

Mathematical proofs in complement of the book

Principles of Abstract Interpretation

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1 Mathematical proofs of chapter 4

Proof of Lemma 4.18 The lemma trivially holds if $\text{escape}[\![S]\!] = \text{ff}$. Otherwise $\text{escape}[\![S]\!] = \text{tt}$ and the proof is by induction on the distance $\delta(S)$ of S to the root of the abstract syntax tree of P (where $\delta(P) = 0$).

- For $S_l ::= S_l' S$, $\delta(S_l') = \delta(S) = \delta(S_l) + 1$. So, in case $\text{escape}[\![S_l]\!] = \text{tt}$, we have $\text{break-to}[\![S_l]\!] \neq \text{after}[\![S_l]\!]$ by induction hypothesis. By def. $\text{escape}[\![S_l]\!] \triangleq \text{escape}[\![S_l']\!] \vee \text{escape}[\![S]\!]$, there are two subcases.
 - If $\text{escape}[\![S_l']\!] = \text{tt}$ then, on one hand, $S_l \neq \{ \dots \{ \epsilon \} \dots \}$, $\text{after}[\![S_l']\!] = \text{at}[\![S]\!]$, $\text{break-to}[\![S_l']\!] \triangleq \text{break-to}[\![S_l]\!]$, $\text{at}[\![S]\!] \in \text{in}[\![S]\!]$ by Lemma 4.15, so $\text{after}[\![S_l']\!] \in \text{in}[\![S]\!]$.
On the other hand $\text{break-to}[\![S_l']\!] \notin \text{in}[\![S]\!]$ since otherwise $\text{break-to}[\![S_l]\!] = \text{break-to}[\![S_l']\!] \in \text{in}[\![S]\!] \subseteq \text{in}[\![S_l]\!]$ in contradiction with Lemma 4.17, proving $\text{break-to}[\![S_l']\!] \neq \text{after}[\![S_l']\!]$;
 - If $\text{escape}[\![S]\!] = \text{tt}$ then $S \neq \{ \dots \{ \epsilon \} \dots \}$, $\text{after}[\![S]\!] = \text{after}[\![S_l]\!]$, $\text{break-to}[\![S]\!] \triangleq \text{break-to}[\![S_l]\!]$, $\text{break-to}[\![S_l]\!] \neq \text{after}[\![S_l]\!]$ by induction hypothesis, so $\text{break-to}[\![S]\!] \neq \text{after}[\![S]\!]$.
- If $S ::= \text{if } \ell \text{ (B) } S_t$ then $\text{escape}[\![S_t]\!] = \text{escape}[\![S]\!] = \text{tt}$, $\text{after}[\![S_t]\!] = \text{after}[\![S]\!]$, $\text{break-to}[\![S_t]\!] = \text{break-to}[\![S]\!]$, and $\text{break-to}[\![S]\!] \neq \text{after}[\![S]\!]$ by induction hypothesis since $\delta(S_t) = \delta(S) + 1$, so $\text{break-to}[\![S_t]\!] \neq \text{after}[\![S_t]\!]$.
- The proof is similar for $S ::= \text{if } \ell \text{ (B) } S_t \text{ else } S_f$ and $S ::= \{ S_l \}$. □

2 Mathematical proofs of chapter 41

Proof of Theorem 41.24 • For the *statement list* $\mathbf{sl} ::= \mathbf{sl}' \mathbf{s}$, by (17.3) (following (6.13), and (6.14)), we have $\mathcal{S}^*[\mathbf{sl}] = \mathcal{S}^*[\mathbf{sl}'] \cup \{ \langle \pi_1, \pi_2 \hat{\cdot} \pi_3 \rangle \mid \langle \pi_1, \pi_2 \rangle \in \mathcal{S}^*[\mathbf{sl}'] \wedge \langle \pi_1 \hat{\cdot} \pi_2, \pi_3 \rangle \in \mathcal{S}^*[\mathbf{s}] \}$.

- A first case is when $\mathbf{sl}' = \epsilon$ is empty. Then,

$$\begin{aligned}
& \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{sl}](\mathcal{S}^*[\mathbf{sl}]) L_b, L_e \\
&= \bigcup \{ \alpha_{\text{use}, \text{mod}}^l[\epsilon \mathbf{s}] L_b, L_e \langle \pi_0, \pi_1 \rangle \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\epsilon \mathbf{s}] \} \\
& \quad \quad \quad (\text{def. (41.3) of } \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{s}] \text{ for } \mathbf{sl} ::= \epsilon \mathbf{s}) \\
&= \bigcup \{ \alpha_{\text{use}, \text{mod}}^l L_b, L_e \langle \pi_0^\ell, \pi_1 \rangle \mid \langle \pi_0^\ell, \pi_1 \rangle \in \mathcal{S}^*[\epsilon] \cup \{ \langle \pi_0^\ell, \pi_2 \hat{\cdot} \pi_3 \rangle \mid \langle \pi_0^\ell, \pi_2 \rangle \in \mathcal{S}^*[\epsilon] \wedge \langle \pi_0^\ell \hat{\cdot} \pi_2, \pi_3 \rangle \in \mathcal{S}^*[\mathbf{s}] \} \} \\
& \quad \quad \quad (\text{def. } \mathcal{S}^*[\epsilon \mathbf{s}]) \\
&= \bigcup \{ \alpha_{\text{use}, \text{mod}}^l L_b, L_e \langle \pi_0, \pi_1 \rangle \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\mathbf{s}] \} \\
& \quad \quad \quad (\text{(6.15) so that } \mathcal{S}^*[\epsilon] = \{ \langle \pi_0 \text{at}[\mathbf{s}], \text{at}[\mathbf{s}] \rangle \mid \pi_0 \text{at}[\mathbf{s}] \in \mathbb{T}^+ \} \text{ and } \langle \pi_0 \text{at}[\mathbf{s}], \text{at}[\mathbf{s}] \rangle \in \mathcal{S}^*[\mathbf{s}] \text{ by (6.11)}) \\
&= \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{sl}](\mathcal{S}^*[\mathbf{s}]) L_b, L_e \quad \quad \quad (\text{def. (41.3) of } \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{s}]) \\
&= \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{s}](\mathcal{S}^*[\mathbf{s}]) L_b, L_e \\
& \quad \quad \quad (\text{(41.3) since } \text{after}[\mathbf{sl}] = \text{after}[\mathbf{s}], \text{escape}[\mathbf{sl}] = \text{escape}[\mathbf{s}], \text{ and } \text{break-to}[\mathbf{sl}] = \text{break-to}[\mathbf{s}] \text{ when } \mathbf{sl}' = \epsilon) \\
&\subseteq \widehat{\mathcal{S}}^{\exists l}[\mathbf{s}] L_b, L_e \quad \quad \quad (\text{ind. hyp. for Theorem 41.24}) \\
&= \widehat{\mathcal{S}}^{\exists l}[\mathbf{s}] L_b, (\widehat{\mathcal{S}}^{\exists l}[\epsilon] L_b, L_e) \quad \quad \quad (\text{since } \widehat{\mathcal{S}}^{\exists l}[\epsilon] L_b, L_e \triangleq L_e \text{ by (41.22)})
\end{aligned}$$

proving (41.22) when $\mathbf{sl}' = \epsilon$.

- A second case is when $\mathbf{s} = \{ \dots \{ \epsilon \} \dots \}$ is empty. Then, as required by (41.22), we have, by induction hypothesis, $\alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{sl}] L_b, L_e = \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{sl}'] L_b, L_e \subseteq \widehat{\mathcal{S}}^{\exists l}[\mathbf{sl}'] L_b, (\widehat{\mathcal{S}}^{\exists l}[\mathbf{s}] L_b, L_e) \triangleq \widehat{\mathcal{S}}^{\exists l}[\mathbf{sl}] L_b, L_e$ since $\widehat{\mathcal{S}}^{\exists l}[\mathbf{s}] L_b, L_e = L_e$ when \mathbf{s} is empty.
- Otherwise, $\mathbf{sl}' \neq \epsilon$ and $\mathbf{s} \neq \{ \dots \{ \epsilon \} \dots \}$ so, by Lemma 4.16, $\text{after}[\mathbf{s}] \notin \text{in}[\mathbf{s}]$. In that case, let us calculate

$$\begin{aligned}
& \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{sl}] L_b, L_e \\
&= \bigcup \{ \alpha_{\text{use}, \text{mod}}^l[\mathbf{sl}] L_b, L_e \langle \pi_0, \pi_1 \rangle \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\mathbf{sl}] \} \quad \quad \quad (\text{def. (41.3) of } \alpha_{\text{use}, \text{mod}}^{\exists l}[\mathbf{s}])
\end{aligned}$$

$$\begin{aligned}
&= \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, n-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i]\} \cup (\ell_n = \text{after}[\text{SL}] \text{ ? } \\
&\quad L_e \text{ : } \emptyset) \cup (\text{escape}[\text{SL}] \wedge \ell_n = \text{break-to}[\text{SL}] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{SL}] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} \ell_n \} \\
&\quad \text{ (By Lemma 41.8, omitting the useless parameters of use and mod)} \\
&= \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, n-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i]\} \cup (\ell_n = \text{after}[\text{S}] \text{ ? } \\
&\quad L_e \text{ : } \emptyset) \cup (\text{escape}[\text{SL}'] \wedge \ell_n = \text{break-to}[\text{SL}'] \text{ ? } L_b \text{ : } \emptyset) \cup (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } \\
&\quad L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{SL}'] \cup \{ \langle \pi_0 \frown \pi_2, \pi_2 \frown \pi_3 \rangle \mid \langle \pi_0, \pi_2 \rangle \in \mathcal{S}^+[\text{SL}'] \wedge \langle \pi_0 \frown \pi_2, \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \} \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} \ell_n \} \\
&\quad \text{ (def. } \mathcal{S}^*[\text{SL}], \text{ after}[\text{SL}] = \text{after}[\text{S}] \text{ in Section 4.2.2, escape}[\text{SL}] \triangleq \text{escape}[\text{SL}'] \vee \text{escape}[\text{S}], \text{ and break-to}[\text{SL}'] \triangleq \text{break-to}[\text{S}] \triangleq \text{break-to}[\text{SL}] \text{ in Section 4.2.4)} \\
&= \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, n-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i]\} \cup (\ell_n = \text{after}[\text{S}] \text{ ? } \\
&\quad L_e \text{ : } \emptyset) \cup (\text{escape}[\text{SL}'] \wedge \ell_n = \text{break-to}[\text{SL}'] \text{ ? } L_b \text{ : } \emptyset) \cup (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } \\
&\quad L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{SL}'] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} \ell_n \} \cup \\
&\quad \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, n-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i]\} \cup (\ell_n = \text{after}[\text{S}] \text{ ? } \\
&\quad L_e \text{ : } \emptyset) \cup (\text{escape}[\text{SL}'] \wedge \ell_n = \text{break-to}[\text{SL}'] \text{ ? } L_b \text{ : } \emptyset) \cup (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } L_b \text{ : } \\
&\quad \emptyset) \mid \langle \pi_0, \pi_2 \rangle \in \mathcal{S}^+[\text{SL}'] \wedge \langle \pi_0 \frown \pi_2, \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \wedge \pi_2 \frown \pi_3 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} \ell_n \} \\
&\quad \text{ (def. } \cup \text{ and def. } \in \text{ so } \langle \pi_0, \pi_1 \rangle = \langle \pi_0 \frown \pi_2, \pi_2 \frown \pi_3 \rangle) \\
&\subseteq \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, m-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i]\} \cup (\text{escape}[\text{SL}'] \wedge \ell_m = \\
&\quad \text{break-to}[\text{SL}'] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{SL}'] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \} \cup \\
&\quad \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, n-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i]\} \cup (\ell_n = \\
&\quad \text{after}[\text{S}] \text{ ? } L_e \text{ : } \emptyset) \cup (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^+[\text{SL}'] \wedge \langle \pi'_0, \\
&\quad \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \wedge \ell_m = \text{after}[\text{SL}'] \wedge \pi_3 = \ell_m \xrightarrow{a_m} \ell_{m+1} \xrightarrow{a_{m+1}} \dots \xrightarrow{a_{n-1}} \ell_n \} \\
&\quad \text{ (— For the first term, } \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{SL}'], \pi_1 \text{ ends in } \ell_n, \text{ and } \ell_n = \text{after}[\text{S}] \text{ is impos-} \\
&\quad \text{sible since } \text{SL}' \text{ and } \text{S} \text{ are not empty. Moreover, if } \ell_n = \text{break-to}[\text{S}] = \text{break-to}[\text{SL}'] \\
&\quad \text{then } a_{n-1} \text{ is a break, so } \text{escape}[\text{SL}'] \text{ holds. } L_b \text{ is included in } (\text{escape}[\text{SL}'] \wedge \ell_n = \\
&\quad \text{break-to}[\text{SL}'] \text{ ? } L_b \text{ : } \emptyset) \text{ and so } (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } L_b \text{ : } \emptyset) \text{ is re-} \\
&\quad \text{dundant. Finally, renaming } n \leftarrow m. \text{)} \\
&\quad \text{ (— For the second term, if } \ell_n = \text{break-to}[\text{SL}'] = \text{break-to}[\text{S}] \text{ then } a_{n-1} \text{ is a break, so} \\
&\quad \text{escape}[\text{S}] \text{ holds. } L_b \text{ is included in } (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } L_b \text{ : } \emptyset) \text{ and so} \\
&\quad (\text{escape}[\text{SL}'] \wedge \ell_n = \text{break-to}[\text{SL}'] \text{ ? } L_b \text{ : } \emptyset) \text{ is redundant. Moreover, } \pi_2 \frown \pi_3 = \\
&\quad \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} \ell_n \text{ is decomposed into } \pi_2 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \\
&\quad \text{and } \pi_3 = \ell_m \xrightarrow{a_m} \ell_{m+1} \xrightarrow{a_{m+1}} \dots \xrightarrow{a_{n-1}} \ell_n \text{ where, by } \langle \pi_0, \pi_2 \rangle \in \mathcal{S}^+[\text{SL}'] \text{ and} \\
&\quad \langle \pi_0 \frown \pi_2, \pi_3 \rangle \in \mathcal{S}^*[\text{S}], \ell_m = \text{after}[\text{SL}'] = \text{at}[\text{S}]. \text{ Moreover, } \pi_0 \frown \pi_2 \text{ is generalized to} \\
&\quad \pi'_0 \text{ (whence inclusion) and } \pi_2 \text{ is renamed into } \pi_1. \text{)}
\end{aligned}$$

$$\begin{aligned}
&= \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, m-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i] \} \cup (\text{escape}[\text{sl}'] \wedge \ell_m = \\
&\quad \text{break-to}[\text{sl}'] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{sl}'] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \} \cup \\
&\quad \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [m, n-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i] \} \cup (\ell_n = \\
&\quad \text{after}[\text{S}] \text{ ? } L_e \text{ : } \emptyset) \cup (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^+[\text{sl}'] \wedge \langle \pi'_0, \\
&\quad \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \wedge \ell_m = \text{after}[\text{sl}'] \wedge \pi_3 = \ell_m \xrightarrow{a_m} \\
&\quad \ell_{m+1} \xrightarrow{a_{m+1}} \dots \xrightarrow{a_{n-1}} \ell_n \} \\
&\quad \text{[since the case } i \in [1, m-1] \text{ of the second term is already incorporated in the first} \\
&\quad \text{term]} \\
&= \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, m-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i] \} \cup (\ell_m = \\
&\quad \text{after}[\text{sl}'] \text{ ? } (\bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [m, n-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i] \} \cup (\ell_n = \text{after}[\text{S}] \text{ ? } \\
&\quad L_e \text{ : } \emptyset) \cup (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi'_0, \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \wedge \pi_3 = \ell_m \xrightarrow{a_m} \\
&\quad \ell_{m+1} \xrightarrow{a_{m+1}} \dots \xrightarrow{a_{n-1}} \ell_n \} \text{ : } \emptyset) \cup (\text{escape}[\text{sl}'] \wedge \ell_m = \text{break-to}[\text{sl}'] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \\
&\quad \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \wedge \pi_3 = \ell_m \xrightarrow{a_m} \ell_{m+1} \xrightarrow{a_{m+1}} \dots \xrightarrow{a_{n-1}} \ell_n \} \text{ : } \emptyset) \cup (\text{escape}[\text{sl}'] \wedge \ell_m = \\
&\quad \text{break-to}[\text{sl}'] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{sl}'] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \} \\
&\quad \text{[incorporating the second term in the first term, in case } \ell_m = \text{after}[\text{sl}'] \text{]} \\
&\subseteq \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, m-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i] \} \cup (\ell_m = \text{after}[\text{sl}'] \text{ ? } \\
&\quad (\bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [m, n-1] . \forall j \in [m, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i] \} \cup (\ell_n = \text{after}[\text{S}] \text{ ? } \\
&\quad L_e \text{ : } \emptyset) \cup (\text{escape}[\text{S}] \wedge \ell_n = \text{break-to}[\text{S}] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi'_0, \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \wedge \pi_3 = \ell_m \xrightarrow{a_m} \\
&\quad \ell_{m+1} \xrightarrow{a_{m+1}} \dots \xrightarrow{a_{n-1}} \ell_n \} \text{ : } \emptyset) \cup (\text{escape}[\text{sl}'] \wedge \ell_m = \text{break-to}[\text{sl}'] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \\
&\quad \pi_1 \rangle \in \mathcal{S}^*[\text{sl}'] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \} \\
&\quad \text{[dropping the test } \forall j \in [1, m-1] . x \notin \text{mod}[a_j] \text{]} \\
&= \bigcup \{ \{x \in \mathcal{V} \mid \exists i \in [1, m-1] . \forall j \in [1, i-1] . x \notin \text{mod}[a_j] \wedge x \in \text{use}[a_i] \} \cup (\ell_m = \text{after}[\text{sl}'] \text{ ? } \\
&\quad (\bigcup \{ \alpha_{\text{use}, \text{mod}}^l[\text{S}] \text{ } L_b, L_e \langle \pi'_0, \pi_3 \rangle \mid \langle \pi'_0, \pi_3 \rangle \in \mathcal{S}^*[\text{S}] \} \text{ : } \emptyset) \cup (\text{escape}[\text{sl}'] \wedge \ell_m = \\
&\quad \text{break-to}[\text{sl}'] \text{ ? } L_b \text{ : } \emptyset) \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{sl}'] \wedge \pi_1 = \ell_1 \xrightarrow{a_1} \ell_2 \xrightarrow{a_2} \dots \xrightarrow{a_{m-1}} \ell_m \} \\
&\quad \text{[Lemma 41.8]} \\
&\subseteq \bigcup \{ \alpha_{\text{use}, \text{mod}}^l[\text{sl}'] \text{ } L_b, (\mathcal{S}^{\exists}[\text{S}] \text{ } L_b, L_e) \langle \pi_0, \pi_1 \rangle \mid \langle \pi_0, \pi_1 \rangle \in \widehat{\mathcal{S}}^*[\text{sl}'] \} \\
&\quad \text{[Lemma 41.8 and (41.3)]} \\
&= \alpha_{\text{use}, \text{mod}}^{\exists l}[\text{sl}'] (\mathcal{S}^*[\text{sl}']) \text{ } L_b, (\widehat{\mathcal{S}}^{\exists}[\text{S}] \text{ } L_b, L_e) \quad \text{[def. (41.3) of } \alpha_{\text{use}, \text{mod}}^{\exists l} \text{]} \\
&\subseteq \widehat{\mathcal{S}}^{\exists l}[\text{sl}'] \text{ } L_b, (\widehat{\mathcal{S}}^{\exists}[\text{S}] \text{ } L_b, L_e) \\
&\quad \text{[ind. hyp. of Theorem 41.24: } \alpha_{\text{use}, \text{mod}}^{\exists l}[\text{sl}'] (\widehat{\mathcal{S}}^*[\text{sl}']) \text{ } L_b, (\widehat{\mathcal{S}}^{\exists}[\text{S}] \text{ } L_b, L_e) \subseteq \\
&\quad \widehat{\mathcal{S}}^{\exists}[\text{sl}'] \text{ } L_b, (\widehat{\mathcal{S}}^{\exists}[\text{S}] \text{ } L_b, L_e) \text{, Q.E.D.]}
\end{aligned}$$

- For the *empty statement list* $\text{sl} ::= \epsilon$, we have $\mathcal{S}^*[\text{sl}] = \{\langle \pi_0^\ell, \ell \rangle\}$ by (6.15), where $\ell = \text{at}[\text{sl}]$ and so

$$\begin{aligned}
& \alpha_{\text{use}, \text{mod}}^{\exists l}[\text{sl}] (\mathcal{S}^*[\text{sl}]) L_b, L_e \\
&= \bigcup \{ \alpha_{\text{use}, \text{mod}}^l[\text{sl}] L_b, L_e \langle \pi_0, \pi_1 \rangle \mid \langle \pi_0, \pi_1 \rangle \in \mathcal{S}^*[\text{sl}] \} \quad \text{? (41.3)} \\
&= \bigcup \{ \alpha_{\text{use}, \text{mod}}^l[\text{sl}] L_b, L_e \langle \pi_0, \pi_1 \rangle \mid \langle \pi_0, \pi_1 \rangle \in \{ \langle \pi_0^\ell, \ell \rangle \} \} \quad \text{? def. } \mathcal{S}^*[\text{sl}] \\
&= \alpha_{\text{use}, \text{mod}}^{\exists l}[\text{sl}] L_b, L_e \langle \pi_0^\ell, \ell \rangle \quad \text{? def. } \epsilon \text{ and } \cup \\
&= \{ \mathbf{x} \in \mathbb{V} \mid (\ell = \text{after}[\text{sl}] \wedge \mathbf{x} \in L_e) \vee (\text{escape}[\text{sl}] \wedge \ell = \text{break-to}[\text{sl}] \wedge \mathbf{x} \in L_b) \} \quad \text{? (41.3)} \\
&= L_e \quad \text{? } \ell = \text{at}[\text{sl}] = \text{after}[\text{sl}] \text{ in Appendix 4.2.1 and } \text{escape}[\text{sl}] = \text{ff} \text{ in 4.2.4 when } \text{sl} = \epsilon \text{ }
\end{aligned}$$

□

Proof of Theorem 41.27 The proof is by structural induction and essentially consists in applying De Morgan laws for complement. For example,

$$\begin{aligned}
& \widehat{\mathcal{S}}^{\forall d}[\text{if } (B) S_t] D_b, D_e \\
&= \neg \widehat{\mathcal{S}}^{\exists l}[\text{if } (B) S_t] \neg D_b, \neg D_e \quad \text{? definition of } \widehat{\mathcal{S}}^{\forall d}[\text{S}] \text{ as dual of } \widehat{\mathcal{S}}^{\exists l}[\text{S}] \\
&= \neg (\text{use}[B] \cup \neg D_e \cup \widehat{\mathcal{S}}^{\exists l}[S_t] \neg D_b, \neg D_e) \quad \text{? (41.22)} \\
&= \neg \text{use}[B] \cap \neg \neg D_e \cap \neg \widehat{\mathcal{S}}^{\exists l}[S_t] \neg D_b, \neg D_e \quad \text{? De Morgan laws} \\
&= \neg \text{use}[B] \cap D_e \cap \widehat{\mathcal{S}}^{\forall d}[S_t] D_b, D_e \quad \text{? structural induction hypothesis}
\end{aligned}$$

All other cases are similar. □

3 Mathematical proofs of chapter 44

Proof of Theorem 44.38 • In case (44.41) of an empty temporal specification ϵ , we have

$$\begin{aligned}
& \mathcal{M}^+[\text{S}] \langle \underline{\rho}, \epsilon \rangle \\
&\triangleq \mathcal{M}^+ \langle \underline{\rho}, \epsilon \rangle (\widehat{\mathcal{S}}_s^*[\text{S}]) \quad \text{? (44.26)} \\
&= \{ \langle \pi, R' \rangle \mid \pi \in \widehat{\mathcal{S}}_s^*[\text{S}] \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t \langle \rho, \epsilon \rangle \pi \} \quad \text{? (44.25)} \\
&= \{ \langle \pi, \epsilon \rangle \mid \pi \in \widehat{\mathcal{S}}_s^*[\text{S}] \} \quad \text{? since } \mathcal{M}^t \langle \underline{\rho}, \epsilon \rangle \pi \triangleq \langle \text{tt}, \epsilon \rangle \text{ by (44.24)} \\
&\triangleq \widehat{\mathcal{M}}^+[\text{S}] \langle \underline{\rho}, \epsilon \rangle \quad \text{? (44.41)}
\end{aligned}$$

- In case (44.43) of an empty statement list $\text{sl} ::= \epsilon$

$$\begin{aligned}
& \mathcal{M}^+[\text{sl}] \langle \underline{\rho}, R \rangle \\
&= \mathcal{M}^+ \langle \underline{\rho}, R \rangle (\widehat{\mathcal{S}}_s^*[\text{sl}]) \quad \text{? (44.26)} \\
&= \{ \langle \pi, R' \rangle \mid \pi \in \widehat{\mathcal{S}}_s^*[\text{sl}] \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t \langle \underline{\rho}, R \rangle \pi \} \quad \text{? (44.25)}
\end{aligned}$$

$$\begin{aligned}
&= \{ \langle \pi, R' \rangle \mid \pi \in \{ \langle \text{at}[\underline{S}], \rho \rangle \mid \rho \in \mathbb{E}\mathbb{V} \} \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t \langle \underline{Q}, R \rangle \pi \} && \text{? (42.10)} \\
&= \{ \langle \langle \text{at}[\underline{S}], \rho \rangle, R' \rangle \mid \rho \in \mathbb{E}\mathbb{V} \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t \langle \underline{Q}, R \rangle (\langle \text{at}[\underline{S}], \rho \rangle) \} && \text{? (def. } \in \text{)} \\
&= \{ \langle \langle \text{at}[\underline{S}], \rho \rangle, R' \rangle \mid \rho \in \mathbb{E}\mathbb{V} \wedge \langle L : B, R' \rangle = \text{fstnxt}(R) \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L : B] \} \\
&\hspace{15em} \text{? (44.24) with } \mathcal{M}^t \langle \underline{Q}, R' \rangle \ni \langle \text{tt}, R' \rangle \} \\
&= \widehat{\mathcal{M}}^+[\underline{S}](\underline{Q}, R) && \text{? (44.43)}
\end{aligned}$$

- In case (44.44) of a skip statement $S ::= ;$

$$\begin{aligned}
&\mathcal{M}^+[\underline{S}](\underline{Q}, R) \\
&= \{ \langle \pi, R' \rangle \mid \pi \in \widehat{\mathcal{S}}^*[\underline{S}] \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t \langle \underline{Q}, R \rangle \pi \} && \text{? (44.26) and (44.25)} \\
&= \{ \langle \pi, R' \rangle \mid \pi \in \{ \langle \text{at}[\underline{S}], \rho \rangle \mid \rho \in \mathbb{E}\mathbb{V} \} \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t \langle \underline{Q}, R \rangle \pi \} && \text{? (42.11)} \\
&= \{ \langle \langle \text{at}[\underline{S}], \rho \rangle, R' \rangle \mid \rho \in \mathbb{E}\mathbb{V} \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t \langle \underline{Q}, R \rangle (\langle \text{at}[\underline{S}], \rho \rangle) \} && \text{? (def. } \in \text{)} \\
&= \{ \langle \langle \text{at}[\underline{S}], \rho \rangle, R' \rangle \mid \rho \in \mathbb{E}\mathbb{V} \wedge \langle L : B, R' \rangle = \text{fstnxt}(R) \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L : B] \} \\
&\hspace{15em} \text{? (44.24) with } \mathcal{M}^t \langle \underline{Q}, R' \rangle \ni \langle \text{tt}, R' \rangle \} \\
&= \widehat{\mathcal{M}}^+[\underline{S}](\underline{Q}, R) && \text{? (44.44)}
\end{aligned}$$

- In case (44.49) of an iteration statement $S ::= \text{while } \ell(B) S_b$, we apply Corollary 18.31 so we have to calculate the abstract transformer that satisfies the commutation property for an iterate X of the concrete transformer $\mathcal{F}_S^*[\underline{S}]$ (which traces must be of the form $\pi \langle \text{at}[\underline{S}], \rho \rangle$).

$$\begin{aligned}
&\mathcal{M}^+ \langle \underline{Q}, R \rangle (\mathcal{F}_S^*[\underline{S}] X) \\
&= \mathcal{M}^+ \langle \underline{Q}, R \rangle (\{ \langle \ell, \rho \rangle \mid \rho \in \mathbb{E}\mathbb{V} \} \cup \{ \pi_2 \langle \ell', \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle \mid \pi_2 \langle \ell', \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge \ell' = \ell \} \\
&\quad \cup \{ \pi_2 \langle \ell', \rho \rangle \langle \text{at}[\underline{S}_b], \rho \rangle \cdot \pi_3 \mid \pi_2 \langle \ell', \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{tt} \wedge \langle \text{at}[\underline{S}_b], \rho \rangle \cdot \pi_3 \in \widehat{\mathcal{S}}^*[\underline{S}_b] \wedge \ell' = \ell \}) \\
&\hspace{15em} \text{? (42.6)} \\
&= \mathcal{M}^+ \langle \underline{Q}, R \rangle (\{ \langle \ell, \rho \rangle \mid \rho \in \mathbb{E}\mathbb{V} \}) \cup \mathcal{M}^+ \langle \underline{Q}, R \rangle (\{ \pi_2 \langle \ell', \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle \mid \pi_2 \langle \ell', \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge \ell' = \ell \}) \\
&\quad \cup \mathcal{M}^+ \langle \underline{Q}, R \rangle (\{ \pi_2 \langle \ell', \rho \rangle \langle \text{at}[\underline{S}_b], \rho \rangle \cdot \pi_3 \mid \pi_2 \langle \ell', \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{tt} \wedge \langle \text{at}[\underline{S}_b], \rho \rangle \cdot \pi_3 \in \widehat{\mathcal{S}}^*[\underline{S}_b] \wedge \ell' = \ell \}) \\
&\hspace{15em} \text{? (Galois connection (44.30), so that, by Lemma 11.37, } \mathcal{M}^+ \langle \underline{Q}, R \rangle \text{ preserves joins)}
\end{aligned}$$

To avoid repeating (44.41), we assume that $R \notin \mathcal{R}_\varepsilon$ so we can let $\langle L' : B', R' \rangle = \text{fstnxt}(R)$. There are three subcases.

