Question 1. (20 pts) MÁY TÍNH : RADIAN *

Given the following data. Find the regression equation $y = a + bx + cx^2$

$\Rightarrow n=5$

x	0.78	1.56	2.34	3.12	3.81
y=(x)	2.5	1.2	1.12	2.25	4.28

Plot the residual w.r.t. x.

* Find the regression

vế trái: tổng y ; vế phải: tổng x

do a K° có biến x nên x thêm cho n

$$\sum y = n.a + b\sum x + c\sum x^2$$
$$\sum xy = a.\sum x + b\sum x^2 + c\sum x^3$$
$$\sum x^2y = a.\sum x^2 + b\sum x^3 + c\sum x^4$$

x_i	y_i	x_iy_i	x_i^2	x_i^3	x_i^4	x_i^2y
0,78	2,5	1,95	0,6084	0,47455	0,37015	1,521
1,56	1,2	1,872	2,4336	3,79642	5,92241	2,92032
2,34	1,12	2,6208	5,4756	12,81290	29,98219	6,13267
3,12	2,25	7,02	9,7344	30,37133	94,75854	21,9024
3,81	4,28	16,3068	14,5161	55,30634	210,71715	62,12891
Total: 11,61	11,35	29,7696	32,7681	102,76154	341,75044	94,6053

$$\begin{pmatrix} 5 & 11,61 & 32,7681 \\ 11,61 & 32,7681 & 102,76154 \\ 32,7681 & 102,76154 & 341,75044 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11,35 \\ 29,7696 \\ 94,6053 \end{pmatrix}$$

$$\begin{cases} 5a + 11,61b + 32,7681c = 11,35 \\ 11,61a + 32,7681b + 102,76154c = 29,7696 \\ 32,7681a + 102,76154b + 341,75044c = 94,6053 \end{cases}$$

$$\Rightarrow \begin{cases} a = 5{,}022 \\ b = -4{,}014 \\ c = 1{,}002 \end{cases}$$

$$\Rightarrow f(x) = y = 5{,}022 - 4{,}014x + 1{,}002x^2$$

x	0.78	1.56	2.34	3.12	3.81
y=(x)	2.5	1.2	1.12	2.25	4.28

Plot the residual w.r.t. x.

* Plot the residual w.r.t.x

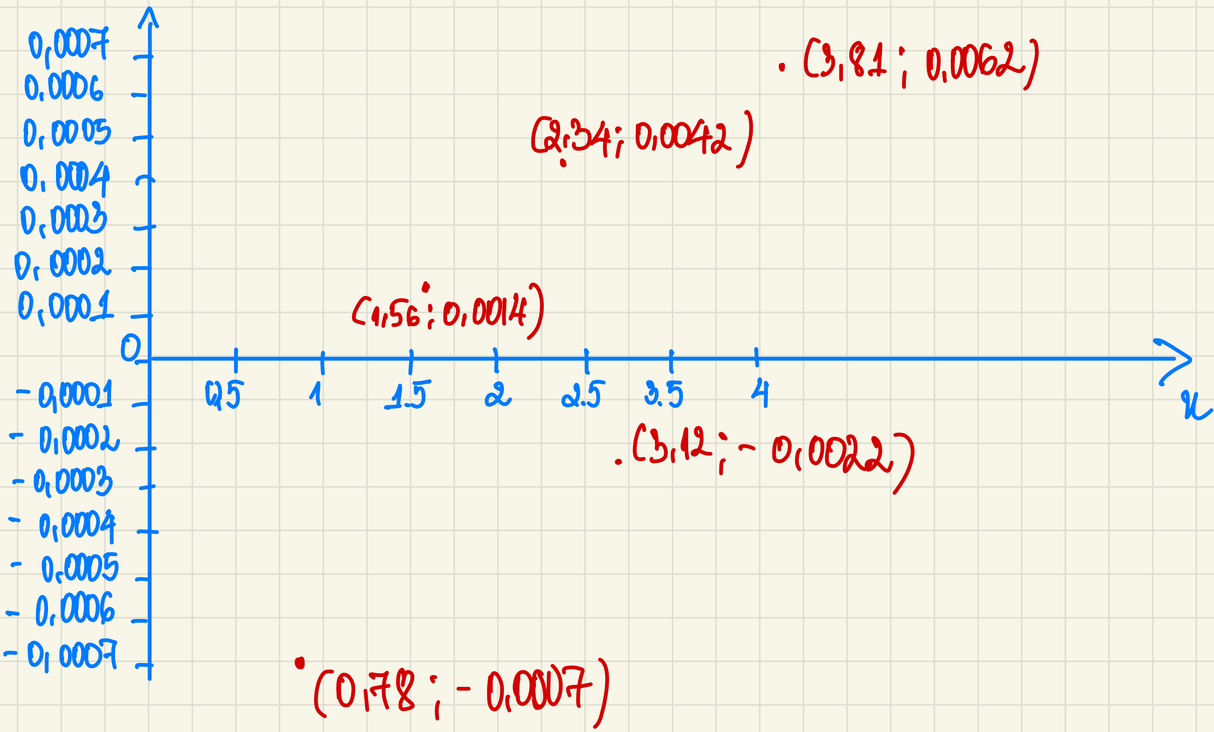
$$f(x) = y = 5{,}022 - 4{,}014x + 1{,}002x^2$$

