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Homework of Calculus

亲爱的助教老师（学长），由于题量较大且全半角字符切换比较麻烦，这份作业我想尝试用我并不好的英语完成，您见谅:-)

1 Exercise 2.1

问题 1 2.1(5)$P(x_0, f(x_0)), f(x) = x^2 + 2x + 3$,tangent line on point p is a parallel line of $y = 4x - 1$, find x_0 and the equation of tangent line & normal line.

解答

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} f(x) &= 2x + 2 \\ 2x_0 + 2 &= 4 \\ x_0 &= 1 \\ \textit{tangent line equation}: \quad y &= 4(x-1) + 1 + 2 + 3 = 4x + 2 \\ \textit{normal line equation}: \quad y &= -\frac{1}{4}(x-1) + 6 = -\frac{1}{4}x + \frac{23}{4} \end{aligned}$$

问题 2 2.1(8)Find the derivatives of the following function.

解答

1. $\frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)(x-1)\tan x\right) = \left(x^2-1\right)\sec^2 x + 2x\tan x.$
2. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{9x+x^2}{5x+6}\right) = \frac{2x+9}{5x+6} - \frac{5\left(x^2+9x\right)}{(5x+6)^2} = \frac{234}{5(5x+6)^2} + \frac{1}{5}.$
3. $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\cos x + \frac{\sin x}{x}\right) = -\frac{\sin x}{x^2} - x\sin x + \frac{\cos x}{x} + \cos x.$
4. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp(x)\sin x\right) = \exp(x)(\sin x + \cos x)$

问题 3 If $P(x) = (x-x_0)^k g(x), g(x) \neq 0$, prove $P'(x) = (x-x_0)^{k-1}h(x), h(x) \neq 0$.

证明 We can take the derivative of $P(x)$ and get:

$$P'(x) = k(x-x_0)^{k-1}g(x) + (x-x_0)^k g'(x) = (x-x_0)^{k-1}(kg(x) + (x-x_0)g'(x))$$

Define $h(x) = kg(x) + (x-x_0)g'(x)$ and the problem is solved. ■

问题 4 $f(x)$ is even, prove $f'(0) = 0$ as $f'(0)$ exists.

证明 When $f'(0)$ exists, we can take the derivative of $g(x) = f(x) - f(-x)$ which always equal to 0 forall $x \in (-a, a)$. Thus $g'(x) = 0 = f'(x) - (-1)f'(-x)$,$f'(x) + f'(-x) = 0$, so when $x = -x = 0$, $f'(0) = 0$. ■

问题 5 Find the derivative of the following function:

$$f(x) = \begin{cases} \dfrac{x}{1+\exp(x^{-1})}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

解答 $f'(x)$ exists forall $x \neq 0$, for the function is a limited composit of basic functions:

$$f'(x) = \frac{x + \exp(x^{-1})(x+1)}{\left(\exp(x^{-1})+1\right)^2 x}, \quad x \neq 0$$

Noticed that $\lim\limits_{x\to 0+0} f'(x) = 0 \neq 1 = \lim\limits_{x\to 0-0} f'(x)$, $f'(0)$ doesn't exist.

2 Exercise 2.2

问题 6 2.2(3)Find the derivatives of the following function:

解答

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos^5\sqrt{1+x^2}\right) = -\dfrac{5x}{\sqrt{x^2+1}}\sin\left(\sqrt{x^2+1}\right)\cos^4\left(\sqrt{x^2+1}\right)$.
2. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left|\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right|\right) = \dfrac{1}{2}\csc\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\sec\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right) = \sec(x)$.
3. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|\right) = \dfrac{1}{2a}\left(\dfrac{1}{x-a}+\dfrac{1}{x+a}\right) = \dfrac{1}{x^2-a^2}$.