- The first case is that of an observation of the execution that stops at loop entry $\ell = \text{at}[\underline{S}]$. This is similar to the above proof *e.g.* of (44.44) for a skip statement, and we get

$$\begin{aligned}
&\mathcal{M}^+ \langle \underline{Q}, R \rangle (\{ \langle \text{at}[\underline{S}], \rho \rangle \mid \rho \in \mathbb{E}\mathbb{V} \}) \\
&= \{ \langle \langle \text{at}[\underline{S}], \rho \rangle, R' \rangle \mid \rho \in \mathbb{E}\mathbb{V} \wedge \langle L' : B', R' \rangle = \text{fstnxt}(R) \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L' : B'] \}
\end{aligned}$$

- The second case is that of the loop exit

$$\begin{aligned}
& \mathcal{M}^+(\underline{Q}, R)(\{\pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle \mid \pi_2\langle \text{at}[\underline{S}], \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff}\}) \\
&= \{\langle \pi, R' \rangle \mid \pi \in \{\pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle \mid \pi_2\langle \text{at}[\underline{S}], \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff}\} \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t(\underline{Q}, R)\pi\} \quad \text{[(44.25)]} \\
&= \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, R' \rangle \mid \pi_2\langle \text{at}[\underline{S}], \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge \langle \text{tt}, R' \rangle = \mathcal{M}^t(\underline{Q}, R)(\pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle)\} \quad \text{[def. } \in \text{]} \\
&= \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, R' \rangle \mid \pi_2\langle \text{at}[\underline{S}], \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge \exists R'' \in \mathcal{R} . \mathcal{M}^t(\underline{Q}, R)(\pi_2\langle \text{at}[\underline{S}], \rho \rangle) = \langle \text{tt}, R'' \rangle \wedge \mathcal{M}^t(\underline{Q}, R'')(\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle) = \langle \text{tt}, R' \rangle\} \quad \text{[Lemma 44.37]} \\
&= \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, R' \rangle \mid \pi_2\langle \text{at}[\underline{S}], \rho \rangle, R'' \rangle \in \{\langle \pi, R'' \rangle \mid \pi \in X \wedge \langle \text{tt}, R'' \rangle = \mathcal{M}^t(\underline{Q}, R)\pi\} \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge \mathcal{M}^t(\underline{Q}, R'')(\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle) = \langle \text{tt}, R' \rangle\} \\
&\quad \text{[} X \text{ is an iterate of the concrete transformer } \mathcal{F}_{\mathbb{S}}^*[\underline{S}] \text{ so its traces must be of the form } \pi\langle \text{at}[\underline{S}], \rho \rangle \text{]} \\
&= \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, R' \rangle \mid \langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle, R'' \rangle \in \mathcal{M}^+(\underline{Q}, R)X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge \mathcal{M}^t(\underline{Q}, R'')(\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle) = \langle \text{tt}, R' \rangle\} \quad \text{[(44.25)]} \\
&= \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, \varepsilon \rangle \mid \langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle, \varepsilon \rangle \in \mathcal{M}^+(\underline{Q}, R)X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff}\} \cup \\
&\quad \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, R' \rangle \mid \langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle, R'' \rangle \in \mathcal{M}^+(\underline{Q}, R)X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge R'' \notin \mathcal{R}_{\varepsilon} \wedge \mathcal{M}^t(\underline{Q}, R'')(\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle) = \langle \text{tt}, R' \rangle\} \\
&\quad \text{[case analysis and } \mathcal{M}^t(\underline{Q}, \varepsilon)\pi \triangleq \langle \text{tt}, \varepsilon \rangle \text{ in (44.24)]} \\
&= \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, \varepsilon \rangle \mid \langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle, \varepsilon \rangle \in \mathcal{M}^+(\underline{Q}, R)X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff}\} \cup \\
&\quad \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, \varepsilon \rangle \mid \langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle, R'' \rangle \in \mathcal{M}^+(\underline{Q}, R)X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge R'' \notin \mathcal{R}_{\varepsilon} \wedge \langle L' : B', R' \rangle = \text{fstnxt}(R'') \wedge R' \in \mathcal{R}_{\varepsilon} \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L' : B']\} \cup \\
&\quad \{\langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle, R' \rangle \mid \langle \pi_2\langle \text{at}[\underline{S}], \rho \rangle, R'' \rangle \in \mathcal{M}^+(\underline{Q}, R)X \wedge \mathcal{B}[\underline{B}] \rho = \text{ff} \wedge R'' \notin \mathcal{R}_{\varepsilon} \wedge \langle L' : B', R''' \rangle = \text{fstnxt}(R'') \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L' : B'] \wedge R''' \notin \mathcal{R}_{\varepsilon} \wedge \langle L'' : B'', R' \rangle = \text{fstnxt}(R''') \wedge \langle \underline{Q}, \langle \text{after}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L'' : B'']\} \\
&\quad \text{[since } (\langle \text{tt}, R' \rangle = \mathcal{M}^t(\underline{Q}, R'')(\langle \text{at}[\underline{S}], \rho \rangle \langle \text{after}[\underline{S}], \rho \rangle)) \Leftrightarrow (\langle L' : B', R' \rangle = \text{fstnxt}(R'') \wedge R' \in \mathcal{R}_{\varepsilon} \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L' : B']) \vee (\langle L' : B', R''' \rangle = \text{fstnxt}(R'') \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L' : B'] \wedge R''' \notin \mathcal{R}_{\varepsilon} \wedge \langle L'' : B'', R' \rangle = \text{fstnxt}(R''') \wedge \langle \underline{Q}, \langle \text{after}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[L'' : B'']) \text{ as shown above while proving the second term in case (44.46) of a conditional statement } S ::= \text{if } \ell \text{ (B) } S_t \text{]}
\end{aligned}$$

— The third and last case is that of an iteration executing the loop body.

$$\mathcal{M}^+(\underline{Q}, R)(\{\pi_2\langle \text{at}[\underline{S}], \rho \rangle \langle \text{at}[\underline{S}_b], \rho \rangle \cdot \pi_3 \mid \pi_2\langle \text{at}[\underline{S}], \rho \rangle \in X \wedge \mathcal{B}[\underline{B}] \rho = \text{tt} \wedge \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3 \in \widehat{\mathcal{S}}_{\mathbb{S}}^*[\underline{S}_b]\})$$

$$\begin{aligned}
&= \{ \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \mathbf{R}' \rangle \mid \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle, \mathbf{R}'' \rangle \in \mathcal{M}^+ \langle \underline{Q}, \mathbf{R} \rangle X \wedge \mathcal{B}[\underline{B}] \rho = \text{tt} \wedge \mathbf{R}'' \notin \\
&\quad \mathcal{R}_\varepsilon \wedge \langle \underline{L} : \underline{B}, \mathbf{R}''' \rangle = \text{fstnxt}(\mathbf{R}'') \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[\underline{L} : \underline{B}] \wedge \mathcal{M}^t \langle \underline{Q}, \mathbf{R}''' \rangle \langle \text{at}[\underline{S}_b], \rho \rangle = \langle \text{tt}, \\
&\quad \mathbf{R}''' \rangle \wedge \langle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \mathbf{R}' \rangle \in \mathcal{M}^+[\underline{S}_b] \langle \underline{Q}, \mathbf{R}''' \rangle \} \quad \wr (44.24) \}
\end{aligned}$$

There are two subsubcases, depending on whether \mathbf{R}''' is empty or not.

– If $\mathbf{R}''' \in \mathcal{R}_\varepsilon$ then, as shown before, $\mathcal{M}^t \langle \underline{Q}, \mathbf{R}''' \rangle \langle \text{at}[\underline{S}_b], \rho \rangle = \langle \text{tt}, \mathbf{R}''' \rangle$ implies that $\mathbf{R}''' \in \mathcal{R}_\varepsilon$ and so $\langle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \mathbf{R}' \rangle \in \mathcal{M}^+[\underline{S}_b] \langle \underline{Q}, \mathbf{R}''' \rangle$ if and only if $\mathbf{R}' \in \mathcal{R}_\varepsilon$ and $\langle \text{at}[\underline{S}_b], \rho \rangle \pi_3 \in \widehat{\mathcal{S}}_\varepsilon^*[\underline{S}_b]$. We get

$$\begin{aligned}
&= \{ \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \varepsilon \rangle \mid \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle, \mathbf{R}'' \rangle \in \mathcal{M}^+ \langle \underline{Q}, \mathbf{R} \rangle X \wedge \mathcal{B}[\underline{B}] \rho = \text{tt} \wedge \mathbf{R}'' \notin \\
&\quad \mathcal{R}_\varepsilon \wedge \langle \underline{L} : \underline{B}, \varepsilon \rangle = \text{fstnxt}(\mathbf{R}'') \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[\underline{L} : \underline{B}] \wedge \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3 \in \widehat{\mathcal{S}}_\varepsilon^*[\underline{S}_b] \} \\
&\quad \wr (44.24) \}
\end{aligned}$$

– Otherwise $\mathbf{R}''' \notin \mathcal{R}_\varepsilon$.

$$\begin{aligned}
&= \{ \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \mathbf{R}' \rangle \mid \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle, \mathbf{R}'' \rangle \in \mathcal{M}^+ \langle \underline{Q}, \mathbf{R} \rangle X \wedge \mathcal{B}[\underline{B}] \rho = \text{tt} \wedge \mathbf{R}'' \notin \\
&\quad \mathcal{R}_\varepsilon \wedge \langle \underline{L} : \underline{B}, \mathbf{R}''' \rangle = \text{fstnxt}(\mathbf{R}'') \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[\underline{L} : \underline{B}] \wedge \mathbf{R}''' \notin \mathcal{R}_\varepsilon \wedge \mathcal{M}^t \langle \underline{Q}, \\
&\quad \mathbf{R}''' \rangle \langle \text{at}[\underline{S}_b], \rho \rangle = \langle \text{tt}, \mathbf{R}''' \rangle \wedge \langle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \mathbf{R}' \rangle \in \mathcal{M}^+[\underline{S}_b] \langle \underline{Q}, \mathbf{R}''' \rangle \} \\
&= \{ \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \mathbf{R}' \rangle \mid \langle \pi_2 \langle \text{at}[\underline{S}], \rho \rangle, \mathbf{R}'' \rangle \in \mathcal{M}^+ \langle \underline{Q}, \mathbf{R} \rangle X \wedge \mathcal{B}[\underline{B}] \rho = \text{tt} \wedge \mathbf{R}'' \notin \\
&\quad \mathcal{R}_\varepsilon \wedge \langle \underline{L} : \underline{B}, \mathbf{R}''' \rangle = \text{fstnxt}(\mathbf{R}'') \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}], \rho \rangle \rangle \in \mathcal{S}^r[\underline{L} : \underline{B}] \wedge \mathbf{R}''' \notin \mathcal{R}_\varepsilon \wedge \langle \underline{L}' : \underline{B}', \\
&\quad \mathbf{R}''' \rangle = \text{fstnxt}(\mathbf{R}''') \wedge \langle \underline{Q}, \langle \text{at}[\underline{S}_b], \rho \rangle \rangle \in \mathcal{S}^r[\underline{L}' : \underline{B}'] \wedge \langle \langle \text{at}[\underline{S}_b], \rho \rangle \pi_3, \mathbf{R}' \rangle \in \mathcal{M}^+[\underline{S}_b] \langle \underline{Q}, \\
&\quad \mathbf{R}''' \rangle \} \\
&\quad \wr (44.24) \}
\end{aligned}$$

— Grouping all cases together we get the term (44.50) defining $\widehat{\mathcal{F}}^+[\underline{S}] \langle \underline{Q}, \mathbf{R} \rangle (\mathcal{M}^+ \langle \underline{Q}, \mathbf{R} \rangle X)$ and so Corollary 18.31 and the commutation condition $\mathcal{M}^+ \langle \underline{Q}, \mathbf{R} \rangle (\mathcal{F}_\varepsilon^*[\underline{S}](X)) = \widehat{\mathcal{F}}^+[\underline{S}] \langle \underline{Q}, \mathbf{R} \rangle (\mathcal{M}^+ \langle \underline{Q}, \mathbf{R} \rangle (X))$ for the iterates X of $\mathcal{F}_\varepsilon^*[\underline{S}]$ yield $\widehat{\mathcal{M}}^+[\underline{S}] \langle \underline{Q}, \mathbf{R} \rangle \triangleq \text{lfpc}(\widehat{\mathcal{F}}^+[\underline{S}] \langle \underline{Q}, \mathbf{R} \rangle)$ that is (44.49).

- In case (44.48) of a break statement $\mathbf{S} ::= \ell \text{ break ;}$

$$\begin{aligned}
&\mathcal{M}^+[\underline{S}] \langle \underline{Q}, \mathbf{R} \rangle \\
&= \{ \langle \pi, \mathbf{R}' \rangle \mid \pi \in \widehat{\mathcal{S}}_\varepsilon^*[\underline{S}] \wedge \langle \text{tt}, \mathbf{R}' \rangle = \mathcal{M}^t \langle \underline{Q}, \mathbf{R} \rangle \pi \} \quad \wr (44.26) \text{ and } (44.25) \} \\
&= \{ \langle \pi, \mathbf{R}' \rangle \mid \pi \in \{ \langle \ell, \rho \rangle \mid \rho \in \mathbb{E}\mathbf{v} \} \cup \{ \langle \ell, \rho \rangle \langle \text{break-to}[\underline{S}], \rho \rangle \mid \rho \in \mathbb{E}\mathbf{v} \} \wedge \langle \text{tt}, \mathbf{R}' \rangle = \mathcal{M}^t \langle \underline{Q}, \mathbf{R} \rangle \pi \} \\
&\quad \wr (42.14) \} \\
&= \{ \langle \langle \ell, \rho \rangle, \mathbf{R}'' \rangle \mid \langle \text{tt}, \mathbf{R}'' \rangle = \mathcal{M}^t \langle \underline{Q}, \mathbf{R} \rangle \langle \ell, \rho \rangle \} \cup \{ \langle \langle \ell, \rho \rangle \langle \text{break-to}[\underline{S}], \rho \rangle, \mathbf{R}'' \rangle \mid \langle \text{tt}, \mathbf{R}'' \rangle = \mathcal{M}^t \langle \underline{Q}, \\
&\quad \mathbf{R} \rangle \langle \langle \ell, \rho \rangle \langle \text{break-to}[\underline{S}], \rho \rangle \rangle \} \quad \wr \text{def. } \cup \text{ and } \in \}
\end{aligned}$$

$$\begin{aligned}
&= \text{let } \langle L : B, R' \rangle = \text{fstnxt}(R) \text{ in } \{ \langle \langle \ell, \rho \rangle, R' \rangle \mid \langle \underline{q}, \langle \ell, \rho \rangle \rangle \in \mathcal{S}^r \llbracket L : B \rrbracket \} \cup \{ \langle \langle \ell, \rho \rangle \langle \text{break-to} \llbracket S \rrbracket, \\
&\quad \rho \rangle, \varepsilon \rangle \mid R' \in \mathcal{R}_\varepsilon \wedge \langle \underline{q}, \langle \ell, \rho \rangle \rangle \in \mathcal{S}^r \llbracket L : B \rrbracket \} \cup \{ \langle \langle \ell, \rho \rangle \langle \text{break-to} \llbracket S \rrbracket, \rho \rangle, R'' \rangle \mid R' \notin \mathcal{R}_\varepsilon \wedge \langle \underline{q}, \\
&\quad \langle \ell, \rho \rangle \rangle \in \mathcal{S}^r \llbracket L : B \rrbracket \wedge \langle L' : B', R'' \rangle = \text{fstnxt}(R') \wedge \langle \underline{q}, \langle \text{break-to} \llbracket S \rrbracket, \rho \rangle \rangle \in \mathcal{S}^r \llbracket L' : B' \rrbracket \} \\
&\quad \wr R \notin \mathcal{R}_\varepsilon, \text{ case analysis on } R' \in \mathcal{R}_\varepsilon, \text{ and (44.24)} \quad \square
\end{aligned}$$

4 Mathematical proofs of chapter 47

Proof (47.47) There are three cases depending on whether the program label ℓ is at or after statement S , or in the true branch S_t .

$$\begin{aligned}
&\text{--- (1) ---} \quad \text{The cases } \ell = \text{at} \llbracket S \rrbracket \text{ was handled in (47.41) and } \ell \notin \text{labx} \llbracket S \rrbracket \text{ in (47.42).} \\
&\text{--- (2) ---} \quad \text{Assume } \ell = \text{after} \llbracket S \rrbracket. \\
&\quad \alpha^d(\{\mathcal{S}^{+\infty} \llbracket S \rrbracket\}) \text{ after} \llbracket S \rrbracket \\
&= \alpha^d(\{\mathcal{S}^* \llbracket S \rrbracket\}) \text{ after} \llbracket S \rrbracket \quad \wr \text{Lemma 47.23} \\
&= \{ \langle x', y \rangle \mid \mathcal{S}^* \llbracket S \rrbracket \in \mathcal{D}(\text{after} \llbracket S \rrbracket) \langle x', y \rangle \} \quad \wr \text{def. (47.25) of } \alpha^d \\
&= \{ \langle x', y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \mathcal{S}^* \llbracket S \rrbracket . (\forall z \in \mathcal{V} \setminus \{x'\} . \mathcal{Q}(\pi_0)z = \mathcal{Q}(\pi'_0)z) \wedge \\
&\quad \text{diff}(\text{seqval} \llbracket y \rrbracket (\text{after} \llbracket S \rrbracket)(\pi_0, \pi_1), \text{seqval} \llbracket y \rrbracket (\text{after} \llbracket S \rrbracket)(\pi'_0, \pi'_1)) \} \quad \wr \text{def. (47.19) of } \mathcal{D}^\ell \langle x', y \rangle \\
&= \{ \langle x', y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \{ \langle \pi \text{at} \llbracket S \rrbracket, \text{at} \llbracket S \rrbracket \xrightarrow{\neg(B)} \text{after} \llbracket S \rrbracket} \mid \mathcal{B} \llbracket B \rrbracket \mathcal{Q}(\pi \text{at} \llbracket S \rrbracket) = \\
&\quad \text{ff} \} \cup \{ \langle \pi \text{at} \llbracket S \rrbracket, \text{at} \llbracket S \rrbracket \xrightarrow{B} \text{at} \llbracket S_t \rrbracket \pi' \text{after} \llbracket S \rrbracket \} \mid \mathcal{B} \llbracket B \rrbracket \mathcal{Q}(\pi \text{at} \llbracket S \rrbracket) = \text{tt} \wedge \text{at} \llbracket S_t \rrbracket \pi' \text{after} \llbracket S \rrbracket \in \\
&\quad \widehat{\mathcal{S}}^{+\infty} \llbracket S_t \rrbracket (\pi \text{at} \llbracket S \rrbracket \xrightarrow{B} \text{at} \llbracket S_t \rrbracket) \} . (\forall z \in \mathcal{V} \setminus \{x'\} . \mathcal{Q}(\pi_0)z = \mathcal{Q}(\pi'_0)z) \wedge \\
&\quad \text{diff}(\text{seqval} \llbracket y \rrbracket (\text{after} \llbracket S \rrbracket)(\pi_0, \pi_1), \text{seqval} \llbracket y \rrbracket (\text{after} \llbracket S \rrbracket)(\pi'_0, \pi'_1)) \} \\
&\quad \wr \text{def. } \mathcal{S}^* \llbracket S \rrbracket \text{ in (6.9), (6.19), and (6.18) so that } \text{after} \llbracket S \rrbracket = \text{after} \llbracket S_t \rrbracket \\
&= \{ \langle x', y \rangle \mid \exists \langle \pi_0, \pi_1 \text{after} \llbracket S \rrbracket \rangle, \langle \pi'_0, \pi'_1 \text{after} \llbracket S \rrbracket \rangle \in \{ \langle \pi \text{at} \llbracket S \rrbracket, \text{at} \llbracket S \rrbracket \xrightarrow{\neg(B)} \text{after} \llbracket S \rrbracket} \mid \\
&\quad \mathcal{B} \llbracket B \rrbracket \mathcal{Q}(\pi \text{at} \llbracket S \rrbracket) = \text{ff} \} \cup \{ \langle \pi \text{at} \llbracket S \rrbracket, \text{at} \llbracket S \rrbracket \xrightarrow{B} \text{at} \llbracket S_t \rrbracket \pi' \text{after} \llbracket S \rrbracket \} \mid \mathcal{B} \llbracket B \rrbracket \mathcal{Q}(\pi \text{at} \llbracket S \rrbracket) = \\
&\quad \text{tt} \wedge \text{at} \llbracket S_t \rrbracket \pi' \text{after} \llbracket S \rrbracket \in \widehat{\mathcal{S}}^{+\infty} \llbracket S_t \rrbracket (\pi \text{at} \llbracket S \rrbracket \xrightarrow{B} \text{at} \llbracket S_t \rrbracket) \} \wedge (\forall z \in \mathcal{V} \setminus \{x'\} . \mathcal{Q}(\pi_0)z = \\
&\quad \mathcal{Q}(\pi'_0)z) \wedge \text{diff}(\mathcal{Q}(\pi_0 \frown \pi_1 \text{after} \llbracket S \rrbracket)y, \mathcal{Q}(\pi'_0 \frown \pi'_1 \text{after} \llbracket S \rrbracket)y) \} \\
&\quad \wr \text{def. } \in \text{ so that } \pi_1 \text{ and } \pi'_1 \text{ must end with } \text{after} \llbracket S \rrbracket \text{ and def. (47.16) of } \text{seqval} \llbracket y \rrbracket \\
&= \{ \langle x', y \rangle \mid \exists \pi_0 \text{at} \llbracket S \rrbracket, \pi_1 \text{after} \llbracket S \rrbracket, \pi'_0 \text{at} \llbracket S \rrbracket, \pi'_1 \text{after} \llbracket S \rrbracket \in \{ \pi \text{at} \llbracket S \rrbracket \xrightarrow{\neg(B)} \text{after} \llbracket S \rrbracket} \mid \\
&\quad \mathcal{B} \llbracket B \rrbracket \mathcal{Q}(\pi \text{at} \llbracket S \rrbracket) = \text{ff} \} \cup \{ \pi \text{at} \llbracket S \rrbracket \xrightarrow{B} \text{at} \llbracket S_t \rrbracket \pi' \text{after} \llbracket S \rrbracket \mid \mathcal{B} \llbracket B \rrbracket \mathcal{Q}(\pi \text{at} \llbracket S \rrbracket) = \text{tt} \wedge \\
&\quad \text{at} \llbracket S_t \rrbracket \pi' \text{after} \llbracket S \rrbracket \in \widehat{\mathcal{S}}^{+\infty} \llbracket S_t \rrbracket (\pi \text{at} \llbracket S \rrbracket \xrightarrow{B} \text{at} \llbracket S_t \rrbracket) \} \wedge (\forall z \in \mathcal{V} \setminus \{x'\} . \mathcal{Q}(\pi_0 \text{at} \llbracket S \rrbracket)z = \\
&\quad \mathcal{Q}(\pi'_0 \text{at} \llbracket S \rrbracket)z) \wedge \text{diff}(\mathcal{Q}(\pi_0 \text{at} \llbracket S \rrbracket \pi_1 \text{after} \llbracket S \rrbracket)y, \mathcal{Q}(\pi'_0 \text{at} \llbracket S \rrbracket \pi'_1 \text{after} \llbracket S \rrbracket)y) \}
\end{aligned}$$

$$\begin{aligned}
& \text{[def. } \in \text{ and trace concatenation } \cdot \text{]} \\
= & \{ \langle x', y \rangle \mid \exists \pi_0 \text{at}[S] \pi_1 \text{after}[S], \pi'_0 \text{at}[S] \pi'_1 \text{after}[S] \in \{ \pi \text{at}[S] \xrightarrow{\neg(B)} \text{after}[S] \mid \\
& \mathcal{B}[B] \mathcal{Q}(\pi \text{at}[S]) = \text{ff} \} \cup \{ \pi \text{at}[S] \xrightarrow{B} \text{at}[S_t] \pi' \text{after}[S] \mid \mathcal{B}[B] \mathcal{Q}(\pi \text{at}[S]) = \\
& \text{tt} \wedge \text{at}[S_t] \pi' \text{after}[S] \in \mathcal{S}^{+\infty}[S_t](\pi \text{at}[S] \xrightarrow{B} \text{at}[S_t]) \} \wedge (\forall z \in \mathcal{V} \setminus \{x'\} . \\
& \mathcal{Q}(\pi_0 \text{at}[S])z = \mathcal{Q}(\pi'_0 \text{at}[S])z) \wedge (\mathcal{Q}(\pi_0 \text{at}[S] \pi_1 \text{after}[S])y \neq \mathcal{Q}(\pi'_0 \text{at}[S] \pi'_1 \text{after}[S])y) \} \\
& \text{[def. (47.18) of diff]}
\end{aligned} \tag{1}$$

There are four subcases, depending upon which branch of the conditional is taken by the two executions $\pi_0 \text{at}[S] \pi_1 \text{after}[S]$ and $\pi'_0 \text{at}[S] \pi'_1 \text{after}[S]$.

— (2.a) — If both executions $\pi_0 \text{at}[S] \pi_1 \text{after}[S]$ and $\pi'_0 \text{at}[S] \pi'_1 \text{after}[S]$ are through the false branch, we have,

$$\begin{aligned}
& (1) \\
= & \{ \langle x', y \rangle \mid \exists \pi_0 \text{at}[S] \xrightarrow{\neg(B)} \text{after}[S], \pi'_0 \text{at}[S] \xrightarrow{\neg(B)} \text{after}[S] . \mathcal{B}[B] \mathcal{Q}(\pi_0 \text{at}[S]) = \text{ff} \wedge \\
& \mathcal{B}[B] \mathcal{Q}(\pi'_0 \text{at}[S]) = \text{ff} \wedge (\forall z \in \mathcal{V} \setminus \{x'\} . \mathcal{Q}(\pi_0 \text{at}[S])z = \mathcal{Q}(\pi'_0 \text{at}[S])z) \wedge (\mathcal{Q}(\pi_0 \text{at}[S] \xrightarrow{\neg(B)} \\
& \text{after}[S])y \neq \mathcal{Q}(\pi'_0 \text{at}[S] \xrightarrow{\neg(B)} \text{after}[S])y) \} \\
& \text{[case } \mathcal{B}[B] \mathcal{Q}(\pi_0 \text{at}[S]) = \text{ff} \text{ and } \mathcal{B}[B] \mathcal{Q}(\pi'_0 \text{at}[S]) = \text{ff} \text{]} \\
= & \{ \langle x', y \rangle \mid \exists \pi_0 \text{at}[S], \pi'_0 \text{at}[S] . \mathcal{B}[B] \mathcal{Q}(\pi_0 \text{at}[S]) = \text{ff} \wedge \mathcal{B}[B] \mathcal{Q}(\pi'_0 \text{at}[S]) = \text{ff} \wedge (\forall z \in \mathcal{V} \setminus \{x'\} . \\
& \mathcal{Q}(\pi_0 \text{at}[S])z = \mathcal{Q}(\pi'_0 \text{at}[S])z) \wedge (\mathcal{Q}(\pi_0 \text{at}[S])y \neq \mathcal{Q}(\pi'_0 \text{at}[S])y) \} \\
& \text{[def. (6.6) of } \mathcal{Q} \text{ so that } \mathcal{Q}(\pi_0 \text{at}[S] \xrightarrow{\neg(B)} \text{after}[S])y = \mathcal{Q}(\pi_0 \text{at}[S])y \text{]} \\
= & \{ \langle x', y \rangle \mid \exists \rho, \nu . \mathcal{B}[B] \rho = \text{ff} \wedge \mathcal{B}[B] \rho[x' \leftarrow \nu] = \text{ff} \wedge \rho(y) \neq \rho[x' \leftarrow \nu]y \} \\
& \text{[letting } \rho = \mathcal{Q}(\pi_0 \text{at}[S]), \nu = \mathcal{Q}(\pi'_0 \text{at}[S])x' \text{ so that } \forall z \in \mathcal{V} \setminus \{x'\} . \mathcal{Q}(\pi_0 \text{at}[S])z = \\
& \mathcal{Q}(\pi'_0 \text{at}[S])z \text{ implies } \mathcal{Q}(\pi'_0 \text{at}[S]) = \rho[x' \leftarrow \nu] \text{ and, conversely Exercise 6.8, so that any} \\
& \text{environment } \rho \text{ can be computed as the result } \mathcal{Q}(\pi'_0 \text{at}[S]) \text{ of an appropriate initialization} \\
& \text{trace } \pi'_0 \text{at}[S] \text{ (otherwise, this is } \subseteq \text{)} \text{]} \\
= & \{ \langle x', x' \rangle \mid \exists \rho, \nu . \rho(x') \neq \nu \wedge \mathcal{B}[B] \rho = \text{ff} \wedge \mathcal{B}[B] \rho[x' \leftarrow \nu] = \text{ff} \} \\
& \text{[since } \rho[x' \leftarrow \nu](y) = \rho(y) \text{ when } y \neq x' \text{]} \\
= & \{ \langle x', x' \rangle \mid x' \in \text{nondet}(\neg B, \neg B) \} \quad \text{[def. (47.48) of nondet]} \\
= & \mathbb{1}_{\mathcal{V}} \upharpoonright \text{nondet}(\neg B, \neg B) \quad \text{[def. left restriction]} \\
\subseteq & \mathbb{1}_{\mathcal{V}}
\end{aligned}$$

In words for that first case, the initial value of x' flows to the value of x' by the false branch of the conditional **if** (B) S_t when there are at least two different values of x' for which B is false. (If

$$\begin{aligned}
&\subseteq \{ \langle x', y \rangle \mid \exists \rho, v . \rho(x') \neq v \wedge \mathcal{B}[\![B]\!]\rho = \text{tt} \wedge \mathcal{B}[\![B]\!]\rho[x' \leftarrow v] = \text{tt} \} \cap \{ \langle x', y \rangle \mid \exists \langle \bar{\pi}_0, \bar{\pi}_1 \text{after}[\![S_t]\!]\pi_2 \rangle, \langle \bar{\pi}_0', \bar{\pi}_1' \text{after}[\![S_t]\!]\pi_2' \rangle \in \mathcal{S}^{+\infty}[\![S_t]\!] . (\forall z \in V \setminus \{x'\} . \mathcal{Q}(\bar{\pi}_0)z = \mathcal{Q}(\bar{\pi}_0')z) \wedge \text{after}[\![S_t]\!] \notin \bar{\pi}_1 \wedge \text{after}[\![S_t]\!] \notin \bar{\pi}_1' \wedge \text{diff}(\text{seqval}[\![y]\!](\text{after}[\![S_t]\!])(\bar{\pi}_0 \frown \bar{\pi}_1 \text{after}[\![S_t]\!], \text{after}[\![S_t]\!]\pi_2), \text{seqval}[\![y]\!](\text{after}[\![S_t]\!])(\bar{\pi}_0' \frown \bar{\pi}_1' \text{after}[\![S_t]\!], \text{after}[\![S_t]\!]\pi_2')) \} \\
&\quad \quad \quad (\text{letting } \rho = \mathcal{Q}(\bar{\pi}_0) \text{ and } v = \mathcal{Q}(\bar{\pi}_0')(x')) \\
&= \{ \langle x', y \rangle \mid \exists \rho, v . \rho(x') \neq v \wedge \mathcal{B}[\![B]\!]\rho = \text{tt} \wedge \mathcal{B}[\![B]\!]\rho[x' \leftarrow v] = \text{tt} \} \cap \{ \langle x', y \rangle \mid \mathcal{S}^{+\infty}[\![S_t]\!] \in \mathcal{D}(\text{after}[\![S_t]\!]) \langle x', y \rangle \} \\
&\quad \quad \quad (\text{def. (47.19) of } \mathcal{D}^e \langle x', y \rangle) \\
&= \{ \langle x', y \rangle \mid \exists \rho, v . \rho(x') \neq v \wedge \mathcal{B}[\![B]\!]\rho = \text{tt} \wedge \mathcal{B}[\![B]\!]\rho[x' \leftarrow v] = \text{tt} \} \cap \alpha^d(\{\mathcal{S}^{+\infty}[\![S_t]\!]\}) \text{after}[\![S_t]\!] \\
&\quad \quad \quad (\text{def. } \subseteq \text{ and def. (47.25) of } \alpha^d)
\end{aligned}$$

In words for that second case, the initial value of x' flows to the value of y by the true branch of the conditional **if** (B) S_t when there are at least two different values of x' for which B is true and x' flows to the value of y in S_t .

$$\begin{aligned}
&\subseteq \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_t]\!] \text{after}[\![S_t]\!] \mid \text{nondet}(B, B) \\
&\quad \quad \quad (\text{by structural ind. hyp. , def. (47.48) of } \text{nondet}, \text{ and def. of the left restriction } \mid \text{ of a relation in Section 2.2.2}) \\
&\subseteq \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_t]\!] \text{after}[\![S_t]\!] \quad \quad \quad (\text{A coarse over-approximation ignoring values})
\end{aligned}$$