$$f(0{,}78) = 5{,}022 - 4{,}014 \times 0{,}78 + 1{,}002 \times (0{,}78)^2 = 2{,}5007$$
$$f(1{,}56) = 5{,}022 - 4{,}014 \times 1{,}56 + 1{,}002 \times (1{,}56)^2 = 1{,}1986$$
$$f(2{,}34) = 5{,}022 - 4{,}014 \times 2{,}34 + 1{,}002 \times (2{,}34)^2 = 1{,}1158$$
$$f(3{,}12) = 5{,}022 - 4{,}014 \times 3{,}12 + 1{,}002 \times (3{,}12)^2 = 2{,}2522$$
$$f(3{,}81) = 5{,}022 - 4{,}014 \times 3{,}81 + 1{,}002 \times (3{,}81)^2 = 4{,}2738$$

y đề cho

y mới tính

x	y - y predict	Residual	
0,78	2,5 - 2,5007	-7×10^{-4}	(-0,0007)
1,56	1,2 - 1,1986	$1{,}4 \times 10^{-3}$	(0,0014)
2,34	1,12 - 1,1158	$4{,}2 \times 10^{-3}$	(0,0042)
3,12	2,25 - 2,2522	$-2{,}2 \times 10^{-3}$	(-0,0022)
3,81	4,28 - 4,2738	$6{,}2 \times 10^{-3}$	(0,0062)



Question 2. (20 pts) → true error

Find the error in the derivative of f(x) = cos(x) by computing directly and using the central divided difference approximation at:

ODD: x = 0.8 (rad).

Central divided difference: $f'(x_2) = \dfrac{f(x+h) - f(x-h)}{2h}$

EVEN: x = 0.6 (rad).

step

Determine h so that absolute error is:

Vd: 1 2

step h = 1

a. less than 10^{-3}.

b. less than 10^{-5}.

Justify your answer.

là f(x)

True error: $|E_t| = |V_{true} - V_{app}|$ → v theo formular

Relative true error

$$|\varepsilon_t| = \left| \frac{V_{true} - V_{app}}{V_{true}} \right| \times 100\%$$

Approximate error → Forward differentiation

$$|E_R| = |V_{sau} - V_{trước}|$$

Relative approximate error

$$|E_R| = \left|\frac{V_{trước} - V_{sau}}{V_{trước}}\right| \times 100\%$$

cái cũ

$$E = \left|\frac{f'(x_1) - f'(x_2)}{f'(x_1)}\right| \leq 10^{-3}$$

$$f(x) = \cos x$$

$$\Rightarrow f'(x) = -\sin x$$

$$\Rightarrow f'(0,8) = -\sin(0,8) = -0,71736$$

Central divided difference :

$$f'(x_2) = \frac{f(x+h) - f(x-h)}{2h}$$

$x = 0,8$ (rad)

$$E = |f'(x) - f'(x_2)| \leq 10^{-3}$$

$$= \left|-0,71736 - \frac{\cos(0,8+h) - \cos(0,8-h)}{2h}\right| \leq 10^{-3}$$

h	E
0,1	$1,1989 \times 10^{-3}$
0.05	$3,0277 \times 10^{-4}$
0.025	$4,6563 \times 10^{-6}$

