage: $C[0..k]$

COUNTING-SORT(A, B, n, k)

for $i \leftarrow 0$ **to** k
    do $C[i] \leftarrow 0$
for $j \leftarrow 1$ **to** n
    do $C[A[j]] \leftarrow C[A[j]] + 1$
for $i \leftarrow 1$ **to** k
    do $C[i] \leftarrow C[i] + C[i-1]$
for $j \leftarrow n$ **downto** 1
    do $B[C[A[j]]] \leftarrow A[j]$
        $C[A[j]] \leftarrow C[A[j]] - 1$

Do an example for $A = 2_1, 5_1, 3_1, 0_1, 2_2, 3_2, 0_2, 3_3$

Counting sort is *stable* (keys with same value appear in same order in output as they did in input) because of how the last loop works.

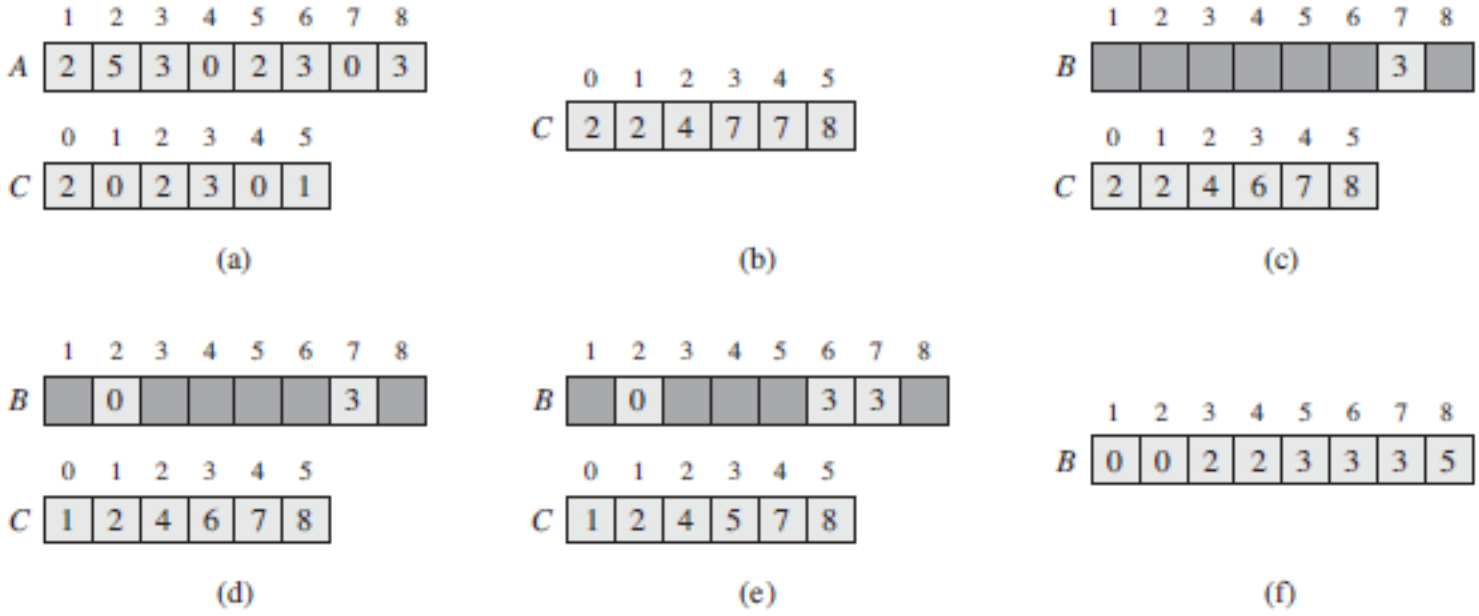


Figure 8.2 The operation of COUNTING-SORT on an input array $A[1..8]$, where each element of A is a nonnegative integer no larger than $k = 5$. **(a)** The array A and the auxiliary array C after line 5. **(b)** The array C after line 8. **(c)–(e)** The output array B and the auxiliary array C after one, two, and three iterations of the loop in lines 10–12, respectively. Only the lightly shaded elements of array B have been filled in. **(f)** The final sorted output array B.

Analysis: $\Theta(n + k)$, which is $\Theta(n)$ if $k = O(n)$.

How big a k is practical?

- Good for sorting 32-bit values? No.
- 16-bit? Probably not.
- 8-bit? Maybe, depending on n.
- 4-bit? Probably (unless n is really small).

Counting sort will be used in radix sort.

Radix sort

How IBM made its money. Punch card readers for census tabulation in early 1900's. Card sorters, worked on one column at a time. It's the algorithm for using the machine that extends the technique to multi-column sorting. The human operator was part of the algorithm!