— (2.c–d) — Otherwise, one execution is through the true branch (say $\pi_0 \text{at}[\![S]\!]\pi_1 \text{after}[\![S]\!]$) and the other is through the false branch (say $\pi'_0 \text{at}[\![S]\!]\pi'_1 \text{after}[\![S]\!]$), we have (the other case is symmetric),

$$\begin{aligned}
&(1) \\
&= \{ \langle x', y \rangle \mid \exists \pi_0 \text{at}[\![S]\!]\pi_1 \text{after}[\![S]\!] \in \{ \pi \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!]\pi' \text{after}[\![S]\!] \mid \mathcal{B}[\![B]\!]\mathcal{Q}(\pi \text{at}[\![S]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi' \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\pi \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!]) \} . \exists \pi'_0 \text{at}[\![S]\!]\pi'_1 \text{after}[\![S]\!] \in \{ \pi \text{at}[\![S]\!] \xrightarrow{\neg(B)} \text{after}[\![S]\!] \mid \mathcal{B}[\![B]\!]\mathcal{Q}(\pi \text{at}[\![S]\!]) = \text{ff} \} . (\forall z \in V \setminus \{x'\} . \mathcal{Q}(\pi_0 \text{at}[\![S]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S]\!])z) \wedge (\mathcal{Q}(\pi_0 \text{at}[\![S]\!]\pi_1 \text{after}[\![S]\!])y \neq \mathcal{Q}(\pi'_0 \text{at}[\![S]\!]\pi'_1 \text{after}[\![S]\!])y) \} \\
&\quad \quad \quad (\text{case } \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0 \text{at}[\![S]\!]) = \text{tt} \text{ and } \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0 \text{at}[\![S]\!]) = \text{ff}) \\
&= \{ \langle x', y \rangle \mid \exists \pi_0, \pi_1, \pi'_0 . \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0 \text{at}[\![S]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\pi_0 \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!]) \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0 \text{at}[\![S]\!]) = \text{ff} \wedge (\forall z \in V \setminus \{x'\} . \mathcal{Q}(\pi_0 \text{at}[\![S]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S]\!])z) \wedge (\mathcal{Q}(\pi_0 \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!])y \neq \mathcal{Q}(\pi'_0 \text{at}[\![S]\!] \xrightarrow{\neg(B)} \text{after}[\![S]\!])y) \} \\
&\quad \quad \quad (\text{def. } \subseteq) \\
&= \{ \langle x', y \rangle \mid \exists \bar{\pi}_0, \pi_1, \pi'_0 . \mathcal{B}[\![B]\!]\mathcal{Q}(\bar{\pi}_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\bar{\pi}_0 \text{at}[\![S_t]\!]) \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0 \text{at}[\![S]\!]) = \text{ff} \wedge (\forall z \in V \setminus \{x'\} . \mathcal{Q}(\bar{\pi}_0 \text{at}[\![S_t]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S]\!])z) \wedge (\mathcal{Q}(\bar{\pi}_0 \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!])y \neq \mathcal{Q}(\pi'_0 \text{at}[\![S]\!])y) \}
\end{aligned}$$

$$\begin{aligned}
& \text{(letting } \bar{\pi}_0 \text{at}[\![S_t]\!] = \pi_0 \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!] \text{ so that by def. (6.6) of } \varrho, \varrho(\pi_0 \text{at}[\![S]\!]) = \\
& \quad \varrho(\bar{\pi}_0 \text{at}[\![S_t]\!]) \text{ so } \mathfrak{B}[\![B]\!]\varrho(\pi_0 \text{at}[\![S]\!]) = \mathfrak{B}[\![B]\!]\varrho(\bar{\pi}_0 \text{at}[\![S_t]\!]) \text{ and } \varrho(\pi'_0 \text{at}[\![S]\!]) \xrightarrow{\neg(B)} \text{after}[\![S]\!])y \\
& \quad = \varrho(\pi'_0 \text{at}[\![S]\!])y \text{)} \\
& = \{ \langle x', y \rangle \mid \exists \bar{\pi}_0, \pi_1, \pi'_0 . \mathfrak{B}[\![B]\!]\varrho(\bar{\pi}_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\bar{\pi}_0 \text{at}[\![S_t]\!]) \wedge \\
& \quad \mathfrak{B}[\![B]\!]\varrho(\pi'_0 \text{at}[\![S]\!]) \xrightarrow{B} \text{at}[\![S_t]\!] = \text{ff} \wedge (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\bar{\pi}_0 \text{at}[\![S_t]\!])z = \varrho(\pi'_0 \text{at}[\![S]\!]) \xrightarrow{B} \text{at}[\![S_t]\!])z) \wedge \\
& \quad (\varrho(\bar{\pi}_0 \text{at}[\![S_t]\!])\pi_1 \text{after}[\![S]\!])y \neq \varrho(\pi'_0 \text{at}[\![S]\!]) \xrightarrow{B} \text{at}[\![S_t]\!])y \} \\
& \quad \text{(by def. (6.6) of } \varrho \text{ so that } \varrho(\pi'_0 \text{at}[\![S]\!]) = \varrho(\pi'_0 \text{at}[\![S]\!]) \xrightarrow{B} \text{at}[\![S_t]\!]) \text{)} \\
& = \{ \langle x', y \rangle \mid \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0 \text{at}[\![S_t]\!])z = \varrho(\pi'_0 \text{at}[\![S_t]\!])z) \wedge \mathfrak{B}[\![B]\!]\varrho(\pi_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \\
& \quad \mathfrak{B}[\![B]\!]\varrho(\pi'_0 \text{at}[\![S_t]\!]) = \text{ff} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\pi_0 \text{at}[\![S_t]\!]) \wedge (\varrho(\pi_0 \text{at}[\![S_t]\!])\pi_1 \text{after}[\![S]\!])y \neq \\
& \quad \varrho(\pi'_0 \text{at}[\![S_t]\!])y \} \\
& \quad \text{(letting } \pi'_0 \text{at}[\![S_t]\!] = \pi'_0 \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!], \text{ commutativity of } \wedge \text{)} \\
& = \{ \langle x', x' \rangle \mid \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0 \text{at}[\![S_t]\!])z = \varrho(\pi'_0 \text{at}[\![S_t]\!])z) \wedge \\
& \quad \mathfrak{B}[\![B]\!]\varrho(\pi_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \mathfrak{B}[\![B]\!]\varrho(\pi'_0 \text{at}[\![S_t]\!]) = \text{ff} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\pi_0 \text{at}[\![S_t]\!]) \wedge \\
& \quad (\varrho(\pi_0 \text{at}[\![S_t]\!])\pi_1 \text{after}[\![S]\!])x' \neq \varrho(\pi'_0 \text{at}[\![S_t]\!])x' \} \\
& \quad \cup \{ \langle x', y \rangle \mid x' \neq y \wedge \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0 \text{at}[\![S_t]\!])z = \varrho(\pi'_0 \text{at}[\![S_t]\!])z) \wedge \\
& \quad \mathfrak{B}[\![B]\!]\varrho(\pi_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \mathfrak{B}[\![B]\!]\varrho(\pi'_0 \text{at}[\![S_t]\!]) = \text{ff} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\pi_0 \text{at}[\![S_t]\!]) \wedge \\
& \quad (\varrho(\pi_0 \text{at}[\![S_t]\!])\pi_1 \text{after}[\![S]\!])y \neq \varrho(\pi'_0 \text{at}[\![S_t]\!])y \} \\
& \quad \text{(since when } x' \neq y, \varrho(\pi'_0 \text{at}[\![S_t]\!])y = \varrho(\pi_0 \text{at}[\![S_t]\!])y \text{)} \\
& \text{In words for that third case, } x' \text{ flows to } x' \text{ if and only if changing } x' \text{ changes the boolean expression } B \text{ and when } B \text{ is true, } S_t \text{ changes } x' \text{ to a value different from that when } B \text{ is false. A counter-example is } \text{if } (x' \neq 1) \ x' = 1 \text{ ;.} \\
& \text{Moreover, } x' \text{ flows to } y \neq x' \text{ if and only if changing } x' \text{ changes the boolean expression } B \text{ and when } B \text{ is true, } S_t \text{ changes } y. \\
& = \{ \langle x', y \rangle \mid \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0 \text{at}[\![S_t]\!])z = \varrho(\pi'_0 \text{at}[\![S_t]\!])z) \wedge \mathfrak{B}[\![B]\!]\varrho(\pi_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \\
& \quad \mathfrak{B}[\![B]\!]\varrho(\pi'_0 \text{at}[\![S_t]\!]) = \text{ff} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\pi_0 \text{at}[\![S_t]\!]) \wedge (\varrho(\pi_0 \text{at}[\![S_t]\!])\pi_1 \text{after}[\![S]\!])y \neq \\
& \quad \varrho(\pi'_0 \text{at}[\![S_t]\!])y \} \quad \text{(grouping cases together)} \\
& = \{ \langle x', y \rangle \mid \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0 \text{at}[\![S_t]\!])z = \varrho(\pi'_0 \text{at}[\![S_t]\!])z) \wedge \mathfrak{B}[\![B]\!]\varrho(\pi_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \\
& \quad \mathfrak{B}[\![B]\!]\varrho(\pi'_0 \text{at}[\![S_t]\!]) = \text{ff} \wedge \text{at}[\![S_t]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S_t]\!](\pi_0 \text{at}[\![S_t]\!]) \wedge (\varrho(\pi_0 \text{at}[\![S_t]\!])\pi_1 \text{after}[\![S]\!])y \neq \\
& \quad \varrho(\pi'_0 \text{at}[\![S_t]\!])y \} \mid \text{nondet}(B, \neg B) \\
& \quad \text{(letting } \rho = \varrho(\pi_0 \text{at}[\![S]\!]), \nu = \varrho(\pi'_0 \text{at}[\![S]\!])x' \text{ so that } \forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0 \text{at}[\![S]\!])z = \\
& \quad \varrho(\pi'_0 \text{at}[\![S]\!])z \text{ implies } \varrho(\pi'_0 \text{at}[\![S]\!]) = \rho[x' \leftarrow \nu]. \text{ It follows that } \exists \rho, \nu . \rho(x') \neq \nu \wedge \\
& \quad \mathfrak{B}[\![B]\!]\rho = \text{tt} \wedge \mathfrak{B}[\![B]\!]\rho[x' \leftarrow \nu] = \text{ff}. \text{ Therefore, by def. (47.48) of nondet, } x' \in \\
& \quad \text{nondet}(B, \neg B) \text{)} \\
& \subseteq \{ \langle x', y \rangle \mid x' \in \text{nondet}(B, \neg B) \wedge y \in \text{mod}[\![S_t]\!] \}
\end{aligned}$$

Since $\{x \mid \exists \pi_0, \pi_1 . \text{at}[\![S]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^*[\![S]\!](\pi_0 \text{at}[\![S]\!]) \wedge \varrho(\pi_0 \text{at}[\![S]\!]\pi_1 \text{after}[\![S]\!])x \neq \varrho(\pi_0 \text{at}[\![S]\!])x\} \subseteq \text{mod}[\![S]\!]$, a simple coarse approximation is to consider the variables y appearing to the left of an assignment in S_t , a necessary condition for y to be modified by the execution of S_t where the set $\text{mod}[\![S]\!]$ of variables that may be modified by the execution of S is syntactically defined as in (47.50). \int

$$= \text{nondet}(\mathbf{B}, \neg \mathbf{B}) \times \text{mod}[\![S_t]\!] \quad \int \text{def. cartesian product} \int$$

$$\subseteq \{\langle x', y \rangle \mid x' \in \text{vars}[\![B]\!] \wedge y \in \text{mod}[\![S_t]\!]\}$$

$\int \text{nondet}(\mathbf{B}, \neg \mathbf{B})$ can be over-approximated by the set of variables x' occurring in the boolean expression \mathbf{B} as defined in Exercise 3.3 \int

Exercise 2 Prove that for all program components $S \in \mathcal{PC}$,

$$\{x \mid \exists \pi_0, \pi_1 . \text{at}[\![S]\!]\pi_1 \text{after}[\![S]\!] \in \widehat{\mathcal{S}}^{+\infty}[\![S]\!](\pi_0 \text{at}[\![S]\!]) \wedge \varrho(\pi_0 \text{at}[\![S]\!]\pi_1 \text{after}[\![S]\!])x \neq \varrho(\pi_0 \text{at}[\![S]\!])x\} \subseteq \text{mod}[\![S]\!]. \quad \square$$

— (3) — Finally, assume $\ell \in \text{in}[\![S_t]\!]$.

$$\begin{aligned} & \alpha^d(\{\mathcal{S}^*[\![S]\!]\})^\ell \\ &= \{\langle x', y \rangle \mid \mathcal{S}^*[\![S]\!] \in \mathcal{D}^\ell(\langle x', y \rangle)\} \quad \int \text{def. (47.25) of } \alpha^d \int \\ &= \{\langle x', y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \mathcal{S}^*[\![S]\!] . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0)z = \varrho(\pi'_0)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0, \pi_1), \text{seqval}[\![y]\!]^\ell(\pi'_0, \pi'_1))\} \quad \int \text{def. (47.19) of } \mathcal{D}^\ell(\langle x', y \rangle) \int \\ &= \{\langle x', y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \{\langle \pi \text{at}[\![S]\!], \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!]\pi' \ell \pi'' \rangle \mid \mathcal{B}[\![B]\!]\varrho(\pi \text{at}[\![S]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi' \ell \pi'' \in \widehat{\mathcal{S}}^*[\![S_t]\!](\pi \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!])\} . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0)z = \varrho(\pi'_0)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0, \pi_1), \text{seqval}[\![y]\!]^\ell(\pi'_0, \pi'_1))\} \quad \int \text{def. (6.19) of } \mathcal{S}^*[\![S]\!] \int \\ &= \{\langle x', y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \{\langle \pi \text{at}[\![S]\!], \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!]\pi' \ell \pi'' \rangle \mid \mathcal{B}[\![B]\!]\varrho(\pi \text{at}[\![S]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi' \ell \pi'' \in \widehat{\mathcal{S}}^*[\![S_t]\!](\pi \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!])\} . (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0)z = \varrho(\pi'_0)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0, \pi_1), \text{seqval}[\![y]\!]^\ell(\pi'_0, \pi'_1))\} \\ & \quad \int \text{since if } \langle \pi_0, \pi_1 \rangle \text{ (or } \langle \pi'_0, \pi'_1 \rangle) \text{ has the form } \langle \pi \text{at}[\![S]\!], \text{at}[\![S]\!] \xrightarrow{\neg(\mathbf{B})} \text{after}[\![S]\!] \rangle \text{ then } \ell \text{ does not appear in } \pi_1 \text{ (resp. } \pi'_1) \text{ so that, by (47.16),} \\ & \quad \text{seqval}[\![y]\!]^\ell(\pi_0, \pi_1) = \ni \text{ (resp. } \text{seqval}[\![y]\!]^\ell(\pi'_0, \pi'_1) = \ni \text{ and therefore, by (47.18),} \\ & \quad \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0, \pi_1), \text{seqval}[\![y]\!]^\ell(\pi'_0, \pi'_1)) \text{ is false} \int \\ &= \{\langle x', y \rangle \mid \exists \pi_0, \pi_1, \pi_2, \pi'_0, \pi'_1, \pi'_2 . \mathcal{B}[\![B]\!]\varrho(\pi_0 \text{at}[\![S]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi_1 \ell \pi_2 \in \widehat{\mathcal{S}}^*[\![S_t]\!](\pi_0 \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!]) \wedge \mathcal{B}[\![B]\!]\varrho(\pi'_0 \text{at}[\![S]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!]\pi'_1 \ell \pi'_2 \in \widehat{\mathcal{S}}^*[\![S_t]\!](\pi'_0 \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!]) \wedge (\forall z \in \mathcal{V} \setminus \{x'\} . \varrho(\pi_0 \text{at}[\![S]\!])z = \varrho(\pi'_0 \text{at}[\![S]\!])z) \wedge \ell \notin \pi_1 \wedge \ell \notin \pi'_1 \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0 \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!]\pi_1 \ell, \ell \pi_2), \text{seqval}[\![y]\!]^\ell(\pi'_0 \text{at}[\![S]\!] \xrightarrow{\mathbf{B}} \text{at}[\![S_t]\!]\pi'_1 \ell, \ell \pi'_2))\} \end{aligned}$$

$$\begin{aligned}
& \wr \text{def. } \in \text{ and if } \ell \text{ has multiple occurrences in } \pi'_1 \ell \pi'_2, \text{ we choose the first one, same for } \pi'_1 \ell \pi'_2 \wr \\
& = \{ \langle x', y \rangle \mid \exists \tilde{\pi}_0, \pi_1, \pi_2, \tilde{\pi}_0', \pi'_1, \pi'_2 . \mathcal{B}[\![B]\!] \mathcal{Q}(\tilde{\pi}_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!] \pi_1 \ell \pi_2 \in \mathcal{S}^*[\![S_t]\!](\tilde{\pi}_0 \text{at}[\![S_t]\!]) \} \wedge \\
& \quad \mathcal{B}[\![B]\!] \mathcal{Q}(\tilde{\pi}_0' \text{at}[\![S_t]\!]) = \text{tt} \wedge \text{at}[\![S_t]\!] \pi'_1 \ell \pi'_2 \in \mathcal{S}^*[\![S_t]\!](\tilde{\pi}_0' \text{at}[\![S_t]\!]) \wedge (\forall z \in V \setminus \{x'\} . \\
& \quad \mathcal{Q}(\tilde{\pi}_0 \text{at}[\![S_t]\!])z = \mathcal{Q}(\tilde{\pi}_0' \text{at}[\![S_t]\!])z) \wedge \ell \notin \pi_1 \wedge \ell \notin \pi'_1 \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\tilde{\pi}_0 \text{at}[\![S_t]\!] \pi_1 \ell, \ell \pi_2), \\
& \quad \text{seqval}[\![y]\!]^\ell(\tilde{\pi}_0' \text{at}[\![S_t]\!] \pi'_1 \ell, \ell \pi'_2)) \} \\
& \quad \wr \text{letting } \tilde{\pi}_0 \text{at}[\![S_t]\!] = \pi_0 \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!], \tilde{\pi}_0' \text{at}[\![S_t]\!] = \pi'_0 \text{at}[\![S]\!] \xrightarrow{B} \text{at}[\![S_t]\!] \text{ so that by} \\
& \quad \text{def. (6.6) of } \mathcal{Q}, \mathcal{Q}(\tilde{\pi}_0 \text{at}[\![S_t]\!]) = \mathcal{Q}(\pi_0 \text{at}[\![S]\!]) \text{ and } \mathcal{Q}(\tilde{\pi}_0' \text{at}[\![S_t]\!]) = \mathcal{Q}(\pi'_0 \text{at}[\![S]\!]) \wr \\
& \subseteq \{ \langle x', y \rangle \mid \exists \pi_0, \pi'_0 . \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi'_0 \text{at}[\![S_t]\!]) = \text{tt} \wedge (\forall z \in \\
& \quad V \setminus \{x'\} . \mathcal{Q}(\pi_0 \text{at}[\![S_t]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S_t]\!])z) \} \cap \{ \langle x', y \rangle \mid \exists \pi_0, \pi_1, \pi_2, \pi'_0, \pi'_1, \pi'_2 . \\
& \quad \text{at}[\![S_t]\!] \pi_1 \ell \pi_2 \in \mathcal{S}^*[\![S_t]\!](\pi_0 \text{at}[\![S_t]\!]) \wedge \text{at}[\![S_t]\!] \pi'_1 \ell \pi'_2 \in \mathcal{S}^*[\![S_t]\!](\pi'_0 \text{at}[\![S_t]\!]) \wedge (\forall z \in V \setminus \{x'\} . \\
& \quad \mathcal{Q}(\pi_0 \text{at}[\![S_t]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S_t]\!])z) \wedge \ell \notin \pi_1 \wedge \ell \notin \pi'_1 \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0 \text{at}[\![S_t]\!] \pi_1 \ell, \ell \pi_2), \\
& \quad \text{seqval}[\![y]\!]^\ell(\pi'_0 \text{at}[\![S_t]\!] \pi'_1 \ell, \ell \pi'_2)) \} \wr \text{def. } \exists \text{ and } \subseteq \wr \\
& = \{ \langle x', y \rangle \mid \exists \rho, v . \rho(x') \neq v \wedge \mathcal{B}[\![B]\!] \rho = \text{tt} \wedge \mathcal{B}[\![B]\!] \rho[x' \leftarrow v] = \text{tt} \} \cap \{ \langle x', y \rangle \mid \mathcal{S}^*[\![S_t]\!] \in \mathcal{D}^\ell(x', y) \} \\
& \quad \wr \text{letting } \rho = \mathcal{Q}(\tilde{\pi}_0), v = \mathcal{Q}(\tilde{\pi}_0')(x') \text{ and def. (47.19) of } \mathcal{D}^\ell(x', y) \wr \\
& = \{ \langle x', y \rangle \mid \exists \rho, v . \rho(x') \neq v \wedge \mathcal{B}[\![B]\!] \rho = \text{tt} \wedge \mathcal{B}[\![B]\!] \rho[x' \leftarrow v] = \text{tt} \} \cap \{ \langle x', y \rangle \mid \{ \mathcal{S}^*[\![S_t]\!] \} \subseteq \\
& \quad \mathcal{D}^\ell(x', y) \} \wr \text{def. } \subseteq \wr \\
& = \{ \langle x', y \rangle \mid \exists \rho, v . \rho(x') \neq v \wedge \mathcal{B}[\![B]\!] \rho = \text{tt} \wedge \mathcal{B}[\![B]\!] \rho[x' \leftarrow v] = \text{tt} \} \cap \alpha^d(\{ \mathcal{S}^*[\![S_t]\!] \})^\ell \\
& \quad \wr \text{def. (47.25) of } \alpha^d \wr \\
& \subseteq \{ \langle x', y \rangle \mid \exists \rho, v . \rho(x') \neq v \wedge \mathcal{B}[\![B]\!] \rho = \text{tt} \wedge \mathcal{B}[\![B]\!] \rho[x' \leftarrow v] = \text{tt} \} \cap \mathcal{S}^d[\![S_t]\!]^\ell \\
& \quad \wr \text{structural induction hypothesis} \wr \\
& = \mathcal{S}^d[\![S_t]\!]^\ell \wr \text{nondet}(B, B) \wr \text{def. (47.48) of } \text{nondet} \wr \\
& \quad \text{In words, the initial value of } x' \text{ flows to the value of } y \text{ at } \ell \text{ in the true branch } S_t \text{ of the conditional} \\
& \quad \text{if } (B) S_t \text{ when there are at least two different values of } x' \text{ for which } B \text{ is true and } x' \text{ flows to} \\
& \quad \text{the value of } y \text{ at } \ell \text{ in } S_t. \\
& \subseteq \mathcal{S}^d[\![S_t]\!]^\ell \\
& \quad \wr \text{A coarse over-approximation ignoring values i.e. that the conditional holds for only one} \\
& \quad \text{value of } x' \wr \quad \square
\end{aligned}$$

Proof of (47.63) By Lemma 47.23, the Definition 47.28 of value dependency using the maximal traces semantics is equivalent to the definition of value dependency for finite prefix traces, as defined by (17.4). So the soundness of (47.63) follows from the following (3):

$$\left[\begin{array}{l} \alpha^d(\mathcal{S}^*[\![S]\!]) = \alpha^d(\text{lfp}^{\subseteq} \mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!]) \\ \subseteq \text{lfp}^{\subseteq} \mathcal{F}^{\text{diff}}[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] = \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S]\!] \end{array} \right] \quad (3)$$

The proof of (3) is an application of Exercise 18.17. $\langle C, \sqsubseteq, \perp, \sqcup \rangle$ is the complete lattice $\langle \wp(\wp(\mathbb{T}^+ \times \mathbb{T}^{+\infty})), \subseteq, \emptyset, \cup \rangle$. $\langle \mathcal{A}, \preceq, 0, \vee \rangle$ is the complete lattice $\langle \mathbb{P}^d, \subseteq^d, \perp^d, \sqcup^d \rangle$. The Galois connection $\langle C, \sqsubseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle \mathcal{A}, \preceq \rangle$ is given by Lemma 47.26. The transformer f is (17.4). It preserves arbitrary non-empty unions so it is continuous. The transformer g is (47.63). It preserves arbitrary non-empty unions pointwise so it is pointwise continuous (i.e. for \subseteq^d and \sqcup^d defined pointwise). The main point of the proof is to check the semi-commutation condition

$$\alpha^d \circ \mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] \subseteq \mathcal{F}^{\text{diff}}[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] \circ \alpha^d. \quad (4)$$

By Exercise 18.17, we need to make the proof only for elements $X \in \mathcal{X}$ where \mathcal{X} is chosen to be exactly the iterates of the transformer $\mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!]$ from \emptyset .

In practice, we have discovered $\mathcal{F}^{\text{diff}}[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!]$ knowing $\mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!]$ and α^d by rewriting until getting a formula of the form $\mathcal{F}^{\text{diff}}[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] \circ \alpha^d$ and using \subseteq -over-approximations to ignore values in the static analysis. By Exercise 18.17, we conclude that

$$\alpha^d(\text{lfp}^{\subseteq} \mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!]) \subseteq \text{lfp}^{\subseteq} \mathcal{F}^{\text{diff}}[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!].$$

The proof of semi-commutation (4) is by calculational design as follows. By def. (47.18) of diff , we do not have to compare futures of prefix traces where one is a prefix of the other.

$$\begin{aligned} & \alpha^d(\{\mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] X\})^{\ell'} \\ &= \{\langle x, y \rangle \mid \mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] X \in \mathcal{D}(\ell')\langle x, y \rangle\} \quad \text{\textit{def. (47.25) of } } \alpha^d \text{\textit{}} \\ &= \{\langle x, y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] X. (\forall z \in \mathcal{V} \setminus \{x\}. \mathbf{q}(\pi_0)z = \mathbf{q}(\pi'_0)z) \wedge \\ & \quad \text{diff}(\text{seqval}[\![y]\!]^{\ell'}(\pi_0, \pi_1), \text{seqval}[\![y]\!]^{\ell'}(\pi'_0, \pi'_1))\} \quad \text{\textit{def. (47.19) of } } \mathcal{D}^{\ell} \text{\textit{}} \\ &= \{\langle x, y \rangle \mid \exists \langle \pi_0^{\ell}, \pi_1^{\ell} \rangle, \langle \pi_0'^{\ell}, \pi_1'^{\ell} \rangle \in \mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!] X. (\forall z \in \mathcal{V} \setminus \{x\}. \\ & \quad \mathbf{q}(\pi_0^{\ell})z = \mathbf{q}(\pi_0'^{\ell})z) \wedge \text{diff}(\text{seqval}[\![y]\!]^{\ell'}(\pi_0^{\ell}, \pi_1^{\ell}), \text{seqval}[\![y]\!]^{\ell'}(\pi_0'^{\ell}, \pi_1'^{\ell}))\} \quad (5) \\ & \quad \text{\textit{since } } \langle \pi_0^{\ell'}, \pi_1^{\ell'} \rangle \notin \mathcal{F}^*[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!](X) \text{ when } \ell' \neq \ell \text{ or } \ell'' \neq \ell \text{\textit{}}} \end{aligned}$$

There are three main cases depending on whether the dependency observation point ℓ' is (1) at the iteration (so $\ell' = \ell = \text{at}[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!]$), (2) is in the loop body (so $\ell' \in \text{in}[\![S_b]\!]$), or (3) is after the iteration (so $\ell' = \text{after}[\![\text{while}^{\ell}(\mathbf{B}) S_b]\!]$).

For each of these case, we have to consider all possible ways the traces π_1 and π'_1 in (5) can go through the dependency observation program point ℓ' . The definition of \mathcal{F}^* below shows all possible choices (A), (B), or (C) of π_1 and π'_1 in (5). Notice that diff in (47.16) is commutative so $\langle \pi_0^{\ell}, \pi_1^{\ell} \rangle$ and $\langle \pi_0'^{\ell}, \pi_1'^{\ell} \rangle$ play symmetric rôles in (5) which reduces the number of cases to be considered.

$$\mathcal{F}^*[\text{while } \ell \text{ (B) } S_b](X) \triangleq \{\langle \pi_0^\ell, \ell \rangle\} \quad (\text{A}) \quad (17.4)$$

$$\cup \{\langle \pi_0^\ell, \ell \pi_2^\ell \xrightarrow{\text{B}} \text{at}[S_b] \pi_3^{\ell''} \rangle \mid \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X \wedge \mathcal{B}[\text{B}]\mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{tt} \quad (\text{B})$$

$$\wedge \langle \pi_0^\ell \pi_2^\ell \xrightarrow{\text{B}} \text{at}[S_b], \text{at}[S_b] \pi_3^{\ell''} \rangle \in \mathcal{S}^*[S_b]\}$$

$$\cup \{\langle \pi_0^\ell, \ell \pi_2^\ell \xrightarrow{\neg(\text{B})} \text{after}[S] \rangle \mid \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X \wedge \mathcal{B}[\text{B}]\mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{ff}\} \quad (\text{C})$$

The case (B) covers essentially 3 subcases depending of where is ℓ'' that is where the prefix observation $\text{at}[S_b] \pi_3^{\ell''}$ of the execution of the body S_b has terminated:

(Ba) within the loop body $\ell'' \in \text{in}[S_b]$;

(Bb) after the loop body $\ell'' = \text{after}[S_b] = \text{at}[S] = \ell$, because of the normal termination of the loop body, and thus at ℓ , just before the next iteration or the loop exit;

(Bc) after the loop $\ell'' = \text{after}[S]$ because of a **break** ; statement in the loop body S_b ; \square

— (1) If the dependency observation point ℓ' is at loop entry then $\ell' = \ell = \text{at}[\text{while } \ell \text{ (B) } S_b]$. There are three subcases, depending on how $\ell' = \ell$ is reached $\ell \pi_1$ by (A), (B), or (C) of $\ell \pi_1$ and $\ell \pi_1'$ in (5).

— (1-A) In the first case $\ell \pi_1 = \ell$ so $\pi_1 = \ni$ in (A). We have $\text{seqval}[\mathbf{y}]^{\ell'}(\pi_0^\ell, \ell) = \mathcal{Q}(\pi_0^\ell) \mathbf{y}$ by (47.16). Whether $\ell \pi_1'$ is determined by (A), (B), or (C) we have in all cases that $\text{seqval}[\mathbf{y}]^{\ell'}(\pi_0^{\ell'}, \ell \pi_1') = \mathcal{Q}(\pi_0^{\ell'}) \mathbf{y} \mathbin{\frown} \sigma$ where σ is a possibly empty sequence of values of \mathbf{y} at $\ell' = \ell$. By def. (47.18) of diff , we don't care about σ since $\text{diff}(\text{seqval}[\mathbf{y}]^{\ell'}(\pi_0^\ell, \ell \pi_1), \text{seqval}[\mathbf{y}]^{\ell'}(\pi_0^{\ell'}, \ell \pi_1'))$ is true if and only if $\mathcal{Q}(\pi_0^\ell) \mathbf{y} \neq \mathcal{Q}(\pi_0^{\ell'}) \mathbf{y}$. In that case, we have

(5)

$$= \{\langle \mathbf{x}, \mathbf{y} \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_1 \rangle, \langle \pi_0^{\ell'}, \ell \pi_1' \rangle \in \mathcal{F}^*[\text{while } \ell \text{ (B) } S_b] X . (\forall \mathbf{z} \in \mathcal{V} \setminus \{\mathbf{x}\} . \mathcal{Q}(\pi_0^\ell) \mathbf{z} = \mathcal{Q}(\pi_0^{\ell'}) \mathbf{z}) \wedge \mathcal{Q}(\pi_0^\ell) \mathbf{y} \neq \mathcal{Q}(\pi_0^{\ell'}) \mathbf{y}\}$$

$$\subseteq \{\langle \mathbf{x}, \mathbf{y} \rangle \mid \exists \pi_0^\ell, \pi_0^{\ell'} . (\forall \mathbf{z} \in \mathcal{V} \setminus \{\mathbf{x}\} . \mathcal{Q}(\pi_0^\ell) \mathbf{z} = \mathcal{Q}(\pi_0^{\ell'}) \mathbf{z}) \wedge (\mathcal{Q}(\pi_0^\ell) \mathbf{y} \neq \mathcal{Q}(\pi_0^{\ell'}) \mathbf{y})\} \quad \text{[def. } \leq \text{]}$$

$$= \{\langle \mathbf{x}, \mathbf{y} \rangle \mid \exists \rho, \nu . \rho(\mathbf{y}) \neq \rho[\mathbf{x} \leftarrow \nu](\mathbf{y})\}$$

[letting $\rho = \mathcal{Q}(\pi_0^\ell)$, $\rho[\mathbf{x} \leftarrow \nu] = \mathcal{Q}(\pi_0^{\ell'})$ and Exercise 6.8]

$$= \{\langle \mathbf{x}, \mathbf{x} \rangle \mid \mathbf{x} \in \mathcal{V}\}$$

[def. (19.10) of the environment assignment]

$$= \mathbb{1}_{\mathcal{V}}$$

[def. identity relation on the set \mathcal{V} of variables in Section 2.2.2]

— (1-Ba/Bc/C) In this second case the trace $\ell \pi_1$ corresponds to one or more iterations of the loop followed by an execution of the loop body or a loop exit.

