Decoding Fourier Analysis

(From a Rigorous Mathematical Perspective)

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Preface

Fourier analysis is a mathematical technique that transforms problems from the time (or spatial) domain into the frequency domain, making them easier to analyze and manipulate. It is a foundational tool in mathematics with widespread applications across physics, engineering, and signal processing. This document, *Decoding Fourier Analysis*, is compiled as part of our collective efforts to understand and present the core concepts of Fourier analysis and distribution theory **from a "pure Math" perspective**. This material can be used as a reference for a first course in Fourier analysis. It lays a foundation to several advanced courses, like wavelets, Harmonic analysis, signal processing, graph neural networks (e.g. Graph Fourier Transform), and so on. It is assumed that the reader holds the basic knowledge of linear algebra, measure theory, functional analysis (for this, we refer the reader to our handbook Decoding Functional Analysis), and complex analysis.

We have taken utmost care to minimze the number of errors, and the presence of any error is completely unintentional. Please feel free to email any suggestions or feedback to pathak.rajdeep@alumni.iith.ac.in.

The reader is strongly encouraged to attempt the exercises on the way. As Prof. James R. Munkres, in the second edition of his book titled "Topology" has said, "Working problems is a crucial part of learning mathematics. No one can learn (topology) by merely poring over the definitions, theorems, and the examples that are worked out in the text. One must work part of it out for oneself. To provide that opportunity is the purpose of the exercises."

We hope that this compilation proves useful for students, scholars, and anyone interested in gaining a solid foundation in functional analysis. The document was compiled when the authors were final-year M.Sc. students in the Department of Mathematics at IIT Hyderabad. Special thanks to Prof. Venku Naidu D for his continuous support and guidance throughout the course. The content of this document is based on his class notes taught under MA5110: Fourier Analysis and Applications in Spring 2025 semester at IIT Hyderabad.

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Notation

We use the standard notation followed by any standard textbook.

1. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: The set of natural numbers, integers, rationals, reals, and complex numbers respectively
2. $\mathscr{P}(X)$: The power set (set of all subsets) of X
3. $\mathbb{F}$: A field (here, either $\mathbb{R}$ or $\mathbb{C}$, unless mentioned otherwise)
4. $\mathbb{F}^n$: $\mathbb{R}^n$ or $\mathbb{C}^n$, the n-dimensional space
5. $\mathcal{P}$: The space of all trigonometric polynomials
6. $\mathcal{P}[a,b]$: The set of all $\mathbb{F}$-valued polynomials over $[a,b]$
7. $C[a,b]$: The set of all $\mathbb{F}$-valued continuous functions $f:[a,b]\to\mathbb{F}$ over $[a,b]$
8. $C^k[a,b]$ or $C^{(k)}[a,b]$: The set of all $\mathbb{F}$-valued k-times continuously differentiable functions over $[a,b]$
9. $C_C(\mathbb{R})$: The set of all compactly supported continuous functions on $\mathbb{R}$
10. $C_C^{(n)}(\mathbb{R})$: The set of all compactly supported n-times continuously differentiable functions on $\mathbb{R}$
11. $\mathscr{D} = C_{2\pi}^{\infty}[-\pi,\pi]$: The set of all 2π-periodic infinitely many times differentiable functions (test functions)
12. $L^p(\mathbb{R}) = \{f:\mathbb{R}\to\mathbb{C} \mid f \text{ is measurable and } \int_{\mathbb{R}} |f(x)|^p\, dx < \infty\}$, for $1 \le p < \infty$
13. $L^\infty(\mathbb{R}) = \{f:\mathbb{R}\to\mathbb{C} \mid f \text{ is measurable and essentially bounded}\}$
14. $L_{2\pi}^p[-\pi,\pi] = \{f:\mathbb{R}\to\mathbb{C} \mid f \text{ is measurable, } 2\pi\text{-periodic, and } \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p\, dx < \infty\}$, for $1 \le p < \infty$
15. $L_{2\pi}^\infty[-\pi,\pi] = \{f:\mathbb{R}\to\mathbb{C} \mid f \text{ is measurable, } 2\pi\text{-periodic, and essentially bounded}\} \subseteq L^\infty(\mathbb{R})$

Contents

1 Preliminaries

Lemma 1.1. *Let $(X, +, \cdot, \|\cdot\|)$ be a normed linear space. Then the maps $+ : X \times X \to X$ given by $(x, y) \mapsto x + y$ and $\cdot : \mathbb{F} \times X \to X$ given by $(\alpha, x) \mapsto \alpha x$ are continuous.*

Proof. 1. **Claim:** $+$ is continuous.

Let $\{(x_n, y_n)\} \in X \times X$ be a sequence such that $(x_n, y_n) \to (x, y)$ as $n \to \infty$ in $X \times X$.

$\implies x_n \to x$ and $y_n \to y$ in X and Y respectively, as $n \to \infty$

Then $\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$ as $n \to \infty$

Thus, $x_n + y_n \to x + y$ as $n \to \infty$

or $+(x_n, y_n) \to +(x, y) = x + y$ as $n \to \infty$

So, $+$ is continuous by the sequential criterion.

2. **Claim:** $\cdot$ is continuous.

Let $\{(\alpha_n, x_n)\} \in \mathbb{F} \times X$ be a sequence such that $(\alpha_n, x_n) \to (\alpha, x)$ as $n \to \infty$

Then $\alpha_n \to \alpha$ in $\mathbb{F}$ and $x_n \to x$ in X as $n \to \infty$

Then $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \leq |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\| \to 0$ as $n \to \infty$

Or $\alpha_n x_n \to \alpha x$ as $n \to \infty$

$\square$

Definition 1.1 (Topological vector space)**.** Let $(X, +, \cdot)$ be a vector space over $\mathbb{F}$. It is called a topological vector space over $\mathbb{F}$ if there exists a topology τ on X such that the maps $+$ and $\cdot$ as defined in lemma 1.1 are continuous.

1.1 Measure Theory

Definition 1.2 (σ-algebra)**.** Let X be a non-empty set. We say that $\mathcal{A} \subset \mathscr{P}(X)$ (set of subsets of X) is a σ-algebra in X if:

1. $\phi \in \mathcal{A}$

2. $\mathcal{A}$ is closed under complements, i.e. if $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$

3. $\mathcal{A}$ is closed under countable unions, i.e. if $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$, then $\cup_i A_i \in \mathcal{A}$

Lemma 1.2. *Let $\mathcal{B} \subseteq \mathscr{P}(X)$ (power set of X). Then there exists smallest σ-algebra $\mathcal{A}$ in X such that $\mathcal{B} \subseteq \mathcal{A}$.*

Proof. Define

$$\mathcal{A} = \bigcap_{\mathcal{B} \subseteq \mathcal{C}} \mathcal{C}$$

for σ-algebras $\mathcal{C}$. Check that $\mathcal{A}$ is a σ-algebra and $\mathcal{B} \subseteq \mathcal{A}$. $\square$

Definition 1.3 (Measure)**.** A set function $\mu : \mathcal{A} \to [0, \infty]$ is called a measure on X if:

1. $\mu(\phi) = 0$

2. $$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i),\ A_i \in \mathcal{A},\ A_i \cap A_j = \phi\ \forall i \neq j$$

Intuitively, a measure on X assigns a positive real number to every subset of X.

Definition 1.4 (Measure space)**.** A measure space is a triplet consisting of a non-empty set X, a σ-algebra $\mathcal{A}$ in X, and a measure μ on X defined on $\mathcal{A}$: denoted by $(X, \mathcal{A}, \mu)$.

Example 1.1 (Measures)**.** The following are a few examples of measures:

1. Let $X \neq \phi$ and $A \subseteq X$. Then a σ-algebra containing A is $\mathcal{A} = \{\phi, A, A^c, X\}$. Define $\mu : \mathcal{A} \to [0, \infty]$ by

   $$\mu(\phi) = 0,\ \mu(A) = \alpha_1,\ \mu(A^c) = \alpha_2,\ \mu(X) = \alpha_1 + \alpha_2$$

   Check that μ is a measure.

2. Let $X = \mathbb{N}$ and $\mathcal{A} = \mathscr{P}(X)$. Define $\mu : \mathcal{A} \to [0, \infty]$ by $\mu(A) = |A|$. Check that μ is a measure on $\mathbb{N}$. As $\mu(A)$ counts the number of elements in A, it is called the **counting measure**.

3. Let $X \neq \phi$ and $\mathcal{A}$ be a σ-algebra on X. Let $x \in X$. Define $\mu_x : \mathcal{A} \to [0, \infty]$ by

   $$\mu_x(A) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$$

   Check that μ_x is a measure, called the **Dirac measure** at $x \in X$.

4. Let $X = \mathbb{N}$ and $\mathcal{A} = \mathscr{P}(\mathbb{N})$. Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\mathbb{N}) = 1$, and $\mu(\{n\}) = \frac{1}{2^n}$, $\forall n \in \mathbb{N}$. Is μ a measure?

Definition 1.5 (Measurable set)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$. Then E is called a measurable set.

Definition 1.6 (Measurable function)**.** A function $f : X \to \mathbb{F}$ is called $\mathcal{A}$-measurable function if $f^{-1}(U) \in \mathcal{A}$, $\forall$ open set U in $\mathbb{F}$.

Remark 1.1. Recall the notions of Lebesgue-measurable sets and Borel sets on $\mathbb{R}$. Recall that $(\mathbb{R}, \mathcal{M}, m)$ is a measure space, where $\mathcal{M}$ is the collection of Lebesgue-measurable sets in $\mathbb{R}$ and m is the Lebesgue measure.

Definition 1.7 (Essentially bounded functions)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function. We say that f is essentially bounded if $\exists M > 0$ such that

$$m\left(\{x \in \mathbb{R} : |f(x)| > M\}\right) = 0$$

That is, $|f(x)| \leq M$ almost everywhere on $\mathbb{R}$.

Definition 1.8 (The space $L^p(\mathbb{R})$)**.** The space of non-periodic functions $L^p(\mathbb{R})$ is defined as:

1. $L^p(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is measurable and } \int_{\mathbb{R}} |f(x)|^p \, dx < \infty\}$, for $1 \leq p < \infty$. The norm $\|\cdot\|_p$ is defined as

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}, \; f \in L^p(\mathbb{R}).$$

2. $L^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is measurable and essentially bounded}\}$. The norm $\|\cdot\|_\infty$ is defined as

$$\|f\|_\infty = \inf\{M > 0 : |f(x)| \leq M \text{ measurable almost everywhere on } \mathbb{R}\}.$$

Remark 1.2. 1. For $1 \leq p < \infty$, $(L^p(\mathbb{R}), \|\cdot\|_p)$ is a separable Banach space.

2. $(L^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a non-separable Banach space.

Exercise 1.1. Let $1 \leq p \leq q \leq infty$. Then show that $L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R})$ and $L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R})$. That is, no L^p space is contained within one another.

Theorem 1.1 (Monotone convergence theorem or MCT). *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let (f_n) be a sequence of non-negative measurable functions on X and $f_n(x) \leq f_{n+1}(x)\ \forall x$ μ-almost everywhere. Then $\exists$ a measurable function f such that $f_n \to f$ μ-almost everywhere on X, i.e. $f(x) = \lim_{n\to\infty} f_n(x)$ μ-almost everywhere on X, and*

$$\begin{aligned} \lim_{n\to\infty} \int_X f_n(x) d\mu(x) &= \int_X \lim_{n\to\infty} f_n(x)\ d\mu(x) \\ \textit{or,}\ \lim_{n\to\infty} \int_X f_n(x) d\mu(x) &= \int_X f(x)\ d\mu(x) \end{aligned}$$

Theorem 1.2 (Dominated Convergence Theorem or DCT). *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let (f_n) be a sequence of non-negative measurable functions on X such that $f(x) = \lim_{n\to\infty} f_n(x)$ μ-almost everywhere on X. If $\exists g \in L^1(X, \mu)$ such that $|f_n(x)| \leq |g(x)|$ μ-almost everywhere on X (i.e. if the sequence (f_n) has an L^1-dominant), then*

$$\lim_{n\to\infty} \int_X f_n(x)\ d\mu(x) = \int_X f(x)\ d\mu(x)$$

The next two theorems, given by Fubini and Tonelli respectively, gives us conditions for exchanging integrals. The only difference between them is their requirements: Fubini's requires the multiple integral of the absolute-valued function to be finite, whereas Tonelli's just requires the function to be non-negative.

Theorem 1.3 (Fubini's theorem). *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. We know that $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ is a product measure space. Let $f : X \times Y \to \mathbb{F}$ be a $\mathcal{A} \times \mathcal{B}$-measurable function. Suppose that*

$$\int_{X\times Y} |f(x,y)|\ d(\mu \times \nu)(x,y) < \infty$$

Then

$$\int_{X\times Y} f(x,y)\ d(\mu \times \nu)(x,y) = \int_Y \int_X f(x,y)\ d\mu(x)\ d\nu(y) = \int_X \int_Y f(x,y)\ d\nu(y)\ d\mu(x)$$

Theorem 1.4 (Tonelli's theorem). *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to \mathbb{R}$ be a non-negative $\mathcal{A} \times \mathcal{B}$-measurable function. Then*

$$\int_{X\times Y} f(x,y)\ d(\mu \times \nu)(x,y) = \int_Y \int_X f(x,y)\ d\mu(x)\ d\nu(y) = \int_X \int_Y f(x,y)\ d\nu(y)\ d\mu(x)$$

1.2 Cesàro and Abel Summations

1.2.1 Cesàro Summation

Let $(a_n) \subseteq \mathbb{R}$ or $\mathbb{C}$. Consider the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

Let $n \in \mathbb{N}$, then the averages of the sequence of partial sums is:

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} S_k$$

Definition 1.9 (Cesàro Summability)**.** Let $(a_n) \subseteq \mathbb{R}$ or $\mathbb{C}$. We say that $\sum_{n=1}^{\infty} a_n$ is Cesàro-summable if (σ_n) is a convergent sequence. If $\lim_{n\to\infty} \sigma_n = L$, then

$$\sum_{n=1}^{\infty} a_n \overset{C}{=} L$$

That is, the sum of the series converges to L in Cesàro sense.

Lemma 1.3. *The averages of the sequence of partial sums of a convergent sequence converges to the same limit. Let $(a_n) \subseteq \mathbb{R}$ or $\mathbb{C}$ such that $\lim_{n\to\infty} a_n = a$. Let*

$$b_n = \frac{1}{n} \sum_{k=1}^{n} a_k, \quad n \in \mathbb{N}$$

Then $b_n \to a$ as $n \to \infty$.

Proof. Given that $a_n \to a$ as $n \to \infty$ (i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for $n \geq N$).
Claim: $b_n \to a$ as $n \to \infty$.

$$\begin{aligned} b_n - a &= \frac{1}{n} \sum_{k=1}^{n} a_k - a \\ &= \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{n} \sum_{k=1}^{n} a \\ &= \frac{1}{n} \sum_{k=1}^{n} (a_k - a) \end{aligned}$$

Let $n \geq N$. Then

$$b_n - a = \frac{1}{n}\sum_{k=1}^{N}(a_k - a) + \frac{1}{n}\sum_{k=N+1}^{n}(a_k - a)$$

$$\implies |b_n - a| \leq \frac{1}{n}\sum_{k=1}^{N}|a_k - a| + \frac{\epsilon}{n}\sum_{k=N+1}^{n} 1$$

Let $M = \sum_{k=1}^{N}|a_k - a|$, so

$$|b_n - a| \leq \frac{M}{n} + \epsilon\left(1 - \frac{N}{n}\right) < 2\epsilon \ \forall n \geq \widetilde{N}.$$

□

Remark 1.3. The converse of the above lemma need not be true. That is, if we know that the averages of the sequence of partial sums converge to a limit, the sequence need not necessarily converge (to the same limit)! Consider the following example. Let $a_n = (-1)^n$. Then

$$b_n = \begin{cases} 0, & n \text{ even} \\ -\frac{1}{n}, & n \text{ odd} \end{cases}$$

The averages $b_n \to 0$, but the sequence (a_n) does not converge.

Example 1.2 (Cesàro summability)**.** Let us look at a series which does not converge in the classical sense, but does in the Cesàro sense. Consider the series $\sum_{n=1}^{\infty}(-1)^n$. We know that this series is not convergent. Observe that the sequence of partial sums, $S_n = \begin{cases} 0, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$

Now,

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} S_k$$

Compute a few terms:

$$\sigma_1 = -1, \quad \sigma_2 = -\frac{1}{2}, \quad \sigma_3 = -\frac{2}{3}, \quad \sigma_4 = -\frac{2}{4}, \quad \sigma_5 = -\frac{3}{5}$$

In general,

$$\sigma_n = \begin{cases} -\frac{1}{2}, & n \text{ even} \\ -\frac{n+1}{2n}, & n \text{ odd} \end{cases}$$

Thus, $\sigma_n \to -\frac{1}{2}$ as $n \to \infty$.

Therefore,

$$\sum_{n=1}^{\infty}(-1)^n \overset{C}{=} -\frac{1}{2}$$

1.2.2 Abel Summation

Definition 1.10 (Abel summability)**.** Let (a_n) be a sequence in $\mathbb{R}$ or $\mathbb{C}$. We say that $\sum_{n=0}^{\infty} a_n$ is **Abel summable** if

1. $\forall r \in (0,1)$, $A_r = \sum_{n=0}^{\infty} a_n r^n$ exists.
2. $\lim_{r\to 1^-} A_r$ exists in $\mathbb{R}$ or $\mathbb{C}$.

If $\lim_{r\to 1^-} A_r = A$, then $\sum_{n=0}^{\infty} a_n$ is Abel summable to A, i.e. $\sum_{n=0}^{\infty} a_n \overset{\text{Abel}}{=} A$.

Lemma 1.4. *Cesàro convergence of a series implies Abel convergence to the same limit. That is,*

$$\sum_{n=0}^{\infty} a_n \overset{C}{=} A \implies \sum_{n=0}^{\infty} a_n \overset{Abel}{=} A$$

Proof. Given that $\sum_{n=0}^{\infty} a_n \overset{C}{=} A$, i.e. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$|\sigma_n - A| < \epsilon \quad \forall n \geq N$$

where,

$$\sigma_n = \frac{1}{n+1}\sum_{i=0}^{n} S_i, \quad S_i = \sum_{j=0}^{i} a_j$$

Claim:

$$\sum_{n=0}^{\infty} a_n \overset{\textbf{Abel}}{=} A$$

Recall: For $r \in (0,1)$,

$$\sum_{n=0}^{\infty} a_n r^n = (1-r)^2 \sum_{n=0}^{\infty} (n+1)\sigma_n \, r^n$$

Also, we have that,

$$\begin{aligned}
\frac{1}{1-r} &= \sum_{n=0}^{\infty} r^n \\
\implies \frac{1}{(1-r)^2} &= \sum_{n=1}^{\infty} nr^{n-1} = \sum_{n=0}^{\infty}(n+1)r^n \\
\implies 1 &= (1-r)^2 \sum_{n=0}^{\infty}(n+1)r^n \qquad (*)
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n r^n - A &= (1-r)^2 \sum_{n=0}^{\infty}(n+1)\sigma_n r^n - A(1-r)^2 \sum_{n=0}^{\infty}(n+1)r^n \text{ (from (*))} \\
&= (1-r)^2 \sum_{n=0}^{\infty}(n+1)(\sigma_n - A)r^n \\
&= (1-r)^2 \sum_{n=0}^{N}(n+1)(\sigma_n - A)r^n + (1-r)^2 \sum_{N+1}^{\infty}(n+1)(\sigma_n - A)r^n \\
\implies \left|\sum_{n=0}^{\infty} a_n r^n - A\right| &\leq (1-r)^2 \sum_{n=0}^{N}(n+1)|\sigma_n - A|r^n + (1-r)^2 \sum_{n=N+1}^{\infty}(n+1)|\sigma_n - A|r^n \\
&\leq (1-r)^2 M + \epsilon(1-r)^2 \underbrace{\sum_{n=N+1}^{\infty}(n+1)r^n}_{\leq 1} \\
&\leq (1-r)^2 M + \epsilon \\
&< \epsilon \text{ } (\exists s > 0 \text{ such that } s < r < 1)
\end{aligned}$$

Therefore, as $r \to 1^-$, $\sum_{n=0}^{\infty} a_n r^n \to A$. □

Remark 1.4. Converse of the above theorem need not be true, i.e. Abel summability need not imply Cesàro summability for a series. For example, check that $\sum_{n=0}^{\infty} n(-1)^n$ is not Cesàro-summable (because $\frac{S_0+\ldots+S_n}{n} \sum \frac{n(n+1)}{2n} \to \infty$).
But, $\sum_{n=0}^{\infty} n(-1)^n \overset{\text{Abel}}{=} -\frac{1}{4}$. This is because:

$$\sum_{n=0}^{\infty}(-1)^n nr^n = -r\sum_{n=0}^{\infty} n(-r)^{n-1} = -\frac{r}{(r+1)^2} \to -\frac{1}{4}$$

Lemma 1.5. *The following identities will be very useful, moving forward:*

1.

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\ dx = 1$$

2.

$$\int_{-\infty}^{\infty} e^{-\pi a x^2}\ dx = \frac{1}{\sqrt{a}}$$

Proof. We have,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi x^2}\ dx &= \int_{-\infty}^{\infty} e^{-\pi y^2}\ dy \\
&= \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\ dx\ dy \\
&= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-\pi r^2} r\ dr\ d\theta \\
&= 1
\end{aligned}$$

□

2 Periodic Fourier Analysis

Definition 2.1 (Periodic function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ is called a periodic function if $\exists \alpha \in \mathbb{R}$ such that $f(x+\alpha) = f(x)\ \forall x \in \mathbb{R}$.

Definition 2.2 (Period of a function)**.** Let f be a periodic function. Then,

$$\beta = \inf\{\alpha > 0 : f(x+\alpha) = f(x)\ \forall x \in \mathbb{R}\}$$

is called the period of f.

Example 2.1 (Periodic functions)**.**

1. $f(x) = \sin x,\ \cos x$ are periodic functions with period 2π.
2. Then $n \in \mathbb{Z}$ and $f(x) = e^{inx},\ x \in \mathbb{R}$ is periodic (can you find its period?)
3. Let $m, n \in \mathbb{N} \setminus \{0\}$. Then $f(x) = \sum_{k=-m}^{n} a_k e^{ikx}$ is periodic. Such functions are called **trigonometric polynomials over $\mathbb{R}$ or $\mathbb{C}$**
4. $f(x) = \{x\}$ (fractional part of x) is periodic with period 1.

Definition 2.3 (Degree of a trigonometric polynomial)**.** Let $m, n \in \mathbb{N}$ and

$$p(\theta) = \sum_{k=-m}^{n} a_k e^{ik\theta}$$

be a trigonometric polynomial. We define the degree of p as

$$deg\ p = \max\{|k| : a_k \neq 0\}$$

Define $S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in [-\pi, \pi)]\}$
Define $\eta : S^1 \to [-\pi, \pi)$ by $\eta(e^{i\theta}) = \theta,\ \forall e^{i\theta} \in S^1$.
Clearly, η is bijective (verify). However, η is not continuous as the image of compact set S^1 under η, which is $[-\pi, \pi)$, is not compact. However, η^{-1} is continuous. Now, let

$$\mathcal{A}_{S^1} = \{\eta^{-1}(E) : E \text{ is measurable set in } [-\pi, \pi)\}$$

As inverse image commutes with unions, intersections, and complements, we have that $\mathcal{A}_{S^1}$ is a σ-algebra on S^1.
Define $\mu : \mathcal{A}_{S^1} \to [0, \infty]$ by

$$\mu\left(\eta^{-1}(E)\right) = \frac{m(E)}{2\pi}\ \forall \eta^{-1}(E) \in \mathcal{A}_{S^1},\ \text{where } m \text{ is the Lebesgue measure.}$$

Verify that μ is a measure on $\mathcal{A}_{S^1}$. Also, note that $\mu(S^1) = 1$. Hence, $(S^1, \mathcal{A}_{S^1}, \mu)$ is a measure space.

Proposition 2.1. *There exists a one-to-one correspondence between 2π-periodic functions on $\mathbb{R}$ and functions on S^1.*

Proof. Let $f : S^1 \to \mathbb{R}$ or $\mathbb{C}$ be a function. Define $g : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ by $g(x) = f(e^{ix})\ \forall x \in \mathbb{R}$. Then g is a periodic function with period 2π.
Conversely, let $g : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ be a function on $\mathbb{R}$ such that $g(x + 2\pi) = g(x)\ \forall x \in \mathbb{R}$. Define $f : S^1 \to \mathbb{R}$ or $\mathbb{C}$ by $f(e^{i\theta}) = g(\theta),\ \forall e^{i\theta}\ \theta \in [-\pi, \pi)$. Clearly, f is a function on S^1. □

2.1 Translations and Modulations

Definition 2.4 (Translation)**.** Let $t \in \mathbb{R}$ and $1 \leq p \leq \infty$. The translation operator is defined as

$$T_t : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \text{ by } f \mapsto T_t f$$

where, $T_t f(x) = f(x-t)\ \forall x \in \mathbb{R}$.

Definition 2.5 (Modulation)**.** Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. The modulation operator is defined as

$$M_s : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \text{ by } f \mapsto M_s f$$

where, $M_s f(x) = e^{2\pi i x s} f(x)\ \forall x \in \mathbb{R}$.