问题 7 2.2(3)Find the derivatives of the following function:

解答

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\arctan x^{-1}\right) = -\dfrac{x^{-2}}{1+x^{-2}} = -\dfrac{1}{x^2+1}$
2. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\arcsin\dfrac{x}{a}\right) = -\dfrac{x^2}{2\sqrt{a^2-x^2}}+\dfrac{1}{2}\sqrt{a^2-x^2}+\dfrac{a}{2\sqrt{1-\dfrac{x^2}{a^2}}} = \sqrt{a^2-x^2}$.
3. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{2}\sqrt{a^2+x^2}+\dfrac{a^2}{2}\ln\dfrac{x+\sqrt{x^a+a^2}}{a}\right) = \sqrt{a^2+x^2}$
4. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2}{\sqrt{a^2-b^2}}\arctan\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right)\right) = \dfrac{1}{a+b\cos x}$

5. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left((1+\sqrt{x})(1+\sqrt{2x})(1+\sqrt{3x})\right) = \dfrac{3\sqrt{6}x+\sqrt{2}+\sqrt{3}+1}{2\sqrt{x}}+\sqrt{2}+\sqrt{3}+\sqrt{6}$(Using function "ln")

6. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{a^a}+a^{x^a}+a^{a^x}\right) = a^a x^{a^a-1}+a^{x^a+1}x^{a-1}\log(a)+a^{a^x+x}\log^2(a)$

Note: the function of Q1 $\arctan x^{-1} = \operatorname{arccot} x$, so the answer is not a surprise. Q2 and Q3 come from the integral of $\cos^2 x$ and $\cosh^2 x$. Q4 comes from of $(a+b\cos x)$, we can let $\cos x = \dfrac{1-t^2}{1+t^2}$ where $t=\tan\dfrac{x}{2}$, $\mathrm{d}t = \dfrac{\sec^2 x/2}{2}\mathrm{d}x = \dfrac{1+t^2}{2}\mathrm{d}x$ and the interal would be easy to calculate.

问题 8 2.2(6)This problem has a graph but I don't want to paste it in my homework so I'll solve it without the description.

解答

$$\begin{aligned} x^2+4-4x\cos\alpha(t) &= 36 \\ 2x\mathrm{d}x - 4\cos\alpha\mathrm{d}x + 4x\sin\alpha\mathrm{d}\alpha &= 0 \\ 2xv-4v\cos\alpha+4x\omega\sin\alpha &= 0 \end{aligned}$$

When $\alpha = \dfrac{\pi}{2}$:

$$\begin{aligned} \sqrt{2}v+16\sqrt{2}\pi &= 0 \\ v &= -16\pi \end{aligned}$$

3 Exercise 2.3

问题 9 When $x\to 0$, $x^{-l}(\sqrt{x+2}-\sqrt{2})\sin x \to l \in \mathbb{R}/\{0\}$, find l.

解答 $x^{-l}(\sqrt{x+2}-\sqrt{2})\sin x \sim x^{-l}\left(\sqrt{2}\left(1+\dfrac{x}{4}\right)-\sqrt{2}\right)x$, thus $l=2$.

问题 10 Sim of $\sqrt[5]{32.16}$.

解答 $\sqrt[5]{32.16} = 2\sqrt[5]{1+0.005} \sim 2(1+0.001) = 2.002$.

问题 11 Find derivatives of the following curves:

解答 1. $x^{2/3}+y^{2/3}=a^{2/3}, x^{-1/3}\mathrm{d}x+y^{-1/3}\mathrm{d}y=0, \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$;(Oh, this one seems not to be in the homework list)

2. $\arctan \frac{y}{x} = \ln \sqrt{x^2+y^2}$, $\left(-\frac{2x}{x^2+y^2} - \frac{y}{x^2\left(\frac{y^2}{x^2}+1\right)}\right)\mathrm{d}x + \left(\frac{1}{x\left(\frac{y^2}{x^2}+1\right)} - \frac{2y}{x^2+y^2}\right)\mathrm{d}y.$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}.$

3. $y\sin x - \cos(x-y) = 0, (\sin x - \sin(x-y))\mathrm{d}y + (y\cos x + \sin(x-y))\mathrm{d}x = 0.$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y\cos x + \sin(x-y)}{\sin x - \sin(x-y)}$