– In case (Ba), we have

$$\text{seqval}[\mathbf{y}]^{\ell'}(\pi_0^\ell, \ell \pi_1)$$

$$\begin{aligned}
&= \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_2^\ell \xrightarrow{B} \text{at}[\mathbb{S}_b]\pi_3^{\ell''}) \text{ where } \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \wedge \mathfrak{B}[\mathbb{B}]\mathfrak{Q}(\pi_0^\ell\pi_2^\ell) = \text{tt} \wedge \\
&\quad \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[\mathbb{S}_b], \text{at}[\mathbb{S}_b]\pi_3^{\ell''} \rangle \in \mathcal{S}^*[\mathbb{S}_b] \quad \text{! (B) with } \ell'' \in \text{in}[\mathbb{S}_b] \text{!} \\
&= \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_2^\ell) \text{ where } \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \wedge \mathfrak{B}[\mathbb{B}]\mathfrak{Q}(\pi_0^\ell\pi_2^\ell) = \text{tt} \\
&\quad \text{! def. (47.16) of seqval}[\mathbb{Y}] \text{ since } \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[\mathbb{S}_b], \text{at}[\mathbb{S}_b]\pi_3^{\ell''} \rangle \in \mathcal{S}^*[\mathbb{S}_b] \text{ with } \ell'' \in \\
&\quad \text{in}[\mathbb{S}_b] \text{ so that } \ell \text{ cannot appear in the trace at}[\mathbb{S}_b]\pi_3^{\ell''} \text{!}
\end{aligned}$$

– In case (Bc), we have

$$\begin{aligned}
&\text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_1) \\
&= \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_2^\ell \xrightarrow{B} \text{at}[\mathbb{S}_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{break-to}[\mathbb{S}]) \text{ where } \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in \\
&\quad X \wedge \mathfrak{B}[\mathbb{B}]\mathfrak{Q}(\pi_0^\ell\pi_2^\ell) = \text{tt} \wedge \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[\mathbb{S}_b], \text{at}[\mathbb{S}_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{break-to}[\mathbb{S}] \rangle \in \mathcal{S}^*[\mathbb{S}_b] \\
&\quad \text{! (B) with } \ell'' \in \text{breaks-of}[\mathbb{S}] \text{ and } \text{break-to}[\mathbb{S}] = \text{after}[\mathbb{S}] \text{!} \\
&= \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_2^\ell) \text{ where } \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \wedge \mathfrak{B}[\mathbb{B}]\mathfrak{Q}(\pi_0^\ell\pi_2^\ell) = \text{tt} \\
&\quad \text{! def. (47.16) of seqval}[\mathbb{Y}] \text{ since } \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[\mathbb{S}_b], \text{at}[\mathbb{S}_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{break-to}[\mathbb{S}] \rangle \in \\
&\quad \mathcal{S}^*[\mathbb{S}_b] \text{ so that } \ell \text{ cannot appear in the trace at}[\mathbb{S}_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{break-to}[\mathbb{S}] \text{!}
\end{aligned}$$

– In case (C), we have

$$\begin{aligned}
&\text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_1) \\
&= \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_2^\ell \xrightarrow{\neg(B)} \text{after}[\mathbb{S}]) \text{ where } \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \wedge \mathfrak{B}[\mathbb{B}]\mathfrak{Q}(\pi_0^\ell\pi_2^\ell) = \text{ff} \quad \text{! (C) !} \\
&= \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_2^\ell) \text{ where } \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \wedge \mathfrak{B}[\mathbb{B}]\mathfrak{Q}(\pi_0^\ell\pi_2^\ell) = \text{ff} \quad \text{! def. (47.16) of seqval}[\mathbb{Y}] \text{!}
\end{aligned}$$

In all of these cases, the future observation $\text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_1)$ is the same so we can handle all cases (1-Ba/Bc/C) as follows:

$$\begin{aligned}
&(5) \\
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_1 \rangle, \langle \pi_0'^\ell, \ell\pi_1' \rangle \in \mathfrak{F}^*[\text{while } \ell(B) \mathbb{S}_b] X . (\forall z \in \mathcal{V} \setminus \{x\} . \mathfrak{Q}(\pi_0^\ell)z = \\
&\quad \mathfrak{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_1), \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0'^\ell, \ell\pi_1')) \} \\
&\subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X . \exists \langle \pi_0'^\ell, \ell\pi_1' \rangle \in \mathfrak{F}^*[\text{while } \ell(B) \mathbb{S}_b] X . (\forall z \in \mathcal{V} \setminus \{x\} . \\
&\quad \mathfrak{Q}(\pi_0^\ell)z = \mathfrak{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0^\ell, \ell\pi_2^\ell), \text{seqval}[\mathbb{Y}]^{\ell'}(\pi_0'^\ell, \ell\pi_1')) \} \\
&\quad \text{! abstracting away the value of the conditions !}
\end{aligned}$$

The possible choices for $\langle \pi_0'^\ell, \ell\pi_1' \rangle \in \mathfrak{F}^*[\text{while } \ell(B) \mathbb{S}_b] X$ are given by (A), (B), and (C) and are considered below.

- (1-Ba/Bc/C-A) This case is the symmetric of (1-A), and so has already been considered.
- (1-Ba/Bc/C-Ba/Bc/C) In this case the above reasoning that we have done in (1-Ba/Bc/C) for the first trace $\ell\pi_1$ is also valid for the second trace $\ell\pi_1'$, and so we get

$$\begin{aligned}
(6) \quad &= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X . \exists \langle \pi_0'^\ell, \ell \pi_1' \rangle \in \mathcal{F}^* [\text{while}^\ell(\mathbf{B}) S_b] X . (\forall z \in \mathcal{V} \setminus \{x\} . \mathbf{q}(\pi_0^\ell)z = \mathbf{q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^{\ell'}(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^{\ell'}(\pi_0'^\ell, \ell \pi_1')) \} \\
&\subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X . \exists \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X . (\forall z \in \mathcal{V} \setminus \{x\} . \mathbf{q}(\pi_0^\ell)z = \mathbf{q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^{\ell'}(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^{\ell'}(\pi_0'^\ell, \ell \pi_2'^\ell)) \} \\
&\quad \quad \quad \wr \text{abstracting away the value of the conditions} \wr \\
&\subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi_0', \pi_1' \rangle \in X . (\forall z \in \mathcal{V} \setminus \{x\} . \mathbf{q}(\pi_0)z = \mathbf{q}(\pi_0')z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0, \pi_1), \text{seqval}[\![y]\!]^\ell(\pi_0', \pi_1')) \} \\
&\quad \quad \quad \wr \text{letting } \pi_0 \leftarrow \pi_0^\ell, \pi_1 \leftarrow \ell \pi_2^\ell, \pi_0' \leftarrow \pi_0'^\ell, \pi_1' \leftarrow \ell \pi_2'^\ell, \text{ and } \ell' = \ell \text{ in case (1)} \wr \\
&= \{ \langle x, y \rangle \mid X \in \{ \Pi \in \wp(\mathbb{T}^+ \times \mathbb{T}^{+\infty}) \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi_0', \pi_1' \rangle \in \Pi . (\forall z \in \mathcal{V} \setminus \{x\} . \mathbf{q}(\pi_0)z = \mathbf{q}(\pi_0')z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0, \pi_1), \text{seqval}[\![y]\!]^\ell(\pi_0', \pi_1')) \} \} \\
&\quad \quad \quad \wr \text{def. (5)} \wr \\
&= \{ \langle x, y \rangle \mid X \in \mathcal{D}^\ell \langle x, y \rangle \} \quad \quad \quad \wr \text{def. (47.19) of } \mathcal{D}^\ell \langle x', y \rangle \wr \\
&= \alpha^d(\{X\})^\ell \quad \quad \quad \wr \text{def. (47.25) of } \alpha^d \wr \\
&- \text{ (1-Ba/Bc/C-Bb) } \quad \text{In this case we are in case (1-Ba/Bc/C) for the first prefix observation trace } \ell \pi_1 \text{ corresponding to one or more iterations of the loop followed by an execution of the loop body or a loop exit and in case Bb for the second trace } \ell \pi_1' \text{ so that, after zero or more executions, the loop body has terminated normally at } \ell'' = \text{after}[\![S_b]\!] = \text{at}[\![S]\!] = \ell \text{ and the prefix observation stops there, just before the next iteration or the loop exit. We have}
\end{aligned}$$

$$\begin{aligned}
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X . \exists \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle . \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X \wedge \\
&\quad \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0'^\ell \pi_2'^\ell) = \text{tt} \wedge \langle \pi_0'^\ell \pi_2'^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \\
&\quad \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell, \ell \pi_2'^\ell)) \} \\
&\quad \cup \\
&\{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X . \exists \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle . \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X \wedge \\
&\quad \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0'^\ell \pi_2'^\ell) = \text{tt} \wedge \langle \pi_0'^\ell \pi_2'^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \\
&\quad \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell \pi_2'^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^\ell)) \} \\
&\quad \text{(By def. (47.16) of seqval}[\![y]\!] \text{ and (47.18) of diff, there is an instant of } \ell \text{ in } \ell \pi_2^\ell \\
&\quad \text{and one in } \ell \pi_2'^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \text{ where the values of } y \text{ while being the same be-} \\
&\quad \text{fore. So there are two possible cases whether this } \ell \text{ is in } \ell \pi_2'^\ell \xrightarrow{B} \text{at}[\![S_b]\!] \text{ or} \\
&\quad \text{in } \text{at}[\![S_b]\!]\pi_3^\ell. \text{ So we have } \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell, \ell \pi_2'^\ell \xrightarrow{B} \\
&\quad \text{at}[\![S_b]\!]\pi_3^\ell)) = \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell, \ell \pi_2'^\ell)) \vee \\
&\quad \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell \pi_2'^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^\ell)) \}) \\
&\subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X . \exists \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle . \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X \wedge (\forall z \in V \setminus \{x\} . \\
&\quad \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell, \ell \pi_2'^\ell)) \} \\
&\quad \cup \\
&\{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2''^\ell \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \rangle . \langle \pi_0^\ell, \ell \pi_2''^\ell \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0^\ell \pi_2''^\ell) = \text{tt} \wedge \\
&\quad \langle \pi_0^\ell \pi_2''^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3'^\ell \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge \exists \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle . \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X \wedge \\
&\quad \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0'^\ell \pi_2'^\ell) = \text{tt} \wedge \langle \pi_0'^\ell \pi_2'^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \\
&\quad \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell \pi_2''^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3'^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell \pi_2'^\ell \xrightarrow{B} \\
&\quad \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^\ell)) \} \\
&\quad \text{(for the second term, we are in the case } \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X \text{ with } \ell \pi_2^\ell = \ell \pi_1 \text{ correspond-} \quad (7) \\
&\quad \text{ing to one or more iterations of the loop (so } \ell \pi_2^\ell \neq \ell \text{ since otherwise we would be in} \\
&\quad \text{case (1-A)), } X \text{ is an iterate of } \mathcal{F}^*[\![\text{while } \ell (B) S_b]\!], \text{ and so, by (17.4), can be writ-} \\
&\quad \text{ten in the form } \ell \pi_2^\ell = \ell \pi_2''^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \text{ (where } \ell \pi_2''^\ell \text{ may be reduced to } \ell \text{ for} \\
&\quad \text{the first iteration) with } \ell \pi_2''^\ell \in X, \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0^\ell \pi_2''^\ell) = \text{tt} \text{ and } \langle \pi_0^\ell \pi_2''^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \\
&\quad \text{at}[\![S_b]\!]\pi_3'^\ell \rangle \in \mathcal{S}^*[\![S_b]\!]. \text{ Moreover if the difference on } y \text{ is in } \ell \pi_2''^\ell, \text{ the case is cov-} \\
&\quad \text{ered by the first term.)} \\
&\subseteq \alpha^d(\{X\})^\ell \\
&\quad \cup
\end{aligned}$$

$$\begin{aligned}
& \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi''_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \rangle . \langle \pi_0^\ell, \ell \pi''_2 \rangle \in X \wedge \langle \pi_0^\ell \pi''_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi'_3 \rangle \in \\
& \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \wedge \exists \langle \pi'_0, \ell \pi'_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \rangle . \langle \pi'_0, \ell \pi'_2 \rangle \in X \wedge \\
& \langle \pi'_0 \ell \pi'_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3 \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \\
& \mathcal{Q}(\pi'_0)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell \pi''_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi''_3 \rangle, \text{seqval}[\![y]\!]^\ell(\pi'_0 \ell \pi'_2 \rangle \xrightarrow{B} \\
& \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3 \rangle)) \\
& \quad \quad \quad \wr \text{since } \mathcal{Q}(\pi) = \mathcal{Q}(\pi \xrightarrow{B} \text{at}[\![S_b]\!]) \wr \\
& = \alpha^d(\{X\})^\ell \cup \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi''_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \rangle . \langle \pi_0^\ell, \ell \pi''_2 \rangle \in X \wedge \langle \pi_0^\ell \pi''_2 \rangle, \\
& \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \rangle \in \{ \langle \pi_0^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi \rangle \mid \langle \pi_0^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi \rangle \in \{ \langle \pi, \\
& \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \} \wedge \exists \langle \pi'_0, \ell \pi'_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \rangle . \langle \pi'_0, \ell \pi'_2 \rangle \in X \wedge \langle \pi'_0 \ell \pi'_2 \rangle, \\
& \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \rangle \in \{ \langle \pi_0^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi \rangle \mid \langle \pi_0^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi \rangle \in \{ \langle \pi, \pi' \rangle \in \\
& \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi'_0)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell \pi''_2 \rangle, \ell \xrightarrow{B} \\
& \text{at}[\![S_b]\!]\pi''_3 \rangle, \text{seqval}[\![y]\!]^\ell(\pi'_0 \ell \pi'_2 \rangle, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \rangle)) \} \\
& \quad \quad \quad \wr \text{def. } \in, \text{ def. (47.18) of diff, and def. (47.16) of seqval}[\![y]\!] \text{ with } \ell \neq \text{at}[\![S_b]\!] \wr \\
& \subseteq \alpha^d(\{X\})^\ell \cup \{ \langle x, y \rangle \mid \exists \pi_0^{\ell_0} \pi_1^{\ell'} \pi_2^{\ell} \pi_3, \pi'_0 \ell_0 \pi'_1 \ell' \pi'_2 \ell \pi'_3 . \langle \pi_0^{\ell_0}, \ell_0 \pi_1^{\ell'} \rangle \in X \wedge \langle \pi_0^{\ell_0} \pi_1^{\ell'} \rangle, \ell' \pi_2^{\ell} \pi_3 \rangle \in \\
& \{ \langle \pi_0^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi \rangle \mid \langle \pi_0^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \} \wedge \\
& \langle \pi_0^{\ell_0}, \ell_0 \pi_1^{\ell'} \rangle \in X \wedge \langle \pi'_0 \ell_0 \pi'_1 \ell', \ell' \pi'_2 \ell \pi'_3 \rangle \in \{ \langle \pi_0^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi \rangle \mid \langle \pi_0^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \\
& \text{at}[\![S_b]\!]\pi \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^{\ell_0})z = \mathcal{Q}(\pi'_0)z) \wedge \\
& \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^{\ell_0} \pi_1^{\ell'} \pi_2^{\ell}, \ell \pi_3), \text{seqval}[\![y]\!]^\ell(\pi'_0 \ell_0 \pi'_1 \ell' \pi'_2 \ell, \ell \pi'_3)) \} \\
& \quad \quad \quad \wr \text{by letting } \pi_0^{\ell_0} \leftarrow \pi_0^\ell, \ell_0 \pi_1^{\ell'} \leftarrow \ell \pi''_2, \ell' \pi_2^{\ell} \leftarrow \ell, \ell \pi_3 \leftarrow \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi''_3, \text{ and similarly} \\
& \quad \quad \quad \text{for the second trace } \wr \\
& \subseteq \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \circ \alpha^d(\{ \langle \pi_0^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi \rangle \mid \langle \pi_0^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi \rangle \in \{ \langle \pi, \pi' \rangle \in \\
& \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \})^\ell) \\
& \quad \quad \quad \wr \text{Lemma 47.59 with } \mathcal{S} \leftarrow X \text{ and } \mathcal{S}' \leftarrow \{ \langle \pi_0^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi \rangle \mid \langle \pi_0^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \\
& \text{at}[\![S_b]\!]\pi \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \} \} \wr \\
& = \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \circ \alpha^d(\{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathfrak{B}[\![B]\!]\mathcal{Q}(\pi) \})^\ell) \\
& \quad \quad \quad \wr \text{def. (47.25) of } \alpha^d, \text{ (47.18) of diff, and (47.16) of seqval}[\![y]\!] \text{ with } \ell \neq \wr \\
& = \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \circ (\alpha^d(\{ \mathcal{S}^*[\![S_b]\!] \})^\ell \wr \text{nondet}(\mathbf{B}, \mathbf{B}))) \quad \wr \text{Lemma 47.62} \wr \\
& = \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \circ (\alpha^d(\{ \mathcal{S}^{+\infty}[\![S_b]\!] \})^\ell \wr \text{nondet}(\mathbf{B}, \mathbf{B}))) \quad \wr \text{Lemma 47.23} \wr \\
& \subseteq \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \circ (\widehat{\mathcal{S}_{\text{diff}}^{\exists}}[\![S_b]\!]^\ell \wr \text{nondet}(\mathbf{B}, \mathbf{B}))) \wr \text{ind. hyp. (47.32), } \circ \text{ and } \wr \text{ are } \subseteq\text{-increasing} \wr
\end{aligned}$$

— (1-Bb) In this third and last case for (1), we have $\ell\pi_1 = \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell$ so the prefix observation ends after the normal termination of the loop body at $\text{after}[\![S_b]\!] = \text{at}[\![S]\!] = \ell$ (just before the next iteration or the loop exit).

The possible choices for $\langle \pi_0^\ell, \ell\pi_1' \rangle \in \mathcal{F}^*[\![\text{while } \ell(B) S_b]\!] X$ are given by (A), (B), and (C) and are considered below.

- (1-Bb-A) This case is the symmetric of (1-A), and so has already been considered.
- (1-Bb-Ba/Bc/C) This case is the symmetric of (1-Ba/Bc/C-Bb), and so has already been considered.
- (1-Bb-Bb) This is the case when the prefix observation traces $\langle \pi_0^\ell, \ell\pi_1 \rangle$ and $\langle \pi_0'^\ell, \ell\pi_1' \rangle$ in (5) both end after the normal termination of the loop body at $\text{after}[\![S_b]\!] = \text{at}[\![S]\!] = \ell$ and so belong to $\{\langle \pi_0^\ell, \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle \mid \langle \pi_0^\ell, \ell\pi_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0^\ell\pi_2) = \text{tt} \wedge \langle \pi_0^\ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \mathcal{S}^*[\![S_b]\!]\}$. In that case, we have

$$\begin{aligned}
(5) &= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_1 \rangle, \langle \pi_0'^\ell, \ell\pi_1' \rangle \in \{ \langle \pi_0^\ell, \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle \mid \langle \pi_0^\ell, \ell\pi_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0^\ell\pi_2) = \text{tt} \wedge \langle \pi_0^\ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \mathcal{S}^*[\![S_b]\!]\} \cdot (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]\ell'(\pi_0^\ell, \ell\pi_1), \text{seqval}[\![y]\!]\ell'(\pi_0'^\ell, \ell\pi_1')) \} \\
&\quad \text{case (1-Bb-Bb)} \\
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle . \langle \pi_0^\ell, \ell\pi_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0^\ell\pi_2) = \text{tt} \wedge \langle \pi_0^\ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \mathcal{S}^*[\![S_b]\!]\} \wedge \exists \langle \pi_0'^\ell, \ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \rangle . \langle \pi_0'^\ell, \ell\pi_2' \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0'^\ell\pi_2') = \text{tt} \wedge \langle \pi_0'^\ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \rangle \in \mathcal{S}^*[\![S_b]\!]\} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]\ell'(\pi_0^\ell, \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell), \text{seqval}[\![y]\!]\ell'(\pi_0'^\ell, \ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell)) \} \\
&\quad \text{def.} \in \} \\
&\subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle . \langle \pi_0^\ell, \ell\pi_2 \rangle \in X \wedge \langle \pi_0^\ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathcal{B}[\![B]\!]\mathcal{Q}(\pi) \} \wedge \exists \langle \pi_0'^\ell, \ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \rangle . \langle \pi_0'^\ell, \ell\pi_2' \rangle \in X \wedge \langle \pi_0'^\ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathcal{B}[\![B]\!]\mathcal{Q}(\pi) \} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]\ell'(\pi_0^\ell, \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell), \text{seqval}[\![y]\!]\ell'(\pi_0'^\ell, \ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell)) \} \\
&\quad \text{def.} \in \} \\
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle . \langle \pi_0^\ell, \ell\pi_2 \rangle \in X \wedge \langle \pi_0^\ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathcal{B}[\![B]\!]\mathcal{Q}(\pi) \} \wedge \exists \langle \pi_0'^\ell, \ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \rangle . \langle \pi_0'^\ell, \ell\pi_2' \rangle \in X \wedge \langle \pi_0'^\ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell \rangle \in \{ \langle \pi, \pi' \rangle \in \mathcal{S}^*[\![S_b]\!] \mid \mathcal{B}[\![B]\!]\mathcal{Q}(\pi) \} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]\ell(\pi_0^\ell, \ell\pi_2), \text{seqval}[\![y]\!]\ell(\pi_0'^\ell, \ell\pi_2')) \} \\
&\quad \text{by def. (47.18) of diff, and def. (47.16) of seqval}[\![y]\!] \text{ since in case (1), } \ell' = \ell \text{ does not appear} \\
&\quad \text{in } \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \text{ and the value of } y \text{ is the same at } \ell \text{ after } \pi_0^\ell\pi_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^\ell \text{ and at } \ell \\
&\quad \text{after } \pi_0^\ell\pi_2. \text{ Same for } \pi_0'^\ell\pi_2' \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3'^\ell. \}
\end{aligned}$$

$$\begin{aligned}
&\subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X, \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X . (\forall z \in V \setminus \{x\} . \varrho(\pi_0'^\ell)z = \varrho(\pi_0^\ell)z) \wedge \\
&\quad \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell, \ell \pi_2'^\ell)) \} \quad \text{\textit{\textup{def.}} } \subseteq \} \\
&\subseteq \alpha^d(\{X\})^\ell \quad \text{\textit{\textup{def.}} } (47.25) \text{ of } \alpha^d \}
\end{aligned}$$

— Summing up for case **(1)** we get $(5) \subseteq \mathbb{1}_V \cup \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell ; \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_b]\!]^\ell) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B})$ which yields (47.63.a) of the form

$$(\ell' = \ell ; \mathbb{1}_V \cup X(\ell) \cup (X(\ell) ; ((\widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_b]\!]^\ell) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B}))) ; \emptyset .$$

However, the term $X(\ell)$ does not appear in (47.63.a) since it can be simplified thanks to Exercise 15.8.

— **(2)** Else, if the dependency observation point ℓ' on prefix traces is in the loop body S_b after zero or more loop iterations. So the two traces $\ell \pi_1$ and $\ell \pi_1'$ in (5) cannot be generated by (17.4.A). The case $\ell' = \ell = \text{after}[\![S_b]\!] = \text{at}[\![S]\!]$ has already been considered in case **(1)** (for subcases involving **(B)** and **(C)**). By def. (47.16) of $\text{seqval}[\![y]\!]$ the case $\ell' = \text{at}[\![S_b]\!]$ is equivalent to $\ell' = \text{at}[\![S]\!]$ already considered in **(1)** since the evaluation of boolean expressions has no side effect so the value of variables y at $\text{at}[\![S_b]\!]$ and $\text{at}[\![S]\!]$ are the same. Similarly, the value of variables y before a **break ;** statement at labels in $\text{breaks-of}[\![S_b]\!]$ that can escape the loop body S_b is the same as the value at $\text{break-to}[\![S_b]\!] = \text{after}[\![S]\!]$ and will be handled with case **(3)**.

It follows that in this case **(2)** we only have to consider the case $\ell' \in \text{in}[\![S_b]\!] \setminus (\{\text{at}[\![S_b]\!], \text{after}[\![S_b]\!]\} \cup \text{breaks-of}[\![S_b]\!])$ and the two traces $\ell \pi_1$ and $\ell \pi_1'$ in (5) are generated by **(B)** or **(C)**. There are three cases to consider.

— **(2-B-B)** The dependency observation point ℓ' on the two prefix observation traces $\ell \pi_1$ and $\ell \pi_1'$ in (5) is in the loop body S_b after zero or more loop iterations and the observation along these two traces stops in the loop body.

$$\begin{aligned}
&(5) \\
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_1 \rangle, \langle \pi_0'^\ell, \ell \pi_1' \rangle \in \{ \langle \pi_0^\ell, \ell \pi_2^\ell \xrightarrow{\mathbf{B}} \text{at}[\![S_b]\!]\pi_3^{\ell''} \rangle \mid \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X \wedge \\
&\quad \mathcal{B}[\![B]\!]\varrho(\pi_0^\ell \pi_2^\ell) = \text{tt} \wedge \langle \pi_0^\ell \pi_2^\ell \xrightarrow{\mathbf{B}} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^{\ell''} \rangle \in \mathcal{S}^*[\![S_b]\!] \} . (\forall z \in V \setminus \{x\} . \\
&\quad \varrho(\pi_0^\ell)z = \varrho(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^{\ell'}(\pi_0^\ell, \ell \pi_1), \text{seqval}[\![y]\!]^{\ell'}(\pi_0'^\ell, \ell \pi_1')) \} \quad \text{\textit{\textup{case 2-B-B}}} \} \\
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \xrightarrow{\mathbf{B}} \text{at}[\![S_b]\!]\pi_3^{\ell''} \rangle . \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X \wedge \mathcal{B}[\![B]\!]\varrho(\pi_0^\ell \pi_2^\ell) = \text{tt} \wedge \\
&\quad \langle \pi_0^\ell \pi_2^\ell \xrightarrow{\mathbf{B}} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^{\ell''} \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge \exists \langle \pi_0'^\ell, \ell \pi_2'^\ell \xrightarrow{\mathbf{B}} \text{at}[\![S_b]\!]\pi_3^{\ell'''} \rangle . \langle \pi_0'^\ell, \\
&\quad \ell \pi_2'^\ell \rangle \in X \wedge \mathcal{B}[\![B]\!]\varrho(\pi_0'^\ell \pi_2'^\ell) = \text{tt} \wedge \langle \pi_0'^\ell \pi_2'^\ell \xrightarrow{\mathbf{B}} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^{\ell'''} \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\forall z \in V \setminus \{x\} . \\
&\quad \varrho(\pi_0^\ell)z = \varrho(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^{\ell'}(\pi_0^\ell, \ell \pi_2^\ell \xrightarrow{\mathbf{B}} \text{at}[\![S_b]\!]\pi_3^{\ell''}), \text{seqval}[\![y]\!]^{\ell'}(\pi_0'^\ell, \ell \pi_2'^\ell \xrightarrow{\mathbf{B}} \\
&\quad \text{at}[\![S_b]\!]\pi_3^{\ell'''})) \} \quad \text{\textit{\textup{def.}} } \in \}
\end{aligned}$$

$$= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_1 \rangle, \langle \pi_0'^\ell, \ell \pi_1' \rangle \in \mathcal{F}^*[\text{while}^\ell(\mathbf{B}) \mathbf{S}_b] X . (\forall z \in \mathbb{V} \setminus \{x\} . \mathbf{q}(\pi_0^\ell)z = \mathbf{q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\mathbf{y}](\text{after}[\mathbf{S}])(\pi_0^\ell, \ell \pi_1), \text{seqval}[\mathbf{y}](\text{after}[\mathbf{S}'])(\pi_0'^\ell, \ell \pi_1')) \}$$

$$\begin{aligned}
= & \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_1 \rangle, \langle \pi_0'^\ell, \ell \pi_1' \rangle \in \{ \langle \pi_0^\ell, \ell \pi_2^\ell \xrightarrow{\neg(B)} \text{after}[\![S]\!] \rangle \mid \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in \\
& X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{ff} \} \cup \{ \langle \pi_0^\ell, \ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \mid \langle \pi_0^\ell, \\
& \ell \pi_2^\ell \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi_0^\ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!] \rangle, \\
& \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \} . \quad (\forall z \in \mathbb{V} \mid \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \\
& \text{diff}(\text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^\ell, \ell \pi_1), \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0'^\ell, \ell \pi_1')) \} \}
\end{aligned} \tag{8}$$

{The only cases in (17.4) where $\ell' = \text{after}[\mathbb{S}]$ is reachable is either via (C) for normal termination after zero or more iterations or via (B) through a **break** ; in the loop body S_b during the first or later iteration}

There are now three subcases, depending on whether the observation prefix traces $\ell\pi_1$ and $\ell\pi'_1$ are both from a normal exit, a both from a break, or one is from a break and the other from a normal exit.

— (3-C-C) This is the case when the observation prefix traces $\ell\pi_1$ and $\ell\pi'_1$ are both from a normal exit.

(8)

$$\begin{aligned}
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \pi_2^\ell \rangle \xrightarrow{\neg(B)} \text{after}[\![S]\!] \} \cdot \langle \pi_0^\ell, \pi_2^\ell \rangle \in X \wedge \mathfrak{B}[\![B]\!]\mathfrak{q}(\pi_0^\ell \pi_2^\ell) = \\
&\quad \text{ff} \wedge \exists \langle \pi_0' \ell, \pi_2' \ell \rangle \xrightarrow{\neg(B)} \text{after}[\![S]\!] \} \cdot \langle \pi_0' \ell, \pi_2' \ell \rangle \in X \wedge \mathfrak{B}[\![B]\!]\mathfrak{q}(\pi_0' \ell \pi_2' \ell) = \text{ff} \wedge \\
&\quad (\forall z \in V \setminus \{x\} \cdot \mathfrak{q}(\pi_0^\ell)z = \mathfrak{q}(\pi_0' \ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!](\text{after}[\![S]\!]) (\pi_0^\ell, \pi_2^\ell) \xrightarrow{\neg(B)} \\
&\quad \text{after}[\![S]\!]), \text{seqval}[\![y]\!](\text{after}[\![S]\!]) (\pi_0' \ell, \pi_2' \ell) \xrightarrow{\neg(B)} \text{after}[\![S]\!])) \} \quad \text{[def. } \in \text{ and } \ell' = \text{after}[\![S]\!]\text{]} \\
&\subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \pi_2^\ell \rangle, \langle \pi_0' \ell, \pi_2' \ell \rangle \in X \wedge \mathfrak{B}[\![B]\!]\mathfrak{q}(\pi_0^\ell \pi_2^\ell) = \text{ff} \wedge \mathfrak{B}[\![B]\!]\mathfrak{q}(\pi_0' \ell \pi_2' \ell) = \\
&\quad \text{ff} \wedge (\forall z \in V \setminus \{x\} \cdot \mathfrak{q}(\pi_0^\ell)z = \mathfrak{q}(\pi_0' \ell)z) \wedge \mathfrak{q}(\pi_0^\ell \pi_2^\ell)y \neq \mathfrak{q}(\pi_0' \ell \pi_2' \ell)y) \}
\end{aligned}$$

(X is an iterate of $\mathcal{F}^*[\text{while}^\ell(\mathbf{B}) S_b]$) so $\ell\pi_2^\ell$ and $\ell\pi_2'^\ell$ are iterates of the loop body. By definition of the labelling in Section 4.2, $\text{after}[\mathbf{S}]$ appears neither in $\ell\pi_2^\ell$ nor in $\ell\pi_2'^\ell$. It follows by def. (47.18) of diff and (47.16) of $\text{seqval}[\mathbf{y}]\text{after}[\mathbf{S}]$ that $\text{diff}(\text{seqval}[\mathbf{y}](\text{after}[\mathbf{S}])(\pi_0^\ell, \ell\pi_2^\ell) \xrightarrow{\neg(\mathbf{B})} \text{after}[\mathbf{S}]), \text{seqval}[\mathbf{y}](\text{after}[\mathbf{S}])(\pi_0'^\ell, \ell\pi_2'^\ell) \xrightarrow{\neg(\mathbf{B})} \text{after}[\mathbf{S}])) = \text{diff}(\text{seqval}[\mathbf{y}](\text{after}[\mathbf{S}])(\pi_0^\ell \pi_2^\ell \xrightarrow{\neg(\mathbf{B})} \text{after}[\mathbf{S}], \text{after}[\mathbf{S}]), \text{seqval}[\mathbf{y}](\text{after}[\mathbf{S}])(\pi_0'^\ell \pi_2'^\ell \xrightarrow{\neg(\mathbf{B})} \text{after}[\mathbf{S}], \text{after}[\mathbf{S}])) = \text{diff}(\text{seqval}[\mathbf{y}](\ell)(\pi_0^\ell \pi_2^\ell, \ell), \text{seqval}[\mathbf{y}](\ell)(\pi_0'^\ell \pi_2'^\ell, \ell)) = \mathbf{q}(\pi_0^\ell \pi_2^\ell) \mathbf{y} \neq \mathbf{q}(\pi_0'^\ell \pi_2'^\ell) \mathbf{y}$)

From there on, the development is very similar to the cases (2.a), (2.b), and (2.c–d) of the condi-

tional with execution traces that may go through the true branch (here entering the loop) or the false branch (here not entering the iteration). There are four subcases (three by symmetry).