Lemma 2.1. *T_t and M_s are isometries and surjective linear transformations on $L^p(\mathbb{R})$.*

Proof. First, we show that T_t and M_s are linear transformations.
Let $f, g \in L^p(\mathbb{R})$ an $\alpha \in \mathbb{R}$. Then,

$$T_t(\alpha f + g) = T_t(\alpha f + g)(x) = (\alpha f + g)(x-t) = \alpha f(x-t) + g(x-t) = \alpha T_t f + T_t g$$

$$M_s(\alpha f + g) = M_s(\alpha f + g)(x)$$

$$= e^{2\pi i x s}(\alpha f + g)(x) = e^{2\pi i x s}[\alpha f(x) + g(x)] = \alpha e^{2\pi i x s} f(x) + e^{2\pi i x s} g(x) = \alpha M_s f(x) + M_s g(x)$$

Now, we show that they are isometries. Clearly for $f \in L^p(\mathbb{R})$,

$$\|T_t f\|_p = \|f(x-t)\|_p = \left(\int_{\mathbb{R}} |f(x-t)|^p \ dx\right)^{1/p} = \left(\int_{\mathbb{R}} |f(x)|^p \ dx\right)^{1/p} = \|f\|_p$$

$$\|M_s f\|_p = \left\|e^{2\pi i x s} f(x)\right\|_p = \left(\int_{\mathbb{R}} \left|e^{2\pi i x s} f(x)\right|^p \ dx\right)^{1/p} = \|f\|_p \ (\text{since } |e^{2\pi i x s}| = 1)$$

To show that they are onto, let $f \in L^p(\mathbb{R})$ (the codomain). Then $T_t f(x+t) = f(x)$, meaning that $g(x)$ defined by $g(x) := f(x+t)\ \forall x \in \mathbb{R}$ is the pre-image of f. Again, $g(x) = e^{-2\pi i x s} f(x) \in L^p(\mathbb{R})$ is the pre-image of f under M_s, because $M_s(g) = f$. Thus, T_t and M_s are surjective. □

Theorem 2.1. *Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Then $\|T_t f - f\|_p \to 0$ as $t \to 0$, i.e., the map from $\mathbb{R} \to L^p(\mathbb{R})$ given by $t \mapsto T_t f$ is continuous.*

Proof. We shall prove this in two steps. First, recall that $C_c(\mathbb{R})$ (the space of all compactly supported continuously functions on $\mathbb{R}$) is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Step 1: Let $g \in C_c(\mathbb{R})$. Then there exists a compact set $K \subseteq \mathbb{R}$, such that $g(x) = 0 \ \forall x \in \mathbb{R} \setminus K$.
We know that continuous functions on compact sets are uniformly continuous. Hence, for a given $\varepsilon > 0, \ \exists \delta > 0$ such that

$$|g(x) - g(y)| < \varepsilon \text{ whenever } |x - y| < \delta, \ x, y \in \mathbb{R}$$

From this, observe that $|T_t g(x) - g(x)| = |g(x-t) - g(x)| < \varepsilon$ whenever $|t| < \delta, \ \forall x \in \mathbb{R}$. Then

$$\begin{aligned}
\|T_t g - g\|_p^p &= \int_{\mathbb{R}} |(T_t g)(x) - g(x)|^p \, dx = \int_{\mathbb{R}} \underbrace{|g(x-t) - g(x)|}_{\exists \text{ compact } \widetilde{K} \text{ outside which this is } 0} \, dx \\
&= \int_{\widetilde{K}} |g(x-t) - g(x)|^p \, dx \\
&< \varepsilon^p \int_{\widetilde{K}} 1 \, dx, \ \text{if } |t| < \delta \\
&= \varepsilon^p \ m(\widetilde{K}), \ \text{if } |t| < \delta, \ \text{where } m \text{ is the Lebesgue measure}
\end{aligned}$$

As $\widetilde{K}$ is compact, $m(\widetilde{K}) < \infty$

$$\implies \|T_t g - g\|_p < \varepsilon \ m(\widetilde{K})^{\frac{1}{p}}, \ \text{if } |t| < \delta$$

$$\implies \|T_t g - g\|_p \to 0 \text{ as } t \to 0$$

So far, we have proved the theorem for $C_c(\mathbb{R})$ functions only. Now, we extend the proof for $L^p(\mathbb{R})$ functions. **Step 2:** Let $f \in L^p(\mathbb{R})$ and $\varepsilon > 0$. Then $\exists g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \varepsilon$, due to density. Observe that,

$$\begin{aligned}
\|T_t f - f\|_p &= \|T_t f - T_t g + T_t g - g + g - f\|_p \\
&\leq \|T_t f - T_t g\|_p + \|T_t g - g\|_p + \|g - f\|_p \\
&= \|T_t (f - g)\|_p + \|T_t g - g\|_p + \|g - f\|_p \\
&= 2 \|f - g\|_p + \|T_t g - g\|_p \\
&< 3\varepsilon, \ \text{if } |t| < \delta
\end{aligned}$$

□

Exercise 2.1. Show that for $p = \infty$, the above theorem fails.

Now, we shall define translations and modulations for functions in $L^p_{2\pi}[-\pi,\pi]$.

Definition 2.6. Let $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

1. The translation operator is defined as

$$T_t : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi] \text{ by } f \mapsto T_t f$$

   where, $T_t f(x) = f(x-t)\ \forall x \in [-\pi,\pi]$.

2. The modulation operator is defined as

$$M_n : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi] \text{ by } f \mapsto M_n f$$

   where, $M_n f(x) = e^{inx}\ f(x)\ \forall x \in [-\pi,\pi]$.

Exercise 2.2. Let $1 \leq p < \infty$ and $f \in L^p_{2\pi}[-\pi,\pi]$. Then show that the map from $[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi]$ given by $t \mapsto T_t f$ is continuous.

Hint: The proof follows a similar structure as theorem 2.1.

2.2 Convolutions of Periodic Functions

Definition 2.7 (Convolutions on $L^1_{2\pi}$)**.** Let $f \in L^1_{2\pi}[-\pi,\pi]$. We define the convolution of f and g as

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(\theta - t)\ dt,\ \ \theta \in [-\pi,\pi]$$

Remark 2.1. Since $L^\infty_{2\pi}[-\pi,\pi] \subset \ldots \subset L^p_{2\pi}[-\pi,\pi] \ldots \subset L^1_{2\pi}[-\pi,\pi]\ \forall 1 < p < \infty$, the definition of convolutions of functions in $L^p_{2\pi}[-\pi,\pi]$ is the same. We have defined it for the largest space, i.e. $L^1_{2\pi}[-\pi,\pi]$.

Lemma 2.2. *Let $f, g \in L^1_{2\pi}[-\pi,\pi]$. Then*

1. $f * g \in L^1_{2\pi}[-\pi,\pi]$

2. $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$

Proof. Observe that,

$$\begin{aligned}
\|f * g\|_1 &= \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} |(f*g)(\theta)| \ d\theta \\
&= \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} \left|\frac{1}{2\pi}\int_{t=-\pi}^{\pi} f(t)g(\theta-t)\ dt\right| \ d\theta \\
&\leq \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} \frac{1}{2\pi}\int_{t=-\pi}^{\pi} |f(t)||g(\theta-t)|\ dt\ d\theta \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} |f(t)|\frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} |g(\theta-t)|\ d\theta\ dt \text{ (Using Tonelli's theorem)} \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} |f(t)|\ \|g\|_1 \ dt \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

hence, $f * g \in L^1_{2\pi}[-\pi,\pi]$ ($f*g$ is defined finitely almost everywhere), and $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$. □

The next proposition is interesting; it says that if you integrate a 2π-periodic function, the value remains same if you translate the limits by the same quantity, as long as the difference between the upper and lower limits is 2π. We shall first prove it, and then make use of it to prove the next lemma.

Proposition 2.2. *Let $f \in L^1_{2\pi}[-\pi,\pi]$ and $\theta \in \mathbb{R}$. Then,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ dt = \frac{1}{2\pi}\int_{-\pi+\theta}^{\pi+\theta} f(t)\ dt = \frac{1}{b-a}\int_a^b f(t)\ dt, \ \textit{where } b-a = 2\pi,\ a,b\in\mathbb{R}.$$

Proof. Define $g : \mathbb{R} \to \mathbb{C}$ by $g(\theta) = \frac{1}{2\pi}\int_{-\pi+\theta}^{\pi+\theta} f(t)\ dt$. We claim that g is constant, so that $g(0) = g(\theta)\ \forall\theta \in \mathbb{R}$, and we have our desired result.
Indeed, $g'(\theta) = \frac{1}{2\pi}\left(f(\pi+\theta) - f(-\pi+\theta)\right) = 0$, since $f(\pi+\theta) = f(-\pi+\theta)$ as f is 2π-periodic. Thus, $g'(\theta) = 0\ \forall\theta\in\mathbb{R}$. □

Lemma 2.3 (Convolutions are commutative and associative)**.** *Let $f, g, h \in L^1_{2\pi}[-\pi,\pi]$. Then,*

1. $f * g = g * f$

2. $f * (g * h) = (f * g) * h$

Proof. 1.

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(\theta - t)\ dt$$

Applying change of variable $t \mapsto \theta - t$,

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{\theta+\pi}^{\theta-\pi} f(\theta - t)g(t)\ (-dt)\\
&= \frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} f(\theta - t)g(t)\ dt\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)g(t)\ dt\\
&= (g * f)(\theta),\ \forall\theta
\end{aligned}$$

2. By definition, we have:

$$\begin{aligned}
((f * g) * h)(\theta) &= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} (f * g)(t)\ h(\theta - t)\ dt\\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} \left(\frac{1}{2\pi}\int_{s=-\pi}^{\pi} f(s)g(t-s)\ ds\right)\ h(\theta - t)\ dt\\
&= \frac{1}{2\pi}\int_{s=-\pi}^{\pi} f(s)\frac{1}{2\pi}\int_{t=-\pi}^{\pi} g(t-s)h(\theta - t)\ dt\ ds \text{ (by Fubini's theorem)}
\end{aligned}$$

We know by Proposition 2.2 that for a 2π-periodic function f, $\int_{-\pi}^{\pi} f(x)\ dx = \int_{-\pi+\theta}^{\pi+\theta} f(x)\ dx\ \forall\theta \in \mathbb{R}$. Now, let $t - s = u$. Then,

$$\begin{aligned}
((f * g) * h)(\theta) &= \frac{1}{2\pi}\int_{s=-\pi}^{\pi} f(s)\left(\frac{1}{2\pi}\int_{u=-\pi-s}^{\pi-s} g(u)h((\theta - s) - u)\ du\right)\ ds\\
&= \frac{1}{2\pi}\int_{s=-\pi}^{\pi} f(s)\ (g * h)(\theta - s)\ ds\\
&= (f * (g * h))(\theta),\ \forall\theta
\end{aligned}$$

□

What follows now is some discussion on the continuity and differentiability of convolutions.

Lemma 2.4. *Let $f, g \in L^1_{2\pi}[-\pi, \pi]$. Then*

1. *$f * g$ is continuous (i.e., $f * g \in C_{2\pi}[-\pi, \pi]$) if one of f or g is continuous.*

2. *$f * g$ is differentiable if one of f or g is continuously differentiable. Moreover, if g is*

differentiable (without loss of generality), then $(f*g)' = f*g'$. If both f and g are differentiable, then $(f*g)' = f'*g = f*g'$.

Proof. 1. Assume without loss of generality that f is continuous. We claim that $f*g$ is also continuous.

Let (x_n) be a sequence in $[-\pi,\pi]$ such that $x_n \to x$ as $n\to\infty$. Then we have to show that $(f*g)(x_n) \to (f*g)(x)$ as $n\to\infty$, so that $f*g$ is continuous by the sequential criterion. For,

$$\begin{aligned}(f*g)(x_n) - (f*g)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_n-t)\ g(t)\ dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\ g(t)\ dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x_n-t)-f(x-t)]\ g(t)\ dt\end{aligned}$$

We have to show that the above term goes to 0 as $n\to\infty$. Let $h_n(t) = [f(x_n-t) - f(x-t)]\ g(t),\ t\in[-\pi,\pi]$. Clearly, $h_n(t)\to 0$ as $n\to\infty$ almost everywhere on $[-\pi,\pi]$, because $f(x_n-t)\to f(x-t)$, f being continuous.

Also, $|h_n(t)| \le 2\,\|f\|_\infty\,|g(t)| \in L^1_{2\pi}[-\pi,\pi]$ almost everywhere.

Thus, we have shown that the term inside the integral converges to 0 and has an L^1-dominant. By Dominated Convergence Theorem or DCT,

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} h_n(t)\ dt \to 0 \text{ as } n\to\infty \\ \implies\ &(f*g)(x_n)\to(f*g)(x) \text{ as } n\to\infty\end{aligned}$$

Hence, $f*g$ is continuous.

2. Without loss of generality, assume that g is continuously differentiable. By 1, we know that $f*g$ is continuous. We claim that $f*g$ is differentiable and $(f*g)' = f*g'$. Let $x\in[-\pi,\pi]$. Then,

$$\frac{(f*g)(x+h)-(f*g)(x)}{h} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{g(x+h-t)-g(x-t)}{h}\ dt$$

For $h\neq 0$, let $\varphi_h(t) = f(t)\frac{g(x+h-t)-g(x-t)}{h},\ t\in[-\pi,\pi]$. Observe that $\varphi_h(t)\to f(t)g'(x-$

t) pointwise as $h \to 0$ almost everywhere on $[-\pi, \pi]$, because g is differentiable. Also, $|\varphi_h(t) \le M|f(t)| \in L^1_{2\pi}[-\pi, \pi]$ almost everywhere for some constant M.

Hence, by Dominated Convergence Theorem or DCT,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_h(t)\ dt \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ g'(x-t)\ dt$$
$$\implies \lim_{h\to 0} \frac{(f*g)(x+h)-(f*g)(x)}{h} = (f*g')(x)$$

$\implies$ $f*g$ is differentiable and $(f*g)' = f*g'$ on $[-\pi, \pi]$.

Moreover, $f*g \in C^1_{2\pi}[-\pi, \pi]$ (the derivative of $f*g$ is also continuous) by (1) because g' is continuous.

□

Corollary 2.1. *Let $f, g \in L^1_{2\pi}[-\pi, \pi]$ and $f \in C^{(k)}_{2\pi}[-\pi, \pi]$ (the space of all k-times continuously differentiable 2π-periodic functions). Then,*

1. *$f*g \in C^{(k)}_{2\pi}[-\pi, \pi]$ (the convolution is also k-times continuously differentiable).*
2. *$(f*g)^{(k)} = f^{(k)}*g$ (the k^{th} derivative of the convolution).*
3. *$f*g \in C^{\infty}_{2\pi}[-\pi, \pi]$ if either f or $g \in C^{\infty}_{2\pi}[-\pi, \pi]$.*

Lemma 2.5. *Let $f \in L^p_{2\pi}[-\pi, \pi]$ and $g \in L^q_{2\pi}[-\pi, \pi]$ such that $\frac{1}{p}+\frac{1}{q} = 1$. Then,*

1. *$f*g \in L^{\infty}_{2\pi}[-\pi, \pi]$.*
2. *$\|f*g\|_\infty \le \|f\|_p \|g\|_q$.*

Proof. We know that $(f*g)(x) = \int_{-\pi}^{\pi} f(t)\ g(x-t)\ dt,\ x \in [-\pi, \pi]$.

By Hölder's inequality,

$$|f*g(x)| \le \|f\|_p \|g\|_q \ \forall x \in [-\pi, \pi]$$
$$\implies \|f*g\|_\infty \le \|f\|_p \|g\|_q \ (\because \|f*g\|_\infty \text{ is the supremum over all } x)$$

□

Next, we state and prove **Minkowski's Integral Inequality** and use it to prove **Young's inequality**.

Theorem 2.2 (Minkowski's Integral Inequality). *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $1 \leq p < \infty$. Let $f : X \times Y \to \mathbb{R}$ or $\mathbb{C}$ be a non-negative $\mathcal{A} \times \mathcal{B}$-measurable function. Then,*

$$\left(\int_Y \left|\int_X f(x,y)\ d\mu(x)\right|^p\ d\nu(y)\right)^{1/p} \leq \int_X \left(\int_Y |f(x,y)|^p\ d\nu(y)\right)^{1/p}\ d\mu(x)$$

Or equivalently,

$$\left(\int_Y \|f(\cdot,y)\|_1^p\ d\nu(y)\right)^{1/p} \leq \int_X \|f(x,\cdot)\|_p\ d\mu(x)$$

Proof. Let $h(y) = \int_X f(x,y)\ d\mu(x),\ y \in Y$. Check that h is measurable. Let $1 \leq p < \infty$ and $q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $g \in L^q(Y, \nu)$.
Define $T_h : L^q(Y, \nu) \to \mathbb{R}$ or $\mathbb{C}$ by $T_h(g) := \int_Y g(y)h(y)\ d\nu(y)\ \forall g \in L^q(Y, \nu)$.

Observe that,

$$\begin{aligned}
|T_h(g)| &= \left|\int_Y g(y)h(y)\ d\nu(y)\right| \\
&\leq \int_Y |g(y)|h(y)\ d\nu(y) \\
&= \int_Y |g(y)| \int_X f(x,y)\ d\mu(x)\ d\nu(y) \\
&= \int_X \left(\int_Y f(x,y)\ |g(y)|\ d\nu(y)\right)\ d\mu(x) \text{ (by Tonelli's theorem)} \\
&\leq \int_X \left(\left(\int_Y |f(x,y)|^p\ d\nu(y)\right)^{1/p} \left(\int_Y |g(y)|^q\ d\nu(y)\right)^{1/q}\right)\ d\mu(x) \text{ (by Hölder's inequality)} \\
&= \int_X \left(\int_Y |f(x,y)|^p\ d\nu(y)\right)^{1/p} \|g\|_q\ d\mu(x) \\
&= \|g\|_q \int_X \left(\int_Y |f(x,y)|^p\ d\nu(y)\right)^{1/p}\ d\mu(x)
\end{aligned}$$

Thus,

$$
\begin{aligned}
&|T_h(g)| \leq c\,\|g\|_q \quad \forall g \in L^q(Y,\nu) \\
&\implies T_h \in (L^q(Y,\nu))^* \ \ \text{(dual space)} \\
&\implies \|T_h\|_p = \|h\|_p \leq \int_X \left(\int_Y |f(x,y)|^p \, d\nu(y) \right)^{1/p} d\mu(x)
\end{aligned}
$$

$$
\implies \left(\int_Y \left| \int_X f(x,y)\, d\mu(x) \right|^p d\nu(y) \right)^{1/p} \leq \int_X \left(\int_Y |f(x,y)|^p \, d\nu(y) \right)^{1/p} d\mu(x)
$$

□

Remark 2.2. Minkowski's Integral Inequality still holds true if the non-negativity condition on f is dropped. The proof requires using Fubini's theorem and Tonelli's theorem together.

Theorem 2.3 (Young's inequality)**.** *Let* $1 \leq p \leq \infty$ *and* $f \in L^p_{2\pi}[-\pi,\pi]$ *and* $g \in L^1_{2\pi}[-\pi,\pi]$. *Then,*

1. $f * g \in L^p_{2\pi}[-\pi,\pi]$
2. $\|f * g\|_p \leq \|f\|_p \, \|g\|_1$

Proof. Given that $f \in L^p_{2\pi}[-\pi,\pi]$ and $g \in L^1_{2\pi}[-\pi,\pi]$.
We know, $(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\; f(x-t)\; dt,\; x \in [-\pi,\pi]$. Observe that,

$$
\begin{aligned}
\|f * g\|_p &= \left(\frac{1}{2\pi} \int_{s=-\pi}^{\pi} |(f*g)(s)|^p \; ds \right)^{1/p} \\
&= \left(\frac{1}{2\pi} \int_{s=-\pi}^{\pi} \left| \frac{1}{2\pi} \int_{t=-\pi}^{\pi} g(t)\; f(s-t)\; dt \right|^p ds \right)^{1/p} \\
&\leq \frac{1}{2\pi} \int_{t=-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{s=-\pi}^{\pi} |f(s-t)\; g(t)|^p \; ds \right)^{1/p} dt \ \text{(using Minkowski's Integral Inequality)} \\
&= \frac{1}{2\pi} \int_{t=-\pi}^{\pi} |g(t)| \left(\frac{1}{2\pi} \int_{s=-\pi}^{\pi} |f(s-t)|^p \; ds \right)^{1/p} dt \\
&= \frac{1}{2\pi} \int_{t=-\pi}^{\pi} |g(t)| \; \|f\|_p \; dt \\
&= \|f\|_p \; \|g\|_1
\end{aligned}
$$

Hence, we have that $f * g \in L^p_{2\pi}[-\pi,\pi]$ and $f * g(x) < \infty$ almost everywhere on $[-\pi,\pi]$, and $\|f * g\|_p \leq \|f\|_p \; \|g\|_1$. □

Definition 2.8 (Approximation to the identity)**.** A sequence $(g_n)_{n=1}^{\infty} \subseteq L^1_{2\pi}[-\pi, \pi]$ is called an approximation to the identity if the following conditions are satisfied:

1. $\frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)\ dt = 1\ \forall n \in \mathbb{N}$.

2. $\exists M > 0$ such that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_n(t)|\ dt \leq M\ \forall n \in \mathbb{N}$, i.e., $\|g_n\|_1$'s are all uniformly bounded.

3. $\forall \delta > 0$, $\frac{1}{2\pi}\int_{\delta \leq |t| \leq \pi} |g_n(t)|\ dt \to 0$ as $n \to \infty$.

Theorem 2.4. *Let $(g_n) \subseteq L^1_{2\pi}[-\pi, \pi]$ be an approximation to the identity. Then we have the following:*

1. *If f is continuous, i.e. $f \in C_{2\pi}[-\pi, \pi]$, then $f * g_n \to f$ uniformly on $[-\pi, \pi]$ as $n \to \infty$ (sup-norm convergence).*

2. *If $f \in L^p_{2\pi}[-\pi, \pi]$, $1 \leq p < \infty$, then $f * g_n \to f$ in $L^p_{2\pi}[-\pi, \pi]$ as $n \to \infty$ (L^p-norm convergence).*

Proof. (1) Given that (g_n) is an approximation to the identity. So, we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)\ dt = 1$. Let $f \in C_{2\pi}[-\pi, \pi]$. Then,

$$\begin{aligned}
(f * g_n)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\ g_n(t)\ dt - f(x) \cdot 1 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, g_n(t)\ dt - f(x) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x-t) - f(x)]\, g_n(t)\ dt
\end{aligned}$$

Since f is continuous on compact $[-\pi, \pi]$, f is uniformly continuous.
$\implies \forall \varepsilon > 0,\ \exists \delta > 0$ such that

$$|f(x-t) - f(x)| < \varepsilon \quad \text{whenever} \quad |t| < \delta \quad \forall x$$

Then,

$$\begin{aligned}
|(f * g_n)(x) - f(x)| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)| \cdot |g_n(t)| dt \\
&= \frac{1}{2\pi}\int_{|t|<\delta} |f(x-t) - f(x)| \cdot |g_n(t)|\, dt + \frac{1}{2\pi}\int_{\delta \leq |t| \leq \pi} |f(x-t) - f(x)| \cdot |g_n(t)|\, dt \\
&\leq \varepsilon \frac{1}{2\pi}\int_{|t|<\delta} |g_n(t)| dt + 2\widetilde{M}\frac{1}{2\pi}\int_{\delta \leq |t| \leq \pi} |g_n(t)| dt \\
&\text{(where } \widetilde{M} > 0 \text{ is such that } |f(t)| \leq \widetilde{M}, \forall x \in [-\pi, \pi]) \\
&\leq M\varepsilon + 2\widetilde{M}\frac{1}{2\pi}\int_{\delta \leq |t| \leq \pi} |g_n(t)| dt
\end{aligned}$$

Since (g_n) is an approximation to the identity, $\exists N \in \mathbb{N}$ such that:

$$|(f * g_n)(x) - f(x)| < \varepsilon(M + 2\widetilde{M}) \;\forall n \geq N \quad \forall x \in [-\pi, \pi]$$

Hence, $\|f * g_n - f\|_\infty \leq \varepsilon' \quad \forall n \geq N$ $(\varepsilon' = \varepsilon(M + 2\widetilde{M}))$.
Thus, $f * g_n \to f$ uniformly on $[-\pi, \pi]$ as $n \to \infty$ (sup-norm convergence).

2) Let $f \in L^p_{2\pi}[-\pi, \pi]$, $1 \leq p < \infty$. Like previous, we have

$$(f * g_n)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x-t) - f(x)]\, g_n(t)\, dt$$

$$\begin{aligned}
\|f * g_n - f\|_p &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f * g_n(x) - f(x)|^p dx\right)^{1/p} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-t) - f(x))\;\, g_n(t)\;\, dt\right|^p \;\, dx\right)^{1/p} \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_n(t)| \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|^p\;\, dt\right) dx \text{ (using Minkowski's Integral Inequality)} \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_n(t)|\;\, \|T_t f - f\|_p\;\, dt
\end{aligned}$$

Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\|T_t f - f\|_p < \varepsilon$ whenever $|t| < \delta$.
Then,

$$
\begin{aligned}
\|f * g_n - f\|_p &\leq \frac{1}{2\pi}\int_{|t|<\delta} |g_n(t)|\ \|T_t f - f\|_p\ dt + \frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} |g_n(t)|\ \|T_t f - f\|_p\ dt \\
&\leq \varepsilon \cdot \frac{1}{2\pi}\int_{|t|<\delta} |g_n(t)|\ dt + 2\|f\|_p \cdot \frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} |g_n(t)|\ dt \\
&\leq \varepsilon M + 2\|f\|_p \cdot \frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} |g_n(t)| dt \\
&\leq \varepsilon(M + 2\|f\|_p), \quad \forall n \geq N \\
\implies\ & f * g_n \to f \text{ in } L^p_{2\pi}[-\pi,\pi] \text{ as } n \to \infty.
\end{aligned}
$$

□

2.3 Fourier Coefficients and Fourier Series

We are now equipped with the firepower to formally define and work with Fourier coefficients and Fourier series.

