Order of Common Operations

Operations on numbers are a fundamental process in mathematics. These include, but are not limited to, addition (+), subtraction (-), multiplication ($\times$), and division ($\div$).

While simple arithmetic involving just one of these operations is simple enough to carry out (for example, 3 x 3), problems usually arise as these are compounded with one another. The expression 6 ÷ 2(2+1), for example, has stumped many netizens, as it can be interpreted as giving either 9 or 1. To resolve such ambiguities, the **order of operations** is taken into consideration, which gives priority to some operations over another.

Most commonly, this order is referred to by the acronym **GEMDAS**, or sometimes PEMDAS or even BODMAS. GEMDAS highlights that priority between operations goes as groups (parentheses, brackets, curly brackets), followed by exponents (as well as roots), followed by multiplication OR division, and finally, addition OR subtraction. This allows one to determine which operation to be used first before another. In the same priority bracket (MD or AS), operations are carried out from left to right. Here we present an example of GEMDAS at work.

Example:

Evaluate $(2^{2+1} + 3) - [(5+2) * 4] * (4 - \frac{6}{3})$

A. - 45
B. 14
C. - 23
D. 37

As stated, grouping, or the use of parentheses, brackets, and the likes, must be prioritized first. It is recommended that one take advantage of this and simplify the notations inside the brackets first as a result. For instance, (a-b)(c-d) is NOT supposed to be multiplied immediately, as the result ac - ad - bc + bd is not particularly pretty.

For the group $(2^{2+1} + 3)$, the first thing that should be tackled is the exponent. Since there is an addition operation in the exponent, one should simply add them together, and the expression simplifies $(2^3 + 3)$, which will produce $(8+3)$. Then, since addition is the only operation in the bracket, there is no need to refer to the GEMDAS order. Hence, 11 is the answer for the first grouping.

For the other groups, it is recommended that one solve from inside to out. In saying this, the parenthetical expression found in the **[]** brackets must be simplified first. The rest can then be done from left to right.

For $[(5+2) * 4]$, by treating the inner expression as an isolated one, we can simply use basic GEMDAS. Analyze the inner expression first, which simplifies the entire thing to $[7 * 4]$, which finally gives 28.

As for the last bracket, $(4-\frac{6}{3})$, we can use the same technique as earlier. Division has the highest priority in the parentheses, which simplifies it to, $(4-2)$, which then finally gives 2.

Now, what's left is three numbers, $(11) - (28)(2)$. On to the final stretch! It should be noted that as there are no more inner operations, we can treat the latter expression as multiplication, resulting in $11-56$. Then the lowest priority in GEMDAS, subtraction, is the final ingredient to solving this elegant question, producing a final answer of -45.

Binary Operations

Operations such as those cited above are referred to as **binary operations.** The vast field, however, contains those that are possibly more unorthodox and perhaps even conjured by the problem makers themselves, usually involving the "fundamental operations" (GEMDAS) in one way or another. While not commonly seen in high school curricula, the CETs heavily incorporates these in their questions. More often than not, however, no knowledge is required beyond simply plugging into these operations.

Example:

1. If $a\$b = \frac{a}{a+b}$, then what is $3\$4$?

   A. $\frac{3}{4}$
   B. $\frac{3}{7}$
   C. $\frac{4}{7}$
   D. $\frac{4}{3}$

To the untrained eye, the problem seems rather confusing due to the use of the dollar sign to represent an operation. It is to be understood that solving this problem simply uses the fact that a = 3 and b = 4, then plugging into the more familiar GEMDAS expression on the right.

Substituting in the values,

$$3\$4 = \frac{3}{3+4} = \frac{3}{7}$$

The answer is therefore **B**.

Properties of Equality (Binary Operations)

It seems intuitively obvious that 7 + 2 = 2 + 7, but also that the same cannot be said for 7 / 2 and 2 / 7. To this extent, we hope to explore the properties that govern binary operations, as well as cite the binary operations that are components of GEMDAS that exhibit such properties. While properties of numbers are not often tackled in classrooms, the topic is still an essential concept that underlies all sections of math. These properties allows for proof and justification in higher maths. For reference, we use the symbol # and $ to mean any binary operation.

Closure under Real Numbers

If A and B are real numbers, and A # B is also a real number, then # is closed under real numbers. For example, the binary operation A # B = A + Bi, where i is the imaginary number sqrt(-1), is not closed under real numbers as 2 # 3 does not equate to a real number. Addition, subtraction, multiplication, and division (which we call the four fundamental operations) are all closed under real numbers.

Commutativity

If an operation is commutative, then for any numbers A and B, A # B = B # A. This means that the order can be switched around carefreely. Subtraction is not commutative as 3 - 2 is not 2 - 3, for example, as with division.

Associativity

If an operation is associative, then for any three numbers A, B, and C, we have that A # (B # C) = (A # B) # C. Subtraction and division are not associative; 3 / (2/4) is not equal to (3/2) / 4.

Distributivity

If an operation # is distributive, then for any three numbers A, B, and C, we have that A # (B + C) = A # B + A # C. Of the four fundamental operations, only multiplication is distributive.

Properties of Equality (Numbers)

Reflexivity

The reflexive property states that A = A. This is usual the final statement of proving something to be true, as the result is obvious once the reflexive property is observed.

Symmetry

The symmetric property of numbers states that A = B, then B = A.

Transitivity

The transitive property of numbers states that if A = B and B = C, then A = C.

Identity number

In an operation #, the identity number is x such that A # x = x # A = A. For example, in multiplication, this is 1, as A * 1 = 1 * A = A and in addition, this is 0, as A + 0 = 0 + A = A. There is no identity number for division and subtraction.

Inverse number

The inverse number under an operation # is the number y corresponding to A such that A # y is the identity element. In multiplication, the **multiplicative inverse** is called the **reciprocal**, while in addition, the **additive inverse** is the negative of a number.

Properties of Numbers and Operations	Example
Closure Property of Addition a + b is a real number if a and b are real numbers.	1 + 2 = 3
Closure Property of Multiplication a * b is a real number if a and b are real numbers.	2 * 3 = 6
Commutative Property of Addition a + b = b + a	1 + 2 = 2 + 1
Commutative Property of Multiplication a * b = b * a	2 * 3 = 3 * 2
Associative Property of Addition a + (b + c) = (a + b) + c	1 + (2 + 3) = (1 + 2) + 3
Associative Property of Multiplication a * (b * c) = (a * b) * c	2 * (3 * 4) = (2 * 3) * 4
Distributive Property a * (b + c) = ab + ac	2 * (1 + 2) = 2*1 + 2*2
Reflexive Property a = a	1 = 1
Symmetric Property If a = b, then b = a	If e = 2.72, then 2.72 = e
Zero: Identity of Addition 0 + a = a + 0 = a	0 +1 = 1 + 0 = 1
One: Identity of Multiplication 1 * a = a * 1 = a	1 * 2 = 2 * 1 = 2
Additive Inverse a + (-a) = 0	1 + (-1) = 0
Multiplicative Inverse a * (1/a) = 1 where a $\neq$ 0	2 * (½) = 1