问题 12 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

解答 1. $\begin{cases} x = t\ln t, \\ y = e^t; \end{cases}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^t}{1+\ln t}, \quad t \neq e^{-1};$

2. $\begin{cases} x = \arccos \frac{1}{\sqrt{1+t^2}}, \\ y = \arcsin \frac{t}{\sqrt{1+t^2}}; \end{cases}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = (sgn)\ t, \quad t \neq 0.$

Q2: Consider a point $P(1,t)$, then $|x| = |y| = \arg \overrightarrow{OP}$.

4 Exercise 2.4

问题 13 $y = (1-x)(2x+1)^2(3x-1)^3$, calculate $y^{(6)}, y^{(7)}$.

解答 Noticed that the highest term of y is 6, $y^{(7)}$ must be 0.
It's trivial that forall $n < 6$, $\frac{\mathrm{d}^6}{\mathrm{d}x^6}x^n = 0$, so: $\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} = -1 \times 2^2 \times 4^3 \times 6! = -77760$.

问题 14 $y = \exp(\lambda x), \ddot{y} + p\dot{y} + qy = 0$, find λ.

解答 **Actually, the upon equation is the differential equation of harmonic vibration**, $p = 2\gamma$ refers to consistent, $q = \omega^2$ refers to the angle frequency.

$$\lambda^2 + p\lambda + q = 0$$
$$\lambda = \frac{-p \pm \sqrt{p^2-4q}}{2}$$

in which λ is the eigenvalue of the equation. In multi-degree vibration, λ becomes a series of value that declears base vibration mode of the system.

问题 15 Let $y = x^2\ln(1+x)$. Calculate $y^{(50)}$.

解答

$$
\begin{aligned}
y^{(50)} &= \sum_{i=0}^{50} \binom{50}{i} x^{2(i)} \ln^{(50-i)}(1+x) \\
&= \binom{50}{0} x^2 \ln^{(50)}(1+x) + 2\binom{50}{1} x \ln^{(49)}(1+x) + 2\binom{50}{2} \ln^{(48)}(1+x) \\
&= -\frac{49!x^2}{(1+x)^{50}} + \frac{100 \times 48!x}{(1+x)^{49}} - \frac{50 \times 49 \times 47!}{(1+x)^{48}} \\
&= -\frac{50!x^2}{50(1+x)^{50}} + \frac{2 \times 50!x}{49(1+x)^{49}} - \frac{50!}{48(1+x)^{48}} \\
&= \frac{50!}{(1+x)^{50}} \left(-\frac{x^2}{50} + \frac{2x(1+x)}{49} - \frac{(1+x)^2}{48} \right)
\end{aligned}
$$

5 Exercise 2.5

问题 **16** Calculate:

解答 1. $a\int \sec^2 t \ \mathrm{d}t = a\tan t + C.$

2. $\int \tan^2 t\mathrm{d}t = \int \sec^2 t - 1 \ \mathrm{d}t = \tan t - t + C.$

3. $\int \cot^2 t \ \mathrm{d}t = \int \frac{1}{\sin^2 t} - 1 \ \mathrm{d}t = -\cot t - t + C.$

4. $\int \frac{3}{\sqrt{x}} + \frac{4}{\sqrt{1-x^2}} \ \mathrm{d}x = 6\sqrt{x} + 4\arcsin x + C.$

5. $\int (1+\cos^2 x)\sec^2 x \ \mathrm{d}x = \tan x + x + C.$

6. $\int \frac{1-x}{1-\sqrt[3]{x}}\mathrm{d}x \xlongequal{x=t^3} \int \frac{1-t^3}{1-t} 3t^2\mathrm{d}t = 3\int t^2(1+t+t^2)\mathrm{d}t = t^3 + \frac{3}{4}t^4 + \frac{3}{5}t^5 + C,$