Definition 2.9 (Fourier coefficient)**.** Let $f \in L^1_{2\pi}[-\pi,\pi]$ and $n \in \mathbb{Z}$. Define

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx$$

$\widehat{f}(n)$ is called the n-th Fourier coefficient of f.

Definition 2.10 (Fourier series)**.** Let $f \in L^1_{2\pi}[-\pi,\pi]$. Then the formal series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}$$

$x \in [-\pi,\pi]$ is called the Fourier series of f.

Hence,

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi,\pi]$$

Remark 2.3. The Fourier series of a trigonometric polynomial is itself. Let $m, n \in \mathbb{N}$ and

$$p(\theta) = \sum_{k=-m}^{n} a_k e^{ik\theta}$$

Then $p \in L^1_{2\pi}[-\pi, \pi]$, and

$$\widehat{p}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)e^{-il\theta}d\theta$$

$$\begin{aligned}\widehat{p}(l) &= \sum_{k=-m}^{n} a_k \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)\theta}d\theta \\ &= a_l\end{aligned}$$

where,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)\theta}d\theta = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

Thus,

$$\widehat{p}(l) = \begin{cases} a_l, & -m \leq l \leq n \\ 0, & \text{otherwise} \end{cases}$$

We now ask the following three questions, which we shall answer moving forward:

1. Does this series converge?
2. If so, does it converge to f?
3. If so, in what sense?

We shall look at three means of the Fourier series: the classical means, the Cesàro means, and the Abel means. Before that, let us recall the notion of **integral operators** and **kernels**.

Definition 2.11 (Integral operator)**.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let $1 \leq p, q \leq \infty$. Let $K : X \times Y \to \mathbb{R}$ or $\mathbb{C}$ be a measurable function on $(X \times Y,\ \mathcal{A} \times \mathcal{B},\ \mu \times \nu)$. A linear transformation $T : L^p(X, \mu) \to L^q(Y, \nu)$ defined by

$$(Tf)(y) = \int_X f(x)K(x, y)\ d\mu(x)$$

is called an integral operator from $L^p(X, \mu)$ to $L^q(Y, \nu)$. K is called the kernel of the linear transformation T.

2.3.1 Classical Means of Fourier Series and Dirichlet Kernel

Definition 2.12 (N^{th} partial sum of Fourier series)**.** Let $f \in L^1_{2\pi}[-\pi,\pi]$ and let $f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta}$, $\theta \in [-\pi,\pi]$. Let $N \in \mathbb{N}$. Define the Nth partial sum of the Fourier series of f by

$$(S_N f)(\theta) = \sum_{n=-N}^{N} \widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi,\pi]$$

Remark 2.4. $S_N f$ is a trigonometric polynomial.

Observe that by the definition of the Fourier coefficient $\widehat{f}(n)$, we have:

$$\begin{aligned} S_N f(\theta) &= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}\, dt\, e^{in\theta}, \quad \theta \in [-\pi,\pi] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(\sum_{n=-N}^{N} e^{-int}e^{in\theta} \right) dt, \quad \theta \in [-\pi,\pi] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(\sum_{n=-N}^{N} e^{i(\theta-t)n} \right) dt, \quad \theta \in [-\pi,\pi] \qquad (*) \end{aligned}$$

Definition 2.13 (N^{th} Dirichlet kernel)**.** Let $N \in \mathbb{N}$, define

$$D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}, \quad \theta \in [-\pi,\pi]$$

D_N is called the N^{th} Dirichlet kernel.

Now,

$$\begin{aligned}
D_N(\theta) &= e^{-iN\theta}(1 + e^{i\theta} + e^{2i\theta} + \ldots + e^{2N\theta}) \\
&= e^{-iN\theta} \sum_{n=0}^{2N} e^{in\theta}, \quad \theta \in [-\pi, \pi] \\
&= e^{-iN\theta} \cdot \frac{1 - e^{i(2N+1)\theta}}{1 - e^{i\theta}} \\
&= \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{e^{-i\theta/2} - e^{i\theta/2}} \\
&= \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\theta/2} - e^{i\theta/2}} \\
&= \frac{2i\sin((2N+1)\frac{\theta}{2})}{2i\sin(\theta/2)} \\
&= \frac{\sin((2N+1)\frac{\theta}{2})}{\sin(\theta/2)}, \quad \theta \in [-\pi, \pi]
\end{aligned}$$

Using $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, we have:

$$\boxed{D_N(\theta) = \frac{\sin\left((2N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}} \quad \theta \in [-\pi, \pi]$$

The equation $(*)$ can be written as

$$\boxed{(S_N f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ D_N(\theta - t)\, dt} \quad \theta \in [-\pi, \pi]$$

Let $N \in \mathbb{N}$ and $\mathcal{P}$ be the set of all trigonometric polynomials. Then $S_N : L^1_{2\pi}[-\pi, \pi] \to \mathcal{P}$ is a linear transformation, and it is an integral transform with Dirichlet kernel.

Moreover, let $1 \leq p, q \leq \infty$. $S_N : L^p_{2\pi}[-\pi, \pi] \to L^q_{2\pi}[-\pi, \pi]$ is an integral transform defined by

$$(S_N f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(\theta - t) d\theta, \quad \theta \in [-\pi, \pi]$$

for each $N \in \mathbb{N}$.

We know that a series converges if the sequence of its partial sums converges. Hence, we can rewrite our previous questions:

1. Does $S_N f$ converge in some sense?

2. If so, does it converge to f?

Lemma 2.6. *The sequence of Dirichlet kernels, $\{D_N\}_{N=1}^{\infty} \subset L^1_{2\pi}[-\pi,\pi]$ is not an Approximation to the identity.*

Proof. We need to check all three properties of definition 2.8 to verify that $\{D_N\}_{N=1}^{\infty}$ is not an approximation to the identity. In fact, we show that the second condition fails, i.e., the 1-norms $\|D_N\|_1$ are not uniformly bounded.

1. $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-N}^{N} e^{ik\theta}d\theta = \sum_{k=-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}d\theta = 1$$

   (since $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}d\theta = 1$ when $k = 0$ and 0 otherwise). Thus, the first condition is satisfied.

2. We have to check whether $\|D_N\|_1$'s are uniformly bounded.

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin\left((2N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}\right| d\theta\\ &= \frac{1}{\pi}\int_{0}^{\pi}\left|\frac{\sin\left((2N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}\right| d\theta\end{aligned}$$

Break $[0,\pi]$ into intervals:

$$\left\{0, \frac{\pi}{2N+1}, \frac{2\pi}{2N+1}, \ldots, \frac{(2N+1)\pi}{2N+1} = \pi\right\}$$

Thus,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|d\theta = \frac{1}{\pi}\sum_{k=0}^{2N}\int_{\frac{k\pi}{2N+1}}^{\frac{(k+1)\pi}{2N+1}}\left|\frac{\sin\left((2N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}\right| d\theta$$

Recall: $|\sin\theta| \le |\theta|$, $\theta\in\mathbb{R}$.

$$\begin{aligned}\implies \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|d\theta &\ge \frac{1}{\pi}\sum_{k=0}^{2N}\int_{\frac{k\pi}{2N+1}}^{\frac{(k+1)\pi}{2N+1}}\left|\frac{\sin\left((2N+1)\frac{\theta}{2}\right)}{\theta/2}\right| d\theta\\ &\ge \frac{1}{\pi}\sum_{k=0}^{2N}\int_{\frac{k\pi}{2N+1}}^{\frac{(k+1)\pi}{2N+1}}\left|\sin\left((2N+1)\frac{\theta}{2}\right)\right|\frac{2}{(k+1)\pi}(2N+1)\,d\theta\\ &= \frac{1}{\pi^2}2(2N+1)\sum_{k=0}^{2N}\frac{1}{k+1}\int_{\frac{k\pi}{2N+1}}^{\frac{(k+1)\pi}{2N+1}}\left|\sin\left((2N+1)\frac{\theta}{2}\right)\right|\,d\theta\end{aligned}$$

Let $(2N+1)\frac{\theta}{2} = t \implies d\theta = \frac{2}{2N+1}dt, \quad t : \frac{k\pi}{2} \to \frac{(k+1)\pi}{2}$

$$\begin{aligned}
\implies \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|d\theta &\geq \frac{1}{\pi^2}2(2N+1)\sum_{k=0}^{2N}\frac{1}{k+1}\int_{k\pi/2}^{(k+1)\pi/2} |\sin\theta| \; \frac{2}{2N+1} \; d\theta \\
&= \frac{4}{\pi^2}\sum_{k=0}^{2N}\frac{1}{k+1}\underbrace{\int_{k\pi/2}^{(k+1)\pi/2} |\sin\theta| \; d\theta}_{=1} \\
&= \frac{4}{\pi^2}\sum_{k=0}^{2N}\frac{1}{k+1} \\
\implies \|D_N\|_1 &\geq \frac{4}{\pi^2}\sum_{k=0}^{2N}\frac{1}{k+1}
\end{aligned}$$

$$\Rightarrow \nexists M > 0 \text{ s.t. } \|D_N\|_1 \leq M \;\forall N \in \mathbb{N}$$

Hence, the second condition fails.

Thus, $\{D_N\}_{N=1}^{\infty}$ is not an approximation to the identity.

□

2.3.2 Cesàro Means of Fourier series and Fejér Kernel

Let $f \in L^1_{2\pi}[-\pi,\pi]$ and

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi,\pi]$$

be the Fourier series of f.

Let $N \in \mathbb{N}$, and $S_N f(\theta) = \sum_{n=-N}^{N} \widehat{f}(n)e^{in\theta}$ be the N^{th} partial sum of Fourier series of f.

Let $N \in \mathbb{N}$. Define the averages of the sequence of partial sums as:

$$\begin{aligned}
(\sigma_N f)(\theta) &= \frac{1}{N+1}\sum_{n=0}^{N}(S_n f)(\theta), \quad \theta \in [-\pi,\pi] \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(\theta - t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\;\frac{1}{N+1}\sum_{n=0}^{N} D_n(\theta - t)\,dt \qquad (**)
\end{aligned}$$

Definition 2.14 (Fejér kernel)**.** Let $N \in \mathbb{N}$,

$$F_N(\theta) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(\theta) \quad \theta \in [-\pi, \pi]$$

is called the **Fejér kernel**.

Observe:

$$\begin{aligned}
F_N(\theta) &= \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin\left((2n+1)\frac{\theta}{2}\right)}{\sin(\theta/2)} \\
&= \frac{1}{(N+1)\sin(\theta/2)}\sum_{n=0}^{N} \sin\left((2n+1)\frac{\theta}{2}\right) \\
&= \frac{1}{(N+1)\sin(\theta/2)}\sum_{n=0}^{N} \operatorname{Im}\left[e^{i(2n+1)\theta/2}\right] \\
&= \frac{1}{(N+1)\sin(\theta/2)} \operatorname{Im}\left(\sum_{n=0}^{N} e^{i(2n+1)\theta/2}\right) \qquad [\text{Im is linear}] \\
&= \frac{1}{(N+1)\sin(\theta/2)} \operatorname{Im}\left(e^{i\theta/2}\sum_{n=0}^{N} e^{in\theta}\right) \\
&= \frac{1}{(N+1)\sin(\theta/2)} \operatorname{Im}\left(e^{i\theta/2}\cdot\frac{1-e^{i(N+1)\theta}}{1-e^{i\theta}}\right) \\
&= \frac{1}{(N+1)\sin(\theta/2)} \operatorname{Im}\left(\frac{1-\cos((N+1)\theta) - i\sin((N+1)\theta)}{e^{-i\theta/2}-e^{i\theta/2}}\right) \\
&= \frac{1}{(N+1)\sin(\theta/2)} \operatorname{Im}\left(\frac{1-\cos((N+1)\theta)}{2i\sin(\theta/2)}\right) \\
&= \frac{1}{(N+1)\sin(\theta/2)} \left(\frac{1-\cos((N+1)\theta)}{2\sin(\theta/2)}\right) \\
&= \frac{1}{(N+1)\sin(\theta/2)} \left(\frac{2\sin^2((N+1)\frac{\theta}{2})}{2\sin(\theta/2)}\right), \quad \theta \in [-\pi, \pi] \\
&= \frac{1}{N+1}\left(\frac{\sin\left((N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}\right)^2, \quad \theta \in [-\pi, \pi]
\end{aligned}$$

From $(**)$, we have the following:

$$\boxed{(\sigma_N f)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ F_N(\theta - t)\, dt} \quad \theta \in [-\pi, \pi] \tag{2.1}$$

where,

$$\boxed{F_N(\theta) = \frac{1}{N+1}\left(\frac{\sin\left((N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}\right)^2} \tag{2.2}$$

Let $1 \le p, q \le \infty$. $\sigma_N : L^p_{2\pi}[-\pi, \pi] \to L^q_{2\pi}[-\pi, \pi]$ defined by

$$\sigma_N f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) F_N(\theta - t)\, dt$$

is a linear integral operator.

Lemma 2.7. *The sequence of Fejér kernels $\{F_N\}_{N=1}^{\infty}$ is an Approximation to the identity.*

Proof. 1.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(\theta)\, d\theta &= \frac{1}{2\pi}\frac{1}{N+1}\int_{-\pi}^{\pi}\sum_{n=0}^{N} D_n(\theta)\, d\theta \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta)\, d\theta}_{=1 \text{ by Lemma 2.6 (1)}} \\
&= \frac{1}{N+1}\sum_{n=0}^{N} 1 = 1
\end{aligned}$$

2. Recall from Equation 2.2 that $F_N \ge 0$, so

$$\|F_N\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F_N(x)|\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\, dx = 1 \quad \forall N \in \mathbb{N}$$

Which implies that the 1-norms are uniformly bounded.

3. Let $\delta > 0$,

$$\begin{aligned}
\frac{1}{2\pi}\int_{|\theta|\ge\delta}^{\pi} F_N(\theta)\, d\theta &= \frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} \frac{1}{N+1}\left(\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)}\right)^2 d\theta \\
&\le \frac{1}{N+1}\frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} \frac{1}{\sin^2(\theta/2)}\, d\theta \\
&\le \frac{1}{N+1}\frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} \frac{1}{\sin^2(\delta/2)}\, d\theta \quad (\because \sin^2(\delta/2) \le \sin^2(\theta/2) \text{ when } \delta \le |\theta|) \\
&\le \frac{1}{\sin^2(\delta/2)} \cdot \frac{1}{N+1} \to 0 \text{ as } N \to \infty.
\end{aligned}$$

Hence, $\{F_N\}_{N=1}^{\infty}$ is an approximation to the identity.

□

2.3.3 Abel Means of Fourier Series and Poisson Kernel

Let $f \in L^1_{2\pi}[-\pi, \pi]$ and $f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi, \pi]$ be the Fourier series of f. Let $r \in (0, 1)$.

Define:

$$(A_r f)(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\, r^{|n|}\, e^{in\theta}, \quad \theta \in [-\pi, \pi]$$

Observe: $|\widehat{f}(n)| \leq \|f\|_1$, $\forall n \in \mathbb{Z}$.

Exercise 2.3. Show that $A_r f$ is continuous for each $r \in (0, 1)$.

Hint:

$$M \sum_{n=-\infty}^{\infty} r^{|n|} = 2M \sum_{n=0}^{\infty} r^n = \frac{2M}{1-r} < \infty$$

Therefore, the series converges uniformly $\implies$ the limit function $A_r f$ is continuous.

Observe that:

$$\begin{aligned}
(A_r f)(\theta) &= \sum_{n=-\infty}^{\infty} \widehat{f}(n) r^{|n|} e^{in\theta}, \quad \theta \in [-\pi, \pi] \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \right) r^{|n|} e^{in\theta} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)} dt \qquad (***)
\end{aligned}$$

Definition 2.15 (Poisson kernel)**.** Let $r \in (0, 1)$. Define:

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}\, e^{in\theta}, \quad \theta \in [-\pi, \pi]$$

P_r is called the Poisson kernel.

Observe that,

$$\begin{aligned}
P_r(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|}\ e^{in\theta} \\
&= \sum_{n=-\infty}^{-1} r^{|n|}\ e^{in\theta} + \sum_{n=0}^{\infty} r^{|n|} e^{in\theta} \\
&= \sum_{n=1}^{\infty} r^n e^{-in\theta} + \sum_{n=0}^{\infty} r^n e^{in\theta} \\
&= \sum_{n=0}^{\infty} (re^{-i\theta})^n + \sum_{n=0}^{\infty} (re^{i\theta})^n - 1 \\
&= \frac{1}{1-re^{-i\theta}} + \frac{1}{1-re^{i\theta}} - 1 \\
&= \frac{1-re^{i\theta}+1-re^{-i\theta}-(1-re^{-i\theta})(1-re^{i\theta})}{(1-re^{-i\theta})(1-re^{i\theta})} \\
&= \frac{1-r^2}{1-2r\cos\theta+r^2} \qquad \forall \theta \in [-\pi,\pi]
\end{aligned}$$

Remark 2.5. $\frac{1-r^2}{1-2r\cos\theta+r^2} = \Re\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right), \quad \theta \in [-\pi,\pi],\ r \in (0,1)$

Then $(***)$ can be written as

$$\boxed{A_r f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ P_r(\theta - t)dt} \quad \theta \in [-\pi,\pi] \tag{2.3}$$

where,

$$\boxed{P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}}$$

Remark 2.6. $\forall\, r \in (0,1)$, $A_r : L^p_{2\pi}[-\pi,\pi] \to L^q_{2\pi}[-\pi,\pi]$ is a linear integral operator with integral kernel as Poisson kernel.

Lemma 2.8. *The sequence of Poisson kernels* $\{P_r\}_{r\in(0,1)}$ *is an Approximation to the identity.*

Proof. For $r \in (0,1)$:

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}, \quad \theta \in [-\pi,\pi]$$

$-1 \leq \cos\theta \leq 1$

$$\begin{aligned}
\implies -2r &\leq 2r\cos\theta \leq 2r \\
\implies 1 - 2r + r^2 &\leq 1 - 2r\cos\theta + r^2 \leq 1 + 2r + r^2 \\
\implies (1-r)^2 &\leq 1 - 2r\cos\theta + r^2 \leq (1+r)^2
\end{aligned}$$

Thus, the numerator and denominator of $P_r(\theta)$ are both positive. Hence, $P_r(\theta) \geq 0 \quad \forall\theta \in [-\pi, \pi]$.

1.
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}d\theta$$

   Recall: if $f_n \in \mathcal{R}[a,b]$ (class of all Riemann-integrable functions on $[a,b]$) and $f_n \to f$, then $\int_a^b f_n \to \int_a^b f$ (i.e., we can move the integral operator inside the summation operator).

$$\implies \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}d\theta = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}\int_{-\pi}^{\pi} e^{in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\cdot d\theta = 1$$

2. As $P_r(\theta) \geq 0$, we have:

$$\|P_r\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|P_r(\theta)|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta)\,d\theta = 1, \quad \forall r \in (0,1)$$

3. Let $\delta > 0$. Then,

$$\frac{1}{2\pi}\int_{\delta\leq|\theta|\leq\pi} P_r(\theta)\,d\theta = \frac{1}{2\pi}\int_{\delta\leq|\theta|\leq\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}d\theta$$

   Observe: $\cos\delta \geq \cos\theta, \forall\theta \in (-\pi,-\delta)\cup(\delta,\pi)$. Therefore,

$$\begin{aligned}
1 - 2r\cos\theta + r^2 &\geq 1 - 2r\cos\delta + r^2 \\
\implies \frac{1}{2\pi}\int_{\delta\leq|\theta|\leq\pi} P_r(\theta)\,d\theta &\leq \frac{1}{2\pi}\cdot\frac{1-r^2}{1-2r\cos\delta+r^2}\int_{\delta\leq|\theta|\leq\pi} 1\ d\theta \\
&\leq \frac{1-r^2}{1-2r\cos\delta+r^2} \to 0 \text{ as } r \to 1^-.
\end{aligned}$$

   Thus, $\{P_r\}_{r\in(0,1)}$ is an approximation to the identity.

□

Summary

Let $f \in L^1_{2\pi}[-\pi,\pi]$ and

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}, \quad \theta \in [-\pi,\pi]$$

be the Fourier series of f.

1. **Classical means (sequence of partial sums)**:

$$S_N f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_N(\theta - t)dt = f * D_N(\theta), \quad \forall \theta \in [-\pi,\pi]$$

where,

$$D_N(\theta) = \frac{\sin\left((2N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)} \quad \text{(Dirichlet kernel)}$$

2. **Cesàro means (averages of S_N)**:

$$\sigma_N f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) F_N(\theta - t)dt = f * F_N(\theta), \quad \theta \in [-\pi,\pi]$$

where,

$$F_N(\theta) = \frac{1}{N+1}\left(\frac{\sin\left((N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}\right)^2 \quad \text{(Fejér kernel)}$$

3. **Abel means**:

$$A_r f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) P_r(\theta - t)dt = f * P_r(\theta), \quad \theta \in [-\pi,\pi]$$

where,

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} \quad \text{(Poisson kernel)}$$

2.4 Cesàro and Abel Convergence of Fourier Series

In this section, we prove that the Fourier series of a function $f \in C_{2\pi}[-\pi,\pi]$ (or $f \in L^p_{2\pi}[-\pi,\pi]$) for $1 \le p < \infty$ converges to f in Cesàro sense and Abel sense (i.e., $\sigma_N f \to f$ and $A_r f \to f$) uniformly (or in $L^p_{2\pi}$-norm). We cannot make any comment yet on whether the Fourier series of f converges to f uniformly in the classical sense (i.e. whether $S_N f \to f$ uniformly). We shall answer this later as we build on the theory.

Theorem 2.5 (Cesàro summability of Fourier series).

1. *Let $f \in C_{2\pi}[-\pi, \pi]$. Then $\sigma_N f \to f$ uniformly as $N \to \infty$. That is, the Fourier series of a 2π-periodic continuous function f represents f uniformly in Cesàro sense.*

2. *Let $1 \leq p < \infty$ and $f \in L^p_{2\pi}[-\pi, \pi]$. Then $\sigma_N f \to f$ uniformly in $L^p_{2\pi}[-\pi, \pi]$ (i.e. with respect to L^p-norm) as $N \to \infty$. That is, Fourier series of $f \in L^p_{2\pi}[-\pi, \pi]$ represents f with respect to $\|\cdot\|_p$ in Cesàro sense.*

Proof. Recall from Equation 2.1, the definition of σ_N:

$$\sigma_N f(\theta) = f * F_N(\theta), \quad \theta \in [-\pi, \pi]$$

From Lemma 2.7, we have that the sequence of Fejér kernels $\{F_N\}_{N=1}^{\infty}$ is an Approximation to the identity.

Thus, using Theorem 2.4, we have:

1. If $f \in C_{2\pi}[-\pi, \pi]$, then $\sigma_N f = f * F_N \to f$ uniformly as $n \to \infty$.

2. If $f \in L^p_{2\pi}[-\pi, \pi]$ for $1 \leq p < \infty$, then $\sigma_N f = f * F_N \to f$ in $L^p_{2\pi}[-\pi, \pi]$.

□

Corollary 2.2. *The above theorem can be used to prove the following:*

1. *Trigonometric polynomials are dense in the space of continuous functions $C_{2\pi}[-\pi, \pi]$.*

2. *Let $1 \leq p < \infty$. Then trigonometric polynomials are dense in $L^p_{2\pi}[-\pi, \pi]$.*

Proof. Recall that $\sigma_N f$ is a trigonometric polynomial for $f \in C_{2\pi}[-\pi, \pi]$ or $f \in L^p_{2\pi}[-\pi, \pi]$. For any $f \in C_{2\pi}[-\pi, \pi]$, we thus have a sequence of trigonometric polynomials $\{\sigma_N f\}_N$ that converges uniformly to f (i.e. $\sigma_N f \to f$ uniformly). Thus, trigonometric polynomials are dense in $C_{2\pi}[-\pi, \pi]$. Similarly, they are also dense in $L^p_{2\pi}[-\pi, \pi]$. □

Corollary 2.3. *Let $f \in L^1_{2\pi}[-\pi, \pi]$ be such that $\widehat{f}(n) = 0$ $\forall n \in \mathbb{Z}$. Then $f = 0$ almost everywhere on $[-\pi, \pi]$. That is, if the Fourier coefficients of a function are 0, then the function is identically 0 almost everywhere.*

Proof. Given that $f \in L^1_{2\pi}[-\pi,\pi]$ and $\widehat{f}(n) = 0\ \forall n \in \mathbb{Z}$. Then

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n)\, e^{inx} = 0\ \forall x \in [-\pi,\pi]$$
$$\implies \sigma_N f(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n f(x) = 0\ \forall x \in [-\pi,\pi]$$

By theorem 2.5, $\sigma_N f \to f$ in $L^1_{2\pi}[-\pi,\pi]$ as $n \to \infty$
$\implies f = 0$ m-almost everywhere on $[-\pi,\pi]$. □

Corollary 2.4. *Let $f, g \in L^1_{2\pi}[-\pi,\pi]$ be such that $\widehat{f}(n) = \widehat{g}(n)\ \forall n \in \mathbb{Z}$. Then $f = g$ m-almost everywhere on $[-\pi,\pi]$. That is, if the Fourier coefficients of two functions are the same, then the functions themselves coincide almost everywhere.*

Proof. Given that $f, g \in L^1_{2\pi}[-\pi,\pi]$ and $\widehat{f}(n) = \widehat{g}(n)\ \forall n \in \mathbb{Z}$.
Let $h(x) = (f-g)(x) = f(x) - g(x),\ x \in [-\pi,\pi]$
As $\widehat{f}(n) - \widehat{g}(n) = 0\ \forall n \in \mathbb{Z}$, we have

$$\widehat{h}(n) = \widehat{(f-g)}(n) = \widehat{f}(n) - \widehat{g}(n) = 0\ \forall n \in \mathbb{Z}$$

(since $\wedge$ is linear by theorem 2.7)
$\implies h(x) = 0$ m-almost everywhere on $[-\pi,\pi]$, by corollary 2.3
$\implies f(x) = g(x)$ m-almost everywhere. □

Theorem 2.6 (Abel summability of Fourier series).