$(< 10^{-3}) \Rightarrow h = 0,05$

$(< 10^{-5})$

Question 3. (20 pts)

Given an integration:

$$\int_{2}^{3.5} \frac{1+x}{1-x}dx$$

b

a

Vd: 1 2 3 4

có 3 space -> 4 gtrị

$h = \frac{b-a}{n}$

=> 12 space -> 13 gtrị

tính f $x_0 \to x_{12}$

n

a. Approximate the integration using Simpson formula, divide [2; 3.5] into 12 equal spaces. Show your work.

b. What is the minimum absolute error we can get using Simpson formula for the integration. Justify your answer.

a) $n = 12$; $h = \frac{b-a}{n} = \frac{3,5-2}{12} = 0,125$

bấm máy : $x = x + 0.125 : y = \frac{1+x}{1-x}$

$f(x) = \frac{1+x}{1-x}$

$f(x_0) = f(2) = \frac{1+2}{1-2} = -3$

$f(x_1) = f(2 + 0,125) = f(2,125) = -25/9$

$f(x_2) = f(2,125 + 0,125) = f(2,25) = -2,6$

$f(x_3) = f(2,25 + 0.125) = f(2.375) = -27/11$

$f(x_4) = f(2.375 + 0.125) = f(2,5) = -7/3$

$f(x_5) = f(2,5 + 0.125) = f(2,625) = -29/13$

$f(x_6) = f(2,625 + 0.125) = f(2.75) = -15/7$

$f(x_7) = f(2.75 + 0.125) = f(2.875) = -31/15$

$f(x_8) = f(2.875 + 0.125) = f(3) = -2$

$f(x_9) = f(3 + 0.125) = f(3.125) = -33/17$

$f(x_{10}) = f(3.125 + 0.125) = f(3.25) = -17/9$

$f(x_{11}) = f(3.25 + 0.125) = f(3,375) = -35/19$

$f(x_{12}) = f(3,375 + 0.125) = f(3,5) = -9/5$

Simpson formular

$$I = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{n-1} f(x_i) + 2\sum_{i=2}^{n-2} f(x_i) + f(x_n)\right]$$

* Hệ số trước $f(x_0)$ và $f(x_n)$ là hệ số trừ đ' bắt đầu và đ' kết thúc } là 1

$4 \times$ lẻ $(1, 3, 5, 7, 9, 11)$
$2 \times$ chẵn $(2, 4, 6, 8, 10)$

$$\Rightarrow I = \frac{0{,}125}{3}\Big[f(2) + 4\times\big(f(2{,}125) + f(2{,}375) + f(2{,}625) + f(2{,}875) + f(3.125) + f(3.375)\big) + 2\times\big(f(2{,}25) + f(2{,}5) + f(2{,}75) + f(3) + f(3{,}25)\big) + f(3{,}5)\Big]$$

$$= \frac{1}{24}\times\Big[-3 + 4\times\Big(-\frac{25}{9} - \frac{27}{11} - \frac{29}{13} - \frac{31}{15} - \frac{33}{17} - \frac{35}{19}\Big) + 2\times\Big(-2{,}6 - \frac{7}{3} - \frac{15}{7} - 2 - \frac{17}{9}\Big) + \Big(-\frac{9}{5}\Big)\Big] = -3{,}33259$$

b) b. What is the minimum absolute error we can get using Simpson formula for the integration. Justify your answer.