Key idea: Sort *least* significant digits first.

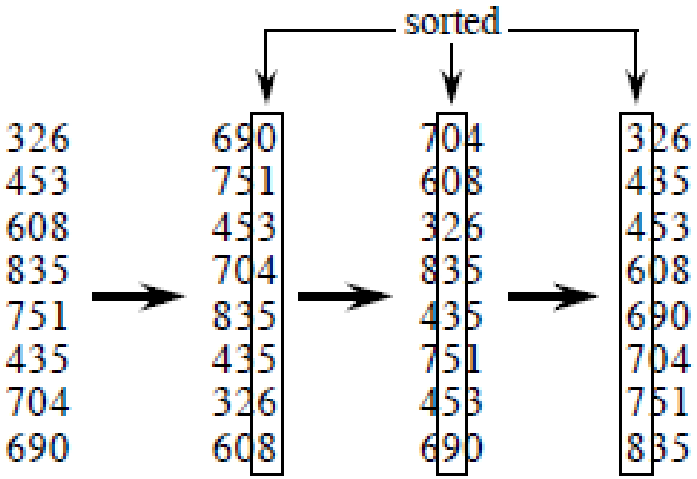
To sort d digits:

RADIX-SORT(A, d)

for $i \leftarrow 1$ **to** d

do use a stable sort to sort array A on digit i

Example:



Correctness:

- Induction on number of passes (i in pseudocode).
- Assume digits $1, 2, \ldots, i-1$ are sorted.
- Show that a stable sort on digit i leaves digits $1, \ldots, i$ sorted:
  - If 2 digits in position i are different, ordering by position i is correct, and positions $1, \ldots, i-1$ are irrelevant.
  - If 2 digits in position i are equal, numbers are already in the right order (by inductive hypothesis). The stable sort on digit i leaves them in the right order.

This argument shows why it's so important to use a stable sort for intermediate sort.

Analysis: Assume that we use counting sort as the intermediate sort.

- $\Theta(n+k)$ per pass (digits in range $0, \ldots, k$)
- d passes
- $\Theta(d(n+k))$ total
- If $k = O(n)$, time $= \Theta(dn)$.

How to break each key into digits?

- n words.
- b bits/word.
- Break into r-bit digits. Have $d = \lceil b/r \rceil$.
- Use counting sort, $k = 2^r - 1$.

  Example: 32-bit words, 8-bit digits. $b = 32$, $r = 8$, $d = \lceil 32/8 \rceil = 4$, $k = 2^8 - 1 = 255$.
- Time $= \Theta\left(\frac{b}{r}(n + 2^r)\right)$.

How to choose r? Balance b/r and $n + 2^r$. Choosing $r \approx \lg n$ gives us $\Theta\left(\frac{b}{\lg n}(n+n)\right) = \Theta(bn/\lg n)$.

- If we choose $r < \lg n$, then $b/r > b/\lg n$, and $n + 2^r$ term doesn't improve.
- If we choose $r > \lg n$, then $n + 2^r$ term gets big. Example: $r = 2\lg n \Rightarrow 2^r = 2^{2\lg n} = (2^{\lg n})^2 = n^2$.

So, to sort 2^{16} 32-bit numbers, use $r = \lg 2^{16} = 16$ bits. $\lceil b/r \rceil = 2$ passes.

Compare radix sort to merge sort and quicksort:

- 1 million (2^{20}) 32-bit integers.
- Radix sort: $\lceil 32/20 \rceil = 2$ passes.
- Merge sort/quicksort: $\lg n = 20$ passes.
- Remember, though, that each radix sort "pass" is really 2 passes—one to take census, and one to move data.

How does radix sort violate the ground rules for a comparison sort?

- Using counting sort allows us to gain information about keys by means other than directly comparing 2 keys.
- Used keys as array indices.

Bucket sort

Assumes the input is generated by a random process that distributes elements uniformly over $[0, 1)$.

Idea:

- Divide $[0, 1)$ into n equal-sized *buckets*.
- Distribute the n input values into the buckets.
- Sort each bucket.
- Then go through buckets in order, listing elements in each one.

Input: $A[1..n]$, where $0 \le A[i] < 1$ for all i.
Auxiliary array: $B[0..n-1]$ of linked lists, each list initially empty.