1. *Let $f \in C_{2\pi}[-\pi,\pi]$. Then $A_r f \to f$ uniformly as $r \to 1^-$. That is, the Fourier series of a 2π-periodic continuous function f represents f uniformly in Abel sense.*
2. *Let $1 \le p < \infty$ and $f \in L^p_{2\pi}[-\pi,\pi]$. Then $A_r f \to f$ uniformly in $L^p_{2\pi}[-\pi,\pi]$ (i.e. with respect to L^p-norm) as $r \to 1^-$. That is, Fourier series of $f \in L^p_{2\pi}[-\pi,\pi]$ represents f with respect to $\|\cdot\|_p$ in Abel sense.*

Proof. Recall from Equation 2.3, the definition of A_r:

$$A_r f(\theta) = f * P_r(\theta), \quad \theta \in [-\pi,\pi],\ r \in (0,1)$$

From Lemma 2.8, we have that the sequence of Poisson kernels $\{P_r\}_{r\in(0,1)}$ is an Approximation to the identity.

Thus, using Theorem 2.4, we have:

1. If $f \in C_{2\pi}[-\pi, \pi]$, then $A_r f = f * P_r \to f$ uniformly as $r \to 1^-$.
2. If $f \in L^p_{2\pi}[-\pi, \pi]$ for $1 \leq p < \infty$, then $A_r f = f * P_r \to f$ in $L^p_{2\pi}[-\pi, \pi]$ as $r \to 1^-$.

$\square$

Corollary 2.5. $\left(C_{2\pi}[-\pi, \pi], \|\cdot\|_p\right)$ *is dense in* $L^p_{2\pi}[-\pi, \pi]$ *for* $1 \leq p < \infty$.

Proof. We know that $A_r f$ is continuous for $f \in L^1_{2\pi}[-\pi, \pi]$ (Exercise 2.3). Hence, for any $f \in L^1_{2\pi}[-\pi, \pi]$, we obtain a sequence of continuous functions $A_r f$ such that $A_r f \to f$ as $r \to 1^-$ in $L^p_{2\pi}$. $\square$

2.5 Periodic Fourier Transform

Definition 2.16 (Periodic Fourier transform)**.** A map $\wedge : L^1_{2\pi}[-\pi, \pi] \to \ell^\infty(\mathbb{Z})$ defined by $\wedge(f) = \left(\widehat{f}(n)\right)_{n \in \mathbb{Z}}$ is called Periodic Fourier transform.

Remark 2.7. We have defined Fourier transform for functions in $L^1_{2\pi}[-\pi, \pi]$. However, it is defined in the same way for functions in $L^p_{2\pi}[-\pi, \pi]$ $\forall 1 \leq p < \infty$. This is because $L^p_{2\pi}[-\pi, \pi] \subset L^q_{2\pi}[-\pi, \pi]$ if $p > q$, or $L^1_{2\pi}[-\pi, \pi]$ is the largest space.

Theorem 2.7.

1. *$\wedge$ is a bounded linear transformation and $\| \wedge \| = 1$.*
2. *$\wedge$ is one-one.*
3. *Let $f \in L^1_{2\pi}[-\pi, \pi]$ and $l \in \mathbb{Z}$. The modulation of f by l is defined as $(M_l f)(\theta) = e^{il\theta} f(\theta)$. Then*
$$\widehat{M_l f}(n) = \widehat{f}(n - l) \;\forall n \in \mathbb{Z}$$
That is, modulation goes to translation under the Fourier transform.
4. *Let $f \in L^1_{2\pi}[-\pi, \pi]$ and $t \in [-\pi, \pi]$. The translation of f by t is defined as $(T_t f)(\theta) = f(\theta - t)$. Then*
$$\widehat{T_t f}(n) = e^{-int} \widehat{f}(n) \;\forall n \in \mathbb{Z}$$
That is, translation goes to modulation under the Fourier transform.

5. *Let* $f \in L^1_{2\pi}[-\pi, \pi]$. *Then*

$$\widehat{f * g}(n) = \widehat{f}(n) \cdot \widehat{g}(n)$$

That is, convolution goes to product under the Fourier transform. In the signal processing literature, this theorem is popularly known as the ***Convolution Theorem****.*

Proof. 1. **Claim 1:** $\wedge$ is linear.

Let $f, g \in L^1_{2\pi}[-\pi, \pi]$ and $\alpha \in \mathbb{F}$. We will show that $\widehat{\alpha f + g} = \alpha \widehat{f} + \widehat{g}$. For,

$$\begin{aligned}
\widehat{\alpha f + g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\alpha f + g)(\theta)\ e^{-in\theta}\ d\theta \\
&= \alpha \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\ e^{-in\theta}\ d\theta \\
&= \alpha \widehat{f}(n) + \widehat{g}(n)\ \forall n \in \mathbb{Z} \\
\implies \widehat{\alpha f + g} &= \alpha \widehat{f} + \widehat{g}
\end{aligned}$$

Claim 2: $\wedge$ is bounded and $\| \wedge \| = 1$. Indeed,

$$| \wedge (f)| = |\widehat{f}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ e^{-inx}\ dx \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\ dx = \|f\|_1$$

$\forall n \in \mathbb{Z}$. This means,

$$\|\wedge(f)\|_\infty = \sup_{n \in \mathbb{Z}} \{|\widehat{f}(n)|\} \leq \|f\|_1 \quad \forall f \in L^1_{2\pi}[-\pi, \pi]$$

$\implies$ $\wedge$ is a bounded linear transformation and $\| \wedge \| \leq 1$. To show that $\| \wedge \| = 1$, it is enough to show that $\exists$ a function f such that $\|\wedge f\|_\infty = 1$.

Consider $f(x) = 1\ \forall x \in [-\pi, \pi]$. Clearly, $f \in L^1_{2\pi}[-\pi, \pi]$ and

$$\widehat{f}(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$\implies$ $\|f\|_1 = 1$ and $\|\wedge f\|_\infty = 1$, $\implies$ $\| \wedge \| = 1$.

2.

$$\wedge(f) = 0 \implies \widehat{f}(n) = 0\ \forall n \in \mathbb{Z} \implies f = 0\ m\text{-almost everywhere (using by 2.3)}$$

Thus, $ker \wedge = 0 \implies \wedge$ is one-one.

3. **Claim:** Translation goes to modulation under the Fourier transform.

$$\begin{aligned}
\widehat{T_t f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(T_t f)(\theta)\ e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)\ e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in(\theta+t)}\ d\theta \text{ (change of variables, } \theta \mapsto \theta + t) \\
&= e^{-int}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta \\
&= e^{-int}\ \widehat{f}(n)\ \forall n \in \mathbb{Z}.
\end{aligned}$$

4. **Claim:** Modulation goes to translation under the Fourier transform.

$$\begin{aligned}
\widehat{M_l f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} M_l f(\theta)\ e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{il\theta} f(\theta)\ e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-i(n-l)\theta}\ d\theta \\
&= \widehat{f}(n-l)\ \forall n \in \mathbb{Z}.
\end{aligned}$$

5. **Claim:** Convolution goes to product under the Fourier transform (Convolution theorem).

$$\begin{aligned}
\widehat{(f * g)}(n) &= \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi}(f * g)(\theta)\ e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{t=-\pi}^{\pi} f(\theta - t)g(t)\ dt\right)\ e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} g(t)\left(\frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} f(\theta - t)\ e^{-in\theta}\ d\theta\right) dt \text{ (by Fubini's theorem)} \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} g(t)\left(\frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} f(\theta)\ e^{-in(\theta+t)}\ d\theta\right)\ dt \text{ (change of variables, } \theta \mapsto \theta + t) \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} g(t)\ e^{-int}\left(\frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta\right) dt \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} g(t)\ e^{-int}\ \widehat{f}(n)\ dt \\
&= \widehat{f}(n)\widehat{g}(n)\ \forall n \in \mathbb{Z}
\end{aligned}$$

$\square$

Theorem 2.8 (Riemann-Lebesgue lemma). *Let $f \in L^1_{2\pi}[-\pi,\pi]$. Then $|\widehat{f}(n)| \to 0$ as $n \to \infty$.*

Proof. **Method 1:**
Let $\varepsilon > 0$ be given and P be a trigonometric polynomial such that $\|f - p\|_1 < \varepsilon$ (since trigonometric polynomials are dense in $L^1_{2\pi}[-\pi,\pi]$).
We know that P would be of finite degree, and the Fourier coefficients of a trigonometric polynomial are itself. Without loss of generality, assume that $\widehat{P}(n) = 0\ \forall |n| \geq N$. Then,

$$\begin{aligned} |\widehat{f}(n)| &= |\widehat{f}(n) - \widehat{P}(n) + \widehat{P}(n)| \\ &\leq |\widehat{f}(n) - \widehat{P}(n)| + |\widehat{P}(n)| \\ &\leq \|f - P\|_1 + |\widehat{P}(n)| \\ &< \varepsilon\ \forall n \geq N \\ \implies |\widehat{f}(n)| &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Method 2:
We know,

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\theta + \frac{\pi}{n}\right)\ e^{-in\theta}\ d\theta,\ \forall n \in \mathbb{Z} \\ \implies 2\widehat{f}(n) &= \widehat{f}(n) + \widehat{f}(n) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\theta + \frac{\pi}{n}\right)\ e^{-in\theta}\ d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(\theta) - f\left(\theta + \frac{\pi}{n}\right)\right)\ e^{-in\theta}\ d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f - T_{-\frac{\pi}{n}} f\right)(\theta)\ e^{-in\theta}\ d\theta \\ \implies 2|\widehat{f}(n)| &\leq \left\|f - T_{-\frac{\pi}{n}} f\right\|_1 < \epsilon\ \forall |n| \geq N \\ \implies |\widehat{f}(n)| &\to 0 \text{ as } n \to \infty. \end{aligned}$$

$\square$

Corollary 2.6. *The map* $\wedge : L^1_{2\pi}[-\pi, \pi] \to C_0(\mathbb{Z})$ *defined by* $f \mapsto \left\{\widehat{f}(n)\right\}_{n\in\mathbb{Z}}$ *is a one-one bounded linear transformation such that* $\| \wedge \| = 1$.

Remark 2.8. $\wedge$ above is not onto. If $\wedge$ is onto, then by the Bounded Inverse Theorem, we would have $\wedge^{-1} : C_0(\mathbb{Z}) \to L^1_{2\pi}[-\pi, \pi]$ is a bounded linear transformation, i.e., $\exists\, c > 0$ such that

$$\left\|\wedge^{-1}((x_n))\right\|_1 \leq c\ \|x_n\|_\infty, \ \forall (x_n) \in C_0(\mathbb{Z}) \qquad (*)$$

Recall the Fourier coefficient of Dirichlet kernel $D_\ell(\theta) = \sum_{n=-\ell}^{\ell} e^{in\theta}$:

$$\widehat{D_\ell}(n) = \begin{cases} 1, & -\ell \leq n \leq \ell \\ 0, & \text{otherwise} \end{cases}$$

Let $\ell \in \mathbb{Z}$, define

$$x_n^{(\ell)} = \begin{cases} 1, & -\ell \leq n \leq \ell \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\wedge^{-1}\left((x_n)\right) = D_\ell \quad (\text{since } \wedge (D_\ell) = (x_n))$$

In particular, from $(*)$, we get

$$\|D_\ell\|_1 \leq c\ \left\|x_n^{(\ell)}\right\|_\infty = c, \quad \forall \ell \in \mathbb{Z}$$
$$\text{or, } \|D_\ell\|_1 \leq c \quad \forall \ell \in \mathbb{Z}$$

Which is a contradiction, because from Lemma 2.6 (2), we have that:

$$\|D_\ell\|_1 \geq \frac{1}{4\pi^2} \sum_{k=0}^{2\ell} \frac{1}{k+1}, \quad \forall \ell \in \mathbb{N}$$

Hence, $\wedge$ as defined in Corollary 2.6 is not onto.

2.5.1 Smoothness of f and Decay of $\widehat{f}$

High smoothness of f implies high decay rate of $\widehat{f}$:

- If f is 2π-periodic and continuously differentiable ($f \in C^1_{2\pi}[-\pi, \pi]$), then its Fourier coefficients $\left(\widehat{f}(n)\right)_n$ has a linear decay rate (decay of degree 1).

- If f is 2π-periodic and m-times continuously differentiable ($f \in C_{2\pi}^m[-\pi,\pi]$), then its Fourier coefficients $\left(\widehat{f}(n)\right)_n$ has a decay of degree m.

Conversely, high decay rate of $\widehat{f}$ implies high smoothness of f:

- If f is a 2π-periodic L^1 function (i.e. $f \in L_{2\pi}^1[-\pi,\pi]$) such that its Fourier coefficients are in $\ell^1(\mathbb{Z})$ (i.e., $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$), then f is continuous ($f \in C_{2\pi}[-\pi,\pi]$) m-almost everywhere on $[-\pi,\pi]$.

- Now, let us give a higher decay rate to the Fourier coefficients. If $f \in L_{2\pi}^1[-\pi,\pi]$ such that for fixed $\varepsilon > 0$,
$$\left|\widehat{f}(n)\right| \leq \frac{c}{|n|^{2+\epsilon}} \ \forall n \in \mathbb{Z}\setminus\{0\}$$
for $c > 0$, then f is continuously differentiable ($f \in C_{2\pi}^1[-\pi,\pi]$) m-almost everywhere on $[-\pi,\pi]$.

- Even higher decay rate for $\left(\widehat{f}(n)\right)_n$ now: Let $0 < \varepsilon < 1$ and $k \in \mathbb{N}$. Let $f \in L_{2\pi}^1[-\pi,\pi]$ such that
$$\left|\widehat{f}(n)\right| \leq \frac{c}{|n|^{k+1+\epsilon}} \ \forall n \in \mathbb{Z}\setminus\{0\}$$
Then f is k times continuously differentiable ($f \in C_{2\pi}^k[-\pi,\pi]$) m-almost everywhere on $[-\pi,\pi]$.

We shall establish these through the following theorems and lemmas. First, we assume smoothness conditions on the function and check the decay rate of its Fourier coefficients.

Remark 2.9. Let $f, g \in C^1[a,b]$. Then using by-parts integration, we have

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

If $f(b)g(b) = f(a)g(a)$ (f and g are periodic), then

$$\int_a^b f(x)g'(x)dx = -\int_a^b f'(x)g(x)$$

That is, we can shift the derivative to the other function inside the integral keeping a negative sign outside.

Lemma 2.9 (Fourier coefficients of the derivative of f)**.** *Let $f \in C_{2\pi}^1[-\pi,\pi]$. Then*

$$\widehat{f'}(n) = in\widehat{f}(n) \ \forall n \in \mathbb{Z}$$

That is, if f is continuously differentiable, then its Fourier coefficients $\widehat{f}(n)$ follow a linear decay rate, as $\widehat{f}(n) = \frac{\widehat{f'}(n)}{in}$ $\forall n \in \mathbb{Z} \setminus \{0\}$.

Proof. By definition, we have

$$\begin{aligned}
\widehat{f'}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx \\
&= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, (-in) e^{-inx}\, dx \text{ (using remark 2.9, as the boundary terms are 0 because of } 2\pi\text{-perio} \\
&= in \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \\
&= in\widehat{f}(n)\ \forall n \in \mathbb{Z}
\end{aligned}$$

□

Corollary 2.7 (Fourier coefficients of the m^{th} derivative of f)**.** *Let $m \in \mathbb{N}$ and $f \in C^m_{2\pi}[-\pi, \pi]$. Then*

$$\widehat{f^{(m)}}(n) = (in)^m\ \widehat{f}(n)\ \forall n \in \mathbb{Z}$$

That is, if f is m-times continuously differentiable, then $\widehat{f}(n)$ has a decay of degree m, as $\widehat{f}(n) = \frac{\widehat{f^{(m)}}(n)}{(in)^m}$, which implies that

$$\left|\widehat{f}(n)\right| \leq \frac{c_m}{|n|^m}\ \forall n \in \mathbb{Z} \setminus \{0\},\ c_m > 0$$

Proof. The proof follows from the above lemma by applying the by-parts formula m times.

□

Corollary 2.8. *Let $f \in C^\infty_{2\pi}[-\pi, \pi]$. Then $\forall m \in \mathbb{N}$, $\exists c_m > 0$ such that*

$$\left|\widehat{f}(n)\right| \leq \frac{c_m}{|n|^m}\ \forall n \in \mathbb{Z} \setminus \{0\}$$

Now, we will assume decay rate conditions on the Fourier coefficients and check the smoothness of the function. We take the function f to be in $L^1_{2\pi}[-\pi, \pi]$ to begin with, so that its Fourier coefficients are well-defined.

Theorem 2.9. *Let $f \in L^1_{2\pi}[-\pi, \pi]$ such that $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$ (i.e., its Fourier coefficients are in $\ell^1(\mathbb{Z})$). Then*

1. *$\exists$ a continuous function $g \in C_{2\pi}[-\pi, \pi]$ such that*

$$g(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\ e^{in\theta},\ \theta \in [-\pi, \pi] \text{ (uniform representation)}$$

2. *$\widehat{g}(n) = \widehat{f}(n)\ \forall n \in \mathbb{Z}$*

3. *$f = g$ m-almost everywhere on $[-\pi, \pi]$*

So that f is continuous ($f \in C_{2\pi}[-\pi, \pi]$) almost everywhere.

Proof. Given that $f \in L^1_{2\pi}[-\pi, \pi]$ and $\sum_{n \in \mathbb{Z}} |\widehat{f}(n)| < \infty$.

1. Let $\sum_{n=-\infty}^{\infty} \widehat{f}(n)\ e^{inx},\ \theta \in [-\pi, \pi]$ be the Fourier series of f.

   For $N \in \mathbb{N}$, let $S_N f(\theta) = \sum_{n=-N}^{N} \widehat{f}(n)\ e^{inx},\ \theta \in [-\pi, \pi]$ be the N^{th} partial sum of the Fourier series.

   Claim: $\{S_N f\}_{N=0}^{\infty}$ is a uniformly Cauchy sequence in $C_{2\pi}[-\pi, \pi]$.

   Let $N > M$. Then

$$|S_N f(\theta) - S_M f(\theta)| = \left| - \sum_{n=-N}^{-(M+1)} \widehat{f}(n)\ e^{inx} + \sum_{n=(M+1)}^{N} \widehat{f}(n)\ e^{inx} \right|$$

$$\implies |S_N f(\theta) - S_M f(\theta)| \leq \sum_{n=-N}^{-(M+1)} \left|\widehat{f}(n)\right| + \sum_{n=(M+1)}^{N} \left|\widehat{f}(n)\right|$$

$$< \varepsilon\ \forall N > M \geq N_0 \in \mathbb{N} \text{ (using the given hypothesis)}$$

   Thus, $\{S_N f\}_{N=0}^{\infty}$ is a uniformly Cauchy sequence in $C_{2\pi}[-\pi, \pi]$, which is complete.

   So, $\exists g \in C_{2\pi}[-\pi, \pi]$ (limit of 2π-periodic functions is also 2π-periodic) such that $S_N f \to g$ uniformly as $N \to \infty$.

   Or, $g(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\ e^{in\theta},\ \theta \in [-\pi, \pi]$ uniformly.

2. Let $n \in \mathbb{Z}$.

$$\begin{aligned}
\widehat{g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, e^{-in\theta}\, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{\infty} \widehat{f}(m)\, e^{im\theta} \right) e^{-in\theta}\, d\theta \\
&= \sum_{m=-\infty}^{\infty} \widehat{f}(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n-m)\theta}\, d\theta \text{ (as the series is uniformly convergent, we can interchange th}
\end{aligned}$$

and integral)

$$= \widehat{f}(n) \ (\because \text{ the integral is } 2\pi \text{ for } m = n, \text{ and } 0 \text{ for all other values of } m)$$

3. 2) $\implies$ 3) follows from corollary 2.4.

□

We shall now impose higher decay rates on $|\widehat{f}(n)|$ and prove increased smoothness of f.

Theorem 2.10 (Closed Graph Theorem)**.** *Let $(f_n) \subseteq C^1[a,b]$ and $x_0 \in [a,b]$. Suppose*

- *$(f_n(x_0))_{n=1}^{\infty}$ is convergent, and*
- *(f_n') converges uniformly to $g \in C[a,b]$.*

Then $\exists f \in C^1[a,b]$ such that $f_n \to f$ uniformly on $[a,b]$ and $f' = g$ on $[a,b]$, i.e.

$$\lim_n \left(\frac{d}{dx} f_n \right) = \frac{d}{dx} \left(\lim_n f_n \right)$$

Theorem 2.11. *Let $f \in L^1_{2\pi}[-\pi,\pi]$ such that $|\widehat{f}(n)| \leq \frac{c}{|n|^{2+\epsilon}}$ for fixed $\epsilon > 0$, $\forall n \in \mathbb{Z} \setminus \{0\}$. Then:*

1. *$\exists\, g \in C^1_{2\pi}[-\pi,\pi]$ such that $g(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}$ (uniform representation) for $\theta \in [-\pi,\pi]$.*
2. *$\widehat{g}(n) = \widehat{f}(n)$, $\forall n \in \mathbb{Z}$*
3. *$f = g$ almost everywhere on $[-\pi,\pi]$*
4. *$f'(\theta) = \sum_{n=-\infty}^{\infty} in\, \widehat{f}(n) e^{in\theta}, \quad \theta \in [-\pi,\pi]$*

Proof. Given that $f \in L^1_{2\pi}[-\pi,\pi]$ and $|\widehat{f}(n)| \leq \frac{c}{|n|^{2+\epsilon}}$, $\forall n \in \mathbb{Z}\setminus\{0\}$.

Let $f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta}$, $\quad \theta \in [-\pi,\pi]$ be the Fourier series of f.

Let $S_N f(\theta) = \sum_{n=-N}^{N} \widehat{f}(n)e^{in\theta}$ be the N-th partial sum of the Fourier series of f.

1. As S_N is a trigonometric polynomial, $S_N f \in C^1_{2\pi}[-\pi,\pi]$, $\forall N \in \mathbb{N}$.

$$(S_N f)'(\theta) = \sum_{n=-N}^{N} (in)\widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi,\pi]$$

Since $|\widehat{f}(n)| \leq \frac{1}{|n|^{2+\epsilon}}$, we know that $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|e^{in\theta}$ converges uniformly on $[-\pi,\pi]$ $\implies S_N f$ converges pointwise on $[-\pi,\pi]$.

Observe:

$$\left|\sum_{n=-\infty}^{\infty} in\ \widehat{f}(n)e^{in\theta}\right| \leq \sum_{n=-\infty}^{\infty} |n||\widehat{f}(n)| \leq \sum_{n=-\infty}^{\infty} |n|\frac{1}{|n|^{2+\epsilon}} = \sum_{n=-\infty}^{\infty} \frac{1}{|n|^{1+\epsilon}} < \infty$$

Assuming $h_n = in\ \widehat{f}(n)e^{in\theta}$ and $M_n = \frac{1}{|n|^{1+\epsilon}}$, we have:

$$|h_n| \leq M_n \text{ and } \sum_n M_n < \infty$$

$$\implies \sum |h_n| \text{ converges uniformly by M-test.}$$

$$\implies (S_N f)' \ (N^{th} \text{ partial sum of } \sum |h_n|) \ \to \sum_{n=-\infty}^{\infty} in\ \widehat{f}(n)e^{in\theta} \text{ uniformly, } \theta \in [-\pi,\pi]$$

$$\implies (S_N f)' \text{ converges uniformly to } h(\theta) = \sum_{n=-\infty}^{\infty} in\ \widehat{f}(n)e^{in\theta}$$

By Closed Graph Theorem, $\exists\, g \in C^1_{2\pi}[-\pi,\pi]$ such that

$$S_N f \xrightarrow{\text{uniformly}} g \quad \text{and} \quad g' = h \leftarrow (S_N f)' \text{ on } [-\pi,\pi].$$

Hence,

$$g(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta}, \text{ and } g'(\theta) = \sum_{n=-\infty}^{\infty} in\ \widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi,\pi].$$

2.

$$\begin{aligned}
\widehat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty}\widehat{f}(m)\, e^{im\theta}e^{-in\theta}d\theta \\
&= \sum_{m=-\infty}^{\infty}\widehat{f}(m)\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(n-m)\theta}d\theta}_{=1 \text{ when } m=n} \\
&= \widehat{f}(n)
\end{aligned}$$

$\implies \widehat{g}(n) = \widehat{f}(n),\ \forall n \in \mathbb{Z}.$

3. $f = g$ almost everywhere and $g' = \sum_{n=-\infty}^{\infty} in\ \widehat{f}(n)e^{in\theta} = f'$ almost everywhere.

