Proportionality

When we say two pairs of quantities are in proportion to one another, we mean that their respective ratios are equal. For example, if we are to solve for x such that 24:16 and 6:x are proportional to one another, we equate the two ratios to obtain:

$\frac{24}{16} = \frac{6}{x}$

$24x = 96$

$x = 4$

Proportionality is highly practical to most pursuits in the real world, inclusive of which are stoichiometry, the creation of alloys, and even mundane tasks like creating more servings from a given recipe. To further increase its versatility, one can generalise the concept to involve abstract variables. We let these be x and y for the purpose of discussion.

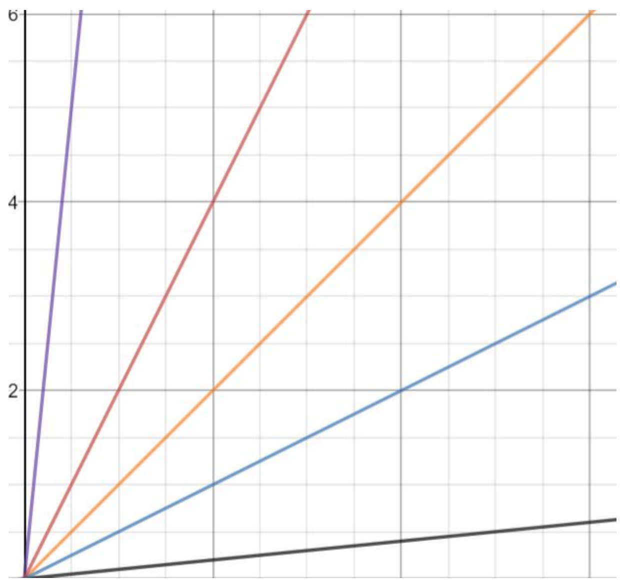
We say that y and x are **directly proportional** to one another if $\frac{y}{x} = k$ for a fixed constant k, meaning the value of k doesn't change as y and x range over their respective values. Conversely, we say that x and y are **inversely proportional to one another** if $xy = k$ for another fixed constant k. We call k the **constant of proportionality.**

The common symbol for proportionality is $\propto$. Therefore, if y is proportional to x, then we can shorthand it as y $\propto$ x.

For example, we say that the set {(1,4), (5,20), (10,40)} of ordered pairs (x,y) has y directly proportional to x for all pairs in the set, with constant of proportionality 4. In contrast, {(5,5), (25,1), (1,25)} is a set where x and y are inversely proportional to one another with constant of proportionality 25. The set {(7,2), (6,3), (5,4)} exhibit neither of these relations, despite the presence of a consistent downward trend.

It can be noted that in direct proportionality, an increase in magnitude of one variable leads to an increase in the other, while in inverse proportionality, as one variable increases, the other decreases. Furthermore, we can manipulate the equation where x and y are inversely proportional to one another to obtain that xy = y/(1/x) = k $xy = \frac{y}{\frac{1}{x}} = k$. This implies that **if x and y are inversely proportional** to one another, **then y and 1/x are directly proportional** to one another. We can therefore write an inverse relationship between x and y as y ∝ $\frac{1}{x}$.

Graphing Proportionality



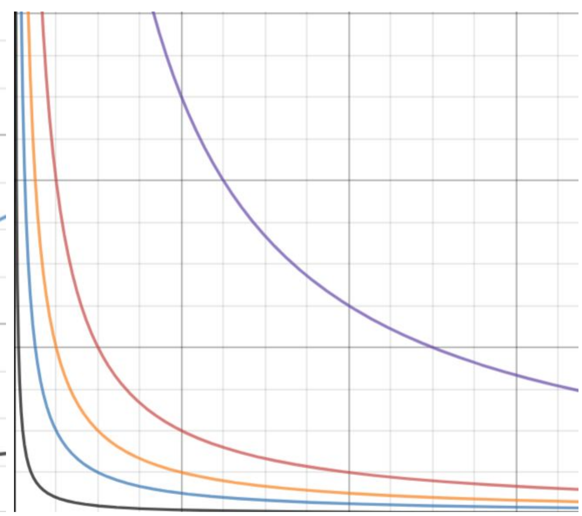


Figure 1: Direct Proportionality (y vs x)