   $= x + \frac{3}{4}x^{4/3} + \frac{3}{5}x^{5/3} + C$

7. $\int 2\cosh x - \sinh x \ \mathrm{d}x = 2\sinh x - \cosh x + C.$

8. $\int \frac{1}{\sin^2 x \cos^2 x}\mathrm{d}x = \int 2\csc^2 2x \ \mathrm{d}2x = -2\cot 2x + C.$

问题 **17** Solve the following differential equation:

$$y''(x) = a + b\exp(-x)$$

解答 To the linear differential equation, we can assume that y can be broken into 2 part: the solution of homogeneous equation $y_0''(x) = 0$, and the special solution of equation $y''^*(x) = a + b\exp(-x)$.
In this example,$y = C_1 x + C_2 + \frac{1}{2}ax^2 + b\exp(-x)$.

问题 18 Solve the following differential equation:

$$xy' + y = x^3 + 1$$

解答 Noticed that $xy' + y = \frac{\mathrm{d}}{\mathrm{d}x}xy(x)$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(xy(x) - \frac{x^4}{4} - x\right)\right) = 0$$

Thus:

$$y(x) = \frac{x^3}{4} + 1 + \frac{C}{x}, \quad x \neq 0$$

6 Exercise 2.6

问题 19 2.6(2)It is a easy problem with complicated description. So I want to answer the quiz without stem.

解答

1. $S = \int_c^d \varphi(y)\ \mathrm{d}y$.
2. $\int_c^d \varphi(y)\ \mathrm{d}y + \int_a^b \psi(y)\ \mathrm{d}y = bd - ac$.

问题 20 Prove $\int_0^1 x^2\ \mathrm{d}x = \frac{1}{3}$.

解答

$$\begin{aligned}
\int_0^1 x^2\ \mathrm{d}x &= \sup \sum_i f(\xi_i)\Delta x \\
&= \sup_n \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} \\
&= \sup_n \frac{n(n+1)(2n+1)}{6n^3} \\
&= \sup_n \frac{1}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right) \\
&= \frac{1}{3}
\end{aligned}$$

问题 21 Prove:

证明 1. $\sin x > 0 \quad \forall x \in (0, \pi/2), I > \dfrac{\pi}{2}$; $\sin x < 1 \quad \forall x \in (0, \pi/2), I < \pi$.

2. $\sqrt{2} < \sqrt{2 + x - x^2} \le \dfrac{3}{2}, \quad \forall x \in (0, 1)$, the $\le$ turns to equal only if $x = \dfrac{1}{2}$.

Note: $\sum \delta x \inf f(x) \le \sup \sum f(\xi_i)\delta x \le \sum \delta x \sup f(x)$. ■

问题 22 Compare:

解答 1. $\displaystyle\int_0^1 e^x \, \mathrm{d}x > \int_0^1 e^{x^2} \, \mathrm{d}x$.

2. $\displaystyle\int_0^{\pi/2} x^2 \mathrm{d}x > \int_0^{\pi/2} \sin^2 x \, \mathrm{d}x$.

3. $\displaystyle\int_0^1 x \, \mathrm{d}x < \int_0^1 \sqrt{1 + x^2} \mathrm{d}x$.

问题 23 The stem is the definition of **riemann integrability**.

证明 So I don't know how to prove a definition. Well, according to squeeze theorem, the traditional internal must satisfied:

$$\sum m_i \delta x_i \le \sum f(\xi_i)\delta x_i \le \sum M_i \delta x_i$$

Take the limit of all three equation:

$$\inf \le \int \le \sup$$

When $\sup = \inf$, $\displaystyle\int$ exists. ■

7 Exercise 2.7

问题 24 Calculate the derivatives of following integrals:

解答 1. $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{1+x^2} \sin t^2 \, \mathrm{d}t = 2x \sin(1 + x^2)^2$.

2. $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 t^2 \cos t \, \mathrm{d}t = -x^2 \cos x$.

3. $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2} \, \mathrm{d}t = 2xe^{-x^4} - e^{-x^2}$.