□

Theorem 2.12. *Let $0 < \epsilon < 1$ and let $k \in \mathbb{N}$. Let $f \in L^1_{2\pi}[-\pi,\pi]$ such that $|\widehat{f}(n)| \leq \frac{c}{|n|^{k+1+\epsilon}}$, $\forall n \in \mathbb{Z}\setminus\{0\}$. Then:*

1. *$\exists$ a function $g \in C^{(k)}_{2\pi}[-\pi,\pi]$ such that $g(\theta) = \sum_{n=-\infty}^{\infty}\widehat{f}(n)e^{in\theta}$, $\theta \in [-\pi,\pi]$ (uniform representation)*
2. *$\widehat{f}(n) = \widehat{g}(n)$, $\forall n \in \mathbb{Z}$*
3. *$f = g$ m-almost everywhere on $[-\pi,\pi]$*
4. *$f^{(k)}(\theta) = g^{(k)}(\theta) = \sum_{n=-\infty}^{\infty}(in)^k\widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi,\pi]$*

Proof. The proof is similar and follows repeated application of Theorem 2.11. □

Remark 2.10. Recall from Corollary 2.7 that $\widehat{f^{(k)}}(n) = (in)^k\widehat{f}(n), \quad k \in \mathbb{N}$.

Corollary 2.9. *Let $0 < \epsilon < 1$ and $f \in L^1_{2\pi}[-\pi,\pi]$. Then $f \in C^{\infty}_{2\pi}[-\pi,\pi]$ iff $\forall k \in \mathbb{N}$, $\exists c_k > 0$ such that $|\widehat{f}(n)| \leq \frac{c_k}{|n|^k}$ for all $n \in \mathbb{Z}\setminus\{0\}$.*

Proof. ($\implies$) Trivial. If $f \in C^{\infty}_{2\pi}[-\pi,\pi]$, then $\forall k \in \mathbb{N}$,

$$|\widehat{f}(n)| = (in)^k\widehat{f}(n),$$

So that $|\widehat{f}(n)| = \frac{\widehat{f^{(k)}}(n)}{|n|^k}$, and the theorem holds with $c_k = \widehat{f^{(k)}}(n)$.

($\Longleftarrow$) For each $k > 0$, $\exists\ c_{k+2} > 0$ such that

$$|\widehat{f}(n)| \leq \frac{c_{k+2}}{|n|^{k+2}}, \ \forall n \in \mathbb{Z} \setminus \{0\}$$

$\implies f \in C_{2\pi}^{(k)}[-\pi, \pi]\ \forall k \in \mathbb{N}$ (by Theorem 2.12) $\implies f \in C_{2\pi}^{\infty}[-\pi, \pi]$ □

Example 2.2. The above theorem can allow us to get infinitely differentiable 2π-periodic functions (i.e., functions in $C_{2\pi}^{\infty}[-\pi, \pi]$) that are not trigonometric polynomials. Consider the following two examples.

1. Consider a function f whose Fourier coefficients are given by $\widehat{f}(n) = e^{-|n|}$. That is, $f(\theta) = \sum_{n=-\infty}^{\infty} e^{-|n|} e^{in\theta}$. Then $f \in C_{2\pi}^{\infty}[-\pi, \pi]$, because:

$$e^{-|n|} \leq \frac{c_k}{|n|^k}$$

for large $|n|$, as $|n|^k e^{|n|} \geq c_k$.

2. Consider a function f whose Fourier coefficients are given by $\widehat{f}(n) = \frac{1}{|n|!}$. It is easy to check that these Fourier coefficients satisfy the high decay rate condition in Corollary 2.9. Hence, $f \in C_{2\pi}^{\infty}[-\pi, \pi]$. To find the explicit form of f:

$$\begin{aligned} f(\theta) &= \sum_{n=-\infty}^{\infty} \frac{1}{|n|!} e^{in\theta} \quad \forall n \in \mathbb{Z} \\ &= \sum_{n=-\infty}^{-1} \frac{1}{|n|!} (e^{i\theta})^n + \sum_{n=0}^{\infty} \frac{1}{n!} (e^{i\theta})^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (e^{-i\theta})^n + \sum_{n=0}^{\infty} \frac{1}{n!} (e^{i\theta})^n - 1 \\ &= e^{e^{-i\theta}} + e^{e^{i\theta}} - 1 \end{aligned}$$

Hence, if you are asked to give an example of an infinitely many times 2π-periodic differentiable function other than a trigonometric polynomial, you can quote $f(\theta) = e^{e^{-i\theta}} + e^{e^{i\theta}} - 1$!

2.5.2 Some Important Theorems

In this section, we shall recall some theorems from functional analysis such as the Riesz-Fischer Theorem, and see some important theorems such as the Interpolation Theorem, Reymond-Bois Theorem, and Dirichlet Theorem.

Remark 2.11. Define the domain and the unit circle:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

Dirichlet Problem: Given a continuous function $f \in C(S^1)$, does there exist a harmonic function u on $\mathbb{D}$, such that $\lim_{r \to 1^-} u(re^{i\theta}) = f(e^{i\theta})\ \forall \theta \in [-\pi, \pi]$?

One can show that $P(re^{i\theta}) = P_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$ is harmonic on $\mathbb{D}$. Because $(P_r)_{r \in (0,1)}$ is an Approximation to the identity (see Lemma 2.8), we have (by Theorem 2.4):

$$u(re^{i\theta}) = f * P_r \xrightarrow{\text{uniformly}} f, \text{ if } f \in C_{2\pi}[-\pi, \pi].$$

Theorem 2.13. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $\{v_n\}_{n \in \mathbb{N}}$ be any orthonormal set in $\mathcal{H}$. Then the following are equivalent:*

1. $\{v_n : n \in \mathbb{N}\}$ *is an orthonormal basis of* $\mathcal{H}$.
2. $\forall x \in \mathcal{H}, \quad x = \sum_{n=1}^{\infty} \langle x, v_n \rangle v_n$
3. $\forall x \in \mathcal{H}, \quad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, v_n \rangle|^2$
4. $\forall x, y \in \mathcal{H}, \quad \langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, v_n \rangle \overline{\langle y, v_n \rangle}$
5. *Let* $x \in \mathcal{H}$, *then* $x = 0 \iff \langle x, v_n \rangle = 0\ \forall n \in \mathbb{N}$

Theorem 2.14 (Riesz-Fischer Theorem)**.** *Let $\mathcal{H}$ be a separable Hilbert space and $\{v_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$. Define*

$$T : \mathcal{H} \to \ell^2(\mathbb{N}), \quad T(x) = (\langle x, v_n \rangle)_{n=1}^{\infty}, \quad \forall x \in \mathcal{H}.$$

Then T is a linear isometric isomorphism.

Remark 2.12.

$$\sum_{i \in I} |x_i| := \sup \left\{ \sum_{i \in F} |x_i| : F \text{ is a finite subset of } I \right\},$$

and $\sum_{i \in I} |x_i| < \infty \implies \exists$ a countable $J \subset I$ such that $x_i = 0\ \forall i \in I \setminus J$ (try).

Remark 2.13. $\ell^2(I) = \{x : \sum_{i \in I} |x_i|^2 < \infty\}$ is non-separable iff I is uncountable.
$\mathcal{H} \cong \ell^2(I)$ for any Hilbert space $\mathcal{H}$ and any index set I.

Remark 2.14. Recall from Theorem 2.5: For $1 \leq p < \infty$, if $f \in L^p_{2\pi}[-\pi, \pi]$, then $\sigma_N f \to f$ in $L^p_{2\pi}[-\pi, \pi]$ as $N \to \infty$.

In particular,

$$\sigma_N f \to f \text{ in } L^2_{2\pi}[-\pi, \pi], \quad \text{for } f \in L^2_{2\pi}[-\pi, \pi]$$
$$\implies \overline{\mathcal{P}_{2\pi}[-\pi, \pi]} = L^2_{2\pi}[-\pi, \pi] \tag{1}$$

That is, the space of 2π-periodic trigonometric polynomials is dense in $L^2_{2\pi}[-\pi, \pi]$.

Let $e_n(\theta) = e^{in\theta}$, $\theta \in [-\pi, \pi]$. Then $\langle e_n, e_m \rangle = \delta_{nm} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$

So $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal set in $L^2_{2\pi}[-\pi, \pi]$ (2).

From (1) and (2), we have that $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis (ONB) of $L^2_{2\pi}[-\pi, \pi]$. L^2 in particular is a very interesting space that attracts attention because of its Hilbert space structure.

Theorem 2.15. *Let $\mathcal{H} = (L^2_{2\pi}[-\pi, \pi], \|\cdot\|_2)$ be a Hilbert space and $\{e_n = e^{in\theta} : n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2_{2\pi}[-\pi, \pi]$. Then:*

1. $\forall f \in L^2_{2\pi}[-\pi, \pi], \quad f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n$
2. $\forall f \in L^2_{2\pi}[-\pi, \pi], \quad \|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2$ *(Plancherel)*
3. $\forall f, g \in L^2_{2\pi}[-\pi, \pi], \quad \langle f, g \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$ *(Parseval)*
4. $f = 0$ *on* $L^2_{2\pi}[-\pi, \pi] \iff \langle f, e_n \rangle = 0 \ \forall n \in \mathbb{Z}$

Corollary 2.10. $L^2_{2\pi}[-\pi, \pi] \cong l^2(\mathbb{Z})$
i.e. $\wedge : L^2_{2\pi}[-\pi, \pi] \to \ell^2(\mathbb{Z})$ *defined by* $\wedge(f) = \{\widehat{f}(n)\}_{n \in \mathbb{Z}}$, $\forall f \in L^2_{2\pi}[-\pi, \pi]$ *is a linear isometric isomorphism.*

Recall: The following maps defined by $f \mapsto \left\{\widehat{f}(n)\right\}_n$ are bounded linear transformations:

- $\wedge : L^2_{2\pi}[-\pi, \pi] \to (c_0(\mathbb{Z}), \|\cdot\|_\infty)$, $\|\wedge\| \leq 1$

- $\wedge : L^2_{2\pi}[-\pi, \pi] \to (\ell^2(\mathbb{Z}), \|\cdot\|_2),\ \|\wedge\| = 1$
- $\wedge : L^1_{2\pi}[-\pi, \pi] \to \ell^\infty(\mathbb{Z}),\ \|\wedge\| \leq 1$

Theorem 2.16 (Interpolation Theorem)**.** *Suppose (X, μ), (Y, γ) are $\sigma-$finite measure spaces and $1 \leq p_1, p_2 \leq \infty,\ 1 \leq q_1, q_2 \leq \infty$. Let us define $T_1 : L^{p_1}(X) \to L^{q_1}(Y)$ be a bounded linear transformation with $\|T_1\| \leq M_1$ and $T_2 : L^{p_2}(X) \to L^{q_2}(Y)$ be a bounded linear transformation with $\|T_2\| \leq M_2$. Then $T : L^p(X) \to L^q(Y)$ is a bounded linear transformation with $\|T\| \leq M_1^\theta M_2^{1-\theta}$. where $\theta \in [0, 1]$ and $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$.*

Theorem 2.17 (Fourier Localization Theorem)**.** *Suppose $f \in L^1_{2\pi}[-\pi, \pi],\ x_0 \in [-\pi, \pi]$ and there exists $\delta > 0$ such that $f(t) = 0,\ \forall t \in (x_0 - \delta, x_0 + \delta) \subseteq [-\pi, \pi]$. Then*

$$0 = f(x_0) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx_0}$$

That is, the Fourier series is 0 on that neighborhood.

Proof. It is given that $f \in L^1_{2\pi}[-\pi, \pi],\ x_0 \in [-\pi, \pi]$ and $f(t) = 0,\ \forall t \in (x_0 - \delta, x_0 + \delta)$. Observe that,

$$\begin{aligned}
S_N f(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ D_N(x_0 - t)\ dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0 - t)\ D_N(t)\ dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0 - t)\ \frac{sin((2N+1)t/2)}{\sin(t/2)}\ dt \\
&= \frac{1}{2\pi} \int_{-\delta}^{\delta} f(x_0 - t)\ \frac{\sin((2N+1)t/2)}{\sin(t/2)}\ dt + \frac{1}{2\pi} \int_{-\delta \leq |t| \leq \pi} f(x_0 - t)\ \frac{\sin((2N+1)t/2)}{\sin(t/2)}\ dt \\
&= 0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[\delta \leq |t| \leq \pi]}(t)\ \frac{f(x_0 - t)}{\sin(t/2)}\ \sin((2N+1)t/2)\ dt
\end{aligned}$$

Now, let us define $g_{x_0}(t) = \chi_{[\delta \leq |t| \leq \pi]}(t)\ \frac{f(x_0 - t)}{\sin(t/2)}$. Clearly, we can see that $g_{x_0} \in L^1_{2\pi}[-\pi, \pi]$. So, we have

$$\begin{aligned}
S_N f(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g_{x_0}(t)\ \sin((2N+1)t/2)\ dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g_{x_0}(t)\ \frac{e^{i(2N+1)t/2} - e^{-i(2N+1)t/2}}{2i}\ dt \\
&= \frac{1}{4\pi i} \int_{-\pi}^{\pi} g_{x_0}(t)\ e^{it/2} e^{iNt}\ dt - \frac{1}{4\pi i e^{it/2}} \int_{-\pi}^{\pi} g_{x_0}(t)\ e^{-it/2} e^{-iNt}\ dt
\end{aligned}$$

Define two functions $g_1,\ g_2$ by $g_1(t) = g_{x_0}(t)\ e^{it/2}e^{iNt}$ and $g_2(t) = g_{x_0}(t)\ e^{-it/2}e^{-iNt}$. Using Riemann-Lebesgue lemma, we have

$$S_N f(x_0) = \frac{1}{2i}\widehat{g_1}(-N) - \frac{1}{2i}\widehat{g_2}(N) \to 0,\ \text{ as } N \to \infty$$

Hence, $S_N f(x_0) \to 0,\ \text{ as } N \to \infty \implies f(x_0) = 0 = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx_0}$. $\square$

Definition 2.17 ($\alpha-$Hölder function)**.** Let $\alpha \in (0,1]$. A function $f : [a,b] \to \mathbb{R}$ or $\mathbb{C}$ is called $\alpha-$Hölder function if $|f(x) - f(y)| \leq c|x-y|^{\alpha}$, for some $c > 0$ and $\forall x, y \in [a,b]$.

Definition 2.18 ($\alpha-$Hölder function in a neighborhood)**.** Let $\alpha \in (0,1]$. A function $f : [a,b] \to \mathbb{R}$ or $\mathbb{C}$ is called $\alpha-$Hölder function in the neighborhoood of $x_0 \in [a,b]$ if $\exists c > 0, \delta > 0$ such that $|f(x_0 - t) - f(x_0)| \leq c|t|^{\alpha}$, $\forall |t| < \delta$.

Remark 2.15. An $\alpha-$Hölder function is always uniformly continuous but need not be differentiable.

Theorem 2.18. *Let $f \in L^1_{2\pi}[-\pi,\pi]$ and $x_0 \in [-\pi,\pi]$. Suppose f is $\alpha-$Hölder function in the neighborhood of x_0. Then $S_N f(x_0) \to f(x_0)$ pointwise as $N \to \infty$. That is, $f(x_0) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\ e^{inx_0}$.*

Proof. It is given that f is $\alpha-$Hölder function in the neighborhood of x_0. Then $\exists c > 0$ and $\delta > 0$ such that $|f(x_0 - t) - f(x_0)| \leq c|t|^{\alpha}$, $\forall |t| < \delta$. Now, we can observe that

$$\begin{aligned}
S_N f(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0 - t)D_N(t)\ dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0)D_N(x_0 - t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0 - t)D_N(t)\ dt - \frac{1}{2\pi}f(x_0)\int_{-\pi}^{\pi} D_N(t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x_0 - t) - f(x_0)D_N(t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x_0 - t) - f(x_0)\frac{\sin(2N+1)t/2}{\sin t/2}\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(f(x_0 - t) - f(x_0))}{\sin t/2}\ \sin(2N+1)t/2\ dt
\end{aligned}$$

Consider $g_{x_0} = \frac{(f(x_0-t)-f(x_0)}{\sin t/2}$, $\forall t \in [-\pi, \pi]$, and we can also see that

$$|g_{x_0}(x)| = \begin{cases} \frac{c|t|^\alpha}{|\sin t/2|}, & \forall |t| < \delta \\ \frac{|f(x_0-t)-f(x_0)|}{|\sin t/2|}, & \text{if } \delta \le |t| \le \pi. \end{cases} \tag{2.4}$$

$$\text{or, } |g_{x_0}(x)| \le \begin{cases} \frac{\tilde{c}|t|^\alpha}{|t|}, & \forall |t| < \tilde{\delta}(\le \delta) \\ \frac{|f(x_0-t)-f(x_0)|}{|\sin t/2|}, & \text{if } \tilde{\delta} \le |t| \le \pi. \end{cases} \tag{2.5}$$

So, we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_{x_0}(t)|\ dt &= \frac{1}{2\pi}\int_{-\tilde{\delta}}^{\tilde{\delta}} |g_{x_0}(t)|\ dt + \frac{1}{2\pi}\int_{\tilde{\delta}\le|t|\le\pi} |g_{x_0}(t)|\ dt \\ &\le \frac{\tilde{c}}{2\pi}\int_{-\tilde{\delta}}^{\tilde{\delta}} |t|^{\alpha-1}\ dt + \frac{1}{2\pi}\int_{\tilde{\delta}\le|t|\le\pi} |\frac{|f(x_0-t)-f(x_0)|}{|\sin t/2|}|\ dt \\ &\le \frac{\tilde{c}}{\pi}\int_{0}^{\tilde{\delta}} |t|^{\alpha-1}\ dt + \frac{1}{2\pi}\frac{1}{|\sin\delta/2|}\int_{\tilde{\delta}\le|t|\le\pi} |f(x_0-t)-f(x_0)|\ dt \end{aligned}$$

Now, since $f \in L^1_{2\pi}[-\pi, \pi]$, hence

$$\int_0^{\tilde{\delta}} |t|^{\alpha-1}\ dt \le \infty \text{ and } \int_{\tilde{\delta}\le|t|\le\pi} |f(x_0-t)-f(x_0)|\ dt \le \infty$$

Thus the function $g_{x_0} \in L^1_{2\pi}[-\pi, \pi]$ an so,

$$\begin{aligned} S_N f(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g_{x_0}(t)\ \sin(2N+1)t/2\ dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g_{x_0}(t)\ \frac{e^{i(2N+1)t/2} - e^{-i(2N+1)t/2}}{2i} \\ &= \frac{1}{4i\pi}\int_{-\pi}^{\pi} g_{x_0}(t)e^{it/2}\ e^{iNt}\ dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} g_{x_0}(t)e^{-it/2}e^{-iNt}\ dt \end{aligned}$$

Let us define $g_1(t) = g_{x_0}(t)e^{it/2}$ and $g_2(t) = g_{x_0}(t)e^{-it/2}$, obviously these two functions $g_1, g_2 \in L^1_{2\pi}[-\pi, \pi]$. So,

$$\begin{aligned} S_N f(x_0) - f(x_0) &= \frac{1}{2i}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g_1(t)\ e^{iNt}\ dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} g_2(t)\ e^{-iNt}\ dt\right) \\ &= \widehat{g}_1(-N) - \widehat{g}_2(N) \\ &\to 0, \quad \text{as } N \to \infty \text{ (by the Riemann-Lebesgue lemma)} \end{aligned}$$

Hence, we have proved $S_N f(x_0) \to f(x_0)$ as $N \to \infty$. □

Corollary 2.11. *Let $f \in L^1_{2\pi}[-\pi, \pi]$ and f is $\alpha-$Hölder function on $[-\pi, \pi]$, then $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}, \ \forall x \in [-\pi, \pi]$.*

Remark 2.16. For any $k \in \mathbb{N}$ and $\alpha, \beta \in (0, 1]$ with $\beta \leq \alpha$ we have, $C^{\infty}_{2\pi}[-\pi, \pi] \subseteq C^k_{2\pi}[-\pi, \pi] \subseteq C^1_{2\pi}[-\pi, \pi] \subseteq Lip_{2\pi}[-\pi, \pi] \subseteq H^{\alpha}_{2\pi}[-\pi, \pi] \subseteq H^{\beta}_{2\pi}[-\pi, \pi] \subseteq C_{2\pi}[-\pi, \pi]$

Remark 2.17. Recall that the Lipschitz class of 2π-periodic functions is defined as

$$Lip_{2\pi}[-\pi, \pi] = \{f : [-\pi, \pi] \to \mathbb{C} \mid \exists k > 0 \text{ such that } |f(x) - f(y)| \leq k|x - y|\}$$

Exercise 2.4. Let $f \in Lip_{2\pi}[-\pi, \pi]$. Then show that $f(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \ e^{in\theta}, \theta \in [-\pi, \pi]$ (uniform representation).

Theorem 2.19 (Reymond-Bois Theorem)**.** *There exists a continuous function $f \in C_{2\pi}[-\pi, \pi]$ whose Fourier series diverges at $x = 0$.*

Proof. Let $N \in \mathbb{N}$ and define $T_N : C_{2\pi}[-\pi, \pi] \to \mathbb{F}$ by $T_N(f) = S_N f(0)$, i.e.

$$T_N f = (f * D_N)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)D_N(-t) \ dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)D_N(t) \ dt$$

We can clearly see that T_N is a linear functional and

$$|T_N f| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| \ |D_N(-t)| \ dt \leq \|f\|_\infty \|D_N\|_1, \ f \in C_{2\pi}[-\pi, \pi]$$

Thus T_N is bounded linear functional, i.e. $T_N \in (C_{2\pi}[-\pi, \pi])^*$ and $\|T_N\| \leq \|D_N\|_1$.
We claim that $\|T_N\| = \|D_N\|_1$.
Let $f(t) = \operatorname{sgn}(D_N(t)), \ \forall t \in [-\pi, \pi]$. We can observe that $|f(t)| \leq 1, \ \forall t \in [-\pi, \pi]$. So $f \in L^1_{2\pi}[-\pi, \pi]$. Now, let us choose $(f_n) \subseteq C_{2\pi}[-\pi, \pi]$ such that $|f_n(t)| \leq 1$ and $f_n \to f$ pointwise almost everywhere in $[-\pi, \pi]$. Hence,

$$f_n(t)D_N(t) \to f(t)D_N(t), \ \text{ pointwise a.e. on } [-\pi, \pi] \text{ as } n \to \infty, \ \forall N \in \mathbb{N}$$

$$\implies f_n(t)D_N(t) \to |D_N(t)|, \ \text{ pointwise a.e. on } [-\pi, \pi] \text{ as } n \to \infty, \ \forall N \in \mathbb{N}$$

and also we have $|f_n(t)D_N(t)| \leq |D_N(t)|, \ \forall n \in \mathbb{N}$. Hence, by the dominated convergence theorem, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)D_N(t)\ dt \to \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|\ dt$$

That is, $T_N(f_n) \to \|D_N\|_1$ as $n \to \infty$. And so, $\|T_N\| = \|D_N\|_1$. Thus, we have that $\{T_N : N \in \mathbb{N}\}$ is not a uniformly bounded sequence in $(C_{2\pi}[-\pi,\pi])^*$. By uniform boundedness principle, we can conclude that $\exists f \in C_{2\pi}[-\pi,\pi]$ such that $(T_N f)$ is not bounded sequence in $\mathbb{F}$, i.e. $(S_N f(0))$ is not bounded sequence in $\mathbb{F}$. So, there exists a continuous function whose Fourier series diverges at 0. $\square$

Next, we will state and prove Dirichlet theorem. Prior to that, let us recall some definitions.

Definition 2.19 (Piecewise continuity)**.** Let $f : [a,b] \to \mathbb{F}$ be a function. We call it a piecewise continuous function if the following are satisfied:

1. There exists $\alpha_i \in [a,b],\ i \in \{1,2,\ldots,n\}$ with $a \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le b$ such that f is continuous on $[a,b] \setminus \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$.

2. $f(\alpha_i^+), f(\alpha_i^-)$ exists, for all $1 \le i \le n$. If $\alpha_1 = a$ or $\alpha_n = b$ then we will only consider $f(\alpha_1^+)$ or $f(\alpha_n^-)$ respectively.

Definition 2.20 (Piecewise smooth)**.** Let $f : [a,b] \to \mathbb{F}$ be a function. We call it a piecewise smooth function if the following are satisfied:

1. f is a piecewise continuous function.

2. f' exists on $[a,b]$ except at finitely many points, say $\{\beta_1,\ldots,\beta_m\} \subseteq [a,b]$, and $f'(\beta_i^+), f'(\beta_i^-)$ exists for all $1 \le i \le m$.