Figure 2: Inverse Proportionality (y vs x)

A lot of information is to be had in graphing direct proportionalities. The equation $y = kx$ takes on the form of the more general linear equation $y = mx + b$, where m is representative of the line's slope and b is its y-intercept. As we have $y = kx + 0$, we can infer that the constant of proportionality, **k, is actually the graph's slope**, and the graph **passes through the origin, (0,0).** In figure 1, the purple line has equation $y = 10x$ and is therefore the steepest, while the black has equation $y = 0.1x$ and is therefore the least steep.

As for graphing inversely proportional variables, we can see that they are from the **parent function y = 1/x**, which we know to be asymptotic to both axes. Increasing the constant of proportionality, k, increases the distance from the axes at any given point. In Figure 2, the purple curve has equation $y = \frac{10}{x}$ and therefore seems the "furthest", while the black has equation $y = \frac{1}{10x}$

The discussion thus far is generally clear-cut, but is a little too simplistic, especially for 12th graders. By our definitions above, we are limited to two variables x and y, whereas in the real world, multiple variables are often involved (pressure, volume, and temperature in thermodynamics). Moreover, these variables often contain complex exponents and radicals, and it is difficult to model on a purely linear assumption that was the foundation of our discussion earlier. We therefore introduce a highly similar concept to proportionality: **variation.**

Variation

Variation, simply put, is **an extension of proportionality** that allows for the inclusion of a higher number of more complex variables. The terminology is also similar; we say that two expressions **vary directly** with one another if $y \propto$ x or $y = kx$ for some constant *k,* and **vary inversely** with one another if they are of the form $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$ for some constant *k.*

It is important to note that y and x here are not representative of singular variables. In variation, it is very much plausible that complex expressions like w^2x^3z vary directly with y^3, for example. For this example, we can write the equation as $w^2x^3z = ky^3$, where k is the constant of proportionality. In words, we can state this as follows:

"The product of the square of w, cube of x, and z is directly proportional to the cube of y."

Where the underlined phrase is interchangeable with the phrase *varies directly.*

Variation also allows for complex combinations of direct and inverse proportionalities. For example, if we have the following statement in words:

"*The square of x is inversely proportional to the square of y and proportional to the cube root of z*"

We may write it as $x^2 = k\frac{\sqrt[3]{z}}{y^2}$. This is obtained from first writing the proportionality statements, namely $x^2 \propto \sqrt[3]{z}$ and $x^2 \propto \frac{1}{y^2}$, and then combining them together, which gives $x^2 \propto \frac{\sqrt[3]{z}}{y^2}$, equivalent to the equation obtained earlier.

As an exercise, translate the following into the word/expression counterpart.

A. $xy^2z \propto \frac{\sqrt[3]{w}}{v^2}$
B. "The cube root of the product of the square root of x, fourth root of y, and fifth root of z is inversely proportional to the quotient obtained by dividing the cube of a over the square of b, and proportional to the fourth power of c."

Conclusion: Solving for concrete values in proportionality statements

We stated earlier that a key advantage of having a proportionality statement is that one can either obtain the constant of proportionality, which we termed k, or the value of either y or x immediately. For example, if we have that $x^2y \propto z$ and that when $x = y = 4$, $z = 12$, we can obtain the equation as follows:

$x^2y = kz$. Isolating k, we obtain

$k = x^2\frac{y}{z}$. Plugging in the values shown earlier, we can finally get that

$k = 4^2 * \frac{4}{12} = \frac{16}{3}$

We can therefore rewrite our equation as $x^2y = \frac{16z}{3}$. As for solving for one value given the others, given x = 5 and y = 10 and the equation above, for example, one can determine the value of z immediately:

$x^2y = \frac{16z}{3}$. Isolating z, we obtain

$z = 3x^2 * \frac{y}{16}$. Plugging in values for x and y, we obtain z $= (3)(5^2)(\frac{10}{16}) = \frac{375}{8}$, or approximately 46.875.

These once more show the versatility of proportionality and variation in fields like Physics were variables seem to intertwine with one another.