BUCKET-SORT(A, n)
for $i \leftarrow 1$ **to** n
    do insert $A[i]$ into list $B[\lfloor n \cdot A[i] \rfloor]$
for $i \leftarrow 0$ **to** $n - 1$
    do sort list $B[i]$ with insertion sort
concatenate lists $B[0], B[1], \ldots, B[n-1]$ together in order
return the concatenated lists

Correctness: Consider $A[i]$, $A[j]$. Assume without loss of generality that $A[i] \leq A[j]$. Then $\lfloor n \cdot A[i] \rfloor \leq \lfloor n \cdot A[j] \rfloor$. So $A[i]$ is placed into the same bucket as $A[j]$ or into a bucket with a lower index.

- If same bucket, insertion sort fixes up.
- If earlier bucket, concatenation of lists fixes up.

Analysis:

- Relies on no bucket getting too many values.
- All lines of algorithm except insertion sorting take $\Theta(n)$ altogether.
- Intuitively, if each bucket gets a constant number of elements, it takes $O(1)$ time to sort each bucket $\Rightarrow$ $O(n)$ sort time for all buckets.
- We "expect" each bucket to have few elements, since the average is 1 element per bucket.
- But we need to do a careful analysis.

Define a random variable:

- n_i = the number of elements placed in bucket $B[i]$.

Because insertion sort runs in quadratic time, bucket sort time is

$$T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2) .$$

Take expectations of both sides:

$$\begin{aligned}
\mathrm{E}[T(n)] &= \mathrm{E}\left[\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)\right] \\
&= \Theta(n) + \sum_{i=0}^{n-1} \mathrm{E}[O(n_i^2)] && \text{(linearity of expectation)} \\
&= \Theta(n) + \sum_{i=0}^{n-1} O(\mathrm{E}[n_i^2]) && (\mathrm{E}[aX] = a\mathrm{E}[X])
\end{aligned}$$

Claim

$\mathrm{E}[n_i^2] = 2 - (1/n)$ for $i = 0, \ldots, n-1$.

Proof of claim

Define indicator random variables:

- $X_{ij} = \mathrm{I}\{A[j] \text{ falls in bucket } i\}$
- $\Pr\{A[j] \text{ falls in bucket } i\} = 1/n$
- $n_i = \sum_{j=1}^{n} X_{ij}$

Then

$$\begin{aligned}
\mathrm{E}[n_i^2] &= \mathrm{E}\left[\left(\sum_{j=1}^{n} X_{ij}\right)^2\right] \\
&= \mathrm{E}\left[\sum_{j=1}^{n} X_{ij}^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} X_{ij}X_{ik}\right] \\
&= \sum_{j=1}^{n} \mathrm{E}\left[X_{ij}^2\right] + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \mathrm{E}\left[X_{ij}X_{ik}\right] \quad \text{(linearity of expectation)}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}\left[X_{ij}^2\right] &= 0^2 \cdot \Pr\{A[j] \text{ doesn't fall in bucket } i\} + 1^2 \cdot \Pr\{A[j] \text{ falls in bucket } i\} \\
&= 0 \cdot \left(1 - \frac{1}{n}\right) + 1 \cdot \frac{1}{n} \\
&= \frac{1}{n}
\end{aligned}
$$

$\mathrm{E}[X_{ij} X_{ik}]$ for $j \neq k$: Since $j \neq k$, X_{ij} and X_{ik} are independent random variables

$$
\begin{aligned}
\Rightarrow \mathrm{E}[X_{ij} X_{ik}] &= \mathrm{E}[X_{ij}]\,\mathrm{E}[X_{ik}] \\
&= \frac{1}{n} \cdot \frac{1}{n} \\
&= \frac{1}{n^2}
\end{aligned}
$$

Therefore:

$$\begin{aligned}
\mathrm{E}[n_i^2] &= \sum_{j=1}^{n} \frac{1}{n} + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \frac{1}{n^2} \\
&= n \cdot \frac{1}{n} + 2\binom{n}{2}\frac{1}{n^2} \\
&= 1 + 2 \cdot \frac{n(n-1)}{2} \cdot \frac{1}{n^2} \\
&= 1 + \frac{n-1}{n} \\
&= 1 + 1 - \frac{1}{n} \\
&= 2 - \frac{1}{n}
\end{aligned}$$

■ (claim)

Therefore:

$$\begin{aligned}
\mathrm{E}[T(n)] &= \Theta(n) + \sum_{i=0}^{n-1} O(2 - 1/n) \\
&= \Theta(n) + O(n) \\
&= \Theta(n)
\end{aligned}$$

- Again, not a comparison sort. Used a function of key values to index into an array.
- This is a *probabilistic analysis*—we used probability to analyze an algorithm whose running time depends on the distribution of inputs.
- Different from a *randomized algorithm*, where we use randomization to *impose* a distribution.
- With bucket sort, if the input isn't drawn from a uniform distribution on [0, 1), all bets are off (performance-wise, but the algorithm is still correct).