Remark 2.18. Recall the following facts about Dirichlet kernel:

- $$D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta} = \frac{e^{i(N+1)\theta} - e^{iN\theta}}{e^{i\theta}-1} = \frac{\sin\left((2N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}, \quad \theta \in [-\pi,\pi]$$

- $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)\ d\theta = 1$$

- D_N is an even function, so

$$\frac{1}{\pi}\int_{0}^{\pi} D_N(\theta)\ d\theta = \frac{1}{\pi}\int_{-\pi}^{0} D_N(\theta)\ d\theta = 1$$

Theorem 2.20 (Dirichlet theorem)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π periodic and piecewise smooth function on $[-\pi, \pi]$. Then the Fourier series converges to the average:*

$$\sum_{n=\infty}^{\infty} \widehat{f}(n)e^{in\theta} = \frac{1}{2}(f(\theta+) + f(\theta-)), \ \forall \theta \in [-\pi, \pi]$$

Proof. Let $\theta \in [-\pi, \pi]$, $N \in \mathbb{N}$ then

$$\begin{aligned}
S_N f(\theta) - \frac{f(\theta+) + f(\theta-)}{2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+t)D_N(t)\ dt - \frac{f(\theta+)}{2} - \frac{f(\theta-)}{2} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+t)D_N(t)\ dt - \frac{f(\theta+)}{2\pi}\int_{0}^{\pi} D_N(t)\ dt - \frac{f(\theta-)}{2\pi}\int_{-\pi}^{0} D_N(t)\ dt \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \big(f(\theta+t) - f(\theta+)\big)D_N(t)\ dt + \frac{1}{2\pi}\int_{-\pi}^{0} \big(f(\theta+t) - f(\theta-)\big)D_N(t)\ dt \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \frac{f(\theta+t) - f(\theta+)}{e^{it}-1}\big(e^{i(N+1)t} - e^{iNt}\big)\ dt + \frac{1}{2\pi}\int_{-\pi}^{0} \frac{f(\theta+t) - f(\theta-)}{e^{it}-1}\big(e^{i(N
\end{aligned}$$

Now, let us consider the function

$$g_\theta(t) = \begin{cases} \frac{f(\theta+t)-f(\theta+)}{e^{it}-1}, & \text{if } 0 \le t \le \pi \\ \frac{f(\theta+t)-f(\theta-)}{e^{it}-1}, & \text{if } -\pi \le t \le 0 \end{cases} \tag{2.6}$$

We can observe that, the function is piecewise continuous on $[-\pi, \pi]$ (using L'Hospital rule). Which gives us $g_\theta \in L^1_{2\pi}[-\pi, \pi]$. Hence we have

$$S_N f(\theta) - \frac{f(\theta+) + f(\theta-)}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_\theta(t)\chi_{[0,\pi]}(t)\ \big(e^{i(N+1)t} - e^{iNt}\big)\ dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} g_\theta(t)\chi_{[-\pi,0]}(t)\ \big(e^{i(N+1)t} - e^{i} \tag{2.7}$$

And now if we consider functions

$$g_{\theta,1}(t) = g_\theta(t)\chi_{[0,\pi]}(t) \text{ and } g_{\theta,2}(t) = \int_{-\pi}^{\pi} g_\theta(t)\chi_{[-\pi,0]}(t)$$

We can easily prove that both functions are in $L^1_{2\pi}[-\pi, \pi]$. So, from the last equation (2.7), we got

$$S_N f(\theta) - \frac{f(\theta+) + f(\theta-)}{2} = \widehat{g_{\theta,1}}(-(N+1)) - \widehat{g_{\theta,1}}(-N) + \widehat{g_{\theta,2}}(-(N+1)) - \widehat{g_{\theta,2}}(-N)$$
$$\to 0 \text{ as } N \to \infty \text{ (by the Riemann-Lebesgue lemma)}$$

Thus,

$$S_N f(\theta) = \sum_{n=\infty}^{\infty} \widehat{f}(n) e^{in\theta} = \frac{1}{2}(f(\theta+) + f(\theta-)), \ \forall \theta \in [-\pi, \pi]$$

□

Remark 2.19. For non-continuous $L^{\infty}_{2\pi}$ functions, the Fourier series does not converge uniformly. This is because $\forall f \in L^{\infty}_{2\pi}[-\pi,\pi] \setminus C_{2\pi}[-\pi,\pi]$, if $S_N f \to f$ uniformly, then the limit function f must be continuous (because $S_N f$'s are continuous, being trigonometric polynomials) - but f is not continuous!

We are now marching towards knowing whether the Fourier series of functions in $L^p_{2\pi}[-\pi,\pi]$ $(1 \leq p < \infty)$ converge to the corresponding functions (in classical sense), and if so, how (whether uniformly, pointwise, or in $L^p_{2\pi}$-norm). The following theorem gives us a necessary and sufficient condition for the Fourier series of functions in $L^p_{2\pi}[-\pi,\pi]$ $(1 \leq p < \infty)$ to converge in $L^p_{2\pi}$-norm.

Theorem 2.21. *Let $1 \leq p < \infty$. The following are equivalent:*

1. *$\forall f \in L^p_{2\pi}[-\pi,\pi]$, $S_N f \to f$ in $L^p_{2\pi}[-\pi,\pi]$.*

2. *$\exists M > 0$ such that $\|S_N\| \leq M$, $\forall N \in \mathbb{N}$, where $S_N : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi]$ defined by $S_N f = f * D_N$ $\forall f \in L^p_{2\pi}[-\pi,\pi]$ is a bounded operator.*

Proof. $(\Rightarrow)$ It is obvious by using the uniform boundedness principle.

$(\Leftarrow)$ Let us recall that for a trigonometric polynomial p of degree N we have $S_M(p) = 0$, $\forall M \geq N$. We also know that for any $f \in L^p_{2\pi}[-\pi,\pi]$, $\sigma_N(f)$ is a trigonometric polynomial of degree N. Now, we are assuming $\exists M > 0$ such that $\|S_N\| \leq M$, $\forall N \in \mathbb{N}$ and let

$f \in L^p_{2\pi}[-\pi, \pi]$ then,

$$\begin{aligned}
\|S_N f - f\|_p &= \|S_N f - \sigma_N f + \sigma_N f - f\|_p \\
&= \|S_N f - S_N(\sigma_N f) + \sigma_N f - f\|_P \\
&= \|S_N(f - \sigma_N f) + \sigma_N f - f\|_p \\
&\leq \|S_N(f - \sigma_N f)\|_p + \|\sigma_N f - f\|_p \\
&\leq (\|S_N\| + 1)\ \|\sigma_N f - f\|_p \\
&\leq (M + 1)\ \|\sigma_N f - f\|_p \\
&\to 0 \ \text{ as } N \to \infty \text{ (Since } \sigma_N f \to f \text{ as } N \to \infty \text{ by Theorem 2.5)}
\end{aligned}$$

Hence, we have proved (2) $\implies$ (1) (classical convergence). □

Remark 2.20. We can observe that the partial sums $S_N : (C_{2\pi}[-\pi, \pi], \|\cdot\|_\infty) \to (C_{2\pi}[-\pi, \pi], \|\cdot\|_\infty)$ defined by $S_N f - f * D_N$, $\forall f \in (C_{2\pi}[-\pi, \pi]$ is a bounded operator. But there does not exist any $M > 0$ such that $\|S_N\| \leq M$, $\forall N \in \mathbb{N}$.

Assume the contrary: $\exists M > 0$ such that $\|S_N\| \leq M$ $\forall N \in \mathbb{N}$. We know that $\forall f \in C_{2\pi}[-\pi, \pi]$, $\sigma_N f \to f$ as $N \to \infty$ (Theorem 2.5). Then we would have $S_N f \to f$ uniformly $\forall f \in C_{2\pi}[-\pi, \pi]$, which contradicts the Reymond-Bois Theorem, as there exists a continuous function whose Fourier series diverges at 0.

Theorem 2.22. *Let $N \in \mathbb{N}$ and $S_N : L^1_{2\pi}[-\pi, \pi] \to L^1_{2\pi}[-\pi, \pi]$ defined by $S_N f = f * D_N$, $\forall f \in L^1_{2\pi}[-\pi, \pi]$. Then $\{S_N\} \subseteq \mathcal{B}(L^1_{2\pi}[-\pi, \pi])$ (i.e. S_N is a bounded linear operator), but the family is not uniformly bounded.*

Proof. For each $N \in \mathbb{N}$ we can clearly see that $\|S_N f\|_1 = \|f * D_N\|_1 \leq \|f\|_1\ \|D_N\|_1$ (using Young's inequality). Thus, each S_N is a bounded linear operator on $L^1_{2\pi}[-\pi, \pi]$.

Now, consider the sequence of Fejér kernels $\{F_M\}_{M=1}^\infty \subseteq L^1_{2\pi}[-\pi, \pi]$ with $\|F_M\|_1 = 1\ \forall M \in \mathbb{N}$. Observe that,

$$S_N(F_M) = F_M * D_N \to D_N$$

(by Theorem 2.4, since $\{F_M\}$ is an Approximation to the identity (see Lemma 2.7) and $D_N \in L^1_{2\pi}[-\pi, \pi]$ for fixed N).

This implies that $\|S_N F_M\|_1 \to \|D_N\|_1$, as $M \to \infty$.

$$\text{i.e. } \|S_N\|_= \|D_N\|_1,\ \forall N \in \mathbb{N}$$

Hence, $\{S_N\} \subseteq \mathcal{B}(L^1_{2\pi}[-\pi,\pi])$ and $\{S_N\}$ is not uniformly bounded, as $\|D_N\|_1 \to \infty$ as $N \to \infty$ (see Lemma 2.6). $\square$

Remark 2.21. There exists $f \in L^1_{2\pi}[-\pi,\pi]$ such that $S_N f \not\to f$ in $L^1_{2\pi}[-\pi,\pi]$. This is because from the above theorem, we have that the family $S_N : L^1_{2\pi}[-\pi,\pi] \to L^1_{2\pi}[-\pi,\pi]$ is not uniformly bounded. But Theorem 2.21 states that uniform boundedness is necessary for the convergence $S_N f \to f$ in $L^1_{2\pi}[-\pi,\pi]$.

Our ultimate aim for this section is to prove that for a function $f \in L^p_{2\pi}[-\pi,\pi]$ $(1 < p < \infty)$, the Fourier series of f converges to f in classical sense $(S_N f \to f)$ in $L^p_{2\pi}[-\pi,\pi]$. There is no direct method of proving this, so we shall take a roundabout by introducing the Hilbert transform H, and proving that its p-norm is bounded ($\|H\|_p < \infty$). This and Theorem 2.21 will help us in achieving our goal.

2.6 The Hilbert Transform and Applications

Let us recall that the signum function is defined by,

$$sgn(n) = \begin{cases} 1, & n > 0 \\ 0, & n = 0 \\ -1, & n < 0 \end{cases}$$

And now using the signum function, we are going to define the *Hilbert transform.*

Definition 2.21 (Hilbert transform)**.** A linear map $H : L^1_{2\pi}[-\pi,\pi] \to L^1_{2\pi}[-\pi,\pi]$ formally defined by

$$Hf(\theta) := \sum_{n=-\infty}^{\infty} -i\ sgn(n)\widehat{f}(n)e^{in\theta},\ \theta \in [-\pi,\pi],\ f \in L^1_{2\pi}[-\pi,\pi]$$

is called *Hilbert transform.*

Remark 2.22. If f is a trigonometric polynomial, then Hf is also a trigonometric polynomial (i.e., $H(\mathcal{P}) \subseteq \mathcal{P}$).

Remark 2.23. H is a densely defined linear transformation on $L^p_{2\pi}[-\pi,\pi]$, $\forall 1 \leq p \leq \infty$, i.e. $H : \mathcal{P} \subseteq L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi]$ is a linear transformation and $\mathcal{P}$ is dense in $L^p_{2\pi}[-\pi,\pi]$.

Next, we shall define some operators on $L^p_{2\pi}[-\pi,\pi]$, $\forall 1 \leq p \leq \infty$:

- $P_+ : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi]$ defined by $P_+ f(\theta) = \sum_{n=1}^{\infty} \widehat{f}(n)\ e^{in\theta}$ (Positive Riesz projection)
- $P_- : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi]$ defined by $P_- f(\theta) = \sum_{n=-\infty}^{-1} \widehat{f}(n)\ e^{in\theta}$ (Negative Riesz projection)
- $A : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi]$ defined by $Af(\theta) = \sum_{n=0}^{\infty} \widehat{f}(n)\ e^{in\theta}$
- $S_N^+ : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi]$, defined by $S_N^+ f(\theta) = \sum_{n=0}^{2N} \widehat{f}(n)\, e^{in\theta}, \quad \theta \in [-\pi,\pi], \forall f \in L^p_{2\pi}[-\pi,\pi]$

Theorem 2.23. *Let $1 < p < \infty$. Then the following are equivalent:*

1. $\|H\|_p < \infty$ *(H is a bounded linear operator on $L^p_{2\pi}[-\pi,\pi]$).*
2. $\|P_+\|_p < \infty$.
3. *There exists $M > 0$ such that $\|S_N\|_p \leq M$, $\forall N \in \mathbb{N}$.*
4. *There exists $\widetilde{M} > 0$ such that $\|S_N^+\|_p \leq \widetilde{M}$, $\forall N \in \mathbb{N}$.*
5. $\|A\|_p < \infty$.

Proof. **Note:** If $f \in L^p_{2\pi}[-\pi,\pi]$, then

$$|\widehat{f}(0)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\ dx = \|f\|_1 \leq \|f\|_p$$

Or, $|\widehat{f}(0)| \leq \|f\|_p$.

In each case, we prove the claims for trigonometric polynomials first. This is sufficient, because the space of trigonometric polynomials is dense in $L^p_{2\pi}[-\pi,\pi]$ - hence, the proof can be lifted to the entire space.

(1) $\iff$ (2):

Let f be a trigonometric polynomial. Observe that,

$$f = P_{-}f + \widehat{f}(0) + P_{+}f$$

$$\begin{aligned} Hf(\theta) &= \sum_{n=-\infty}^{\infty} -i\ \mathrm{sgn}(n)\ \widehat{f}(n)\ e^{in\theta} \\ &= \sum_{n=-\infty}^{-1} i\widehat{f}(n)\ e^{in\theta} + \sum_{n=1}^{\infty} -i\widehat{f}(n)\ e^{in\theta} \\ &= iP_{-}f(\theta) - iP_{+}f(\theta) \\ &= i(f - \widehat{f}(0) - P_{+}f) - iP_{+}f \\ &= -2iP_{+}f(\theta) + if - i\widehat{f}(0) \end{aligned}$$

Now, assume that $\|H\|_p < \infty$.

$$\begin{aligned} P_{+}f(\theta) &= \frac{i}{2}\left[Hf(\theta) - if + i\widehat{f}(0)\right] \\ \implies \|P_{+}f\|_p &\le \frac{1}{2}(\|Hf\|_p + \|f\|_p + \|f\|_p) \\ &\le \frac{1}{2}(\|H\|_p\|f\|_p + 2\|f\|_p) \\ &= \frac{1}{2}(\|H\|_p + 2)\|f\|_p,\ \forall f \in \mathcal{P}\ \text{(trigonometric polynomials)} \\ \implies \|P_{+}f\|_p &< \infty \end{aligned}$$

Similarly, we can assume that $\|P_{+}f\|_p < \infty$ and conclude that $\|H\|_p < \infty$.
Thus, (1) $\iff$ (2).

(3) $\iff$ (4):

Assume that $\|S_N\|_p \le m\ \forall N \in \mathbb{N}$. Let f be a trigonometric polynomial.

By definition,

$$S_N^+ f(\theta) = \sum_{n=0}^{2N} \widehat{f}(n) e^{in\theta}$$

$$S_N f(\theta) = \sum_{n=-N}^{N} \widehat{f}(n) e^{in\theta} = \sum_{n=0}^{2N} \widehat{f}(n-N) e^{i(n-N)\theta} = e^{-iN\theta} \sum_{n=0}^{2N} \widehat{f}(n-N) e^{in\theta}$$

Let $g(\theta) = e^{iN\theta} f(\theta),\ \theta \in [-\pi, \pi]$.

Then

$$\widehat{g}(n) = \widehat{f}(n-N),\ \forall n \in \mathbb{Z}$$

(since Fourier coefficient of modulation = translation)

and $\|g\|_p = \|f\|_p$. Then,

$$S_N f(\theta) = e^{-iN\theta} \sum_{n=0}^{2N} \widehat{g}(n)\ e^{in\theta} = e^{-iN\theta}\ S_N^+ g(\theta)$$

$$\implies \left\| S_N^+ g \right\|_p = \|S_N f\|_p \le \|S_N\|_p \|f\|_p \le M \|f\|_p = M \|g\|_p$$

for all $g \in \mathcal{P}$ (trigonometric polynomials). Or, $\left\|S_N^+\right\|_p \le \widetilde{M},\ \forall N \in \mathbb{N}$. Hence, (3) $\implies$ (4).

For the reverse implication, assume that $\left\|S_N^+\right\|_p \le \widetilde{M},\ \forall N \in \mathbb{N}$.

$$S_N^+ f(\theta) = \sum_{n=0}^{2N} \widehat{f}(n)\ e^{in\theta} = \sum_{n=-N}^{N} \widehat{f}(n+N)\ e^{i(n+N)\theta} = e^{iN\theta} \sum_{n=-N}^{N} \widehat{f}(n+N)\ e^{in\theta}$$

Let $g(\theta) = e^{-iN\theta}\ f(\theta)$.

Then, $\widehat{g}(n) = \widehat{f}(n+N)$ and $\|g\|_p = \|f\|_p$.

$$\therefore S_N^+ f(\theta) = e^{iN\theta} \sum_{n=-N}^{N} \widehat{g}(n)\ e^{in\theta} = e^{iN\theta}\ S_N g(\theta)$$

$$\implies \|S_N g(\theta)\|_p = \left\|S_N^+ f(\theta)\right\|_p \le \left\|S_N^+\right\|_p \|f\|_p \le \widetilde{M} \|g\|_p,\ \forall N$$

Hence, we have $\|S_N\|_p \le M\ \forall N$, so that (4) $\implies$ (3).

<u>(4) $\implies$ (5)</u>:

Assume that $\left\|S_N^+\right\|_p \leq M$ for all $N \in \mathbb{N}$. Let f be a trigonometric polynomial. Then,

$$Af(\theta) = \sum_{n=0}^{\infty} \widehat{f}(n)\ e^{in\theta} = \lim_{N\to\infty} S_N^+ f(\theta) \implies |Af(\theta)|^p = \lim_{N\to\infty} |S_N^+ f(\theta)|^p$$

By Fatou's lemma,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \liminf\ |S_N^+ f(\theta)|^p\, d\theta &\leq \liminf \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_N^+ f(\theta)|^p d\theta \\
\implies \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim\ |S_N^+ f(\theta)|^p\, d\theta &\leq \liminf \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_N^+ f(\theta)|^p d\theta \\
\implies \frac{1}{2\pi}\int_{-\pi}^{\pi} |Af(\theta)|^p\ \ d\theta &\leq \liminf \left\|S_N^+ f\right\|_p^p \leq M^p \left\|f\right\|_p^p \\
\implies \left\|Af\right\|_p &\leq M \left\|f\right\|_p\ \ \forall f \in \mathcal{P}
\end{aligned}$$

Therefore, $\|A\|_p < \infty$ and we have (4) $\implies$ (5).

<u>(5) $\implies$ (1)</u>:

Let f be a trigonometric polynomial, and $Af(\theta) = \sum_{n=0}^{\infty} \widehat{f}(n)\ e^{in\theta}$.

$$\begin{aligned}
Hf(\theta) &= \sum_{n=-\infty}^{\infty} -i\ \mathrm{sgn}(n)\, \widehat{f}(n)\, e^{in\theta} \\
&= i \sum_{n=-\infty}^{-1} \widehat{f}(n)\ e^{in\theta} - i \sum_{n=1}^{\infty} \widehat{f}(n)\ e^{in\theta} \\
&= i \sum_{n=1}^{\infty} \widehat{f}(-n)\ e^{-in\theta} - \sum_{n=1}^{\infty} \widehat{f}(n)\ e^{in\theta} \\
&= i \sum_{n=1}^{\infty} \widehat{f}(-n)\ e^{-in\theta} - i\ Af(\theta) + i\widehat{f}(0)
\end{aligned}$$

Let $g(\theta) = f(-\theta)\ \forall n \in \mathbb{Z}$ and $\|g\|_p = \|f\|_p$.

$$\begin{aligned}
Hf(\theta) &= -i\sum_{n=1}^{\infty}\widehat{g}(n)e^{-in\theta} - iAf(\theta) + i\widehat{f}(0) \\
&= -i\sum_{n=0}^{\infty}\widehat{g}(n)e^{-in\theta} - iAf(\theta) + i\widehat{f}(0) + i\widehat{g}(0) \\
&= -i(Ag)(-\theta) - iAf(\theta) + i\widehat{f}(0) + i\widehat{g}(0)
\end{aligned}$$

$$\implies \|Hf\|_p \le \|Ag\|_p + \|Af\|_p + |\widehat{f}(0)| + |\widehat{g}(0)| \le 2\|A\|_p \|f\|_p + 2\|f\|_p = 2(\|A\|_p + 1)\|f\|_p\,,\ \forall f \in \mathcal{P}$$

Thus, $\|H\|_p < \infty$, and (5) $\implies$ (1).

$\underline{(2) \implies (4)}$:

Assume that $\|P_+ f\|_p < \infty$.

Observe:

$$P_+ f(\theta) = \sum_{n=1}^{\infty}\widehat{f}(n)\ e^{in\theta} = \sum_{n=0}^{2N}\widehat{f}(n)\ e^{in\theta} + \sum_{n=2N+1}^{\infty}\widehat{f}(n)\ e^{in\theta} - \widehat{f}(0)$$

$$= S_N^+ f(\theta) + \sum_{n=0}^{\infty}\widehat{f}(n+2N+1)\ e^{i(n+2N+1)\theta} - \widehat{f}(0)$$

$$= S_N^+ f(\theta) + e^{i(2N+1)\theta} + \sum_{n=0}^{\infty}\widehat{f}(n+2N+1)\ e^{in\theta} - \widehat{f}(0)$$

Let $g(\theta) = e^{-i(2N+1)\theta}f(\theta), \quad \theta \in [-\pi,\pi]$.

Then $\widehat{g}(n) = \widehat{f}(n+2N+1), \forall n \in \mathbb{N}$ and $\|g\|_p = \|f\|_p$

So,

$$
\begin{aligned}
P_+f(\theta) &= S_N^+f(\theta) + e^{i(2N+1)\theta}\sum_{n=0}^{\infty}\widehat{g}(n)\ e^{in\theta} - \widehat{f}(0) \\
&= S_N^+f(\theta) + e^{i(2N+1)\theta}\sum_{n=1}^{\infty}\widehat{g}(n)\ e^{in\theta} - \widehat{f}(0) + e^{i(2N+1)}\widehat{g}(0) \\
&= S_N^+f(\theta) + e^{i(2N+1)\theta}P_+g(\theta) - \widehat{f}(0) + e^{i(2N+1)}\widehat{g}(0) \\
\implies S_N^+f(\theta) &= P_+f(\theta) - e^{i(2N+1)\theta}P_+g(\theta) + \widehat{f}(0) - e^{i(2N+1)}\widehat{g}(0) \\
\implies \left\|S_N^+f\right\|_p &\leq \|P_+\|_p\|f\|_p + \|P_+\|_p\|f\|_p + \|f\|_p + \|f\|_p \text{ (using } \|g\|_p = \|f\|_p \text{ and } |\widehat{f}(0)| \leq \|f\|_p) \\
&= 2(\|P_+\|_p + 1)\|f\|_p \quad \forall f \in \mathcal{P} \text{ (trigonometric polynomials)} \\
\implies \|S_N^+\|_p &\leq 2(\|P_+\|_p + 1) \quad \forall N \in \mathbb{N}
\end{aligned}
$$

Thus, we have shown that (2) $\implies$ (4), and with this, we can conclude that (1)-(5) are equivalent. $\square$

From the above theorem, we have seen that if the Hilbert transform is bounded (i.e., if we have $\|H\|_p < \infty$), then (2)-(5) follows (as they are equivalent). Now, we shall prove that the Hilbert transform is indeed bounded (i.e. $\|H\|_p < \infty$ is independently true).

Recall:

$$Hf(\theta) = \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\,\widehat{f}(n)\,e^{in\theta}, \quad \theta \in [-\pi, \pi]$$

Remark 2.24. Hilbert transform takes constant functions to 0:
Let $f(\theta) = a \quad \forall \theta \in [-\pi, \pi]$. Then $Hf(\theta) = 0 \quad \forall \theta$.

Theorem 2.24 (Hilbert Transform is L^p-bounded)**.** *Let $1 < p < \infty$. Then $H : L^p_{2\pi}[-\pi, \pi] \to L^p_{2\pi}[-\pi, \pi]$ is a bounded linear transformation.*

Proof. We prove this in multiple steps, by taking gradually stronger assumptions on $f \in L^p_{2\pi}[-\pi, \pi]$ and proving the theorem for each such f.

Step 1: Let f be a non-zero real-valued trigonometric polynomial with $\widehat{f}(0) = 0$.

Then $\widehat{f}(-n) = \overline{\widehat{f}(n)}\ \forall n \in \mathbb{Z}$, because:

$$\widehat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{in\theta}\,d\theta = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{f(\theta)}\ e^{-in\theta}\,d\theta} = \overline{\widehat{f}(n)} \quad \forall n \in \mathbb{Z}$$

Now, we have:

$$
\begin{aligned}
Hf(\theta) &= \sum_{n=-\infty}^{\infty} -i \ \mathrm{sgn}(n) \ \widehat{f}(n) \ e^{in\theta} \\
&= i \sum_{n=-\infty}^{-1} \widehat{f}(n)\, e^{in\theta} - i \sum_{n=1}^{\infty} \widehat{f}(n)\, e^{in\theta} \\
&= i \sum_{n=1}^{\infty} \widehat{f}(-n) e^{-in\theta} - i \sum_{n=1}^{\infty} \widehat{f}(n) e^{in\theta} \\
&= i \sum_{n=1}^{\infty} \overline{\widehat{f}(n)} e^{-in\theta} - i \sum_{n=1}^{\infty} \widehat{f}(n) e^{in\theta} \\
&= 2\,\mathrm{Re}\left(-i \sum_{n=1}^{\infty} \widehat{f}(n) e^{-in\theta} \right)
\end{aligned}
$$

$\implies$ Hf is a real-valued trigonometric polynomial.