– (3–C–C.a) If none of the executions $\pi_0^\ell \pi_2^\ell$ and $\pi_0'^\ell \pi_2'^\ell$ enter the loop body since in both cases the condition **B** is false, we have $\ell \pi_2^\ell = \ell$ and $\ell \pi_2'^\ell = \ell$.

$$\begin{aligned}
 & (9) \\
 & \subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \rangle, \langle \pi_0'^\ell, \ell \rangle \in X \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell) = \text{ff} \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0'^\ell) = \text{ff} \wedge (\forall z \in \mathcal{V} \setminus \{x\} . \\
 & \quad \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \mathcal{Q}(\pi_0^\ell)y \neq \mathcal{Q}(\pi_0'^\ell)y \} \upharpoonright \text{nondet}(\neg B, \neg B) \quad \text{[case (3–C–C.a)]} \\
 & \subseteq \mathbb{1}_{\mathcal{V}} \upharpoonright \text{nondet}(\neg B, \neg B)
 \end{aligned}$$

(since if $x \notin \text{nondet}(\neg B, \neg B)$ then $x \in \text{det}(\neg B, \neg B)$ so $\mathcal{B}[\![\neg B]\!] \mathcal{Q}(\pi_0^\ell)$ and $\mathcal{B}[\![\neg B]\!] \mathcal{Q}(\pi_0'^\ell)$ would imply $\mathcal{Q}(\pi_0^\ell)x = \mathcal{Q}(\pi_0'^\ell)x$. Therefore $\mathcal{Q}(\pi_0^\ell) = \mathcal{Q}(\pi_0'^\ell)$ in contradiction with $\mathcal{Q}(\pi_0^\ell)y \neq \mathcal{Q}(\pi_0'^\ell)y$.)

– (3–C–C.b) Else, if both executions $\pi_0^\ell \pi_2^\ell$ and $\pi_0'^\ell \pi_2'^\ell$ enter the loop body since in both cases the condition **B** is true, we have $\ell \pi_2^\ell \neq \ell$ and $\ell \pi_2'^\ell \neq \ell$

$$\begin{aligned}
 & (9) \\
 & = \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle, \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{ff} \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0'^\ell \pi_2'^\ell) = \text{ff} \wedge (\forall z \in \\
 & \quad \mathcal{V} \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \mathcal{Q}(\pi_0^\ell \pi_2^\ell)y \neq \mathcal{Q}(\pi_0'^\ell \pi_2'^\ell)y \} \upharpoonright \text{nondet}(B, B) \\
 & \quad \text{[case (3–C–C.b) and } X \text{ belongs to the iterates of } \mathcal{F}^*[\![\text{while } \ell(B) S_b]\!] \text{ so this is possible} \\
 & \quad \text{only when } \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell) = \text{tt} \text{ and } \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0'^\ell) = \text{tt} \text{ and def. (47.48) of nondet} \text{]} \\
 & \subseteq \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X . \exists \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \xrightarrow{B} \text{at}[\![S_b]\!] \pi_3^\ell . \langle \pi_0'^\ell, \ell \pi_2'^\ell \rangle \in X \wedge (\forall z \in \mathcal{V} \setminus \{x\} . \\
 & \quad \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell, \ell \pi_2'^\ell)) \} \\
 & \quad \text{[since } \mathcal{Q}(\pi_0^\ell \pi_2^\ell)y \neq \mathcal{Q}(\pi_0'^\ell \pi_2'^\ell)y \text{ implies } \text{diff}(\text{seqval}[\![y]\!]^\ell(\pi_0^\ell, \ell \pi_2^\ell), \text{seqval}[\![y]\!]^\ell(\pi_0'^\ell, \ell \pi_2'^\ell)) \text{]} \\
 & \subseteq \alpha^d(\{X\})^\ell \quad \text{[def. (47.25) of } \alpha^d \text{]}
 \end{aligned}$$

– (3–C–C.c) Otherwise, one execution enters the loop body (say $\pi_0^\ell \pi_2^\ell$) and the other does not (say $\pi_0'^\ell \pi_2'^\ell$), we have (the other case is symmetric) $\ell \pi_2^\ell \neq \ell$ and $\ell \pi_2'^\ell = \ell$. The calculation is similar to (2.c–d) for the simple conditional.

$$\begin{aligned}
 & (9) \\
 & = \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle, \langle \pi_0'^\ell, \ell \rangle \in X \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell) = \text{tt} \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{ff} \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0'^\ell) = \\
 & \quad \text{ff} \wedge (\forall z \in \mathcal{V} \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \mathcal{Q}(\pi_0^\ell \pi_2^\ell)y \neq \mathcal{Q}(\pi_0'^\ell)y \} \\
 & \quad \text{[case (3–C–C.c) and } X \text{ is included in the iterates of } \mathcal{F}^*[\![\text{while } \ell(B) S_b]\!] \text{ so this is pos-} \\
 & \quad \text{sible only when } \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell) = \text{tt}, \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{ff}, \text{ and } \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0'^\ell) = \text{ff} \text{]} \\
 & = \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell \pi_2^\ell \rangle, \langle \pi_0'^\ell, \ell \rangle \in X \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell) = \text{tt} \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0^\ell \pi_2^\ell) = \text{ff} \wedge \mathcal{B}[\![B]\!] \mathcal{Q}(\pi_0'^\ell) = \\
 & \quad \text{ff} \wedge (\forall z \in \mathcal{V} \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi_0'^\ell)z) \wedge \mathcal{Q}(\pi_0^\ell \pi_2^\ell)y \neq \mathcal{Q}(\pi_0'^\ell)y \} \upharpoonright \text{nondet}(B, \neg B)
 \end{aligned}$$

(since, by def. (47.48) of nondet , if $x \notin \text{nondet}(B, \neg B)$ then $x \in \text{det}(B, \neg B)$ so by (47.48), $\mathcal{B}[B]q(\pi_0^\ell)$ and $\mathcal{B}[\neg B]q(\pi_0'^\ell)$ would imply $q(\pi_0^\ell)x = q(\pi_0'^\ell)x$ and therefore $q(\pi_0^\ell) = q(\pi_0'^\ell)$. X being included in the iterates of $\mathcal{F}^*[\text{while } \ell(B) S_b]$ and, by Exercises 17.13 and 17.21, the language being deterministic, this would imply that $\ell\pi_2^\ell = \ell$, in contradiction with $\mathcal{B}[B]q(\pi_0^\ell) = \text{tt}$ and $\mathcal{B}[B]q(\pi_0^\ell\pi_2^\ell) = \text{ff}$)

$$= \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell \rangle . \langle \pi_0^\ell, \ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell \rangle \in X \wedge \mathcal{B}[B]q(\pi_0^\ell) = \text{tt} \wedge \mathcal{B}[B]q(\pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell) = \text{ff} \wedge \langle \pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3'^\ell \rangle \in \mathcal{S}^*[S_b] \wedge \exists \langle \pi_0'^\ell, \ell \rangle \in X \wedge \mathcal{B}[B]q(\pi_0'^\ell) = \text{ff} \wedge (\forall z \in V \setminus \{x\} . q(\pi_0^\ell)z = q(\pi_0'^\ell)z) \wedge q(\pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell)y \neq q(\pi_0'^\ell)y \} \mid \text{nondet}(B, \neg B)$$

(by the argument (7) that if $\langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X$ corresponds to one or more iterations of the loop then it can be written in the form $\ell\pi_2^\ell = \ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell$ (where $\ell\pi_2''^\ell$ may be reduced to ℓ for the first iteration) with $\ell\pi_2''^\ell \in X$, $\mathcal{B}[B]q(\pi_0^\ell\pi_2''^\ell) = \text{tt}$ and $\langle \pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3'^\ell \rangle \in \mathcal{S}^*[S_b]$)

$$\subseteq \{ \langle x, y \rangle \mid \exists \pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell, \pi_0'^\ell . \langle \pi_0^\ell, \ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell \rangle \in X \wedge \langle \pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3'^\ell \rangle \in \mathcal{S}^*[S_b] \wedge \mathcal{B}[B]q(\pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell) = \text{ff} \wedge \langle \pi_0'^\ell, \ell \rangle \in X \wedge \mathcal{B}[B]q(\pi_0^\ell\pi_2''^\ell) = \text{tt} \wedge \mathcal{B}[B]q(\pi_0'^\ell) = \text{ff} \wedge (\forall z \in V \setminus \{x\} . q(\pi_0^\ell)z = q(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[y]\text{after}[S](\pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell \xrightarrow{\neg B} \text{after}[S], \text{after}[S]), \text{seqval}[y]\text{after}[S](\pi_0'^\ell \xrightarrow{\neg B} \text{after}[S], \text{after}[S])) \} \mid \text{nondet}(B, \neg B)$$

(def. (6.6) of q , def. (47.16) of $\text{seqval}[y]$ and program labelling so that $\text{after}[S]$ does not appear in the trace (in particular $\ell \neq \text{after}[S]$), and def. (47.18) of diff)

$$= \{ \langle x, y \rangle \mid \exists \pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell \xrightarrow{\neg B} \text{after}[S], \pi_0'^\ell \xrightarrow{\neg B} \text{after}[S] . \langle \pi_0^\ell, \ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell \rangle \in X \wedge \langle \pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell, \ell \xrightarrow{\neg B} \text{after}[S] \rangle \in \mathcal{S}' \wedge \langle \pi_0'^\ell, \ell \rangle \in X \wedge \langle \pi_0'^\ell, \ell \xrightarrow{\neg B} \text{after}[S] \rangle \in \mathcal{S}' \wedge (\forall z \in V \setminus \{x\} . q(\pi_0^\ell)z = q(\pi_0'^\ell)z) \wedge \text{diff}(\text{seqval}[y]\text{after}[S](\pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell \xrightarrow{\neg B} \text{after}[S], \text{after}[S]), \text{seqval}[y]\text{after}[S](\pi_0'^\ell \xrightarrow{\neg B} \text{after}[S], \text{after}[S])) \} \mid \text{nondet}(B, \neg B)$$

(where $\mathcal{S}' = \{ \langle \pi_1'^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell, \ell \xrightarrow{\neg B} \text{after}[S] \rangle \mid \mathcal{B}[B]q(\pi_1'^\ell) = \text{tt} \wedge \mathcal{B}[B]q(\pi_0^\ell\pi_2''^\ell \xrightarrow{B} \text{at}[S_b]\pi_3'^\ell) = \text{ff} \wedge \langle \pi_1'^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3'^\ell \rangle \in \mathcal{S}^*[S_b] \} \cup \{ \langle \pi_0'^\ell, \ell \xrightarrow{\neg B} \text{after}[S] \rangle \mid \mathcal{B}[B]q(\pi_0'^\ell) = \text{ff} \}$)

$$\subseteq (\alpha^d(\{X\})^\ell \circ \alpha^d(\{\mathcal{S}'\}) \text{after}[S]) \mid \text{nondet}(B, \neg B)$$

(Lemma 47.59 with $\ell_0 \leftarrow \ell$, $\ell' \leftarrow \ell$, and $\ell \leftarrow \text{after}[S]$)

We have to calculate the second term

$$\alpha^d(\{\mathcal{S}'\}) \text{after}[S] \tag{10}$$

$$\begin{aligned}
&= \{ \langle x, y \rangle \mid \mathcal{S}' \in \mathcal{D}(\text{after}[\![S]\!]) \langle x, y \rangle \} && \text{[def. (47.25) of } \alpha^d \text{]} \\
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \mathcal{S}' . (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0)z = \mathcal{Q}(\pi'_0)z) \wedge \\
&\quad \text{diff}(\text{seqval}[\![y]\!]\text{after}[\![S]\!](\pi_0, \pi_1), \text{seqval}[\![y]\!]\text{after}[\![S]\!](\pi'_0, \pi'_1)) \} && \text{[def. (47.19) of } \mathcal{D}^\ell \langle x, y \rangle \text{]} \\
&= \{ \langle x, y \rangle \mid \exists \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \xrightarrow{\neg B} \text{after}[\![S]\!] . \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_2) = \text{tt} \wedge \langle \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!], \\
&\quad \text{at}[\![S_b]\!]\pi'_3 \rangle \in \mathcal{S}^*[\![S_b]\!] , \exists \pi'_0 . \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0) = \text{ff} \} . (\forall z \in V \setminus \{x\} . \\
&\quad \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 z = \mathcal{Q}(\pi'_0)z \wedge \text{diff}(\text{seqval}[\![y]\!]\text{after}[\![S]\!](\pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3, \pi'_0) \\
&\quad \text{after}[\![S]\!], \text{seqval}[\![y]\!]\text{after}[\![S]\!](\pi'_0, \pi'_2 \xrightarrow{\neg B} \text{after}[\![S]\!])) \} \\
&\quad \text{[def. } \mathcal{S}' \text{ and the other two combinations have already been considered in (3-C-C.a) and (3-C-C.b)]} \\
&= \{ \langle x, y \rangle \mid \exists \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \xrightarrow{\neg B} \text{after}[\![S]\!] . \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_2) = \text{tt} \wedge \langle \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!], \\
&\quad \text{at}[\![S_b]\!]\pi'_3 \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0 \pi''_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 = \text{ff} \wedge \exists \pi'_0 . \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0) = \\
&\quad \text{ff} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 z = \mathcal{Q}(\pi'_0)z) \wedge \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 y \neq \mathcal{Q}(\pi'_0)y) \} \\
&\quad \text{[def. (6.6) of } \mathcal{Q} \text{, def. (47.16) of } \text{seqval}[\![y]\!] \text{ and program labelling so that } \text{after}[\![S]\!] \text{ does not} \\
&\quad \text{appear in the trace (in particular } \ell \neq \text{after}[\![S]\!] \text{), and def. (47.18) of } \text{diff} \text{]} \\
&= \{ \langle x, y \rangle \mid \exists \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \xrightarrow{\neg B} \text{after}[\![S]\!] . \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_2) = \text{tt} \wedge \langle \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!], \\
&\quad \text{at}[\![S_b]\!]\pi'_3 \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0 \pi''_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 = \text{ff} \wedge \exists \pi'_0 . \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0) = \\
&\quad \text{ff} \wedge (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 z = \mathcal{Q}(\pi'_0)z) \wedge \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 y \neq \\
&\quad \mathcal{Q}(\pi'_0)y) \} \mid \text{nondet}(\neg B, \neg B) \\
&\quad \text{[since if } x \notin \text{nondet}(\neg B, \neg B) \text{ then } x \in \text{det}(\neg B, \neg B) \text{ so by (47.48), } \mathcal{B}[\![\neg B]\!]\mathcal{Q}(\pi'_0 \pi''_2) \xrightarrow{B} \\
&\quad \text{at}[\![S_b]\!]\pi'_3, \text{ and } \mathcal{B}[\![\neg B]\!]\mathcal{Q}(\pi'_0), \text{ we would have } \mathcal{Q}(\pi'_0 \pi''_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 = \mathcal{Q}(\pi'_0), \\
&\quad \text{which, with } \forall z \in V \setminus \{x\} . \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 z = \mathcal{Q}(\pi'_0)z, \text{ would imply } \forall z \in V \setminus \\
&\quad \{x\} . \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 = \mathcal{Q}(\pi'_0), \text{ in contradiction with } \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 y \neq \\
&\quad \mathcal{Q}(\pi'_0)y) \text{]} \\
&\subseteq \{ \langle x, y \rangle \mid \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])z) \wedge \langle \pi_0 \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_1 \rangle \in \\
&\quad \mathcal{S}^*[\![S_b]\!] \wedge (\mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])\pi_1 y \neq \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])y) \} \mid \text{nondet}(\neg B, \neg B) \\
&\quad \text{[letting } \pi_0 \text{at}[\![S_b]\!] \leftarrow \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!] \text{ with } \mathcal{Q}(\pi'_2) \xrightarrow{B} \text{at}[\![S_b]\!] = \mathcal{Q}(\pi'_2), \pi_0 \text{at}[\![S_b]\!] \leftarrow \\
&\quad \pi'_2 \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3, \text{ and } \pi_1 \leftarrow \pi'_3 \text{]} \\
&= (\{ \langle x, x \rangle \mid \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])z) \wedge \langle \pi_0 \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_1 \rangle \in \\
&\quad \mathcal{S}^*[\![S_b]\!] \wedge (\mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])\pi_1 x \neq \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])x) \} \\
&\quad \cup \{ \langle x, y \rangle \mid x \neq y \wedge \exists \pi_0, \pi_1, \pi'_0 . (\forall z \in V \setminus \{x\} . \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])z) \wedge \langle \pi_0 \text{at}[\![S_b]\!], \\
&\quad \text{at}[\![S_b]\!]\pi_1 \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])\pi_1 y \neq \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])y) \} \mid \text{nondet}(\neg B, \neg B) \\
&\quad \text{[since when } x \neq y, \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])y = \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])y \text{]}
\end{aligned}$$

$$\begin{aligned}
&= \{ \langle x, y \rangle \mid \exists \pi_0, \pi_1, \pi'_0. (\forall z \in \mathcal{V} \setminus \{x\}. \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])z) \wedge \langle \pi_0 \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_1 \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])\pi_1 \neq \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])y) \mid \text{nondet}(\neg B, \neg B) \} \quad (\text{grouping cases together}) \\
&= \{ \langle x, y \rangle \mid \exists \pi_0, \pi_1, \pi'_0. (\forall z \in \mathcal{V} \setminus \{x\}. \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])z = \mathcal{Q}(\pi'_0 \text{at}[\![S_b]\!])z) \wedge \langle \pi_0 \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_1 \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])\pi_1 \neq \mathcal{Q}(\pi_0 \text{at}[\![S_b]\!])y) \mid \text{nondet}(\neg B, \neg B) \} \\
&\quad (\text{letting } \rho = \mathcal{Q}(\pi_0 \ell), \nu = \mathcal{Q}(\pi'_0 \ell)x \text{ so that } \forall z \in \mathcal{V} \setminus \{x\}. \mathcal{Q}(\pi_0 \ell)z = \mathcal{Q}(\pi'_0 \ell)z \text{ implies } \mathcal{Q}(\pi'_0 \ell) = \rho[x \leftarrow \nu].) \\
&\subseteq (\{ \langle x, x \rangle \mid x \in \mathcal{V} \} \cup \{ \langle x, y \rangle \mid x \in \mathcal{V} \wedge y \in \text{mod}[\![S_b]\!]\}) \mid \text{nondet}(\neg B, \neg B)
\end{aligned}$$

(A coarse approximation is to consider the variables $y \neq x$ appearing to the left of an assignment in S_b , a necessary condition for y to be modified by the execution of S_b where the set $\text{mod}[\![S]\!]$ of variables that may be modified by the execution of S is syntactically defined as in (47.50).)

$$\begin{aligned}
&= \mathbb{1}_{\text{nondet}(\neg B, \neg B)} \cup \text{nondet}(\neg B, \neg B) \times \text{mod}[\![S_b]\!] \quad (\text{def.}) \\
&- \text{Summing up for all subcases of (3-C-C), we get } (5) \subseteq \mathbb{1}_{\text{nondet}(\neg B, \neg B)} \cup \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \mathbin{\&}; (\mathbb{1}_{\text{nondet}(\neg B, \neg B)} \cup \text{nondet}(\neg B, \neg B) \times \text{mod}[\![S_b]\!])) \mid \text{nondet}(B, \neg B). \\
&- \text{(3-B-B) This is the case when the observation prefix traces } \ell\pi_1 \text{ and } \ell\pi'_1 \text{ are both from a } \text{break}; \text{ in the iteration body } S_b.
\end{aligned}$$

$$\begin{aligned}
&(8) \\
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0 \ell, \ell\pi_1 \rangle, \langle \pi'_0 \ell, \ell\pi'_1 \rangle \in \{ \langle \pi_0 \ell, \ell\pi_2 \rangle \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!] \} \mid \langle \pi_0 \ell, \ell\pi_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0 \ell\pi_2 \ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi_0 \ell\pi_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \} . (\forall z \in \mathcal{V} \setminus \{x\}. \mathcal{Q}(\pi_0 \ell)z = \mathcal{Q}(\pi'_0 \ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0 \ell, \ell\pi_1), \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi'_0 \ell, \ell\pi'_1)) \} \quad (\text{case (3-B-B)}) \\
&= \{ \langle x, y \rangle \mid \exists \pi_0 \ell\pi_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \ell'' . \langle \pi_0 \ell, \ell\pi_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0 \ell\pi_2 \ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi_0 \ell\pi_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge \exists \pi'_0 \ell\pi'_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \ell'' . \langle \pi'_0 \ell, \ell\pi'_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0 \ell\pi'_2 \ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi'_0 \ell\pi'_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi'_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\forall z \in \mathcal{V} \setminus \{x\}. \mathcal{Q}(\pi_0 \ell)z = \mathcal{Q}(\pi'_0 \ell)z) \wedge \text{diff}(\text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0 \ell, \ell\pi_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!]), \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi'_0 \ell, \ell\pi'_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!])) \} \quad (\text{def.}) \\
&= \{ \langle x, y \rangle \mid \exists \pi_0 \ell\pi_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \ell'' . \langle \pi_0 \ell, \ell\pi_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi_0 \ell\pi_2 \ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi_0 \ell\pi_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge \exists \pi'_0 \ell\pi'_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \ell'' . \langle \pi'_0 \ell, \ell\pi'_2 \rangle \in X \wedge \mathcal{B}[\![B]\!]\mathcal{Q}(\pi'_0 \ell\pi'_2 \ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi'_0 \ell\pi'_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi'_3 \ell'' \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge (\forall z \in \mathcal{V} \setminus \{x\}. \mathcal{Q}(\pi_0 \ell)z = \mathcal{Q}(\pi'_0 \ell)z) \wedge \mathcal{Q}(\pi_0 \ell\pi_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3 \ell'') \neq \mathcal{Q}(\pi'_0 \ell\pi'_2 \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi'_3 \ell'') \}
\end{aligned}$$

$$\begin{aligned}
& \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \text{ and } X \text{ contains only iterates of } \mathcal{F}^*[\text{while } \ell(B) S_b] \text{ so } \text{after}[S] \neq \\
& \ell \text{ cannot appear in } \ell\pi_2^\ell. \text{ Moreover, } \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \text{ so, by def. Section 4.2 of program labelling, } \text{after}[S] \neq \text{at}[S_b] \\
& \text{cannot appear in } \text{at}[S_b]\pi_3^{\ell''}. \text{ Therefore, by def. (6.6) of } \mathcal{Q} \text{ and (47.16) of } \text{seqval}[y]^\ell, \\
& \text{seqval}[y](\text{after}[S])(\pi_0^\ell, \ell\pi_2^\ell) \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] = \mathcal{Q}(\pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''}). \text{ We conclude by def. (47.18) of } \text{diff} \} \\
= & \bigcup_{\ell'' \in \text{breaks-of}[S_b]} \{ \langle x, y \rangle \mid \exists \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} . \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \wedge \mathcal{B}[B]\mathcal{Q}(\pi_0^\ell\pi_2^\ell) = \\
& \text{tt} \wedge \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \wedge \exists \pi'_0\pi'_2 \xrightarrow{B} \text{at}[S_b]\pi'_3{}^{\ell''} . \\
& \langle \pi'_0\ell, \ell\pi'_2\ell \rangle \in X \wedge \mathcal{B}[B]\mathcal{Q}(\pi'_0\ell\pi'_2\ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[S_b] \wedge \langle \pi'_0\ell\pi'_2\ell \xrightarrow{B} \text{at}[S_b], \\
& \text{at}[S_b]\pi'_3{}^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \wedge (\forall z \in \mathcal{V} \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi'_0\ell)z) \wedge \mathcal{Q}(\pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''}) \neq \mathcal{Q}(\pi'_0\ell\pi'_2\ell \xrightarrow{B} \text{at}[S_b]\pi'_3{}^{\ell''}) \} \\
\subseteq & \bigcup_{\ell'' \in \text{breaks-of}[S_b]} \alpha^d(\{X\})^\ell \circ (\widehat{\mathcal{S}}_{\text{diff}}^{\exists}[S_b] \ell'' \mid \text{nondet}(B, B)) \\
& \text{(by a reasoning similar to the one we did in case (1-Ba/Bc/C-Bb) from (7) on.)} \\
= & \alpha^d(\{X\})^\ell \circ \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[S_b]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[S_b] \ell'' \right) \mid \text{nondet}(B, B) \right) \quad (\text{;} \text{ and } \mid \text{ preserve arbitrary joins}) \\
\text{--- (3-B-C)} & \text{ This is the case when the observation prefix trace } \ell\pi_1 \text{ is from a normal exit of the} \\
& \text{iteration and } \ell\pi'_1 \text{ is from a } \text{break ;} \text{ in the iteration body } S_b. \text{ By symmetry of } \text{diff} \text{ this also covers} \\
& \text{the inverse case.} \\
(8) & \\
= & \{ \langle x, y \rangle \mid \exists \langle \pi_0^\ell, \ell\pi_1 \rangle \in \{ \langle \pi_0^\ell, \ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \mid \langle \pi_0^\ell, \\
& \ell\pi_2^\ell \rangle \in X \wedge \mathcal{B}[B]\mathcal{Q}(\pi_0^\ell\pi_2^\ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[S_b] \wedge \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b], \\
& \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \} . \exists \langle \pi'_0\ell, \ell\pi'_1 \rangle \in \{ \langle \pi'_0\ell, \ell\pi'_2\ell \xrightarrow{\neg(B)} \text{after}[S] \rangle \mid \\
& \langle \pi'_0\ell, \ell\pi'_2\ell \rangle \in X \wedge \mathcal{B}[B]\mathcal{Q}(\pi'_0\ell\pi'_2\ell) = \text{ff} \} . (\forall z \in \mathcal{V} \setminus \{x\} . \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi'_0\ell)z) \wedge \\
& \text{diff}(\text{seqval}[y](\text{after}[S])(\pi_0^\ell, \ell\pi_1), \text{seqval}[y](\text{after}[S])(\pi'_0\ell, \ell\pi'_1)) \} \quad \text{(case (3-B-C))} \\
= & \{ \langle x, y \rangle \mid \exists \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''}\pi'_0\ell\pi'_2\ell . \langle \pi_0^\ell, \ell\pi_2^\ell \rangle \in X \wedge \mathcal{B}[B]\mathcal{Q}(\pi_0^\ell\pi_2^\ell) = \\
& \text{tt} \wedge \ell'' \in \text{breaks-of}[S_b] \wedge \langle \pi_0^\ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \\
& \mathcal{S}^*[S_b] \wedge \langle \pi'_0\ell, \ell\pi'_2\ell \rangle \in X \wedge \mathcal{B}[B]\mathcal{Q}(\pi'_0\ell\pi'_2\ell) = \text{ff} \wedge (\forall z \in \mathcal{V} \setminus \{x\} . \\
& \mathcal{Q}(\pi_0^\ell)z = \mathcal{Q}(\pi'_0\ell)z) \wedge \text{diff}(\text{seqval}[y](\text{after}[S])(\pi_0^\ell, \ell\pi_2^\ell) \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \\
& \text{after}[S]), \text{seqval}[y](\text{after}[S])(\pi'_0\ell, \ell\pi'_2\ell) \xrightarrow{\neg(B)} \text{after}[S]) \} \quad \text{(def.)}
\end{aligned}$$

$$\begin{aligned}
&= \{ \langle x, y \rangle \mid \exists \pi_0^\ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!], \pi_0^{\ell'} \pi_2^{\ell'} \xrightarrow{\neg(B)} \text{after}[\![S]\!] \cdot \\
&\quad \wedge^{\ell''} \in \text{breaks-of}[\![S_b]\!]\langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X \wedge \mathfrak{B}[\![B]\!]\mathfrak{Q}(\pi_0^\ell \pi_2^\ell) = \text{tt} \wedge \langle \pi_0^\ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \\
&\quad \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \wedge \langle \pi_0^{\ell'}, \ell \pi_2^{\ell'} \rangle \in X \wedge \mathfrak{B}[\![B]\!]\mathfrak{Q}(\pi_0^{\ell'} \pi_2^{\ell'}) = \text{ff} \wedge (\forall z \in \\
&\quad \mathcal{V} \setminus \{x\} \cdot \mathfrak{Q}(\pi_0^\ell)z = \mathfrak{Q}(\pi_0^{\ell'})z \wedge \text{diff}(\text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^\ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \\
&\quad \text{after}[\![S]\!], \text{after}[\![S]\!]), \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^{\ell'} \pi_2^{\ell'}) \xrightarrow{\neg(B)} \text{after}[\![S]\!], \text{after}[\![S]\!] \rangle) \} \\
&\quad \wr \langle \pi_0^\ell, \ell \pi_2^\ell \rangle, \langle \pi_0^{\ell'}, \ell \pi_2^{\ell'} \rangle \in X \text{ and } X \text{ contains only iterates of } \mathcal{F}^*[\![\text{while } \ell(B) S_b]\!] \\
&\quad \text{so } \text{after}[\![S]\!] \neq \ell \text{ can appear neither in } \ell \pi_2^\ell \text{ nor in } \ell \pi_2^{\ell'}. \text{ Moreover, } \langle \pi_0^\ell \pi_2^\ell \xrightarrow{B} \\
&\quad \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \text{ so, by def. Section 4.2 of program} \\
&\quad \text{labelling, } \text{after}[\![S]\!] \neq \text{at}[\![S_b]\!] \text{ cannot appear in } \text{at}[\![S_b]\!]\pi_3^{\ell''}. \text{ Therefore, by def. (6.6) of} \\
&\quad \mathfrak{Q} \text{ and (47.16) of } \text{seqval}[\![y]\!]^\ell, \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^\ell, \ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \\
&\quad \text{after}[\![S]\!]) = \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^\ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!], \text{after}[\![S]\!]) \text{ and} \\
&\quad \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^{\ell'}, \ell \pi_2^{\ell'}) \xrightarrow{\neg(B)} \text{after}[\![S]\!] = \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^{\ell'} \pi_2^{\ell'} \xrightarrow{\neg(B)} \\
&\quad \text{after}[\![S]\!], \text{after}[\![S]\!]) \}. \} \\
&= \{ \langle x, y \rangle \mid \exists \pi_0^\ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!], \pi_0^{\ell'} \pi_2^{\ell'} \xrightarrow{\neg(B)} \text{after}[\![S]\!] \cdot \\
&\quad \langle \pi_0^\ell, \ell \pi_2^\ell \rangle \in X \wedge \langle \pi_0^\ell \pi_2^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \{ \langle \pi^\ell, \ell \xrightarrow{B} \\
&\quad \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \mid \mathfrak{B}[\![B]\!]\mathfrak{Q}(\pi^\ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi^\ell \xrightarrow{B} \\
&\quad \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \} \wedge \langle \pi_0^{\ell'}, \ell \pi_2^{\ell'} \rangle \in X \wedge \langle \pi_0^{\ell'} \pi_2^{\ell'}, \\
&\quad \ell \xrightarrow{\neg(B)} \text{after}[\![S]\!] \rangle \in \{ \langle \pi^\ell, \ell \xrightarrow{\neg(B)} \text{after}[\![S]\!] \rangle \mid \mathfrak{B}[\![B]\!]\mathfrak{Q}(\pi^\ell) = \text{ff} \} \wedge (\forall z \in \mathcal{V} \setminus \\
&\quad \{x\} \cdot \mathfrak{Q}(\pi_0^\ell)z = \mathfrak{Q}(\pi_0^{\ell'})z \wedge \text{diff}(\text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^\ell \pi_2^\ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \\
&\quad \text{after}[\![S]\!], \text{after}[\![S]\!]), \text{seqval}[\![y]\!](\text{after}[\![S]\!])(\pi_0^{\ell'} \pi_2^{\ell'}) \xrightarrow{\neg(B)} \text{after}[\![S]\!], \text{after}[\![S]\!] \rangle) \} \quad \wr \text{def. } \epsilon \} \\
&\subseteq \alpha^d(\{X\})^\ell \circ \alpha^d(\{\mathcal{S}'\}) \text{after}[\![S]\!]
\end{aligned}$$

$$\begin{aligned}
&\wr \text{by Lemma 47.59 where } \mathcal{S}' = \{ \langle \pi^\ell, \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \mid \mathfrak{B}[\![B]\!]\mathfrak{Q}(\pi^\ell) = \\
&\quad \text{tt} \wedge \ell'' \in \text{breaks-of}[\![S_b]\!] \wedge \langle \pi^\ell \xrightarrow{B} \text{at}[\![S_b]\!], \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!] \rangle \in \mathcal{S}^*[\![S_b]\!] \} \cup \{ \langle \pi^\ell, \\
&\quad \ell \xrightarrow{\neg(B)} \text{after}[\![S]\!] \rangle \mid \mathfrak{B}[\![B]\!]\mathfrak{Q}(\pi^\ell) = \text{ff} \} \text{ with } \pi_0^{\ell_0} \leftarrow \pi_0^\ell, \ell_0 \pi_1^{\ell'} \leftarrow \ell \pi_2^\ell, \ell \leftarrow \text{after}[\![S]\!], \\
&\quad \ell' \pi_2^{\ell'} \leftarrow \ell \xrightarrow{B} \text{at}[\![S_b]\!]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[\![S]\!], \ell \pi_3 \leftarrow \text{after}[\![S]\!] \text{ so } \pi_3 = \ni, \text{ and } \pi_0^{\ell_0} \leftarrow \pi_0^{\ell'}, \\
&\quad \ell_0 \pi_1^{\ell'} \leftarrow \ell_0 \pi_2^{\ell'}, \ell' \pi_2^{\ell'} \leftarrow \ell \xrightarrow{B} \text{at}[\![S_b]\!], \ell \pi_3 \leftarrow \text{after}[\![S]\!] \text{ so } \pi_3' = \ni \}
\end{aligned}$$