问题 25 Prove $\displaystyle\lim_{x\to a+0} \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t) \, \mathrm{d}t = f(a)$, if $f(x)$ is continuous on $[a, b]$.

证明

$$\lim_{x\to a+0}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\ \mathrm{d}t=\lim_{x\to a+0}f(x)=f(a)$$ ■

问题 26 If $y=f(x)$ has integrability on $[a,b]$, $|f(x)|\le L$, prove: $F(x)=\displaystyle\int_a^x f(x)\ \mathrm{d}x$ satisfies $|F(x_1)-F(x_2)|\le|x_1-x_2|$.

证明 Noticed that:

$$|F(x_1)-F(x_2)|=\left|\int_{x_2}^{x_1}f(x)\ \mathrm{d}x\right|\le\int_M^m|f(x)|\ \mathrm{d}x\le\int_M^m L\ \mathrm{d}x=L|x_1-x_2|$$

where $m=\min(x_1,x_2), M=\max(x_1,x_2)$. ■

问题 27 Calculate:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x\left(e^t\int_0^t\sin z\ \mathrm{d}z\right)\mathrm{d}t$$

解答

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x\left(e^t\int_0^t\sin z\ \mathrm{d}z\right)\mathrm{d}t&=\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\int_0^x\sin z\ \mathrm{d}z\right)\\&=e^x(1-\cos x+\sin x)\end{aligned}$$

8 Exercise 2.8

问题 28 Calculate:

解答

1. $\displaystyle\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n}\sin\frac{k}{n}=\int_0^1\sin x\ \mathrm{d}x=1-\cos 1.$
2. $\displaystyle\lim_{n\to\infty}\sum_{k=1}^n\frac{k^3}{n^4}=\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n}\left(\frac{k}{n}\right)^3=\int_0^1x^3\ \mathrm{d}x=\frac{1}{4}.$
3. $\displaystyle\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n+k}=\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n}\left(1+\frac{k}{n}\right)^{-1}=\int_0^1\frac{\mathrm{d}x}{1+x}=\ln 2.$

问题 29 Calculate:

解答

1. $\displaystyle\int_{-1}^1|x|\mathrm{d}x=1.$
2. $\displaystyle\int_{-1}^1\mathrm{sgn}\ x\ \mathrm{d}x=0.$

3. $\displaystyle\int_0^1 x\left|\frac{1}{2}-x\right|\mathrm{d}x=\int_{1/2}^1 x^2-\frac{1}{2}x\ \mathrm{d}x+\int_0^{1/2}\frac{1}{2}x-x^2\mathrm{d}x=\frac{1}{8}.$

4. $\displaystyle\int_0^{2\pi}|\sin x|\mathrm{d}x=2\int_0^{\pi}\sin x\mathrm{d}x=4.$

5. $\displaystyle\int_0^2 x-[x]\ \mathrm{d}x=1.$

9 Chapter 2 Exercise

问题 **30 Required Questions** Asuume $f(x) \quad D=\mathbb{R}$ and satisfies:

1. $f(a+b)=f(a)\cdot f(b), \quad \forall a,b\in\mathbb{R}.$

2. $f(0)=1.$

3. $f'(0)$ exists.

Prove: $f'(x)=f'(0)\cdot f(x), \quad \forall x\in\mathbb{R}.$

证明 Noticed that $f(x+b)=f(x)\cdot f(b)\Rightarrow f'(x+b)=f'(x)\cdot f(b)$, it's obvious that we can let $b\to 0$ and we will find the target $f'(x)=f'(0)\cdot f(x)$. ■

Here is another proof:

证明

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=f(x)\lim_{\Delta x\to 0}\frac{f(\Delta x)-f(x)}{\Delta x}=f(x)f'(0)$$ ■

I will finish optional questions with another LaTeXdocument few days later. *The only thing I could feel now is tired because its 2:12 (UTC+8) and I need to wake up at 7:30 (UTC+8) this morning.*