Observe that:

$$
\begin{aligned}
(f + iHf)(\theta) &= \sum_{n=-\infty}^{\infty} \widehat{f}(n) \ e^{in\theta} - \sum_{n=-\infty}^{-1} \widehat{f}(n) \ e^{in\theta} + \sum_{n=1}^{\infty} \widehat{f}(n) \ e^{in\theta} \\
&= \sum_{n=-\infty}^{-1} \widehat{f}(n) \ e^{in\theta} + \sum_{n=1}^{\infty} \widehat{f}(n) \ e^{in\theta} - \sum_{n=-\infty}^{-1} \widehat{f}(n) \ e^{in\theta} + \sum_{n=1}^{\infty} \widehat{f}(n) \ e^{in\theta} \\
&= 2 \sum_{n=1}^{\infty} \widehat{f}(n) \ e^{in\theta}, \quad \theta \in [-\pi, \pi]. \quad (\because \widehat{f}(0) = 0)
\end{aligned}
$$

Let $k \in \mathbb{N}$,

$$
(f + iHf)^{2k}(\theta) = 2^{2k} \sum_{n=1}^{\infty} c_n \ e^{in\theta}, \quad \theta \in [-\pi, \pi]
$$

$$
\implies \frac{1}{2\pi} \int_{-\pi}^{\pi} (f + iHf)^{2k}(\theta) d\theta = 0
$$

$$
\implies \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=0}^{2k} \binom{2k}{j} \ i^j \ (Hf)^j(\theta) \ f^{2k-j}(\theta) \ d\theta = 0
$$

Taking the real part (by taking even j), we get:

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=0}^{k} \binom{2k}{2j} \ i^{2j} \ (Hf)^{2j}(\theta) \ f^{2k-2j}(\theta) \ d\theta = 0
$$

$$\implies \frac{1}{2\pi}\int_{-\pi}^{\pi}(Hf)^{2k}(\theta)\ d\theta = -(-1)^k\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{k-1}(-1)^j\binom{2k}{2j}\ (Hf)^{2j}(\theta)\ f^{2k-2j}(\theta)\ d\theta$$

$$\implies \|Hf\|_{2k}^{2k} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{k-1}\binom{2k}{2j}(Hf)^{2j}(\theta)\ f^{2k-2j}(\theta)\ d\theta = \sum_{j=0}^{k-1}\binom{2k}{2j}\left\|(Hf)^{2j}\ f^{2k-2j}\right\|_1$$

Choose $p = \frac{k}{j},\ q = \frac{k}{k-j}$ so that $1/p + 1/q = 1$. By Hölder's inequality,

$$\|Hf\|_{2k}^{2k} \leq \sum_{j=0}^{k-1}\binom{2k}{2j}\left\|(Hf)^{2j}\right\|_{k/j}\left\|f^{2k-2j}\right\|_{\frac{k}{k-j}}$$

Observe that,

$$\left\|(Hf)^{2j}\right\|_{k/j} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|(Hf)^{2j}(\theta)\right|^{k/j}d\theta\right)^{j/k} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(Hf)^{2k}(\theta)\ d\theta\right)^{j/k} = \|Hf\|_{2k}^{2j}$$

Thus,

$$\|Hf\|_{2k}^{2k} \leq \sum_{j=0}^{k-1}\binom{2k}{2j}\left(\|Hf\|_{2k}^{2j}\right)\left(\|f\|_{2k}^{2k-2j}\right) = \sum_{j=0}^{k-1}\binom{2k}{2j}\|Hf\|_{2k}^{2j}\ \ \|f\|_{2k}^{2k}\ \ \|f\|_{2k}^{-2j}$$

$$\implies \left(\frac{\|Hf\|_{2k}}{\|f\|_{2k}}\right)^{2k} \leq \sum_{j=0}^{k-1}\binom{2k}{2j}\left(\frac{\|Hf\|_{2k}}{\|f\|_{2k}}\right)^{2j} \tag{*}$$

Let $R = \frac{\|Hf\|_{2k}}{\|f\|_{2k}}$. Then $(*)$ is

$$R^{2k} \leq \sum_{j=0}^{k-1}\binom{2k}{2j}R^{2j}$$

$$\implies \exists\ c_{2k} > 0 \text{ such that } R \leq c_{2k} \text{ (otherwise } (*) \text{ fails)}$$

$$\implies \|Hf\|_{2k} \leq c_{2k}\,\|f\|_{2k}$$

for all non-zero real-valued trigonometric polynomials with $\widehat{f}(0) = 0$.

Step 2: Let f be a real-valued trigonometric polynomial (with no additional assumptions this time). Then,

$$(Hf)(\theta) = H(f - \widehat{f}(0))(\theta)$$

and $f - \widehat{f}(0) = g$ is a real-valued trigonometric polynomial with $\widehat{g}(0) = 0$. By step 1,

$$\implies \|Hf\|_{2k} \leq c_{2k} \left\|f - \widehat{f}(0)\right\|_{2k} \leq c_{2k} \left(\|f\|_{2k} + \|f\|_{2k}\right)$$

$$\implies \|Hf\|_{2k} \leq 2c_{2k} \ \|f\|_{2k} \qquad \forall f \text{ real-valued trigonometric polynomials.}$$

Step 3: Let $f \in \mathcal{P}$ (space of all trigonometric polynomials). Write $f = P + iQ$, where P, Q are real-valued trigonometric polynomials. Observe that,

$$\|P\|_{2k} \leq \|f\|_{2k}, \text{ and } \|Q\|_{2k} \leq \|f\|_{2k} \text{ (why?)}$$

Then,

$$\begin{aligned}
\|Hf\|_{2k} = \|H(P+iQ)\|_{2k} &\leq \|H(P)\|_{2k} + \|H(Q)\|_{2k} \\
&\leq 2c_{2k} \ \|P\|_{2k} + 2c_{2k} \ \|Q\|_{2k} \\
&\leq 4c_{2k} \ \|f\|_{2k} \\
\implies \|Hf\|_{2k} &\leq 4c_{2k} \ \|f\|_{2k} \quad \forall f \in \mathcal{P}
\end{aligned}$$

i.e. $\forall k \in \mathbb{N}$, $H : L^{2k}_{2\pi}[-\pi, \pi] \to L^{2k}_{2\pi}[-\pi, \pi]$ is a bounded linear transformation (since trigonometric polynomials are dense in $L^{2k}_{2\pi}[-\pi, \pi]$ $\forall k \in \mathbb{N}$).

So,

$$\begin{aligned}
&H : L^{2}_{2\pi}[-\pi, \pi] \to L^{2}_{2\pi}[-\pi, \pi], \\
&H : L^{4}_{2\pi}[-\pi, \pi] \to L^{4}_{2\pi}[-\pi, \pi], \\
&\qquad\qquad \cdots \\
&H : L^{2k}_{2\pi}[-\pi, \pi] \to L^{2k}_{2\pi}[-\pi, \pi] \text{ are bounded linear transformations.}
\end{aligned}$$

By Interpolation Theorem, we can see that $\forall 2 \leq p < \infty$,

$$\|Hf\|_p \leq c_p \|f\|_p \quad \forall f \in L^p_{2\pi}[-\pi, \pi]$$

Thus, so far we have proved that the Hilbert transform is L^p-bounded for $2 \leq p < \infty$, and it remains to show the same for $1 < p < 2$.

Let $1 < p < 2$ and $f \in \mathcal{P}$. Then

$$\|Hf\|_p = \sup\{|\langle Hf, g\rangle| : g \in L^{p}_{2\pi}[-\pi,\pi],\ \|g\|_p \leq 1\} \tag{**}$$

Observe that,

$$\begin{aligned}
\langle Hf, g\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Hf(\theta)\ \overline{g(\theta)}\, d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\, \widehat{f}(n)\ e^{in\theta}\ \overline{g(\theta)}\, d\theta \\
&= \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\, \widehat{f}(n) \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\ e^{-in\theta}\ d\theta}
\end{aligned}$$

$$\begin{aligned}
\implies \langle Hf, g\rangle &= \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\, \widehat{f}(n)\, \overline{\widehat{g}(n)} \\
&= \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\ \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta\right)\ \overline{\widehat{g}(n)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\ \overline{\widehat{g}(n)}\ e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ \overline{\sum_{n=-\infty}^{\infty} i\,\mathrm{sgn}(n)\ \widehat{g}(n)\ e^{in\theta}}\ d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ \overline{(-Hg)(\theta)}\ d\theta \\
&= \langle f, -Hg\rangle = -\langle f, Hg\rangle \\
\implies \langle Hf, g\rangle &= -\langle f, Hg\rangle\ \forall f \in \mathcal{P},\ \forall g \in L^{p'}_{2\pi}[-\pi,\pi]
\end{aligned}$$

From $(**)$,

$$\begin{aligned}
\|Hf\|_p &= \sup\left\{|\langle Hf, g\rangle| : g \in L^{p'}_{2\pi}[-\pi,\pi],\ \|g\|_{p'} \le 1\right\} \\
&= \sup\left\{|\langle f, Hg\rangle| : g \in L^{p'}_{2\pi}[-\pi,\pi],\ \|g\|_{p'} \le 1\right\} \\
&\le \sup\left\{\|f\|_p \|Hg\|_{p'} : g \in L^{p'}_{2\pi}[-\pi,\pi],\ \|g\|_{p'} \le 1\right\} \\
&\le \|f\|_p \ c_{p'} \\
\implies \|Hf\|_p &\le c_{p'} \|f\|_p \quad \forall f \in \mathcal{P}
\end{aligned}$$

Therefore,

$$\forall 1 < p < 2, \quad H : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi] \text{ is a bounded linear transformation}$$

and thus, we have that

$$\forall 1 < p < \infty, \quad H : L^p_{2\pi}[-\pi,\pi] \to L^p_{2\pi}[-\pi,\pi] \text{ is a bounded linear transformation.}$$

□

Now, we are ready to achieve our goal, which is proving the following theorem.

Theorem 2.25. *Let $1 < p < \infty$. For all $f \in L^p_{2\pi}[-\pi,\pi]$, the Fourier series of f converges to f in $L^p_{2\pi}[-\pi,\pi]$ in the classical sense; that is,*

$$S_N f \to f \text{ in } L^p_{2\pi}[-\pi,\pi] \text{ as } N \to \infty$$

Proof. Let $1 < p < \infty$. Recall that the following are equivalent:

1. $\forall f \in L^p_{2\pi}[-\pi,\pi],\ S_N f \to f$ in $L^p_{2\pi}[-\pi,\pi]$.
2. $\exists M > 0$ such that $\|S_N\|_p \le M\ \forall N \in \mathbb{N}$.
3. $\|H\|_p < \infty$.

We have proved (1) $\iff$ (2) (Theorem 2.21), (2) $\iff$ (3) (Theorem 2.23), and that (3) is independently true (Theorem 2.24). Thus, (3) $\implies$ (1), and we have our result. □

3 Periodic Distribution Theory

We shall follow the notation $\mathscr{D} = C_{2\pi}^{\infty}[-\pi, \pi]$ and call functions in $\mathscr{D}$ as **test functions**.

Definition 3.1 (Convergence of test functions in $\mathscr{D}$)**.** Let $(\varphi_n)_n$ be a sequence of test functions and $\varphi \in \mathscr{D}$. We say that $\varphi_n \to \varphi$ in $\mathscr{D}$ as $n \to \infty$ if

$$\forall m > 0, \frac{d^m \varphi_n}{dx^m} \to \frac{d^m \varphi}{dx^m} \text{ uniformly on } [-\pi, \pi] \text{ as } n \to \infty$$

Definition 3.2 (Norm on test functions)**.** Let $N \in \mathbb{N}$. Define

$$\|\varphi\|_N = \sum_{m=0}^{N} \left\| \frac{d^m \varphi}{dx^m} \right\|_\infty \text{ for } \varphi \in C_{2\pi}^{\infty}[-\pi, \pi] = \bigcap_{m=0}^{\infty} C_{2\pi}^{m}[-\pi, \pi]$$

Recall from functional analysis that $C^1[a, b]$ is complete with respect to the norm defined as $\|f\|_* = \|f\|_\infty + \|f'\|_\infty$. The above norm is defined with a similar motivation, with summation of $N + 1$ sup-norms of the function and it's derivatives upto the N^{th} derivative.

Exercise 3.1. Check that for each $N \geq 0$, $\|\cdot\|_N$ is a norm on $\mathscr{D} = C_{2\pi}^{\infty}[-\pi, \pi]$. However, $(C_{2\pi}^{\infty}[-\pi, \pi], \|\cdot\|_N)$ is not a complete normed linear space (Banach).

Definition 3.3 (Metric on $\mathscr{D}$)**.** Let $\varphi, \psi \in \mathscr{D}$. Define

$$D(\varphi, \psi) = \sum_{N=0}^{\infty} \frac{1}{2^N} \frac{\|\varphi - \psi\|_N}{1 + \|\varphi - \psi\|_N}$$

Exercise 3.2.

1. Show that $(C_{2\pi}^{\infty}[-\pi, \pi], D)$ is a complete metric space.
2. Let $(\varphi_n) \subseteq C_{2\pi}^{\infty}[-\pi, \pi]$ (i.e. $(\varphi_n)_n$ is a sequence of test functions) and $\varphi \in C_{2\pi}^{\infty}[-\pi, \pi]$. Then $\varphi_n \to \varphi$ as $n \to \infty$ according to definition 3.1 if and only if $D(\varphi_n, \varphi) \to 0$ as $n \to \infty$.

Remark 3.1. There does not exist any norm on $C_{2\pi}^{\infty}[-\pi, \pi]$ which induces D, because D is a bounded metric. Hence, $C_{2\pi}^{\infty}[-\pi, \pi]$ is not a Banach space with respect to any norm.

Exercise 3.3. Show that $(C_{2\pi}^{\infty}[-\pi, \pi], D)$ is a Topological vector space.

Remark 3.2. Recall that a function whose Fourier coefficients has 'very high decay rate' is 'very smooth' (i.e., infinitely many times differentiable). In particular, by corollary 2.8,

$$\begin{aligned}\varphi \in C^{\infty}_{2\pi}[-\pi,\pi] &\iff \forall k \in \mathbb{N},\ \exists c_k > 0 \text{ such that } |\widehat{\varphi}(n)| \leq \frac{c_k}{|n|^k}\ \forall n \in \mathbb{Z}\setminus\{0\} \\ &\iff \forall k \in \mathbb{N},\ \exists c'_k > 0 \text{ such that } |\widehat{\varphi}(n)| \leq \frac{c_k}{(1+n^2)^{k/2}}\ \forall n \in \mathbb{Z}\end{aligned}$$

Example 3.1 (Test functions)**.** Following are some examples of test functions (functions in $C^{\infty}_{2\pi}[-\pi,\pi]$):

1. $\mathcal{P}$ (trigonometric polynomials) $\subseteq C^{\infty}_{2\pi}[-\pi,\pi]$.
2. $\sum_{n=-\infty}^{\infty} \frac{1}{|n|!} e^{inx} \in C^{\infty}_{2\pi}[-\pi,\pi]$ (because the Fourier coefficients $\widehat{f} = \frac{1}{|n|!}$ satisfies the high decay condition).
3. $\sum_{n=-\infty}^{\infty} \frac{1}{|n|^{|n|}} e^{inx} \in C^{\infty}_{2\pi}[-\pi,\pi]$

Lemma 3.1. *Let $\varphi, \psi \in \mathscr{D}$. Then*

1. *$\widetilde{\varphi} \in \mathscr{D}$, where $\widetilde{\varphi}(x) = \varphi(-x)$ (reflection of φ)*
2. *$\varphi\psi \in \mathscr{D}$*
3. *$\varphi * \psi \in \mathscr{D}$ and $(\varphi * \psi)^{(n)} = \varphi^{(n)} * \psi = \varphi * \psi^{(n)}$*
4. *$T_x\varphi \in \mathscr{D}$, where $T_x(\varphi(t)) = \varphi(t-x)$ (translation by x, $\forall x \in \mathbb{R}$)*
5. *$\overline{\varphi} \in \mathscr{D}$, where $\overline{\varphi}(x) = \overline{\varphi(x)}$ $\forall x \in \mathbb{R}$ (conjugation)*

Definition 3.4 (Rapidly decreasing sequence)**.** A sequence $(a_n)_{n\in\mathbb{Z}}$ is called as a rapidly decreasing sequence in $\mathbb{F}$ if $\forall k \in \mathbb{N},\ \exists c_k > 0$ such that

$$|a_n| \leq \frac{c_k}{(1+n^2)^{k/2}}\ \forall n \in \mathbb{Z}$$

Definition 3.5 (Schwartz sequence space)**.** The collection of all rapidly decreasing sequences in $\mathbb{F}$ is called Schwartz sequence space over $\mathbb{F}$ and is denoted by $S(\mathbb{Z})$. That is,

$$S(\mathbb{Z}) = \{(a_n) : (a_n) \text{ is a rapidly decreasing sequence}\}$$

Remark 3.3. By remark 3.2, $\varphi \in C^{\infty}_{2\pi}[-\pi,\pi] \implies (\widehat{\varphi}(n))_{n\in\mathbb{Z}}$ is a rapidly decreasing sequence, i.e. $\varphi \in C^{\infty}_{2\pi}[-\pi,\pi] \implies (\widehat{\varphi}(n))_{n\in\mathbb{Z}} \in S(\mathbb{Z})$.

Lemma 3.2. *Let $l \in \mathbb{N}$ and $(a_n) \in S(\mathbb{Z})$. Then $|n|^l a_n \to 0$ as $n \to \infty$.*

Proof. We know that as $(a_n) \in S(\mathbb{Z})$, $\forall k \in \mathbb{N},\ \exists c_k > 0$ such that

$$\begin{aligned} |a_n| &\leq \frac{c_k}{(1+n^2)^{k/2}} \ \forall n \in \mathbb{Z} \\ \implies |n|^l |a_n| &\leq \frac{|n|^l\ c_k}{(1+n^2)^{k/2}} \end{aligned}$$

As this holds for all $k \in \mathbb{N}$, choose $k = l+1$. Then

$$|n|^l |a_n| \leq \frac{|n|^l\ c_k}{(1+n^2)^{\frac{l+1}{2}}} \to 0 \text{ as } |n| \to \infty$$

Note that this is why (a_n) is called as 'rapidly decreasing'. $\square$

Definition 3.6 (Norm in $S(\mathbb{Z})$)**.** Let $N \in \mathbb{N}$ and $(a_n) \in S(\mathbb{Z})$. Define

$$\|(a_n)\|_N = \sup_{n\in\mathbb{Z}} \left\{(1+n^2)^{N/2}\ |a_n|\right\}$$

Exercise 3.4. Check that $\|\cdot\|_N$ as defined above is a norm on $S(\mathbb{Z})$ $\forall N \in \mathbb{N}$.

Definition 3.7 (Metric on $S(\mathbb{Z})$)**.** Let $(a_n), (b_n) \in S(\mathbb{Z})$. Define

$$d((a_n),(b_n)) = \sum_{N=0}^{\infty} \frac{1}{2^N} \frac{\|(a_n)-(b_n)\|_N}{1+\|(a_n)-(b_n)\|_N}$$

Lemma 3.3. *$(S(\mathbb{Z}), d)$ is a complete metric space.*

Exercise 3.5. Show that $(S(\mathbb{Z}), d)$ is a Topological vector space and that $C_{00}(\mathbb{Z})$ (space of all eventually zero sequences) is dense in $(S(\mathbb{Z}), d)$.

Definition 3.8 (Convergence in $S(\mathbb{Z})$)**.** Let $(x_n)_{n\in\mathbb{Z}}$ be a sequence in $S(\mathbb{Z})$ (i.e., $(x_n)_{n\in\mathbb{Z}}$ is a sequence of rapidly decreasing sequences). For fixed $n \in \mathbb{Z}$, let $x_n = (x_n(m))_{m=-\infty}^{\infty} \in S(\mathbb{Z})$. We say that $x_n \to 0$ in $S(\mathbb{Z})$ if

$$\forall k \in \mathbb{N},\ \sup_{n\in\mathbb{Z}}\{|m|^k\ |x_n(m)|\} \to 0 \text{ as } n \to \infty$$

Exercise 3.6. Let $(x_n)_{n\in\mathbb{Z}}$ be a sequence in $S(\mathbb{Z})$. Then show that $x_n \to 0$ in $S(\mathbb{Z})$ by the above definition if and only if $d\left((x_n), 0\right) \to 0$ as $n \to \infty$.

Theorem 3.1. *The periodic Fourier transform*

$$\wedge : C^\infty_{2\pi}[-\pi,\pi] \to S(\mathbb{Z}) \text{ defined by } f \mapsto \left(\widehat{f}(n)\right)_{n\in\mathbb{Z}}$$

is a topological isomorphism (i.e., $\wedge$ is a homeomorphism - bijective and bicontinuous).

Proof. We know that if $f \in C^\infty_{2\pi}[-\pi,\pi]$, then $\forall k \in \mathbb{N}$, $\exists c_k > 0$ such that $|\widehat{f}(n)| \leq \frac{c_k}{(1+n^2)^{k/2}}$ $\forall n \in \mathbb{Z}$ by remark 3.3. Thus, $\left(\widehat{f}(n)\right)_{n\in\mathbb{Z}} \in S(\mathbb{Z})$ and the map $\wedge$ is well-defined.

Claim: $\wedge$ is one-one.

We need to show that $\wedge(f) = 0 \iff f = 0\ \forall f \in \mathscr{D}$. Let $f = 0$. Then

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}\ dx = 0\ \forall n \in \mathbb{Z}$$
$$\implies \wedge(f) = \left(\widehat{f}(n)\right)_{n\in\mathbb{Z}} = 0$$

Conversely, let $\wedge(f) = \left(\widehat{f}(n)\right)_{n\in\mathbb{Z}} = 0$. That is, $\widehat{f}(n) = 0\ \forall n \in \mathbb{Z} \implies f = 0$ almost everywhere by corollary 2.3.

Claim: $\wedge$ is onto.

We need to show that for every rapidly decreasing sequence $(a_n) \in S(\mathbb{Z})$, $\exists$ a function $f \in C^\infty_{2\pi}[-\pi,\pi]$ whose Fourier coefficients are $(a_n)_{n\in\mathbb{Z}}$.
Let $(a_n) \in S(\mathbb{Z})$ and define $f(x) = \sum_{n=-\infty}^{\infty} a_n\ e^{inx}$.
We shall show that $f(x) \in C^\infty_{2\pi}[-\pi,\pi]$ and $\widehat{f}(n) = a_n\ \forall n \in \mathbb{Z}$.

First, we claim that the series $\sum_{n=-\infty}^{\infty} a_n\ e^{inx}$ converges uniformly. Indeed, we have that $\forall k \in \mathbb{N}$, $\exists c_k > 0$ such that $|a_n\ e^{inx}| \leq \frac{c_k}{(1+n^2)^{k/2}}$ $\forall n \in \mathbb{Z}$
and by p-test, $\sum_n \frac{c_k}{(1+n^2)^{k/2}} < \infty$ $(k \geq 2)$.
Thus, by Weierstrass' M-test, the series $\sum_{n=-\infty}^{\infty} a_n\ e^{inx}$ converges uniformly.

Now,

$$
\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}\ dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} a_m\ e^{imx}\right)\ e^{-inx}\ dx \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} a_m \int_{-\pi}^{\pi} e^{i(m-n)x}\ dx \ \ \text{(we can interchange summation and integral as the series converges} \\
&= a_n \text{ (the integral is equal to } 2\pi \text{ for } m=n\text{, and 0 otherwise)}
\end{aligned}
$$

Thus, $\forall n \in \mathbb{Z}$,

$$
\begin{aligned}
\widehat{f}(n) &= a_n \\
\implies |\widehat{f}(n)| &\leq \frac{c_k}{(1+n^2)^{k/2}}\ \forall k \in \mathbb{N} \\
\implies f &\in C_{2\pi}^{\infty}[-\pi,\pi] \text{ (by remark 3.2)}
\end{aligned}
$$

That $\wedge$ and $\wedge^{-1}$ are continuous is left to the reader as an exercise.