Similar to the calculation starting at (10), we have to calculate the second term

$$\begin{aligned}
&\alpha^d(\{\mathcal{S}'\}) \text{after}[\![S]\!] \\
&= \{ \langle x, y \rangle \mid \mathcal{S}' \in \mathcal{D}(\text{after}[\![S]\!]) \langle x, y \rangle \} \quad \wr \text{def. (47.25) of } \alpha^d \}
\end{aligned}$$

$$\begin{aligned}
&= \{ \langle x, y \rangle \mid \exists \langle \pi_0, \pi_1 \rangle, \langle \pi'_0, \pi'_1 \rangle \in \mathcal{S}' . (\forall z \in V \setminus \{x\} . \varrho(\pi_0)z = \varrho(\pi'_0)z) \wedge \\
&\quad \text{diff}(\text{seqval}[y]\text{after}[S](\pi_0, \pi_1), \text{seqval}[y]\text{after}[S](\pi'_0, \pi'_1)) \} \quad \text{[def. (47.19) of } \mathcal{D}^\ell(x, y)\text{]} \\
&= \{ \langle x, y \rangle \mid \exists \pi^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S], \pi'^\ell \xrightarrow{\neg(B)} \text{after}[S] . \\
&\quad \mathfrak{B}[B]\varrho(\pi^\ell) = \text{tt} \wedge \ell'' \in \text{breaks-of}[S_b] \wedge \langle \pi^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \\
&\quad \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \wedge \mathfrak{B}[B]\varrho(\pi'^\ell) = \text{ff} \wedge (\forall z \in V \setminus \{x\} . \varrho(\pi^\ell)z = \varrho(\pi'^\ell)z) \wedge \\
&\quad \text{diff}(\text{seqval}[y]\text{after}[S](\pi^\ell, \ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S]), \text{seqval}[y]\text{after}[S](\pi'^\ell, \ell \\
&\quad \xrightarrow{\neg(B)} \text{after}[S])) \} \\
&\quad \text{[def. } \mathcal{S}' \text{ and the other two combinations have already been considered in (3-B-B) and} \\
&\quad \text{(2-C-C)]} \\
&= \{ \langle x, y \rangle \mid \exists \pi^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S], \pi'^\ell \xrightarrow{\neg(B)} \text{after}[S] . \mathfrak{B}[B]\varrho(\pi^\ell) = \text{tt} \wedge \ell'' \in \\
&\quad \text{breaks-of}[S_b] \wedge \langle \pi^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \wedge \mathfrak{B}[B]\varrho(\pi'^\ell) = \\
&\quad \text{ff} \wedge (\forall z \in V \setminus \{x\} . \varrho(\pi^\ell)z = \varrho(\pi'^\ell)z) \wedge \varrho(\pi^\ell) \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S])y \neq \\
&\quad \varrho(\pi'^\ell \xrightarrow{\neg(B)} \text{after}[S])y \} \\
&\quad \langle \pi^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \text{ so, by def. Section 4.2 of} \\
&\quad \text{program labelling, } \text{after}[S] \neq \text{at}[S_b] \text{ cannot appear in } \text{at}[S_b]\pi_3^{\ell''}. \text{ Therefore, by def. (6.6)} \\
&\quad \text{of } \varrho \text{ and (47.16) of } \text{seqval}[y]^\ell, \text{seqval}[y](\text{after}[S])(\pi_0^\ell, \ell\pi_2^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \\
&\quad \text{after}[S]) = \varrho(\pi^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''}) \text{ and } \text{seqval}[y](\text{after}[S])(\pi'^\ell, \ell\pi_2'^\ell \xrightarrow{\neg(B)} \text{after}[S]) = \\
&\quad \varrho(\pi'^\ell \pi_2'^\ell). \text{ We conclude by def. (47.18) of } \text{diff} \} \\
&\subseteq \{ \langle x, y \rangle \mid \exists \pi^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S], \pi'^\ell \xrightarrow{\neg(B)} \text{after}[S] . \ell'' \in \text{breaks-of}[S_b] \wedge \\
&\quad \langle \pi^\ell \xrightarrow{B} \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \wedge (\forall z \in V \setminus \{x\} . \varrho(\pi^\ell)z = \varrho(\pi'^\ell)z) \wedge \\
&\quad \varrho(\pi^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S])y \neq \varrho(\pi'^\ell \xrightarrow{\neg(B)} \text{after}[S])y \} \mid \text{nondet}(B, \neg B) \\
&\quad \text{[since if } x \notin \text{nondet}(B, \neg B) \text{ then } x \in \text{det}(B, \neg B) \text{ so by (47.48), } \mathfrak{B}[B]\varrho(\pi^\ell) = \text{tt} \text{ and} \\
&\quad \mathfrak{B}[\neg B]\varrho(\pi'^\ell) = \text{tt} \text{ imply } \varrho(\pi^\ell)x = \varrho(\pi'^\ell)x \text{ which, together with } \forall z \in V \setminus \{x\} . \\
&\quad \varrho(\pi^\ell)z = \varrho(\pi'^\ell)z, \text{ implies that } \varrho(\pi^\ell) = \varrho(\pi'^\ell), \text{ in contradiction with } \mathfrak{B}[B]\varrho(\pi^\ell) = \text{tt} \\
&\quad \text{and } \mathfrak{B}[\neg B]\varrho(\pi'^\ell) = \text{ff} \} \\
&= \bigcup_{\ell'' \in \text{breaks-of}[S_b]} \{ \langle x, y \rangle \mid \exists \pi^\ell \xrightarrow{B} \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S], \pi'^\ell \xrightarrow{\neg(B)} \text{after}[S] . \langle \pi^\ell \xrightarrow{B} \\
&\quad \text{at}[S_b], \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S] \rangle \in \mathcal{S}^*[S_b] \wedge (\forall z \in V \setminus \{x\} . \varrho(\pi^\ell)z = \varrho(\pi'^\ell)z) \wedge \varrho(\pi^\ell \xrightarrow{B} \\
&\quad \text{at}[S_b]\pi_3^{\ell''} \xrightarrow{\text{break}} \text{after}[S])y \neq \varrho(\pi'^\ell \xrightarrow{\neg(B)} \text{after}[S])y \} \mid \text{nondet}(B, \neg B) \\
&\quad \text{[def. } \cup \text{]}
\end{aligned}$$

$$\subseteq \bigcup_{\ell'' \in \text{breaks-of}[\llbracket S_b \rrbracket]} (\{\langle x, x \rangle \mid x \in \mathcal{V} \} \cup \{\langle x, y \rangle \mid x \in \mathcal{V} \wedge y \in \text{mod}[\llbracket S_b \rrbracket]\}) \upharpoonright \text{nondet}(\mathbf{B}, \neg \mathbf{B})$$

(since if $y \neq x$ then $\mathbf{q}(\pi^\ell)y = \mathbf{q}(\pi'^\ell)y = \mathbf{q}(\pi'^\ell \xrightarrow{\neg(\mathbf{B})} \text{after}[\llbracket S \rrbracket])y$ so for the value of y to be different in $\mathbf{q}(\pi^\ell \xrightarrow{\mathbf{B}} \text{at}[\llbracket S_b \rrbracket]\pi_3\ell'') \xrightarrow{\text{break}} \text{after}[\llbracket S \rrbracket]) = \mathbf{q}(\pi^\ell \xrightarrow{\mathbf{B}} \text{at}[\llbracket S_b \rrbracket]\pi_3\ell'') = \mathbf{q}(\pi'^\ell \xrightarrow{\mathbf{B}} \text{at}[\llbracket S_b \rrbracket]\pi_3\ell'')$, y must be modified during the execution $\text{at}[\llbracket S_b \rrbracket]\pi_3\ell''$ of S_b . A coarse approximation is to consider that variable y appears to the left of an assignment in S_b , a necessary condition for y to be modified by the execution of S_b where the set $\text{mod}[\llbracket S \rrbracket]$ of variables that may be modified by the execution of S is syntactically defined as in (47.50).)

$$(\mathbb{1}_{\mathcal{V}} \cup \{\langle x, y \rangle \mid x \in \mathcal{V} \wedge y \in \text{mod}[\llbracket S_b \rrbracket]\}) \upharpoonright \text{nondet}(\mathbf{B}, \neg \mathbf{B}) \quad (\text{def. identity relation } \mathbb{1} \text{ and } \cup)$$

$$= \mathbb{1}_{\text{nondet}(\mathbf{B}, \neg \mathbf{B})} \cup (\text{nondet}(\mathbf{B}, \neg \mathbf{B}) \times \text{mod}[\llbracket S_b \rrbracket]) \quad (\text{def. } \upharpoonright)$$

– Summing up for cases (3-B-B) and (3-B-C), we get

$$(5) \subseteq \alpha^d(\{X\})^\ell \circ \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\llbracket S_b \rrbracket]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\llbracket S_b \rrbracket] \ell'' \right) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \cup \mathbb{1}_{\text{nondet}(\mathbf{B}, \neg \mathbf{B})} \cup (\text{nondet}(\mathbf{B}, \neg \mathbf{B}) \times \text{mod}[\llbracket S_b \rrbracket]).$$

— Summing up for all subcases of (3) for a dependency observation point $\ell' = \text{after}[\llbracket S \rrbracket]$, we would get a term (47.63.c) of the form

$$\begin{aligned} (\ell' = \text{after}[\llbracket S \rrbracket] \text{ ? } (\mathbb{1}_{\text{nondet}(\neg \mathbf{B}, \neg \mathbf{B})} \cup X(\ell)) \cup & \quad (47.63.c') \\ (X(\ell) \circ (\mathbb{1}_{\text{nondet}(\neg \mathbf{B}, \neg \mathbf{B})} \cup \text{nondet}(\neg \mathbf{B}, \neg \mathbf{B}) \times \text{mod}[\llbracket S_b \rrbracket])) \upharpoonright \text{nondet}(\mathbf{B}, \neg \mathbf{B}) \cup & \\ X(\ell) \circ \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\llbracket S_b \rrbracket]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\llbracket S_b \rrbracket] \ell'' \right) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \cup & \\ \mathbb{1}_{\text{nondet}(\mathbf{B}, \neg \mathbf{B})} \cup (\text{nondet}(\mathbf{B}, \neg \mathbf{B}) \times \text{mod}[\llbracket S_b \rrbracket]) \circ \emptyset. & \end{aligned}$$

that can be simplified as follows (while loosing precision)

(5)

$$\begin{aligned} & \subseteq \mathbb{1}_{\text{nondet}(\neg \mathbf{B}, \neg \mathbf{B})} \cup \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \circ (\mathbb{1}_{\text{nondet}(\neg \mathbf{B}, \neg \mathbf{B})} \cup \text{nondet}(\neg \mathbf{B}, \neg \mathbf{B}) \times \text{mod}[\llbracket S_b \rrbracket])) \upharpoonright \\ & \quad \text{nondet}(\mathbf{B}, \neg \mathbf{B}) \cup \alpha^d(\{X\})^\ell \circ \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\llbracket S_b \rrbracket]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\llbracket S_b \rrbracket] \ell'' \right) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \cup \mathbb{1}_{\text{nondet}(\mathbf{B}, \neg \mathbf{B})} \cup \\ & \quad (\text{nondet}(\mathbf{B}, \neg \mathbf{B}) \times \text{mod}[\llbracket S_b \rrbracket]) \\ & \subseteq \mathbb{1}_{\mathcal{V}} \cup \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \circ (\mathbb{1}_{\mathcal{V}} \cup \mathcal{V} \times \text{mod}[\llbracket S_b \rrbracket])) \cup \alpha^d(\{X\})^\ell \circ \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\llbracket S_b \rrbracket]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\llbracket S_b \rrbracket] \ell'' \right) \upharpoonright \right. \\ & \quad \left. \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \cup \mathbb{1}_{\mathcal{V}} \cup (\mathcal{V} \times \text{mod}[\llbracket S_b \rrbracket]) \end{aligned}$$

(since $\text{nondet}(\mathbf{B}_1, \mathbf{B}_2) \subseteq \mathcal{V}$ so $\mathbb{1}_{\text{nondet}(\mathbf{B}_1, \mathbf{B}_2)} \subseteq \mathbb{1}_{\mathcal{V}}$ and def. \upharpoonright)

$$\begin{aligned}
&\subseteq \mathbb{1}_V \cup \alpha^d(\{X\})^\ell \cup (\alpha^d(\{X\})^\ell \mathbin{\circ} \mathbb{1}_V) \cup (\alpha^d(\{X\})^\ell \mathbin{\circ} V \times \text{mod}[\![S_b]\!]) \cup \alpha^d(\{X\})^\ell \mathbin{\circ} \\
&\quad \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\![S_b]\!]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_b]\!]^{\ell''} \right) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \cup \mathbb{1}_V \cup (V \times \text{mod}[\![S_b]\!]) \\
&\hspace{15em} \{\text{since } \mathbin{\circ} \text{ distributes over } \cup\} \\
&= \mathbb{1}_V \cup \alpha^d(\{X\})^\ell \cup ((\mathbb{1}_V \cup \alpha^d(\{X\})^\ell) \mathbin{\circ} (V \times \text{mod}[\![S_b]\!])) \cup \alpha^d(\{X\})^\ell \mathbin{\circ} \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\![S_b]\!]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_b]\!]^{\ell''} \right) \upharpoonright \right. \\
&\quad \left. \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \hspace{15em} \{\text{idempotency law for } \cup \text{ and } \mathbin{\circ} \text{ distributes over } \cup\}
\end{aligned}$$

After simplification, we get a term (47.63.c) of the form

$$\begin{aligned}
&(\ell' = \text{after}[\![S]\!] \mathbin{\circ} \mathbb{1}_V \cup X(\ell) \cup ((\mathbb{1}_V \cup X(\ell)) \mathbin{\circ} (V \times \text{mod}[\![S_b]\!])) \cup \\
&\quad X(\ell) \mathbin{\circ} \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\![S_b]\!]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_b]\!]^{\ell''} \right) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \mathbin{\circ} \emptyset.
\end{aligned}$$

For fixpoints X of $\mathcal{F}^{\text{diff}}[\![\text{while } \ell \ (\mathbf{B}) \ S_b]\!]$, we have $\mathbb{1}_V \subseteq X(\ell)$ by (47.63.a) so that, by the chaotic iteration theorem [1, 2], $\mathbb{1}_V \cup X(\ell)$ can be replaced by $X(\ell)$. We get a term (47.63.c) of the form

$$\begin{aligned}
&(\ell' = \text{after}[\![S]\!] \mathbin{\circ} X(\ell) \cup (X(\ell) \mathbin{\circ} (V \times \text{mod}[\![S_b]\!])) \cup \\
&\quad X(\ell) \mathbin{\circ} \left(\left(\bigcup_{\ell'' \in \text{breaks-of}[\![S_b]\!]} \widehat{\mathcal{S}}_{\text{diff}}^{\exists}[\![S_b]\!]^{\ell''} \right) \upharpoonright \text{nondet}(\mathbf{B}, \mathbf{B}) \right) \mathbin{\circ} \emptyset.
\end{aligned}$$

— Summing up for all cases (1), (2), and (3) for all dependency observation points, we conclude that

$$\forall \ell' \in \text{labx}[\![S]\!] . \alpha^d(\{\mathcal{F}^*[\![\text{while } \ell \ (\mathbf{B}) \ S_b]\!](X)\})^{\ell'} \subseteq \mathcal{F}^{\text{diff}}[\![\text{while } \ell \ (\mathbf{B}) \ S_b]\!] \alpha^d(\{X\})^{\ell'}$$

proving pointwise semi-commutation. \square

5 Mathematical proofs of chapter 48

Proof of Lemma 48.63 By induction on the sequence of calls to `unify`. We proceed by by calculational design and case analysis on the structure of τ_1 and τ_2 which can be a variable or a structured term and may belong to the domain of \mathcal{V}_0 , or not.

- If $\text{unify}(\tau_1, \tau_2, \vartheta_0) = \Omega_s^r$ in case (48.47.8) of an occur-check, we have $\gamma_s^r(\Omega_s^r) = \emptyset$ by (48.46). By the test (48.47.8), $\alpha \in \text{vars}[\![\tau_2]\!]$. If $\tau_2 = \beta \in \mathcal{V}_t$ were a variable then the test $\alpha \in \text{vars}[\![\tau_2]\!]$ at (48.47.8) would be true only if $\alpha = \beta$ but this case is prevented by the test (48.47.7). By contradiction, $\tau_2 \notin \mathcal{V}_t$ in case (48.47.8). It follows, by def. (48.51) of γ_e that $\gamma_e(\tau_1 \doteq \tau_2) = \gamma_e(\alpha \doteq \tau_2) = \emptyset$ since otherwise, there would be some ϱ such that $\varrho(\tau_1) = \varrho(f(\dots \alpha \dots))$ which would be an infinite object not in \mathbf{P}^v , as shown in Lemma 48.9.

- By Lemma 48.58, **unify** does terminate so that, in case (48.47.6) with $\vartheta_n = \Omega_s^r$ there must be a series of recursive calls ending up in (48.47.8). So τ_1 or τ_2 has a recursive subterm which, again by Lemma 48.9, implies $\gamma_s^r(\text{unify}(\tau_1, \tau_2, \vartheta_0)) = \gamma_s^r(\text{unify}(\tau_1, \tau_2, \vartheta_0)) = \gamma_s^r(\Omega_s^r) = \emptyset$;

- In case (48.47.6) with $\vartheta_n \neq \Omega_s^r$, we have,

$$\begin{aligned}
& \gamma_e(\tau_1 \doteq \tau_2) \cap \gamma_s^r(\vartheta_0) \\
&= \gamma_e(f(\tau_1^1, \dots, \tau_1^n) \doteq g(\tau_2^1, \dots, \tau_2^m)) \cap \gamma_s^r(\vartheta_0) && \text{? test (48.47.1) is tt ?} \\
&= \gamma_e(f(\tau_1^1, \dots, \tau_1^n) \doteq f(\tau_2^1, \dots, \tau_2^n)) \cap \gamma_s^r(\vartheta_0) && \text{? test (48.47.2) is ff ?} \\
&= \{\varrho \in \mathbf{P}^v \mid \varrho(f(\tau_1^1, \dots, \tau_1^n)) = \varrho(f(\tau_2^1, \dots, \tau_2^n))\} \cap \gamma_s^r(\vartheta_0) && \text{? def. (48.51) of } \gamma_e \text{ ?} \\
&= \{\varrho \in \mathbf{P}^v \mid \bigwedge_{i=1}^n \varrho(\tau_i^1) = \varrho(\tau_i^2)\} && \text{? def. (48.7) of assignment application ?} \\
&= \bigcap_{i=1}^n \{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^1) = \varrho(\tau_i^2)\} \cap \gamma_s^r(\vartheta_0) && \text{? def. } \bigcap \text{ ?} \\
&= (\{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^1) = \varrho(\tau_i^2)\} \cap \gamma_s^r(\vartheta_0)) \cap \bigcap_{2=1}^n \{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^1) = \varrho(\tau_i^2)\} \\
& && \text{? } \bigcap \text{ is associative and commutative ?} \\
&= (\gamma_e(\tau_i^1 \doteq \tau_i^2) \cap \gamma_s^r(\vartheta_0)) \cap \bigcap_{2=1}^n \{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^1) = \varrho(\tau_i^2)\} && \text{? def. (48.51)x of } \gamma_e \text{ ?} \\
&= \text{let } \vartheta_1 = \text{unify}(\tau_i^1, \tau_i^2, \vartheta_0) \text{ in} \\
& \quad \bigcap_{2=1}^n \{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^1) = \varrho(\tau_i^2)\} \cap \gamma_s^r(\vartheta_1) && \text{? ind. hyp. and } \bigcap \text{ commutative ?} \\
&= \text{let } \vartheta_1 = \text{unify}((\tau_i^1, \tau_i^2), \vartheta_0) \text{ in} \\
& \quad \dots \\
& \quad \text{let } \vartheta_j = \text{unify}(\tau_i^j, \tau_i^j, \vartheta_{j-1}) \text{ in} \\
& \quad \bigcap_{i=j+1}^n \{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^1) = \varrho(\tau_i^2)\} \cap \gamma_s^r(\vartheta_j) && \text{? recurrence hyp., } j < n \text{ ?} \\
&= \text{let } \vartheta_1 = \text{unify}(\tau_i^1, \tau_i^2, \vartheta_0) \text{ in} \\
& \quad \dots \\
& \quad \text{let } \vartheta_j = \text{unify}(\tau_i^j, \tau_i^j, \vartheta_{j-1}) \text{ in} \\
& \quad \{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^{j+1}) = \varrho(\tau_i^{j+1})\} \cap \gamma_s^r(\vartheta_j) \cap \\
& \quad \bigcap_{i=j+2}^n \{\varrho \in \mathbf{P}^v \mid \varrho(\tau_i^1) = \varrho(\tau_i^2)\} && \text{? } \bigcap \text{ is associative and commutative ?}
\end{aligned}$$

$$\begin{aligned}
&= \text{let } \vartheta_1 = \text{unify}(\tau_i^1, \tau_2^1, \vartheta_0) \text{ in} \\
&\quad \dots \\
&\quad \text{let } \vartheta_j = \text{unify}(\tau_i^j, \tau_2^j, \vartheta_{j-1}) \text{ in} \\
&\quad \text{let } \vartheta_{j+1} = \text{unify}(\tau_i^{j+1}, \tau_2^{j+1}, \vartheta_j) \text{ in} \\
&\quad \bigcap_{i=j+2}^n \{ \varrho \in \mathbf{P}^\nu \mid \varrho(\tau_i^1) = \varrho(\tau_2^1) \} \cap \gamma_s^r(\vartheta_{j+1}) \quad \text{[ind. hyp. and } \bigcap \text{ commutative]} \\
&= \text{let } \vartheta_1 = \text{unify}(\tau_i^1, \tau_2^1, \vartheta_0) \text{ in} \\
&\quad \dots \\
&\quad \text{let } \vartheta_j = \text{unify}(\tau_i^n, \tau_2^n, \vartheta_{n-1}) \text{ in} \\
&\quad \bigcap_{i=n+2}^n \{ \varrho \in \mathbf{P}^\nu \mid \varrho(\tau_i^1) = \varrho(\tau_2^1) \} \cap \gamma_s^r(\vartheta_n) \quad \text{[by recurrence when } j+1 = n] \\
&= \text{let } \vartheta_1 = \text{unify}(\tau_i^1, \tau_2^1, \vartheta_0) \text{ in} \\
&\quad \dots \\
&\quad \text{let } \vartheta_j = \text{unify}(\tau_i^n, \tau_2^n, \vartheta_{n-1}) \text{ in} \\
&\quad \gamma_s^r(\vartheta_n) \\
&\quad \text{[since } \bigcap_{i=n+2}^n \{ \varrho \in \mathbf{P}^\nu \mid \varrho(\tau_i^1) = \varrho(\tau_2^1) \} = \bigcap \emptyset = \mathbf{P}^\nu \text{ is the identity for } \cap \text{]}
\end{aligned}$$

- In case (48.47.7), we have

$$\begin{aligned}
&\gamma_e(\tau_1 \doteq \tau_2) \cap \gamma_s^r(\vartheta_0) \\
&= \gamma_e(\alpha \doteq \alpha) \cap \gamma_s^r(\vartheta_0) \quad \text{[} \alpha \in \mathbb{V}_{\mathbb{t}} \text{ by test (48.47.7)]} \\
&= \{ \varrho \in \mathbf{P}^\nu \mid \varrho(\alpha) = \varrho(\alpha) \} \cap \gamma_s^r(\vartheta_0) \quad \text{[def. (48.51) of } \gamma_e] \\
&= \mathbf{P}^\nu \cap \gamma_s^r(\vartheta_0) \quad \text{[since } \varrho \in \mathbf{P}^\nu \triangleq \mathbb{V}_{\mathbb{t}} \rightarrow \mathbf{T} \text{ by (48.6)]} \\
&= \gamma_s^r(\vartheta_0) \quad \text{[} \mathbf{P}^\nu \text{ is the identity for } \cap \text{]} \\
&= \gamma_s^r(\text{unify}(\tau_1, \tau_2, \vartheta_0)) \quad \text{[def. unify in case (48.47.7)]}
\end{aligned}$$

- In case (48.47.11), we have

$$\begin{aligned}
&\gamma_e(\tau_1 \doteq \tau_2) \cap \gamma_s^r(\vartheta_0) \\
&= \gamma_e(\alpha \doteq \tau_2) \cap \gamma_s^r(\vartheta_0) \\
&\quad \text{[where } \alpha \in \mathbb{V}_{\mathbb{t}} \text{ by test (48.47.9), } \alpha \notin \text{vars}[\tau_2] \text{ since test (48.47.8) is ff, } \alpha \notin \text{dom}(\vartheta_0) \text{ by} \\
&\quad \text{test (48.47.10), and } \tau_2 \notin \mathbb{V}_{\mathbb{t}} \text{ since test (48.47.1) is ff]} \\
&= \{ \varrho \in \mathbf{P}^\nu \mid \varrho(\alpha) = \varrho(\tau_2) \} \cap \gamma_s^r(\vartheta_0) \quad \text{[def. (48.51) of } \gamma_e] \\
&= \{ \varrho \in \mathbf{P}^\nu \mid \varrho(\alpha) = \varrho(\tau_2) \} \cap \{ \varrho \in \mathbf{P}^\nu \mid \forall \beta \in \mathbb{V}_{\mathbb{t}} . \varrho(\beta) = \varrho(\vartheta_0(\beta)) \} \quad \text{[def. (48.52) of } \gamma_s^r] \\
&= \{ \varrho \in \mathbf{P}^\nu \mid \varrho(\alpha) = \varrho(\tau_2) \wedge \forall \beta \in \mathbb{V}_{\mathbb{t}} . \varrho(\beta) = \varrho(\vartheta_0(\beta)) \} \quad \text{[def. } \cap \text{]} \\
&= \{ \varrho \in \mathbf{P}^\nu \mid \forall \beta \in \mathbb{V}_{\mathbb{t}} . \varrho(\beta) = [\beta = \alpha \text{ ? } \varrho(\vartheta_0(\beta)) [\beta \in \text{vars}[\tau_2] \leftarrow \tau_2] : \varrho(\tau_2 [\alpha \leftarrow \vartheta_0(\beta)])] \}
\end{aligned}$$

$$\begin{aligned}
& \text{[def. (48.7) of assignment application where } \boldsymbol{\varrho}(\alpha) \text{ is replaced by its equal } \boldsymbol{\varrho}(\tau_2) \text{ and for} \\
& \quad \beta \in \mathbb{V}_{\neq} \setminus \{\alpha\}, \boldsymbol{\varrho}(\beta) \text{ is replaced by its equal } \boldsymbol{\varrho}(\vartheta_0(\beta))\text{]} \\
= & \{\boldsymbol{\varrho} \in \mathbf{P}^v \mid \forall \beta \in \mathbb{V}_{\neq} . \boldsymbol{\varrho}(\beta) = \llbracket \beta = \alpha \text{ ? } \boldsymbol{\varrho}(\vartheta_0(\beta))[\beta \in \text{vars}[\tau_2] \leftarrow \tau_2] : \boldsymbol{\varrho}(\{\langle \alpha, \tau_2 \rangle\}(\vartheta_0(\beta))) \rrbracket \} \\
& \text{[by Exercise 48.60 where } \tau' = \vartheta_0(\beta)\text{]} \\
= & \{\boldsymbol{\varrho} \in \mathbf{P}^v \mid \forall \beta \in \mathbb{V}_{\neq} . \boldsymbol{\varrho}(\beta) = \llbracket \beta = \alpha \text{ ? } \boldsymbol{\varrho}(\vartheta_0(\tau_2)) : \boldsymbol{\varrho}(\{\langle \alpha, \tau_2 \rangle\}(\vartheta_0(\beta))) \rrbracket \} \text{ [by Exercise 48.62]} \\
= & \{\boldsymbol{\varrho} \in \mathbf{P}^v \mid \forall \beta \in \mathbb{V}_{\neq} . \boldsymbol{\varrho}(\beta) = \boldsymbol{\varrho}(\llbracket \beta = \alpha \text{ ? } \vartheta_0(\tau_2) : (\{\langle \alpha, \tau_2 \rangle\} \circ \vartheta_0)(\beta) \rrbracket \} \\
& \text{[def. conditional and function composition } \circ \text{]} \\
= & \{\boldsymbol{\varrho} \in \mathbf{P}^v \mid \forall \beta \in \mathbb{V}_{\neq} . \boldsymbol{\varrho}(\beta) = \boldsymbol{\varrho}(\llbracket \beta = \alpha \text{ ? } (\{\langle \alpha, \tau_2 \rangle\} \circ \vartheta_0)(\alpha) : (\{\langle \alpha, \tau_2 \rangle\} \circ \vartheta_0)(\beta) \rrbracket \} \\
& \text{[since } X \notin \text{dom}(\vartheta_0) \text{ so } (\{\langle \alpha, \tau_2 \rangle\} \circ \vartheta_0)(\alpha) = \{\langle \alpha, \tau_2 \rangle\}(\vartheta_0(\alpha)) = \{\langle \alpha, \tau_2 \rangle\}(\alpha) = \tau_2\text{]} \\
= & \{\boldsymbol{\varrho} \in \mathbf{P}^v \mid \forall \beta \in \mathbb{V}_{\neq} . \boldsymbol{\varrho}(\beta) = \boldsymbol{\varrho}(\{\langle \alpha, \tau_2 \rangle\} \circ \vartheta_0)(\beta) \text{ [def. conditional]} \\
= & \gamma_s^r(\{\langle \alpha, \tau_2 \rangle\} \circ \vartheta_0) \text{ [def. (48.52) of } \gamma_s^r\text{]} \\
= & \gamma_s^r(\text{unify}(\tau_1, \tau_2, \vartheta_0)) \text{ [(48.47.11)]}
\end{aligned}$$

- In case (48.47.12), we have $\tau_1 = \alpha \in \text{dom}(\vartheta_0)$ by tests (48.47.9) and (48.47.10) and $\tau_2 \notin \mathbb{V}_{\neq}$ since test (48.47.1) is ff.

$$\begin{aligned}
& \gamma_e(\tau_1 \doteq \tau_2) \cap \gamma_s^r(\vartheta_0) \\
= & \gamma_e(\alpha \doteq \tau_2) \cap \gamma_s^r(\vartheta_0) \text{ [}\tau_1 = \alpha\text{]} \\
= & \{\boldsymbol{\varrho} \in \mathbf{P}^v \mid \boldsymbol{\varrho}(\alpha) = \boldsymbol{\varrho}(\tau_2) \wedge \forall \beta \in \mathbb{V}_{\neq} . \boldsymbol{\varrho}(\beta) = \boldsymbol{\varrho}(\vartheta_0(\beta))\} \\
& \text{[def. (48.51) of } \gamma_e, \text{ (48.52) of } \gamma_s^r, \text{ and def. } \cap \text{]} \\
= & \{\boldsymbol{\varrho} \in \mathbf{P}^v \mid \boldsymbol{\varrho}(\vartheta_0(\alpha)) = \boldsymbol{\varrho}(\tau_2) \wedge \forall \beta \in \mathbb{V}_{\neq} . \boldsymbol{\varrho}(\beta) = \boldsymbol{\varrho}(\vartheta_0(\beta))\} \\
& \text{[}\alpha \in \text{dom}(\vartheta_0) \text{ so } \boldsymbol{\varrho}(\alpha) = \boldsymbol{\varrho}(\vartheta_0(\beta)) = \boldsymbol{\varrho}(\tau_2)\text{]} \\
= & \gamma_e(\vartheta_0(\alpha) \doteq \tau_2) \cap \gamma_s^r(\vartheta_0) \text{ [def. (48.51) of } \gamma_e, \text{ (48.52) of } \gamma_s^r, \text{ and def. } \cap \text{]} \\
= & \gamma_s^r(\text{unify}(\vartheta_0(\alpha), \tau_2, \vartheta_0)) \text{ [ind. hyp. of Lemma 48.63]} \\
= & \gamma_s^r(\text{unify}(\tau_1, \tau_2, \vartheta_0)) \text{ [(48.47.12)]}
\end{aligned}$$

- In case (48.47.13) we are back to (48.47.11) or (48.47.12) by the symmetry argument of Remark 48.49. \square

The following Lemma 11 shows that new entries are successively added to the table T_0 .