Question 3. (20 pts)
Given an integration:

$$\int_{2}^{3.5} \frac{1+x}{1-x}dx$$

Nếu max:

$$E \le \frac{(|b-a|)^5}{90 \cdot n^4} \times \max_{x\in[a,b]} |f^{(4)}(x)|$$

Min $E \geq \frac{(|b-a|)^5}{90 \cdot n^4} \times \min_{x \in [a,b]} |f^{(4)}(x)|$ đạo hàm $f(x)$ 4 lần

$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$ | $u^{\alpha} = \alpha . u^{\alpha-1} . u'$

$f(x) = \frac{1+x}{1-x}$

$f'(x) = \frac{1.(1-x) - (1+x).(-1)}{(1-x)^2} = \frac{1-x+1+x}{(1-x)^2} = \frac{2}{(1-x)^2}$

$f^{(2)}(x) = \frac{-2.2.(1-x).(-1)}{(1-x)^4} = \frac{4}{(1-x)^3}$

$f^{(3)}(x) = \frac{-4.3(1-x)^2.(-1)}{(1-x)^6} = \frac{12}{(1-x)^4}$

$f^{(4)}(x) = \frac{-12.4.(1-x)^3.(-1)}{(1-x)^8} = \frac{48}{(1-x)^5}$

xét 2 cận 2 đổ

$f^4(2) = -48 \Rightarrow |f^4(2)| = 48$

$f^4(3,5) = -0,49152 \quad |f^4(3.5)| = 0.49152 \Rightarrow \min$

$\Rightarrow E \geq \frac{(3,5-2)^5}{90 . 12^4} \times |-0,49152| \approx 2 \times 10^{-6}$

Question 5. (10 pts)

Using trapezoidal rules, calculate the following integration using n = 12:

ODD: $\int_{1.3}^{2.5} \ln(\sqrt{x+6})dx$

EVEN: $\int_{2.3}^{4.1} \ln(\sqrt{x+6})dx$

bấm máy: $x = x + 0,1 : y = \ln\sqrt{x+6}$

$n = 12$

$$I = \int_{1.3}^{2.5} \ln\left(\sqrt{x+6}\right) dx$$

$$h = \frac{b-a}{n} = \frac{2{,}5 - 1{,}3}{12} = 0{,}1$$

$$I = \frac{h}{2}\left[f(1.3) + 2\left(\sum_{i=1}^{n-1} f(\underbrace{1{,}3 + i \times 0{,}1}_{x_0 + i \times h})\right) + f(2{,}5)\right]$$

$$= \frac{0{,}1}{2}\Big(f(1{,}3) + 2\big\{ (f(1{,}4) + f(1{,}5) + f(1{,}6) + f(1{,}7)$$
$$+ f(1{,}8) + f(1{,}9) + f(2) + f(2.1) + f(2.2) + f(2.3) +$$
$$f(2.4) \big\} + f(2.5) \Big)$$

$$= 0{,}05 \Big(0{,}9939 + 2\big\{ 1{,}00074 + 1{,}00745 + 1{,}0141 + 1{,}0206 +$$
$$1{,}0271 + 1{,}0334 + 1{,}0397 + 1{,}0459 + 1{,}0521 + 1{,}0581 + 1{,}0641 \big) + 1{,}07 \Big)$$

$$= 1{,}13$$

Question 6: Given an integration: $\int_0^1 \left(e^{-2x} + e^{3x}\right) dx$; $n = 10$

What is the maximum absolute error we can get using Simpson formular for the integration

$$(e^u)' = u' . e^u$$

$$\text{Max}: \quad E \le \frac{(|b-a|)^5}{90 . n^4} \times \max_{x \in [a,b]} \left| f^{(4)}(x) \right|$$

$$f(x) = e^{-2x} + e^{3x}$$

$$f'(x) = -2e^{-2x} + 3e^{3x}$$

$$f^{(2)}(x) = -2.(-2).e^{-2x} + 3.3.e^{3x} = 4e^{-2x} + 9e^{3x}$$

$$f^{(3)}(x) = -8e^{-2x} + 27e^{3x}$$

$f^{(4)}(x) = 16e^{-2x} + 81e^{3x}$

$f^{(4)}(0) = 97$

$f^{(4)}(1) = 1629,09 \quad \Rightarrow \max$

$\Rightarrow E \leq \dfrac{(|1-0|)^5}{90 \cdot 10^4} \times 1629,09 = 1,8101 \times 10^{-3}$