Lemma 11 For all $\tau_1^0, \tau_2^0 \in \mathbf{T}^v$, if $\text{lub}(\tau_1, \tau_2, T_0)$ is (recursively) called from the main call $\text{lcg}(\tau_1^0, \tau_2^0)$ and returns $\langle \tau, T' \rangle = \text{lub}(\tau_1, \tau_2, T_0)$, then

$$\begin{array}{ll}
\text{preinvariant:} & \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbf{T}^\nu \wedge T_0 \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu \\
\text{postinvariant:} & \boldsymbol{\tau} \in \mathbf{T}^\nu \wedge T' \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu \wedge \text{vars}[\![\boldsymbol{\tau}]\!] \subseteq \text{dom}(T') \wedge \\
& \forall \alpha \in \text{dom}(T_0) . T_0(\alpha) = T'(\alpha)
\end{array} \tag{12}$$

Proof of Lemma 11 By induction on the sequence of calls to lub and, for any given call, by recurrence to handle the recursive calls at (48.68.2), ..., (48.68.4), and by case analysis on the conditional.

The first call at (48.68.12) satisfies the preinvariant of (48.39) since $\boldsymbol{\tau}_1^0, \boldsymbol{\tau}_2^0 \in \mathbf{T}^\nu$ by hypothesis and $T_0 = \emptyset \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu$;

Assuming that an intermediate call to $\text{lub}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, T_0)$ satisfies the preinvariant (48.39), the proof that it satisfies the postinvariant (48.39) is by case analysis.

- In case (48.68.5), $\boldsymbol{\tau}_j \in \mathbf{T}^\nu$ by hypothesis on the intermediate call, so $\boldsymbol{\tau}_j^i \in \mathbf{T}^\nu, i = 1, \dots, n, j = 1, 2$, by the test (48.68.1). Then we proceed by recurrence on the recursive calls.
 - For the basis $i = 0$, T_0 satisfies (48.39) by hypothesis on the intermediate call;
 - Assume, by recurrence hypothesis for $i \in [0, n]$, that $T_i \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu \wedge \forall \alpha \in \text{dom}(T_0) . T_0(\alpha) = T_i(\alpha)$. Then, by induction on the sequence of calls to lub, $\boldsymbol{\tau}^{i+1} \in \mathbf{T}^\nu$ and $T_{i+1} \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu \wedge \text{vars}[\![\boldsymbol{\tau}^{i+1}]\!] \subseteq \text{dom}(T_{i+1}) \wedge \forall \alpha \in \text{dom}(T_i) . T_i(\alpha) = T_{i+1}(\alpha)$. By transitivity, $\forall \alpha \in \text{dom}(T_0) . T_0(\alpha) = T_{i+1}(\alpha)$. \square

By recurrence for $i = n$, $T' = T_n$ at (48.68.5) satisfies (48.39) since $\boldsymbol{\tau}^i \in \mathbf{T}^\nu, i = 1, \dots, n$, implies $f(\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n) \in \mathbf{T}^\nu$ and $\text{vars}[\![f(\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n)]\!] = \bigcup_{i=1}^n \text{vars}[\![\boldsymbol{\tau}^i]\!]$;

- The case (48.68.7) is trivial since $\beta \in \mathbf{T}^\nu, T' = T_0$, and $\beta \in \text{dom}(T_0)$;
- In case (48.68.9), $T_0 \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu$ by hypothesis, $\beta \in \mathbf{T}^\nu$, and $\beta \in \mathbb{V}_\ell \setminus \text{dom}(T_0)$ by the test (48.68.8) so $T' = \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle [\beta \leftarrow T_0] \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu$ and for all $\alpha \in \text{dom}(T_0), \alpha \neq \beta$ so $T'(\alpha) = \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle [\beta \leftarrow T_0](\alpha) = T_0(\alpha)$. Moreover $\beta \in \text{vars}[\![\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle [\beta \leftarrow T_0]]\!] = \text{vars}[\![T']\!]$. \square

Remark Lemma 11 shows that T_0 can be declared as a variable local to lcg and global to lub, which would be initialized to \emptyset and updated by an assignment at (48.68.9).

For $T \in \mathbb{V}_\ell \rightarrow \mathbf{T}^\nu \times \mathbf{T}^\nu$, let us define, when $\alpha \in \text{dom}(T)$,

$$\begin{array}{ll}
\bar{\zeta}_1(T)\alpha & \triangleq \text{let } \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle = T(\alpha) \text{ in } \boldsymbol{\tau}_1 \\
\bar{\zeta}_2(T)\alpha & \triangleq \text{let } \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle = T(\alpha) \text{ in } \boldsymbol{\tau}_2
\end{array} \tag{13}$$

(which is undefined when $\alpha \notin \text{dom}(T)$ in which case (48.30) applies, in particular when $T = \emptyset$).

The following Lemma 14 shows that table T_0 maintains two substitutions $\bar{\zeta}_1(T)$ and $\bar{\zeta}_2(T)$ which can be used to instantiate the term resulting from the call to the parameters.

Lemma 14 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2 \in \mathbf{T}^\nu$ and $T_0 \in \wp(\mathbb{V}_{\mathbb{E}} \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, if $\text{lub}(\tau_1, \tau_2, T_0)$ is (recursively) called from the main call $\text{lgc}(\tau_1^0, \tau_2^0)$ and returns $\langle \tau, T' \rangle = \text{lub}(\tau_1, \tau_2, T_0)$, then

$$\bar{\zeta}_1(T')\tau = \tau_1 \quad \text{and} \quad \bar{\zeta}_2(T')\tau = \tau_2 \quad (15) \quad \square$$

Proof of Lemma 14 The preinvariant is \mathbf{tt} . By induction on the sequence of calls to lub and, for any given call, by recurrence to handle the recursive calls at (48.68.2), ..., (48.68.4), and by case analysis for the conditional.

- In case (48.68.5), by recurrence and induction on the sequence of recursive calls to leq , we have $\bar{\zeta}_1(T_i)\tau^i = \tau_1^i$ and $\bar{\zeta}_2(T_i)\tau^i = \tau_2^i$ for all $i \in [1, n]$. By the postinvariant of (48.39), we have $\forall \alpha \in \text{dom}(T_i) . T_0(\alpha) = T_{i+1}(\alpha)$. It follows, by (13) that $\forall \alpha \in \text{vars}[\tau^i] \subseteq \text{dom}(T_i) . T_i(\alpha) = T_{i+1}(\alpha)$. Therefore, by (13), $\forall \alpha \in \text{vars}[\tau^i] . \vartheta_j(T_{i+1})(\tau^i) = \vartheta_j(T_i)(\tau^i)$. It follows by (48.30) that $\vartheta_j(T_n)(f(\tau^1, \tau^2, \dots, \tau^n)) = f(\vartheta_j(T_n)(\tau^1), \vartheta_j(T_n)(\tau^2), \dots, \vartheta_j(T_n)(\tau^n)) = f(\vartheta_j(T_1)(\tau^1), \vartheta_j(T_2)(\tau^2), \dots, \vartheta_j(T_n)(\tau^n)) = f(\tau_1^1, \dots, \tau_1^n) = \tau_j, j = 1, 2$;
- In case (48.68.7), (15) directly follows from $\tau = \beta, T' = T_0, \beta \in \text{dom}(T_0), T_0(\beta) = \langle \tau_1, \tau_2 \rangle$, and (13);
- In case (48.68.9), $\bar{\zeta}_j(T')\tau = \vartheta_j(\langle \tau_1, \tau_2 \rangle[\beta \leftarrow T_0])\beta = \text{if } \beta \in \text{dom}(T) \text{ then let } \langle \tau'_1, \tau'_2 \rangle = \langle \tau_1, \tau_2 \rangle[\beta \leftarrow T_0](\beta) \text{ in } \tau'_j \text{ else } \alpha = \tau_j, j = 1, 2$. \square

$\text{lgc}(\tau_1, \tau_2)$ computes an upper-bound of τ_1 and τ_2 .

Lemma 16 For all $\tau_1, \tau_2 \in \mathbf{T}^\nu$, the lgc algorithm terminates with $[\tau_1]_{\leq^\nu} \leq_{\leq^\nu} [\text{lgc}(\tau_1, \tau_2)]_{\leq^\nu}$ and $[\tau_2]_{\leq^\nu} \leq_{\leq^\nu} [\text{lgc}(\tau_1, \tau_2)]_{\leq^\nu}$. \square

Proof of Lemma 16 The termination proof of $\text{lub}(\tau_1, \tau_2, T_0)$ is by structural induction on τ_1 (or τ_2). So the main call $\text{lub}(\tau_1, \tau_2, \emptyset)$ at (48.68.12) does terminate.

Lemma 16 follows by def. of the infimum $\bar{\mathcal{O}}^\nu$ in cases (48.68.11).

Otherwise, at (48.68.12), $\text{lgc}(\tau_1, \tau_2) = \tau$ where $\langle \tau, T' \rangle = \text{lub}(\tau_1, \tau_2, \emptyset)$. By (48.42), $\bar{\zeta}_j(T')\tau = \tau_j, j = 1, 2$. So by Exercise 48.16, $[\tau_j]_{\leq^\nu} \leq_{\leq^\nu} [\tau]_{\leq^\nu} = [\text{lgc}(\tau_1, \tau_2)]_{\leq^\nu}$. \square

Let $[\tau']_{\leq^\nu}$ be an upper bound of $[\tau_1]_{\leq^\nu}$ and $[\tau_2]_{\leq^\nu}$ i.e. $\tau_1 \leq_{\leq^\nu} \tau'$ and $\tau_2 \leq_{\leq^\nu} \tau'$ so that, by Theorem 48.31, there exists substitutions ϑ_1 and ϑ_2 such that $\vartheta_1(\tau') = \tau_1$ and $\vartheta_2(\tau') = \tau_2$. We must prove that $[\text{lgc}(\tau_1, \tau_2)]_{\leq^\nu} \leq_{\leq^\nu} [\tau']_{\leq^\nu}$ that is, by Theorem 48.31, that there exist a substitution ϑ' such that $\vartheta'(\text{lgc}(\tau_1, \tau_2)) = \tau'$.

We modify the lub algorithm into lub' (which calls lub) as given in Figure 18 to construct this substitution ϑ' given any upper bound τ' .

Example 19 The assumption (17.13) prevents a call like $\text{lub}'(f(a, b), f(b, a), \emptyset, f(\alpha, \alpha), \varepsilon, \emptyset)$ where $f(\alpha, \alpha)$ is not an upper bound of $\{f(a, b), f(b, a)\}$. \square

Example 20 For $\tau_1 = f(g(a), g(g(a)), g(a), b, b)$, $\tau_2 = f(g(b), g(h(b)), g(b), a, a)$ and $\tau' = f(g(\alpha), \beta, g(\alpha), \gamma, U)$, we have

$$\text{lub}'(f(g(a), g(g(a)), g(a), b, b), f(g(b), g(h(b)), g(b), a, a), \emptyset, f(g(\alpha), \beta, g(\alpha), \gamma, U), \varepsilon)$$

$$\text{lub}'(g(a), g(b), \emptyset, g(\alpha), \varepsilon) \quad (17.2b)$$

$$\text{lub}'(a, b, \emptyset, \alpha, \varepsilon) \quad (17.2b)$$

$$= \langle \beta, \{\langle \beta, \langle a, b \rangle\}, \{\langle \alpha, \beta \rangle\} \rangle \quad (17.9)$$

$$= \langle g(\beta), \{\langle \beta, \langle a, b \rangle\}, \{\langle \alpha, \beta \rangle\} \rangle \quad (17.5b)$$

$$\text{lub}'(g(g(a)), g(h(b)), \{\langle \beta, \langle a, b \rangle\}, \beta, \{\langle \alpha, \beta \rangle\}) \quad (17.3b)$$

$$\text{lub}(g(a), h(b), \{\langle \beta, \langle a, b \rangle\}) \quad (17.2a)$$

$$= \langle \gamma, \{\langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\} \rangle$$

$$= \langle g(\gamma), \{\langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle\} \rangle \quad (17.5a)$$

$$\text{lub}'(g(a), g(b), \{\langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, g(\alpha), \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle\}) \quad (17.4b)$$

$$\text{lub}'(a, b, \{\langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \alpha, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle\}) \quad (17.6)$$

$$= \langle \beta, \{\langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle\} \rangle \quad (17.7)$$

$$= \langle g(\beta), \{\langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle\} \rangle \quad (17.5b)$$

$$\text{lub}'(b, a, \{\langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \gamma, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle\}) \quad (17.8)$$

$$= \langle \alpha, \{\langle \alpha, \langle b, a \rangle\}, \langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle, \langle \gamma, \alpha \rangle\} \rangle \quad (17.9)$$

$$\text{lub}'(b, a, \{\langle \alpha, \langle b, a \rangle\}, \langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, U, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle, \langle \gamma, \alpha \rangle\}) \quad (17.8)$$

$$= \langle \alpha, \{\langle \alpha, \langle b, a \rangle\}, \langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle, \langle \gamma, \langle g(a), h(b) \rangle\}, \langle U, \alpha \rangle\} \rangle$$

$$= \langle f(g(\beta), g(\gamma), g(\beta), \alpha, \alpha), \{\langle \alpha, \langle b, a \rangle\}, \langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}, \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle, \langle \gamma, \alpha \rangle, \langle U, \alpha \rangle\} \rangle \quad (17.5b)$$

so that $\tau = f(g(\beta), g(\gamma), g(\beta), \alpha, \alpha)$, $T = \{\langle \alpha, \langle b, a \rangle\}, \langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}$, and $\vartheta' = \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle, \langle \gamma, \alpha \rangle, \langle U, \alpha \rangle\}$. Let us check that

1. $\vartheta'(\tau') = \{\langle \alpha, \beta \rangle, \langle \beta, g(\gamma) \rangle, \langle \gamma, \alpha \rangle, \langle U, \alpha \rangle\}(f(g(\alpha), \beta, g(\alpha), \gamma, U)) = f(g(\beta), g(\gamma), g(\beta), \alpha, \alpha) = \tau$;
2. $\bar{\zeta}_1(T) = \bar{\zeta}_1(\{\langle \alpha, \langle b, a \rangle\}, \langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}) = \{\langle \alpha, b \rangle, \langle \beta, a \rangle, \langle \gamma, g(a) \rangle\}$;
3. $\bar{\zeta}_1(T)(\tau) = \{\langle \alpha, b \rangle, \langle \beta, a \rangle, \langle \gamma, g(a) \rangle\}(f(g(\beta), g(\gamma), g(\beta), \alpha, \alpha)) = f(g(a), g(g(a)), g(a), b, b) = \tau_1$;
4. $\bar{\zeta}_2(T) = \bar{\zeta}_2(\{\langle \alpha, \langle b, a \rangle\}, \langle \beta, \langle a, b \rangle\}, \langle \gamma, \langle g(a), h(b) \rangle\}) = \{\langle \alpha, a \rangle, \langle \beta, b \rangle, \langle \gamma, h(b) \rangle\}$;
5. $\bar{\zeta}_2(T)(\tau) = \{\langle \alpha, a \rangle, \langle \beta, b \rangle, \langle \gamma, h(b) \rangle\}(f(g(\beta), g(\gamma), g(\beta), \alpha, \alpha)) = f(g(b), g(h(b)), g(b), a, a) = \tau_2$. \square

We must show that lub' and lub compute the same result τ .

Lemma 21 For all $\tau_1, \tau_2, \tau, \tau', \tau'' \in \mathbf{T}^\nu$, $T_0, T, T'' \in \wp(\mathbb{V}_\# \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta' \in \mathbb{V}_\# \rightarrow \mathbf{T}^\nu$, if $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ and $\langle \tau'', T'' \rangle = \text{lub}(\tau_1, \tau_2, T_0)$ then $\tau = \tau''$ and $T = T''$. \square

Proof of Lemma 21 Any execution trace of $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ can be abstracted into an execution trace of $\text{lub}(\tau_1, \tau_2, T_0)$ simply by ignoring the input ϑ_0 , the resulting substitution ϑ' , ignoring the program point (17.a) and mapping (17.2a), ..., (17.5a) and (17.2b), ..., (17.5b) to the program point (48.68.2), ..., (48.68.5). The proof is by induction on the calls to lub and lub' which are synchronous in the two traces. The point is that the result $\langle \tau, T \rangle$ of a call $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ does not depend during its computation on the parameters τ' , and ϑ_0 . An exception is the test (17.a) but the two alternative yield the same result. (17.2a), ..., (17.4a) is identical to (48.68.2), ..., (48.68.4) while, by induction on the sequence of calls to lub' (17.2b), ..., (17.4b) is abstracted to that of (48.68.2), ..., (48.68.4). It follows that $\langle \tau, T \rangle$ at (48.68.12) is equal to $\langle \tau, T \rangle$ at (17.14). \square

The following Lemma 22 proves the well-typing of algorithm lub' .

Lemma 22 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau', \tau' \in \mathbf{T}^\nu$, $T_0 \in \wp(\mathbb{V}_\# \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathbb{V}_\# \rightarrow \mathbf{T}^\nu$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau', \varepsilon)$ with hypothesis $\vartheta_1(\tau_1^0) = \tau_1^0 \wedge \vartheta_2(\tau_2^0) = \tau_2^0$, then the case analysis in the definition of lub' is complete (i.e., there is no missing case) and $\exists \gamma \in \mathbb{V}_\# \cdot \tau' = \gamma$ at (17.6) and (17.8). \square

Proof of Lemma 22 Notice that Lemmata 11, 14, and 16 are valid for lub' since they do not involve the extra parameters τ' , ϑ_0 or result ϑ' . The proof is by case analysis.

- For (17.1), the only possible cases for τ' are (17.a) and (17.b), by definition (48.2) of terms with variables.
- For (17.6) and (17.8), the test (17.1) is false so, by the preinvariant of Lemma 11 and def. (48.2) of terms with variables, at least one τ_j , $j = 1, 2$ of τ_1 or τ_2 is a variable. Then τ' must be a variable since otherwise $\tau' = g(\tau'_1, \dots, \tau'_m)$ so that it is impossible that $\vartheta_j(\tau') = \tau_j$ be a variable. \square

The following Lemma 23 shows that variables recorded in T_0 are for non-matching subterms only.

Lemma 23 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2 \in \mathbf{T}^\nu$ and $T_0 \in \wp(\mathbb{V}_\# \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, if $\text{lub}(\tau_1, \tau_2, T_0)$ is (recursively) called from the main call $\text{lub}(\tau_1^0, \tau_2^0, \emptyset)$, then for all $\tau'_1, \tau_1'^1, \dots, \tau_1'^n, \tau'_2, \tau_2'^1, \dots, \tau_2'^n \in \mathbf{T}^\nu$,
if $\exists f \in \mathbf{F}_n \cdot \tau'_1 = f(\tau_1'^1, \dots, \tau_1'^n) \wedge \tau'_2 = f(\tau_2'^1, \dots, \tau_2'^n)$ then $\forall \beta \in \text{dom}(T_0) \cdot T_0(\beta) \neq \langle \tau'_2, \tau'_1 \rangle$. \square

Proof of Lemma 23 Let us prove the contraposition, that is “if $\exists \beta \in \text{dom}(T_0) . T_0(\beta) = \langle \tau'_2, \tau'_1 \rangle$ then $\forall f \in \mathbf{F}_n . \tau'_1 \neq f(\tau_1^1, \dots, \tau_1^n) \vee \tau'_2 \neq f(\tau_2^1, \dots, \tau_2^n)$ ”.

The proof is by induction on the sequence of calls to `lub` and Lemma 23 is obviously true for the initial value of $T_0 = \emptyset$. Then observe that the only modification to the parameter T_0 in calls to `lub` is (48.68.9) for which (48.68.1) is false so that the returned T' is $\langle \tau_1, \tau_2 \rangle[\beta \leftarrow T_0]$ with $\neg(\tau_1 = f(\tau_1^1, \dots, \tau_1^n) \wedge \tau_2 = f(\tau_2^1, \dots, \tau_2^n))$. This property is preserved by the recursive calls (17.2a) to (17.4a) for T_n returned at (17.5a) as well as for the unmodified T_0 returned at (17.7). By induction, Lemma 23 holds for all calls from the main call (17.14). \square

Lemma 24 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau'_0, \tau, \tau' \in \mathbf{T}^v$, $T_0, T \in \mathbb{V}_\epsilon \rightarrow (\mathbf{T}^v \times \mathbf{T}^v)$, and $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta' \in \mathbb{V}_\epsilon \rightarrow \mathbf{T}^v$, if `lub'($\tau_1, \tau_2, T_0, \tau', \vartheta_0$)` is (recursively) called from the main call `lub'($\tau_1^0, \tau_2^0, \emptyset, \tau'_0, \epsilon$)` with hypothesis $\vartheta_1(\tau'_0) = \tau_1^0 \wedge \vartheta_2(\tau'_0) = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then

$$(\exists \beta \in \text{dom}(T_0) . T_0(\beta) = \langle \tau_1, \tau_2 \rangle \wedge \tau' = \gamma) \Rightarrow (\gamma \in \text{dom}(\vartheta_0) \wedge \vartheta_0(\gamma) = \beta) \quad \square$$

Proof of Lemma 24 We prove the stronger property that the following preinvariant and postinvariant do hold for any call $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$.

$$\begin{aligned} \text{preinvariant } & (\exists \beta \in \text{dom}(T_0) . T_0(\beta) = \langle \tau_1, \tau_2 \rangle \wedge \tau' = \gamma) \Rightarrow (\gamma \in \text{dom}(\vartheta_0) \wedge \vartheta_0(\gamma) = \beta) \\ \text{postinvariant } & (\exists \beta \in \text{dom}(T) . T(\beta) = \langle \tau_1, \tau_2 \rangle \wedge \tau' = \gamma) \Rightarrow (\gamma \in \text{dom}(\vartheta') \wedge \vartheta'(\gamma) = \beta) \end{aligned} \quad (25)$$

The proof is by induction on the sequence of calls to `lub'` and, for any given call, by recurrence to handle the recursive calls at (17.2b), (17.3b), ..., (17.4b), and by case analysis for the conditional.

- For the basis, the preinvariant of (25) holds vacuously at the first call (17.14) since $T_0 = \emptyset$;
- For the induction step, we proceed by case analysis.
 - In case (17.5a), there is no recursive call to `lub'` and, by Lemma 23, the premiss of the postinvariant of (25) is **ff** so it does hold vacuously.
 - In case (17.5b), the first recursive call at (17.2a) satisfies the preinvariant because this preinvariant is assumed to hold for the intermediate call at (17).

In case $n = 0$, this is also the postinvariant.

Otherwise $n > 0$. Assume, by recurrence hypothesis, that the preinvariant holds before the call $\langle \tau^i, T_i, \vartheta_i \rangle = \text{lub}'(\tau_1^i, \tau_2^i, T_{i-1}, \tau'_i, \vartheta_{i-1})$. Then, by induction hypothesis on the sequence of calls to `lub'`, the postinvariant (25) holds for T_i and ϑ_i , which is the preinvariant of the next recursive call, if any.

It follows, by recurrence, that the postinvariant of (25) holds at (17.5b) for T_n and ϑ_n .

- In case (17.7), we know by the test (17.6) and Lemma 22 that $\exists \beta \in \text{dom}(T_0) . T_0(\beta) = \langle \tau_1, \tau_2 \rangle \wedge \tau' = \gamma$ so by the preinvariant $\gamma \in \text{dom}(\vartheta_0)$ and $\vartheta_0(\gamma) = \beta$. Since $T = T_0$ and $\vartheta' = \vartheta_0$, we have $\gamma \in \text{dom}(\vartheta') \wedge \vartheta'(\gamma) = \beta$;

- In case (17.9), $\vartheta' = \beta[\gamma \leftarrow \vartheta_0]$, which implies the postinvariant (25). \square

Let us prove the converse of Lemma 24.

Lemma 26 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau_0', \tau', \tau \in \mathbf{T}^\nu$, $T_0, T \in \wp(\mathbb{V}_\# \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta' \in \mathbb{V}_\# \rightarrow \mathbf{T}^\nu$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau_0', \varepsilon)$ with hypothesis $\vartheta_1(\tau_0') = \tau_1^0 \wedge \vartheta_2(\tau_0') = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then

$$\forall \beta, \gamma \in \mathbb{V}_\# . (\gamma \in \text{dom}(\vartheta_0) \wedge \vartheta_0(\gamma) = \beta) \Rightarrow (\beta \in \text{dom}(T_0)). \quad \square$$

Proof of Lemma 26 We prove the stronger property that the following preinvariant and postinvariant do hold for any call $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$.

$$\begin{array}{ll} \text{preinvariant} & \forall \beta, \gamma \in \mathbb{V}_\# . (\gamma \in \text{dom}(\vartheta_0) \wedge \vartheta_0(\gamma) = \beta) \Rightarrow (\beta \in \text{dom}(T_0)) \\ \text{postinvariant} & \forall \beta, \gamma \in \mathbb{V}_\# . (\gamma \in \text{dom}(\vartheta') \wedge \vartheta'(\gamma) = \beta) \Rightarrow (\beta \in \text{dom}(T)) \end{array} \quad (27)$$

The proof is by induction on the sequence of calls to lub' and, for any given call, by recurrence to handle the recursive calls at (17.2b), (17.3b), ..., (17.4b), and by case analysis for the conditional.

- For the basis, $\vartheta_0 = \varepsilon$ so $\text{dom}(\vartheta_0) = \emptyset$ so the preinvariant (27) holds vacuously;
- The induction step is by case analysis.
 - In case (17.5a), there is no recursive call to lub' and $\vartheta' = f(\tau^1, \dots, \tau^n)[\gamma \leftarrow \vartheta_0]$. So if $\alpha \in \text{dom}(\vartheta') \setminus \{\gamma\}$ then the postinvariant follows from the preinvariant. For $\gamma \in \text{dom}(\vartheta')$, we have $\vartheta'(\gamma) = f(\tau^1, \dots, \tau^n) \notin \mathbb{V}_\#$ so that the postcondition holds vacuously;
 - In case (17.5b), the preinvariant of the first recursive call (17.2a) holds by the preinvariant of (27) on the main call (17). Assuming the preinvariant holds for a following recursive call, the postinvariant holds by induction on the sequence of calls to lub' , which is also the preinvariant of the next call. By recurrence the postinvariant of (27) holds for $\vartheta' = \vartheta_n$ and $T = T_n$ after the last call at (17.5b);
 - In case (17.7), we have $\gamma \in \text{dom}(\vartheta') \wedge \vartheta'(\gamma) = \beta$ so the preinvariant (27) on the intermediate call trivially implies the postinvariant;
 - In case (17.9), $T = \langle \tau_1, \tau_2 \rangle[\beta \leftarrow T_0]$ and $\vartheta' = \beta[\gamma \leftarrow \vartheta_0]$. If $\alpha \in \text{dom}(\vartheta') \setminus \{\gamma\}$ and $\vartheta'(\alpha) = \beta'$ then $\alpha \in \text{dom}(\vartheta_0)$ and $\vartheta_0(\alpha) = \beta'$ then, by the preinvariant on the intermediate call, $\beta' \in \text{dom}(T_0) = \text{dom}(T)$. Otherwise, for $\gamma \in \text{dom}(\vartheta')$, we have $\vartheta'(\gamma) = \beta[\gamma \leftarrow \vartheta_0](\gamma) = \beta$ with $\beta \in \text{dom}(\langle \tau_1, \tau_2 \rangle[\beta \leftarrow T_0]) = \text{dom}(T)$. \square

The next Lemma 28 shows how the term variables are used.

Lemma 28 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau'_0, \tau', \tau \in \mathbf{T}^\nu$, $T_0, T \in \wp(\mathcal{V}_\ell \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta_1^0, \vartheta_2^0, \vartheta' \in \mathcal{V}_\ell \rightarrow \mathbf{T}^\nu$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau'_0, \varepsilon)$ with hypothesis $\vartheta_1^0(\tau'_0) = \tau_1^0 \wedge \vartheta_2^0(\tau'_0) = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then

$$\begin{array}{ll} \text{preinvariant} & \text{vars}[\vartheta_0(\mathcal{V}_\ell)] \subseteq \text{dom}(T_0) \\ \text{postinvariant} & \text{vars}[\vartheta'(\mathcal{V}_\ell)] \subseteq \text{dom}(T) \end{array} \quad (29)$$

(where $\vartheta_0(S) = \{\vartheta_0(\alpha) \mid \alpha \in S\}$ and $\text{vars}[S] = \bigcup \{\text{vars}[\tau] \mid \tau \in S\}$.) □

Proof of Lemma 28 The proof is by induction on the sequence of calls to lub' and, for any given call, by recurrence to handle the recursive calls at (17.2b), (17.3b), ..., (17.4b), and by case analysis for the conditional.

- For the first call at (17.14), $\vartheta_0 = \varepsilon$ so $\text{vars}[\vartheta_0(\mathcal{V}_\ell)] = \text{vars}[\emptyset] = \emptyset \subseteq \text{dom}(T_0)$;
- Otherwise the preinvariant of (29) holds for T_0 and ϑ_0 at the first recursive call (17.2b). Assume, by induction hypothesis, that $\text{vars}[\vartheta_{i-1}(\mathcal{V}_\ell)] \subseteq \text{dom}(T_{i-1})$ before the i^{th} call (17.2b), ..., (17.4b), $i \in [1, n]$. By induction hypothesis on the sequence of calls to lub' , we have $\text{vars}[\vartheta_i(\mathcal{V}_\ell)] \subseteq \text{dom}(T_i)$ after that call, which is also the preinvariant of the next call, if any. By recurrence, $\text{vars}[\vartheta'(\mathcal{V}_\ell)] = \text{vars}[\vartheta_n(\mathcal{V}_\ell)] \subseteq \text{dom}(T_n) = \text{dom}(T)$ in case the call (17) to lub' terminates at (17.5b);
- If lub' terminates at (17.5a), there are two cases.
 - $\text{vars}[\vartheta'(\{\gamma\})] = \text{vars}[f(\tau^1, \dots, \tau^n)[\gamma \leftarrow \vartheta_0](\{\gamma\})] = \text{vars}[f(\tau^1, \dots, \tau^n)] = \bigcup_{i=1}^n \text{vars}[\tau^i]$.
By Lemma 11 and 21, we have $\text{vars}[\tau^i] \subseteq \text{dom}(T_i)$, $i = 1, \dots, n$ and $\text{dom}(T_i) \subseteq \text{dom}(T_n)$ so that $\bigcup_{i=1}^n \text{vars}[\tau^i] \subseteq \bigcup_{i=1}^n \text{dom}(T_i) \subseteq \text{dom}(T_n) = \text{dom}(T)$;
 - $\text{vars}[\vartheta'(\mathcal{V}_\ell \setminus \{\gamma\})] = \text{vars}[f(\tau^1, \dots, \tau^n)[\gamma \leftarrow \vartheta_0](\mathcal{V}_\ell \setminus \{\gamma\})] = \text{vars}[\vartheta_0(\mathcal{V}_\ell \setminus \{\gamma\})] \subseteq \text{vars}[\vartheta_0(\mathcal{V}_\ell)]$ which, by the preinvariant (29), is included in $\text{dom}(T_0)$. By Lemma 11 and 21, $\text{dom}(T_{i=1}) \subseteq \text{dom}(T_i)$, $i = 1, \dots, n$ so that, by transitivity, $\text{dom}(T_0) \subseteq \text{dom}(T_n) = \text{dom}(T)$. Therefore $\text{vars}[\vartheta'(\mathcal{V}_\ell \setminus \{\gamma\})] \subseteq \text{dom}(T)$;
 - Since $\vartheta'(\mathcal{V}_\ell) = \vartheta'(\{\gamma\}) \cup \vartheta'(\mathcal{V}_\ell \setminus \{\gamma\})$, we conclude that $\text{vars}[\vartheta'(\mathcal{V}_\ell)] = \text{vars}[\vartheta'(\{\gamma\})] \cup \text{vars}[\vartheta'(\mathcal{V}_\ell \setminus \{\gamma\})] = \text{vars}[\vartheta'(\{\gamma\})] \cup \text{vars}[\vartheta'(\mathcal{V}_\ell \setminus \{\gamma\})] \subseteq \text{dom}(\vartheta') \cup \text{dom}(\vartheta') = \text{dom}(\vartheta')$; □
- If lub' terminates at (17.7) then the postinvariant directly follows from the preinvariant of (29) since $T = T_0$ and $\vartheta' = \vartheta_0$;
- Finally, if lub' terminates at (17.9), there are two subcases.
 - We have $\text{vars}[\vartheta'(\{\gamma\})] = \text{vars}[\beta[\gamma \leftarrow \vartheta_0](\{\gamma\})] = \text{vars}[\{\beta\}] = \{\beta\} \subseteq \text{dom}(\langle \tau_1, \tau_2 \rangle[\beta \leftarrow T_0]) = \text{dom}(T)$;

- Moreover $\text{vars}[\llbracket \vartheta'(\mathbb{V}_{\mathbb{E}} \setminus \{\gamma\}) \rrbracket] = \text{vars}[\llbracket \beta[\gamma \leftarrow \vartheta_0(\mathbb{V}_{\mathbb{E}} \setminus \{\gamma\})] \rrbracket] = \text{vars}[\llbracket \vartheta_0(\mathbb{V}_{\mathbb{E}} \setminus \{\gamma\}) \rrbracket] \subseteq \text{vars}[\llbracket \vartheta_0(\mathbb{V}_{\mathbb{E}}) \rrbracket] \subseteq \text{dom}(T_0)$, by the preinvariant of (29). But $\text{dom}(T_0) \subseteq \text{dom}(T_0) \cup \{\beta\} = \text{dom}(\langle \tau_1, \tau_2 \rangle [\beta \leftarrow T_0]) = \text{dom}(T)$, proving the postinvariant of vars -codom-substitution0 by transitivity;
- We conclude since vars preserves joins. \square

The following series of lemmata aims at proving that the substitution built by lub' is the one allowing us to prove that lub returns the least common generalization.