Tính tại $x = 0$

Question 6. (10 pts)

If f(x) = a^x (a ≠ 0) is given for x = 0, 0.5, 1. $f(x) = a^x$

Prove by numerical differentiation that $f'(0) = 4\sqrt{a} - a - 3$

$f(x) = a^x$

$f(x+1) = a^{x+1}$ tương tự cho $+0,5$

$a^m \cdot a^n = a^{m+n}$

$$f'(x) = \frac{f(x+1) - f(x+0.5)}{\Delta x}$$

$$= \frac{a^{x+1} - a^{x+0.5}}{0,5}$$

$$= \frac{a^x \cdot a - a^x \cdot a^{(0,5)}}{0.5}$$

$$= 2a^x \cdot a - 2a^x \cdot \sqrt{a}$$

$$f'(0) = 2a^0 \cdot a - 2a^0 \cdot \sqrt{a}$$

$$= 2a - 2\sqrt{a}$$

(1)

$$f'(x) = \frac{f(x+0.5) - f(x)}{\Delta x}$$

$$= \frac{a^{x+0.5} - a^{x}}{0.5}$$

$$= 2a^{x} \cdot a^{0.5} - 2a^{x}$$

$$= 2a^{x} \cdot \sqrt{a} - 2a^{x}$$

$$f'(0) = 2a^{0}\sqrt{a} - 2a^{0}$$

$$= 2\sqrt{a} - 2 \qquad (2)$$

$$f'(x) = \frac{f(x+1) - f(x)}{2\Delta x}$$

xét từ 1 → 0

0 ; 0.5 ; 1

có 2 khoảng ⇒ $2\Delta x$

$$\Rightarrow f'(x) = \frac{a^{x+1} - a^{x}}{(1-0)} = \frac{a^{x+1} - a^{x}}{1} = a^{x} \cdot a - a^{x}$$

$$\Rightarrow f'(0) = a - 1 \qquad (3)$$

$$f'(0) = (2) + (3) - (1)$$

$$= 2\sqrt{a} - 2 + a - 1 - (2a - 2\sqrt{a})$$

$$= 2\sqrt{a} + a - 3 - 2a + 2\sqrt{a}$$

$$= 4\sqrt{a} - a - 3$$

Question 6. (10 pts)

If f(x) = a^x (a ≠ 0) is given for x = 0, 0.5, 1.

Prove by numerical differentiation that $f'(0) = 4\sqrt{a} - a - 3$

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y(x_2)$$

$$= \frac{(x^2 - x x_2 - x x_1 + x_1 x_2)}{(x_0-x_1)(x_0-x_2)} y(x_0) + \frac{(x^2 - x x_2 - x_0 x + x_0 x_2)}{(x_1-x_0)(x_1-x_2)} y(x_1) + \frac{(x^2 - x x_1 - x x_0 + x_0 x_1)}{(x_2-x_0)(x_2-x_1)} y(x_2)$$

$$y'(x) = \frac{2x - x_2 - x_1}{(x_0-x_1)(x_0-x_2)} y(x_0) + \frac{2x - x_2 - x_0}{(x_1-x_0)(x_1-x_2)} y(x_1) + \frac{2x - x_1 - x_0}{(x_2-x_0)(x_2-x_1)} y(x_2)$$

$x_0 = 0$
$x_1 = 0.5$
$x_2 = 1$

$$y'(0) = \frac{0-1-0.5}{(0-0.5)(0-1)} y(0) + \frac{0-1-0}{(0.5-0)(0.5-1)} y(0.5) + \frac{0-0.5-0}{(1-0)(1-0.5)} y(1)$$

$$= \frac{-1.5}{0.5} a^0 + \frac{-1}{-0.25} a^{0.5} + \frac{-0.5}{0.5} a^1$$

$$= -3 + 4a^{0.5} - a$$

$$= -3 + 4\sqrt{a} - a$$