Lemma 30 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau_0', \tau' \in \mathbf{T}^\nu$, $T_0, T \in \wp(\mathbb{V}_{\mathbb{E}} \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta_1^0, \vartheta_2^0, \vartheta' \in \mathbb{V}_{\mathbb{E}} \rightarrow \mathbf{T}^\nu$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau_0', \varepsilon)$ with hypothesis $\vartheta_1^0(\tau_0') = \tau_1^0 \wedge \vartheta_2^0(\tau_0') = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then

$$\vartheta_1^0(\tau') = \tau_1 \wedge \vartheta_2^0(\tau') = \tau_2. \quad (31) \quad \square$$

Proof of Lemma 30 For the first call at (17.14), (31) holds by the hypothesis $\vartheta_1^0(\tau_0') = \tau_1^0 \wedge \vartheta_2^0(\tau_0') = \tau_2^0$ on the actual parameters. Assume that $\vartheta_j^0(\tau') = \tau_j$, $j = 1, 2$ before an intermediate call (17). Then (31) holds before the recursive calls (17.2b), ..., (17.4b) since the induction hypothesis $\vartheta_j^0(\tau') = \tau_j$, $\tau' = f(\tau_1', \dots, \tau_n')$ by the test (17.a) which is false, $\tau_j = f(\tau_j^1, \dots, \tau_j^n)$ by the test (17.1) which is true, and (48.30) imply that $\vartheta_j^0(\tau') = \vartheta_j^0(f(\tau_1', \dots, \tau_n')) = f(\vartheta_j^0(\tau_1'), \dots, \vartheta_j^0(\tau_n')) = f(\tau_j^1, \dots, \tau_j^n) = \tau_j$ and therefore $\vartheta_j^0(\tau_i') = \tau_j^i$, $j = 1, \dots, n$. We conclude by induction on the sequence of calls to lub' . \square

Lemma 32 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau_0', \tau', \tau \in \mathbf{T}^\nu$, $T_0, T \in \wp(\mathbb{V}_{\mathbb{E}} \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta_1^0, \vartheta_2^0, \vartheta' \in \mathbb{V}_{\mathbb{E}} \rightarrow \mathbf{T}^\nu$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau_0', \varepsilon)$ with hypothesis $\vartheta_1^0(\tau_0') = \tau_1^0 \wedge \vartheta_2^0(\tau_0') = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then

$$\begin{aligned} \text{preinvariant} \quad & \forall j = 1, 2. \forall \alpha \in \text{dom}(\vartheta_0). \vartheta_j^0(\alpha) = \bar{\zeta}_j(T_0)(\vartheta_0(\alpha)) \\ \text{postinvariant} \quad & \forall j = 1, 2. \forall \alpha \in \text{dom}(\vartheta'). \vartheta_j^0(\alpha) = \bar{\zeta}_j(T)(\vartheta'(\alpha)) \wedge \bar{\zeta}_j(T)(\tau) = \tau_j \end{aligned} \quad (33) \quad \square$$

Proof of Lemma 32 Notice again that Lemma 11, 14, and 16 are valid for lub' since they do not involve the extra parameters τ' , ϑ_0 , or result ϑ' . It follows, by Lemma 14, that the postinvariant of (33) satisfies $\bar{\zeta}_j(T)(\tau) = \tau_j$, $j = 1, 2$. The proof of (33) is by induction on the sequence of calls to lub' and, for any given call, by recurrence to handle the recursive calls at (17.2b), (17.3b), ..., (17.4b), and by case analysis for the conditional.

- For the basis, the preinvariant (33) holds vacuously for the main call (17.14) since $\vartheta_0 = \varepsilon$ so $\text{dom}(\vartheta_0) = \emptyset$;
- Assume that the preinvariant (33) holds before any intermediate call (17) of lub' . We must show that it holds before all recursive calls (17.2b), ..., (17.4b).

By hypothesis on the intermediate call, we have $\forall j = 1, 2 . \forall \alpha \in \text{dom}(\vartheta') . \vartheta_j^0(\alpha) = \bar{\zeta}_j(T_0)(\vartheta'(\alpha))$ at the first recursive call (17.2b).

Assume that $\forall j = 1, 2 . \forall \alpha \in \text{dom}(\vartheta_{i-1}) . \vartheta_j^0(\alpha) = \bar{\zeta}_j(T_{i-1})(\vartheta_{i-1}(\alpha))$ before the i^{th} recursive call. By induction on the sequence of calls to lub' , the postinvariant of (33) holds. Therefore we have $\forall j = 1, 2 . \forall \alpha \in \text{dom}(\vartheta_i) . \vartheta_j^0(\alpha) = \bar{\zeta}_j(T_i)(\vartheta_i(\alpha))$ before the $i + 1^{\text{th}}$ call. By recurrence, all recursive calls do satisfy (33).

We must also show that the intermediate call satisfies the postinvariant of (33). We proceed by cases.

- In case (17.5b), we have $T = T_n$ and ϑ_n which satisfy the postinvariant of (33), as shown above.
- In case (17.5a), the postinvariant is $\forall j = 1, 2 . \forall \alpha \in \text{dom}(f(\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n)[\gamma \leftarrow \vartheta_0]) . \vartheta_j^0(\alpha) = \bar{\zeta}_j(T_n)(f(\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n)[\gamma \leftarrow \vartheta_0](\alpha))$.
 - If $\alpha \in \text{dom}(\vartheta_0) \setminus \{\gamma\}$, we must show that $\vartheta_j^0(\alpha) = \bar{\zeta}_j(T_n)(\vartheta_0(\alpha))$.
By Lemma 11, $\forall \alpha \in \text{dom}(T_{i-1}) . T_{i-1}(\alpha) = T_i(\alpha)$, $i = 1, \dots, n$ so that, by transitivity, $\forall \alpha \in \text{dom}(T_0) . T_0(\alpha) = T_n(\alpha)$. Therefore, by (13), for all $\beta \in \text{dom}(T_0)$, $\bar{\zeta}_j(T_0)\beta \triangleq \text{let } \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle = T_0(\beta) \text{ in } \boldsymbol{\tau}_j = \text{let } \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle = T_n(\beta) \text{ in } \boldsymbol{\tau}_j = \bar{\zeta}_j(T_n)\beta$. By Lemma 28, $\text{vars}[\![\vartheta_0(\mathbb{V}_\ell)]\!] \subseteq \text{dom}(T_0)$ so, in particular, $\forall \alpha \in \text{dom}(\vartheta_0) \setminus \{\gamma\} . \text{vars}[\![\vartheta_0(\alpha)]\!] \subseteq \text{dom}(T_0)$. This implies that $\forall \alpha \in \text{dom}(\vartheta_0) \setminus \{\gamma\} . \forall \beta \in \text{vars}[\![\vartheta_0(\alpha)]\!] . \bar{\zeta}_j(T_0)\beta = \bar{\zeta}_j(T_n)\beta$. By (48.30) and (48.30), we infer that $\forall \alpha \in \text{dom}(\vartheta_0) \setminus \{\gamma\} . \bar{\zeta}_j(T_0)\mathbb{B}_0(\mathbb{B}) = \bar{\zeta}_j(T_n)\mathbb{B}_0(\mathbb{B})$. By the preinvariant of (33), we have $\forall \alpha \in \text{dom}(\vartheta_0) . \vartheta_j^0(\alpha) = \bar{\zeta}_j(T_0)(\vartheta_0(\alpha))$. Therefore, by transitivity, $\vartheta_j^0(\alpha) = \bar{\zeta}_j(T_n)(\vartheta_0(\alpha))$.
 - Otherwise $\alpha = \gamma$, in which case we must show that $\vartheta_j^0(\gamma) = \bar{\zeta}_j(T_n)(f(\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n))$. By Lemma 30, (48.42) of Lemma 48.40, and (17.5a), we have $\vartheta_j^0(\gamma) = \vartheta_j^0(\boldsymbol{\tau}') = \boldsymbol{\tau}_j = \bar{\zeta}_j(T)(\boldsymbol{\tau}) = \bar{\zeta}_j(T)(f(\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n))$.
- In case (17.7), the postinvariant of (31) immediately follows from the preinvariant since $T = T_0$ and $\vartheta' = \vartheta_0$;
- In case (17.9), we must show that $\forall j = 1, 2 . \forall \alpha \in \text{dom}(\beta[\gamma \leftarrow \vartheta_0]) . \vartheta_j^0(\alpha) = \bar{\zeta}_j(\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle[\beta \leftarrow T_0])(\beta[\gamma \leftarrow \vartheta_0](\alpha))$. There are two cases.
 - If $\alpha = \gamma$ then we must prove that $\vartheta_j^0(\gamma) = \bar{\zeta}_j(\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle[\beta \leftarrow T_0])(\beta)$, that is, by (13), $\vartheta_j^0(\gamma) = \boldsymbol{\tau}_j$. It is not possible that $\gamma \in \text{dom}(\vartheta_0)$ since otherwise, we would have $\forall \beta \in \text{dom}(T_0) . T_0(\beta) \neq \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$ since the test (17.6) is ff and $\boldsymbol{\tau}' = \gamma \in \mathbb{V}_\ell$ by Lemma 22, which is in contradiction with (the contrapositive of) Lemma 26. Therefore $\vartheta_0(\gamma) = \gamma$ by (48.30). It follows that we have to prove that $\vartheta_j^0(\vartheta_0(\gamma)) = \boldsymbol{\tau}_j$, which directly follows from the preinvariant of (31);
 - Otherwise, $\alpha \in \text{dom}(\vartheta_0) \setminus \{\gamma\}$ and we must show that $\vartheta_j^0(\alpha) = \bar{\zeta}_j(\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle[\beta \leftarrow T_0])(\vartheta_0(\alpha))$. The test (17.8) implies $\beta \notin \text{dom}(T_0)$ and so $\beta \notin \text{vars}[\![\vartheta_0(\alpha)]\!]$ since $\text{vars}[\![\vartheta_0(\mathbb{V}_\ell)]\!] \subseteq \text{dom}(T_0)$ by (29) of Lemma 28. Therefore, by (13), $\forall \gamma \in \text{vars}[\![\vartheta_0(\alpha)]\!] . \bar{\zeta}_j(T_0)(\gamma) =$

$\bar{\zeta}_j(\langle \tau_1, \tau_2 \rangle [\beta \leftarrow T_0])(\gamma)$. It follows, by (48.30) and (48.30), that $\bar{\zeta}_j(T_0)(\vartheta_0(\alpha)) = \bar{\zeta}_j(\langle \tau_1, \tau_2 \rangle [\beta \leftarrow T_0])(\vartheta_0(\alpha))$. We conclude, by the preinvariant (31) and transitivity that $\bar{\zeta}_j(\langle \tau_1, \tau_2 \rangle [\beta \leftarrow T_0])(\vartheta_0(\alpha)) = \vartheta_j^0(\alpha)$. \square \square

Lemma 34 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau'_0, \tau', \tau \in \mathbf{T}^\nu$, $T_0, T \in \wp(\mathbb{V}_\# \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta' \in \mathbb{V}_\# \rightarrow \mathbf{T}^\nu$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau'_0, \varepsilon)$ with hypothesis $\vartheta_1(\tau'_0) = \tau_1^0 \wedge \vartheta_2(\tau'_0) = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then the following postinvariant holds after the call.

$$\text{dom}(\vartheta') = \text{dom}(\vartheta_0) \cup \text{vars}[\tau'] \quad (35) \quad \square$$

Proof of Lemma 34 The proof of (35) is by induction on the sequence of calls to lub' and, for any given call, by recurrence to handle the recursive calls at (17.2b), (17.3b), ..., (17.4b), and by case analysis for the conditional.

Consider any intermediate call $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau'_0, \varepsilon)$. We proceed by case analysis of the returned values $\langle \tau, T, \vartheta' \rangle$.

- In case (17.5a), we have $\text{dom}(\vartheta') = \text{dom}(f(\tau^1, \dots, \tau^n)[\gamma \leftarrow \vartheta_0]) = \text{dom}(\vartheta_0) \cup \{\gamma\} = \text{dom}(\vartheta_0) \cup \text{vars}[\tau']$ since $\vartheta' = \gamma$ by the test (17.a);
- In case (17.5b), we have $\text{dom}(\vartheta_i) = \text{dom}(\vartheta_{i-1}) \cup \text{vars}[\tau^i]$, $i = 1, \dots, n$, by induction hypothesis on the sequence of calls to lub' . It follows that $\text{dom}(\vartheta') = \text{dom}(\vartheta_n) = \text{dom}(\vartheta_0) \cup \bigcup_{i=1}^n \text{vars}[\tau^i] = \text{dom}(\vartheta_0) \cup \text{vars}[f(\tau^1, \dots, \tau^n)] = \text{dom}(\vartheta_0) \cup \text{vars}[\tau']$;
- In case (17.7), we have $\vartheta' = \beta[\gamma \leftarrow \vartheta_0]$ so $\text{dom}(\vartheta') = \text{dom}(\vartheta_0) \cup \{\gamma\} = \text{dom}(\vartheta_0) \cup \text{vars}[\tau']$ since $\tau' = \gamma$ by Lemma 22;
- Finally, in case (17.9), $\text{dom}(\vartheta') = \text{dom}(\beta[\gamma \leftarrow \vartheta_0]) = \text{dom}(\vartheta_0) \cup \{\gamma\} = \text{dom}(\vartheta_0) \cup \text{vars}[\tau']$ since $\tau' = \gamma$ by Lemma 22. \square \square

Lemma 36 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau'^0, \tau^{n-1}, \tau^n, \tau^{m-1}, \tau^m \in \mathbf{T}^\nu$, $T_n, T_m \in \wp(\mathbb{V}_\# \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, consider any computation trace for the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau'^0, \varepsilon, \emptyset)$ at (17.14) with hypothesis $\vartheta_1(\tau'^0) = \tau_1^0 \wedge \vartheta_2(\tau'^0) = \tau_2^0$. Assume that in this computation trace, a call $\langle \tau^k, T_k \rangle = \text{lub}(\tau_1, \tau_2, T_{k-1})$ is followed by a later call $\langle \tau^m, T_m \rangle = \text{lub}(\tau_1, \tau_2, T_{m-1})$ with the same parameters τ_1 and τ_2 . Then $\tau^k = \tau^m$.

By Lemma 21, this also holds for calls to lub' independently of the other two parameters. \square

Proof of Lemma 36 By (12) in Lemma 11, Lemma 21, (17.2a), ..., (17.4a), and (17.2b), ..., (17.4b) and recurrence, the successive calls of lub and lub' in the trace have parameters T_i and result T_{i+1}

with increasing domains and preservation of the previous values so that $\forall \alpha \in \text{dom}(T_k) . T_k(\alpha) = T_m(\alpha)$.

To prove that $\tau^k = \tau^m$, we consider the calls $\langle \tau^k, T_k \rangle = \text{lub}(\tau_1, \tau_2, T_{k-1})$ and the later $\langle \tau^m, T_m \rangle = \text{lub}(\tau_1, \tau_2, T_{m-1})$ to lub (by Lemma 21, the reasoning is the same for lub'). The only possible executions are the following.

- If one execution follows the true branch of (48.68.1), so does the other since they have the same parameters. By recurrence and induction on the sequence of calls for (48.68.2), ..., (48.68.4) with $\forall \alpha \in \text{dom}(T_{i-1}) . T_{i-1}(\alpha) = T_i(\alpha)$, $i = 1, \dots, n$, we have $\tau^k = f(\tau^{1^k}, \dots, \tau^{n^k}) = f(\tau^{1^m}, \dots, \tau^{n^m}) = \tau^m$;
- If both calls go through (48.68.7) then obviously $\tau^k = \tau^m = \beta$;
- Both calls cannot go through (48.68.9) since the first ones (which is $\langle \tau^k, T_k \rangle = \text{lub}(\tau_1, \tau_2, T_{k-1})$) that goes through (48.68.9) will add β to the $\text{dom}(T_k) \subseteq \text{dom}(T_{m-1})$;
- If $\langle \tau^k, T_k \rangle = \text{lub}(\tau_1, \tau_2, T_{k-1})$ goes through (48.68.9) then the call $\langle \tau^m, T_m \rangle = \text{lub}(\tau_1, \tau_2, T_{m-1})$ must go through (48.68.7) since $\text{dom}(T_k) \subseteq \text{dom}(T_{m-1})$ with $\beta \in \text{dom}(T_{m-1})$ so that $\tau^k = \tau^m = \beta$. \square

Lemma 37 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau_0', \tau', \tau \in \mathbf{T}^v$, $T_0, T \in \wp(\mathbb{V}_{\#} \times \mathbf{T}^v \times \mathbf{T}^v)$, and $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta' \in \mathbb{V}_{\#} \rightarrow \mathbf{T}^v$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau_0', \varepsilon)$ with hypothesis $\vartheta_1(\tau_0') = \tau_1^0 \wedge \vartheta_2(\tau_0') = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then the following postinvariant holds after the call.

$$\forall \alpha \in \text{dom}(\vartheta_0) . \vartheta_0(\alpha) = \vartheta'(\alpha) \quad (38) \quad \square$$

Proof of Lemma 37 The proof of (35) is by induction on the sequence of calls to lub' and, for any given call, by recurrence to handle the recursive calls at (17.2b), (17.3b), ..., (17.4b), and by case analysis for the conditional.

Consider any intermediate call $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau_0', \varepsilon)$. We proceed by case analysis of the returned values $\langle \tau, T, \vartheta' \rangle$.

- In case (17.5a), we have $\forall \alpha \in \text{dom}(\vartheta_0) \setminus \{\gamma\} . \vartheta_0(\alpha) = f(\tau^1, \dots, \tau^n)[\gamma \leftarrow \vartheta_0](\alpha) = \vartheta'(\alpha)$.

It may also be that $\gamma \in \text{dom}(\vartheta_0)$. Since the main call starts with ε and by (35) the domain of ϑ_0 grows along the calls, there must be a previous call that added γ to $\text{dom}(\vartheta_0)$. At that previous call, say $\text{lub}'(\tau_1^k, \tau_2^k, T_0^k, \tau'^k, \vartheta_0^k)$, we had $\tau'^k = \gamma$ since (17.5a) and (17.9) are the two only cases where the domain of ϑ_0^k is extending with γ . By the initial hypothesis and (31) of Lemma 30, $\vartheta_j^0(\tau'^k) = \vartheta_j^0(\gamma) = \tau_j^k$. At the current call $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ where $\tau_0' = \gamma$, we also have, by the initial hypothesis and (31) of Lemma 30, that $\vartheta_j^0(\tau') = \vartheta_j^0(\gamma) = \tau_j$. By transitivity $\tau_j^k = \tau_j$. So the current and previous calls had the same first two parameters. It follows, by Lemma 36, that they have the same results. This implies that necessarily, $\vartheta_0(\gamma) = f(\tau^1, \dots, \tau^n)$.

- In case (17.5b), we have $\forall \alpha \in \text{dom}(\vartheta_{i-1}) . \vartheta_{i-1}(\alpha) = \vartheta_i(\alpha), i = 1, \dots, n$, by induction hypothesis on the sequence of calls to lub' . It follows, by transitivity, that $\forall \alpha \in \text{dom}(\vartheta_0) . \vartheta_0(\alpha) = \vartheta_n(\alpha) = \vartheta'(\alpha)$;
- In case (17.7), for all $\alpha \in \text{dom}(\vartheta_0) \setminus \{\gamma\}$, we have $\vartheta_0(\alpha) = \beta[\gamma \leftarrow \vartheta_0](\alpha) = \vartheta'(\alpha)$. We may also have $\gamma \in \text{dom}(\vartheta_0)$, in which case the test (17.6), Lemma 22, and Lemma 24 imply that $\vartheta_0(\gamma) = \beta$ so $\vartheta_0(\gamma) = \beta = \beta[\gamma \leftarrow \vartheta_0](\gamma) = \vartheta'(\gamma)$;
- Finally, in case (17.9), it is not possible that $\gamma \in \text{dom}(\vartheta_0)$ since otherwise, we would have $\forall \beta \in \text{dom}(T_0) . T_0(\beta) \neq \langle \tau_1, \tau_2 \rangle$ since the test (17.6) is **ff** and $\tau' = \gamma \in \mathbb{V}_{\sharp}$ by Lemma 22, which is in contradiction with (the contrapositive of) Lemma 26. It follows that $\forall \alpha \in \text{dom}(\vartheta_0) . \vartheta_0(\alpha) = \beta[\gamma \leftarrow \vartheta_0](\alpha) = \vartheta'(\alpha)$ since $\alpha \neq \gamma$. \square \square

Lemma 39 For all $\tau_1^0, \tau_2^0, \tau_1, \tau_2, \tau_0', \tau', \tau \in \mathbf{T}^\nu$, $T_0, T \in \wp(\mathbb{V}_{\sharp} \times \mathbf{T}^\nu \times \mathbf{T}^\nu)$, and $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta' \in \mathbb{V}_{\sharp} \rightarrow \mathbf{T}^\nu$, if $\text{lub}'(\tau_1, \tau_2, T_0, \tau', \vartheta_0)$ is (recursively) called from the main call $\text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau_0', \varepsilon)$ with hypothesis $\vartheta_1(\tau_1^0) = \tau_1^0 \wedge \vartheta_2(\tau_2^0) = \tau_2^0$ and returns $\langle \tau, T, \vartheta' \rangle$, then the following postinvariant holds after the call.

$$\vartheta'(\tau') = \tau \quad (40) \quad \square$$

Proof of Lemma 39 The proof of (40) is by induction on the sequence of calls to lub' and, for any given call, by recurrence to handle the recursive calls at (17.2b), (17.3b), ..., (17.4b), and by case analysis for the conditional.

Consider any intermediate call $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1^0, \tau_2^0, \emptyset, \tau_0', \varepsilon)$. We proceed by case analysis of the returned values $\langle \tau, T, \vartheta' \rangle$.

- In case (17.5a), we have $\vartheta'(\tau') = f(\tau^1, \dots, \tau^n)[\gamma \leftarrow \vartheta_0](\gamma) = f(\tau^1, \dots, \tau^n) = \tau$;
- In case (17.5b), we handle (17.2b), ..., (17.4b) by recurrence.
 - For the basis at (17.2b), we have $\text{dom}(\vartheta_1) = \text{dom}(\vartheta_0) \cup \text{vars}[\tau_1']$ by (35) of Lemma 34, and $\vartheta_1(\tau_1') = \tau_1^1$, by induction on the sequence of calls to lub' ;
 - Assume, by recurrence hypothesis, that for the i^{th} call (17.2b), ..., (17.4b), $i \in [1, n]$, we have

$$\begin{aligned} \text{dom}(\vartheta_i) &= \text{dom}(\vartheta_0) \cup \bigcup_{j=1}^i \text{vars}[\tau_j'] \wedge \\ \forall j \in [1, i] . \forall \alpha \in \text{dom}(\vartheta_j) . \vartheta_i(\alpha) &= \vartheta_j(\alpha) \wedge \\ \forall j \in [1, i] . \vartheta_i(\tau_j') &= \vartheta_j(\tau_j') = \tau^j \end{aligned} \quad (41)$$

- At the next $i + 1^{\text{th}}$ call, we have
 1. By (35) of Lemma 34 and recurrence hyp. (41), $\text{dom}(\vartheta_{i+1}) = \text{dom}(\vartheta_i) \cup \text{vars}[\tau_{i+1}'] = \text{dom}(\vartheta_0) \cup \bigcup_{j=1}^i \text{vars}[\tau_j'] \cup \text{vars}[\tau_{i+1}'] = \text{dom}(\vartheta_0) \cup \bigcup_{j=1}^{i+1} \text{vars}[\tau_j']$;

2. By (38) of Lemma 37, we have $\forall \alpha \in \text{dom}(\vartheta_i) . \vartheta_i(\alpha) = \vartheta_{i+1}(\alpha)$ so that by recurrence hyp. (41), $\forall j \in [1, i+1] . \forall \alpha \in \text{dom}(\vartheta_j) . \vartheta_{i+1}(\alpha) = \vartheta_i(\alpha) = \vartheta_j(\alpha)$
3. By (1), $\forall j \in [1, i+1] . \text{vars}[\tau'_j] \subseteq \text{dom}(\vartheta_j) \subseteq \text{dom}(\vartheta_{i+1})$ and by (2), $\forall \alpha \in \text{dom}(\vartheta_j) . \vartheta_{i+1}(\alpha) = \vartheta_j(\alpha)$ so that, by (48.30) and (48.30), $\forall j \in [1, i] . \vartheta_{i+1}(\tau'_j) = \vartheta_i(\tau'_j) = \vartheta_j(\tau'_j) = \tau^j$. Moreover, $\vartheta_{i+1}(\tau'_{i+1}) = \tau^{i+1}$, by induction on the sequence of calls to lub' . Grouping all cases $j \in [1, i]$ and $j = i+1$ together, we have $\forall j \in [1, i+1] . \vartheta_{i+1}(\tau'_j) = \vartheta_j(\tau'_j) = \tau^j$. \square

By recurrence, (41) holds for $i = n$. Therefore $\vartheta'(\tau') = \vartheta_n(f(\tau'_1, \dots, \tau'_n)) = f(\vartheta_n(\tau'_1), \dots, \vartheta_n(\tau'_n)) = f(\tau^1, \dots, \tau^n) = \tau$.

- In case (17.7), we have $\exists \beta \in \text{dom}(T_0) . T_0(\beta) = \langle \tau_1, \tau_2 \rangle \wedge \tau' = \gamma$ so that by Lemma 24, we have $\gamma \in \text{dom}(\vartheta_0) \wedge \vartheta_0(\gamma) = \beta$. It follows that $\vartheta'(\tau') = \vartheta_0(\gamma) = \beta = \tau$.
- Finally, in case (17.9), by (17.9) and Lemma 22, we have $\vartheta'(\tau') = \beta[\gamma \leftarrow \vartheta_0](\gamma) = \beta = \tau$. \square

Proof of Theorem 48.72 By Lemma 16, $[\text{lgc}(\tau_1, \tau_2)]_{\leq^v}$ is a \leq^v -upper-bound of $[\tau_1]_{\leq^v}$ and $[\tau_2]_{\leq^v}$. By Lemma 21, so is $[\text{lgc}'(\tau_1, \tau_2)]_{\leq^v}$.

Now if $[\tau']_{\leq^v}$ is any \leq^v -upper-bound of $[\tau_1]_{\leq^v}$ and $[\tau_2]_{\leq^v}$ then by Exercise 48.16, $\exists \vartheta_1, \vartheta_2 . \vartheta_1(\tau') = \tau_1 \wedge \vartheta_2(\tau') = \tau_2$, which is the precondition (17.13). It follows that the call to $\text{lub}'(\tau_1, \tau_2, \emptyset, \tau', \varepsilon, \emptyset)$ terminates (by Lemma 16 and 21) and returns $\langle \text{lgc}'(\tau_1, \tau_2), T, \vartheta' \rangle$ such that $\vartheta'(\tau') = \text{lgc}'(\tau_1, \tau_2)$ (by (40) of Lemma 39). By Exercise 48.16, this means that $\text{lgc}'(\tau_1, \tau_2) \leq^v [\tau']_{\leq^v}$. This proves by Lemma 21 that $\text{lgc}(\tau_1, \tau_2)$ is the \leq^v -least upper-bound of $[\tau_1]_{\leq^v}$ and $[\tau_2]_{\leq^v}$. \square

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let rec lub'( $\tau_1, \tau_2, T_0, \tau', \vartheta_0$ ) = (17)
  if  $\tau_1 = f(\tau_1^1, \dots, \tau_1^n) \wedge \tau_2 = f(\tau_2^1, \dots, \tau_2^n)$  then (1)
    if  $\tau' = \gamma \in \mathbb{V}_{\mathbb{E}}$  then (a)
      let  $\langle \tau^1, T_1 \rangle = \text{lub}(\tau_1^1, \tau_2^1, T_0)$  in (2a)
      let  $\langle \tau^2, T_2 \rangle = \text{lub}(\tau_1^2, \tau_2^2, T_1)$  in (3a)
      ...
      let  $\langle \tau^n, T_n \rangle = \text{lub}(\tau_1^n, \tau_2^n, T_{n-1})$  in (4a)
       $\langle f(\tau^1, \dots, \tau^n), T_n, f(\tau^1, \dots, \tau^n)[\gamma \leftarrow \vartheta_0] \rangle$  (5a)
    else /*  $\tau' = f(\tau_1', \dots, \tau_n')$  */ (b)
      let  $\langle \tau^1, T_1, \vartheta_1 \rangle = \text{lub}'(\tau_1^1, \tau_2^1, T_0, \tau_1', \vartheta_0)$  in (2b)
      let  $\langle \tau^2, T_2, \vartheta_2 \rangle = \text{lub}'(\tau_1^2, \tau_2^2, T_1, \tau_2', \vartheta_1)$  in (3b)
      ...
      let  $\langle \tau^n, T_n, \vartheta_n \rangle = \text{lub}'(\tau_1^n, \tau_2^n, T_{n-1}, \tau_n', \vartheta_{n-1})$  in (4b)
       $\langle f(\tau^1, \dots, \tau^n), T_n, \vartheta_n \rangle$  (5b)
  elseif  $\exists \beta \in \text{dom}(T_0) . T_0(\beta) = \langle \tau_1, \tau_2 \rangle$  then /*  $\tau' = \gamma \in \mathbb{V}_{\mathbb{E}}$  */ (6)
     $\langle \beta, T_0, \vartheta_0 \rangle$  (7)
  else let  $\beta \in \mathbb{V}_{\mathbb{E}} \setminus \text{dom}(T_0)$  in /*  $\tau' = \gamma \in \mathbb{V}_{\mathbb{E}}$  */ (8)
     $\langle \beta, \langle \tau_1, \tau_2 \rangle[\beta \leftarrow T_0], \beta[\gamma \leftarrow \vartheta_0] \rangle$  (9)
let lcg'( $\tau_1, \tau_2$ ) = (10)
  if  $\tau_1 = \overline{\emptyset}^v$  then  $\tau_2$  (11)
  elseif  $\tau_2 = \overline{\emptyset}^v$  then  $\tau_1$  (12)
  else /* assume  $\exists \vartheta_1, \vartheta_2 . \vartheta_1(\tau') = \tau_1 \wedge \vartheta_2(\tau') = \tau_2$  */ (13)
    let  $\langle \tau, T, \vartheta' \rangle = \text{lub}'(\tau_1, \tau_2, \emptyset, \tau', \varepsilon, \emptyset)$  in  $\tau$  /*  $\vartheta'(\tau') = \tau^*$  */ (14)

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Figure 18: The modified least upper bound algorithm