□

Lemma 3.4. *The following maps are continuous.*

1. $\sim : \mathscr{D} \to \mathscr{D}$ *by* $\varphi \mapsto \widetilde{\varphi}$

2. $: \mathscr{D} \to \mathscr{D}$ *by* $\varphi \mapsto \overline{\varphi}$

3. $u : \mathscr{D} \to \mathscr{D}$ *by* $\varphi \mapsto u\varphi$

4. $* : \mathscr{D} \to \mathscr{D}$ *by* $\varphi \mapsto \varphi * u,\ u \in \mathscr{D}$

5. $\frac{d^m}{dx^m} : \mathscr{D} \to \mathscr{D}$ *by* $\varphi \mapsto \varphi^{(m)},\ \forall m \in \mathbb{N}$

Lemma 3.5. *Let* $\varphi, \psi \in \mathscr{D}$. *Then:*

1. $\widehat{\widetilde{\varphi}}(n) = \widehat{\varphi}(-n)\ \forall n \in \mathbb{Z}$

2. $\widehat{\overline{\varphi}}(n) = \overline{\widehat{\varphi}(-n)}\ \forall n \in \mathbb{Z}$

3. $\widehat{\varphi * \psi}(n) = \widehat{\varphi}(n) \cdot \widehat{\psi}(n)\ \forall n \in \mathbb{Z}$

Proof. 1.

$$\begin{aligned}
\widehat{\widetilde{\varphi}}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{\varphi}(n)\ e^{-inx}\ dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(-x)\ e^{-inx}\ dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)\ e^{inx}\ dx \text{ (applying the transformation } x \mapsto -x) \\
&= \widehat{\varphi}(-n)
\end{aligned}$$

2.

$$\widehat{\overline{\varphi}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\varphi}(x)\ e^{-inx}\ dx = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)\ e^{-inx}\ dx} = \overline{\widehat{\varphi}(-n)}$$

3.

$$\begin{aligned}
\widehat{\varphi * \psi}(n) &= \frac{1}{2\pi}\int_{x=-\pi}^{\pi} (\varphi * \psi)(x)\ e^{-inx}\ dx \\
&= \frac{1}{2\pi}\int_{x=-\pi}^{\pi} \left(\frac{1}{2\pi}\int_{t=-\pi}^{\pi} \varphi(t)\psi(x-t)\ dt\right)\ e^{-inx}\ dx \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} \varphi(t)\ \frac{1}{2\pi}\int_{x=-\pi}^{\pi} \psi(x-t)\ e^{-inx}\ dx\ dt \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} \varphi(t)\ \frac{1}{2\pi}\int_{x=-\pi}^{\pi} \psi(x)e^{-in(x+t)}\ dx\ dt \\
&= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} \varphi(t)e^{-int}dt \cdot\ \frac{1}{2\pi}\int_{x=-\pi}^{\pi} \psi(x)\ e^{-inx}dx \\
&= \widehat{\varphi}(n)\cdot\widehat{\psi}(n)
\end{aligned}$$

□

Exercise 3.7. Show that the Fourier series of a test function converges to the test function in $\mathscr{D}$. That is, for $\varphi \in \mathscr{D}$, we have

$$\varphi(x) \overset{\mathscr{D}}{=} \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\ e^{inx} \overset{\mathscr{D}}{=} \lim_{N\to\infty} \sum_{|n|\leq N} \widehat{\varphi}(n)\ e^{inx}$$

3.1 Periodic Distributions

Definition 3.9 (Periodic distribution)**.** A linear functional $T : C^{\infty}_{2\pi}[-\pi,\pi] \to \mathbb{F}$ is called a periodic distribution if T is a continuous linear functional. That is, any continuous linear functional on $C^{\infty}_{2\pi}[-\pi,\pi]$ is called a periodic distribution.

Remark 3.4. Periodic distrbutions are also called generalized periodic functions on $\mathbb{F}$.

Remark 3.5. $(C^\infty_{2\pi}[-\pi,\pi])^*$, the dual space of $C^\infty_{2\pi}[-\pi,\pi]$ is the space of all periodic distributions or generalized periodic functions. Remember that we are denoting $C^\infty_{2\pi}[-\pi,\pi]$ by $\mathscr{D}$. The notation we shall follow for the space of periodic distributions is $\mathscr{D}' = (C^\infty_{2\pi}[-\pi,\pi])^*$.

Let us look at some examples of periodic distributions. We shall explore periodic distributions generated by functions, measures, and also show that there exists distributions which can be generated by neither functions nor measures!

3.1.1 Periodic Distributions Generated by Functions

Let $f \in L^1_{2\pi}[-\pi,\pi]$. Define $T_f : C^\infty_{2\pi}[-\pi,\pi] \to \mathbb{F}$ by

$$T_f(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\varphi(x)\,dx \qquad \forall \varphi \in C^\infty_{2\pi}[-\pi,\pi]$$

We show that T_f is a periodic distribution. Clearly, T_f is a linear functional, which follows because the integral operator is linear. It remains to show that T_f is continuous, so that it is a distribution.

Let (φ_n) be a sequence in $C^\infty_{2\pi}[-\pi,\pi]$ such that $\varphi_n \to 0$ in $C^\infty_{2\pi}[-\pi,\pi]$. That is,

$$\forall m \in \mathbb{N}\cup\{0\}, \quad \frac{d^m\varphi_n}{dx^m} \xrightarrow{\text{uniformly}} 0 \text{ on } [-\pi,\pi] \text{ as } n \to \infty$$

This means that $\varphi_n \to 0$ uniformly as $n \to \infty$ (taking $m = 0$). Hence, $\varphi_n(x) \to 0$ pointwise $\forall x$, which in turn implies that $\|\varphi_n\|_\infty = \sup_x\{\varphi_n(x)\} \to 0$ as $n \to \infty$. Keeping this in mind, we have:

$$\begin{aligned} T_f(\varphi_n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\varphi_n(x)\,dx \\ |T_f(\varphi_n)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)||\varphi_n(x)|\,dx \\ &\le \|f\|_1\,\|\varphi_n\|_\infty \to 0 \text{ as } n \to \infty \end{aligned}$$

Thus, T_f is a continuous linear functional on $\mathscr{D}$, and hence a periodic distribution ($T_f \in \mathscr{D}'$). Such a distribution is called a **periodic distribution generated by a function** $f \in L^1_{2\pi}[-\pi,\pi]$.

Lemma 3.6. *Let $f, g \in L^1_{2\pi}[-\pi,\pi]$. Then from the above discussion, $T_f,\ T_g \in \mathscr{D}'$. $T_f = T_g$*

iff $f = g$ *almost everywhere on* $[-\pi, \pi]$.

Proof. ($\Longleftarrow$) Suppose $f = g$ almost everywhere on $[-\pi, \pi]$.
$\Longrightarrow$ $f - g = 0$ almost everywhere on $[-\pi, \pi]$.

$$\Longrightarrow (f-g)(x) \cdot \varphi(x) = 0 \text{ almost everywhere on } [-\pi,\pi]\ \forall \varphi \in \mathscr{D}$$
$$\Longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} (f-g)(x)\cdot\varphi(x)\ dx = 0\ \forall \varphi \in \mathscr{D}$$
$$\Longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \varphi(x)\ dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\ \varphi(x)\ dx = 0\ \forall \varphi \in \mathscr{D}$$
$$\Longrightarrow T_f(\varphi) = T_g(\varphi)\ \forall \varphi \in \mathscr{D}$$
$$\Longrightarrow T_f = T_g$$

($\Longrightarrow$) Suppose $T_f = T_g$,

$$\Longrightarrow T_f(\varphi) = T_g(\varphi)\ \forall \varphi \in \mathscr{D}$$

$$\Longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \varphi(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\ \varphi(x)\, dx$$
$$\Longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} (f-g)(x)\,\varphi(x)\, dx = 0\ \forall \varphi \in \mathscr{D}$$

In particular, $\forall n \in \mathbb{Z}$, taking $\varphi(x) = e^{-inx}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (f-g)(x)\ e^{-inx}\ dx = 0 \iff \widehat{f-g}(n) = 0 \Longrightarrow \widehat{f}(n) = \widehat{g}(n)\ \forall n \in \mathbb{Z}$$

$\Longrightarrow$ $f = g$ almost everywhere (by Theorem 2.11). $\square$

Remark 3.6. Let $f \in L^1_{2\pi}[-\pi, \pi]$. Then $f = 0$ almost everywhere on $[-\pi, \pi] \iff T_f = 0$ on $\mathscr{D}$.

Remark 3.7. The map $L^1_{2\pi}[-\pi, \pi] \to \mathscr{D}'$, $f \mapsto T_f$ is one-one. In this way, we can say that $L^1_{2\pi}[-\pi, \pi] \subseteq \mathscr{D}'$. That is, everywhere function in $L^1_{2\pi}[-\pi, \pi]$ can be identified as a distribution that it generates, i.e. $f := T_f$.

A very natural question to ask here is, do we have $L^1_{2\pi}[-\pi, \pi] = \mathscr{D}'$? That is, are all distributions generated by L^1 functions? The answer is no - we shall see in the next section (Proposition 3.1) that the 'Dirac periodic distribution' cannot be generated by an L^1 function. Hence, we have $L^1_{2\pi}[-\pi, \pi] \subset \mathscr{D}'$: or the class of distributions is larger than the class of L^1 functions!

3.1.2 Periodic Distributions Generated by Measure

Let μ be a complex measure or a finite Borel measure such that $\mu(E+2\pi) = \mu(E)\ \forall E \subset \mathscr{B}(\mathbb{R})$ (complex measures are always finite). Define $T_\mu : \mathscr{D} \to \mathbb{F}$ by

$$T_\mu(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)\, d\mu(x), \quad \forall \varphi \in \mathscr{D}$$

Clearly, T_μ is a linear functional (due to the linearity of integral operator) and

$$|T_\mu(\varphi)| \leq \|\varphi\|_\infty \frac{1}{2\pi}\mu([-\pi,\pi]) \quad \forall \varphi \in \mathscr{D}$$

Suppose (φ_j) is a sequence such that $\varphi_j \to 0$ in $\mathscr{D}$ as $j \to \infty$. Then,

$$|T_\mu(\varphi_j)| \leq \|\varphi_j\|_\infty \frac{1}{2\pi}\mu([-\pi,\pi]) \to 0 \text{ as } j \to \infty$$

$$\implies T_\mu \in \mathscr{D}', \text{ or } T_\mu \text{ is a periodic distribution.}$$

Such a periodic distribution is called a periodic distribution generated by a measure. Again, we can identify a measure as the distribution it generates, i.e. $\mu := T_\mu \in \mathscr{D}'$. The space of distributions is larger than the space of all measures, because of the existence of a distribution that cannot be generated by a measure (see Proposition 3.2).

Definition 3.10 (Dirac measure)**.** Let $\mathcal{M}_{[-\pi,\pi]}$ be the collection of all Borel sets in $[-\pi,\pi]$. Define $\delta : \mathcal{M}_{[-\pi,\pi]} \to [0,\infty]$ by

$$\delta(E) = \begin{cases} 2\pi, & 0 \in E \\ 0, & \text{otherwise} \end{cases}$$

δ as is known as Dirac measure (at point 0).

Exercise 3.8. Check that δ is a finite Borel measure on $[-\pi,\pi]$, and $\delta(E+2\pi) = \delta(E)$.

Definition 3.11. The distribution δ generates is known as Dirac periodic distribution ($\delta :=$

$T_\delta \in \mathscr{D}'$). Observe that,

$$T_\delta(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)\ d\delta(x) \quad \forall \varphi \in \mathscr{D} = \varphi(0)$$

i.e., $T_\delta : \mathscr{D} \to \mathbb{F}$ defined by $\varphi \mapsto \varphi(0)$ is called the Dirac periodic distribution.

Remark 3.8. A generalized version of the above distribution is as follows. Let $x \in [-\pi, \pi]$. Define δ_x by

$$\delta_x(E) = \begin{cases} 2\pi, & x \in E \\ 0, & \text{otherwise} \end{cases}$$

$$T_{\delta_x}(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(y)\ d\delta_x(y) \quad \forall \varphi \in \mathscr{D} = \varphi(x)$$

$$T_{\delta_x} := \delta_x \in \mathscr{D}'$$

δ_x is the Dirac measure at point x, and T_δ is the distribution generated by δ_x, which sends $\varphi \in \mathscr{D}$ to $\varphi(x) \in \mathbb{F}$.

Proposition 3.1 (No L^1 function can generate the Dirac distribution)**.** *$\nexists f \in L^1_{2\pi}[-\pi, \pi]$ such that $\delta = T_f$. (The class of periodic distributions are larger than the class of functions, $L^1_{2\pi}[-\pi, \pi] \subset \mathscr{D}'$).*

Proof. Assume for the sake of contradiction, that $\exists f \in L^1_{2\pi}[-\pi, \pi]$ such that $\delta = T_f$. Then by definition,

$$\delta(\varphi) = T_f(\varphi), \quad \forall \varphi \in \mathscr{D}$$

$$\implies \varphi(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\varphi(x)\, dx, \quad \forall \varphi \in \mathscr{D}$$

In particular, taking $\varphi(x) = e^{-inx}$, we have:

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}\, dx \quad \forall n \in \mathbb{Z}$$

$$\implies 1 = \widehat{f}(n) \quad \forall n \in \mathbb{Z}$$

which, as the Riemann-Lebesgue lemma says, is impossible as $|\widehat{f}(n)| \to 0$ as $|n| \to \infty$. $\square$

Define $\delta' : \mathscr{D} \to \mathbb{F}$ by $T(\varphi) = -\varphi'(0)$ for all $\varphi \in \mathscr{D}$. Then T is a linear functional. Observe that $|T(\varphi)| = |\varphi'(0)| \leq \|\varphi\|_N \ \forall N \geq 1,$ and $|T(\varphi)| \leq \|\varphi'\|_\infty$.

Let $(\varphi_j) \subset \mathscr{D}$ such that $\varphi_j \to 0$ in $\mathscr{D}$ as $j \to \infty$. Then $|T(\varphi_j)| \leq \left\|\varphi_j'\right\|_\infty \to 0$ as $j \to \infty$. Thus, $\delta' := T \in \mathscr{D}'$.

Proposition 3.2. *There does not exist any measure μ that generates the distribution δ' (i.e., $\nexists \mu$ such that $\delta' = T_\mu$). Hence, there exist distributions that cannot be generated by measures.*

Proof. Suppose there exists a measure μ such that $\delta' = T_\mu$. That is,

$$\delta'(\varphi) = T_\mu(\varphi) \quad \forall \varphi \in \mathscr{D}$$

$$\implies -\varphi'(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)\, d\mu(x), \quad \forall \varphi \in \mathscr{D}$$

$$\implies |\varphi'(0)| \leq \|\varphi\|_\infty \cdot \frac{1}{2\pi}\mu([-\pi,\pi]), \quad \forall \varphi \in \mathscr{D}$$

But derivatives cannot be bounded by sup-norm; Let $n \in \mathbb{Z}$, let $\varphi_n(x) = e^{inx}$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} |\varphi_n'(0)| &\leq \|\varphi_n\|_\infty \frac{1}{2\pi}\mu([-\pi,\pi]) \\ \implies |n| &\leq \frac{1}{2\pi}\mu([-\pi,\pi]), \quad \forall n \in \mathbb{Z} \end{aligned}$$

which is clearly a contradiction because natural numbers are not bounded. Hence, $\delta' := T_{\delta'}$ cannot be generated by a measure. $\square$

3.2 Operations on Periodic Distributions

Now that we have seen a bunch of examples of periodic distributions, it is time to define operations on them. In this section, we will discuss about translation, reflection, and conjugation of distributions, and also glance upon the multiplication and the convolution of a distribution with a function in $C^\infty_{2\pi}[-\pi,\pi]$.

The definitions of the operations on distributions are very intuitive. We will first apply the corresponding operation on distributions generated by a function $T_u \in \mathscr{D}'$, and see how it works out. From that, we will generalize the definition of the operation for an arbitrary distribution $T \in \mathscr{D}'$.

Remark 3.9. Recall from Lemma 3.1 that for $u \in \mathscr{D}$, the reflection acts as $\widetilde{u}(x) = u(-x)$ for all $x \in [-\pi, \pi]$. As $\widetilde{u} \in \mathscr{D}$, we know $T_u, T_{\widetilde{u}} \in \mathscr{D}'$. By definition, we have

$$T_u(\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\varphi(x)\, dx \quad \forall \varphi \in \mathscr{D}$$

and

$$\begin{aligned} T_{\widetilde{u}}(\varphi) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{u}(x)\varphi(x)\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(-x)\varphi(x)\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\varphi(-x)\, dx \ (x \mapsto -x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\widetilde{\varphi}(x)\, dx, \ \forall \varphi \in \mathscr{D} \end{aligned}$$

Hence, $T_{\widetilde{u}}(\varphi) = T_u(\widetilde{\varphi}) \quad \forall \varphi \in \mathscr{D}$.

The reader must spend some time here to understand what's going on. Given $u \in \mathscr{D}$, we know about the distribution it generates, namely T_u. As such, we have no information about $T_{\widetilde{u}}$. However, we see that we can express $T_{\widetilde{u}}$ as T_u, in the way that $T_{\widetilde{u}}$ acting on φ is the same as T_u acting on $\widetilde{\varphi}$ (the reflection of $\varphi \in \mathscr{D}$). In other words, we have $T_{\widetilde{u}}(\varphi) = T_u(\widetilde{\varphi})\ \forall \varphi \in \mathscr{D}$: we are able to express an operator we don't know ($T_{\widetilde{u}}$) in terms of an operator that we know (T_u). Motivated by this, we define the reflection of an arbitrary distribution as follows.

Definition 3.12 (Reflection of a distribution $T \in \mathscr{D}'$)**.** Let $T \in \mathscr{D}'$. Define

$$\widetilde{T}(\varphi) := T(\widetilde{\varphi}), \ \forall \varphi \in \mathscr{D}.$$

$\widetilde{T}$ is called the reflection of T.

Lemma 3.7. *The reflection of $T \in \mathscr{D}'$, denoted as $\widetilde{T}$, is a periodic distribution. That is, for $T \in \mathscr{D}'$, we have $\widetilde{T} \in \mathscr{D}'$.*

Proof. We have to show that $\widetilde{T}$ defined as $\widetilde{T}(\varphi) = T(\widetilde{\varphi})\ \forall \varphi \in \mathscr{D}$ is a continuous linear functional. First, note that reflection is linear on $\mathscr{D}$. Let $\varphi, \psi \in \mathscr{D}$. Then,

$$\widetilde{\varphi + \psi}(x) = (\varphi + \psi)(-x) = \varphi(-x) + \psi(-x) = \widetilde{\varphi}(x) + \widetilde{\psi}(x)$$

Now, for $\varphi, \psi \in \mathscr{D}$ and $\alpha \in \mathbb{F}$,

$$\widetilde{T}(\alpha\varphi + \psi) = T\left(\widetilde{\alpha\varphi + \psi}\right) = T(\alpha\widetilde{\varphi} + \widetilde{\psi})$$

$$= \alpha T(\widetilde{\varphi}) + T(\widetilde{\psi}) \text{ (as } T \text{ is linear, since } T \in \mathscr{D}') = \alpha\widetilde{T}(\varphi) + \widetilde{T}(\psi)$$

Thus, T is a linear functional. To prove that it is continuous, let (φ_n) be a sequence in $\mathscr{D}$ such that $\varphi_n \to 0$ in $\mathscr{D}$.

This means that $\varphi_n \to 0$ uniformly as $n \to \infty$.

$\implies \varphi_n(x) \to 0$ pointwise $\forall x$ as $n \to \infty$.

$\implies \varphi_n(-x) \to 0\ \forall x$.

$\implies \widetilde{\varphi}_n \to 0$ as $n \to \infty$.

Hence, $\widetilde{T}(\varphi_n) = T(\widetilde{\varphi_n}) \to T(0)$ as $n \to \infty$, by continuity of T. So, $\widetilde{T}$ is a continuous linear functional $\implies \widetilde{T} \in \mathscr{D}'$. □

Definition 3.13 (Conjugation of a distribution $T \in \mathscr{D}'$)**.** Let $T \in \mathscr{D}'$. Define the conjugation of T, denoted as $\overline{T} : \mathscr{D} \to \mathbb{F}$ by

$$\overline{T}(\varphi) = \overline{T(\overline{\varphi})}, \quad \forall \varphi \in \mathscr{D}$$

Exercise 3.9. Show that the conjugation of a distribution T as defined above is a distribution, i.e., $\overline{T} \in \mathscr{D}'$.

Remark 3.10. What we shall do now will motivate us to define the derivative of a distribution. Let $u \in \mathscr{D}$, then $u' \in \mathscr{D}$, and $T_u, T_{u'} \in \mathscr{D}'$.

$$T_{u'}(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u'(x)\varphi(x)\,dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\varphi'(x)\,dx \text{ (using by parts, see Remark 2.9)} = -T_u(\varphi')$$

$\implies T_{u'}(\varphi) = -T_u(\varphi'),\ \forall \varphi \in \mathscr{D}$.

Definition 3.14 (Derivative of a distribution $T \in \mathscr{D}'$)**.** Let $T \in \mathscr{D}'$. Define the derivative of T, denoted as T', by

$$T' : \mathscr{D} \to \mathbb{F}, \quad T'(\varphi) := -T(\varphi'), \quad \forall \varphi \in \mathscr{D}$$

Exercise 3.10. Show that the derivative of a periodic distribution is also a periodic distribution, i.e., $T' \in \mathscr{D}'$

Definition 3.15 (m^{th} Derivative of $T \in \mathscr{D}'$)**.** Let $m \in \mathbb{N}$ and $T \in \mathscr{D}'$. Define the m^{th} derivative of T, denoted $T^{(m)}$, by

$$T^{(m)} : \mathscr{D} \to \mathbb{F}, \quad T^{(m)}(\varphi) := (-1)^m \, T(\varphi^{(m)}), \quad \forall \varphi \in \mathscr{D}$$

Example 3.2 (Distributional derivative)**.** Let $f(x) = |x|$, $-\pi \leq x \leq \pi$, then $T_f \in \mathscr{D}'$. We are interested in finding T_f' and T_f'': the first and second derivatives of the distribution generated by f, namely T_f.

Observe:

$$T_f'(\varphi) = -T_f(\varphi') = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\varphi'(x)\,dx = -\frac{1}{2\pi}\int_0^{\pi} x\varphi'(x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{0} x\varphi'(x)\;dx$$

Integration by parts:

$$\begin{aligned}
&= -\frac{1}{2\pi}\left([x\varphi(x)]_0^{\pi} - \int_0^{\pi} \varphi(x)\;dx\right) + \frac{1}{2\pi}\left([x\varphi(x)]_{-\pi}^{0} - \int_{-\pi}^{0} \varphi(x)\,dx\right) \\
&= -\frac{\pi\varphi(\pi)}{2\pi} + \frac{1}{2\pi}\int_0^{\pi} \varphi(x)\,dx + \frac{1}{2\pi}\pi\varphi(-\pi) - \frac{1}{2\pi}\int_{-\pi}^{0} \varphi(x)\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} \varphi(x)\,dx - \frac{1}{2\pi}\int_{-\pi}^{0} \varphi(x)\,dx
\end{aligned}$$

Let $f_1(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases} \in L^1_{2\pi}[-\pi, \pi]$, extended periodically with period 2π.

$\implies T_f'(\varphi) = \langle f_1, \varphi\rangle = T_{f_1}(\varphi) \quad \forall \varphi \in \mathscr{D}$.

$\implies T_f' = T_{f_1} := f_1 \in \mathscr{D}'$.

For the second distributional derivative T_f'', observe that

$$T_f''(\varphi) = -T_f'(\varphi') = -T_{f_1}(\varphi') = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x)\,\varphi'(x)\,dx$$

Splitting the integral,

$$\begin{aligned}
&= -\frac{1}{2\pi}\int_{-\pi}^{0}(-1)\cdot\varphi'(x)\,dx - \frac{1}{2\pi}\int_{0}^{\pi}\varphi'(x)\,dx \\
&= \frac{1}{2\pi}(\varphi(0)-\varphi(-\pi)) - \frac{1}{2\pi}(\varphi(\pi)-\varphi(0))\,dx \\
&= \frac{1}{\pi}\varphi(0) - \frac{1}{\pi}\varphi(\pi) \\
&= \frac{1}{\pi}\delta_0(\varphi) - \frac{1}{\pi}\delta_\pi(\varphi) \\
&= \frac{1}{\pi}(\delta_0 - \delta_\pi)(\varphi)
\end{aligned}$$

where δ_0, δ_π are Dirac delta distributions at 0 and π, respectively.

$$\implies T_f'' = \frac{1}{\pi}(\delta_0 - \delta_\pi)$$

Lemma 3.8. *Let $T_1, T_2 \in \mathscr{D}'$, and $\alpha \in \mathbb{F}$. Then,*

1. $(T_1 - T_2)' = T_1' - T_2'$
2. $(\alpha T_1)' = \alpha T_1'$
3. *For* $u \in \mathscr{D}$, $(uT)' = u'T + uT'$

Proof.

$$(T_1 - T_2)'(\varphi) = -(T_1 - T_2)(\varphi') = -T_1(\varphi') - (-T_2(\varphi')) = T_1'(\varphi) - T_2'(\varphi),\ \forall \varphi \in \mathscr{D}.$$

$(\alpha T_1)'(\varphi) = -\alpha T_1(\varphi') = \alpha T_1'(\varphi),\ \forall \varphi \in \mathscr{D}.$

$$\begin{aligned}
(uT)'(\varphi) = -(uT)(\varphi') &= -T(u\varphi') \\
&= -T(u\varphi' + u'\varphi - u'\varphi) \\
&= -T((u\varphi)') + T(u'\varphi) \\
&= T'(u\varphi) + (u'T)(\varphi) \\
&= (uT)'(\varphi) + (u'T)(\varphi) \\
&= (u'T + uT')(\varphi) \quad \forall \varphi \in \mathscr{D} \\
\implies (uT)' &= u'T + uT'
\end{aligned}$$

□

Definition 3.16 (Multiplication of $u \in \mathscr{D}$ with $T \in \mathscr{D}'$)**.** Let $u \in \mathscr{D}$ and $T \in \mathscr{D}'$. Define $uT : \mathscr{D} \to \mathbb{F}$ by

$$(uT)(\varphi) := T(u\varphi), \quad \forall \varphi \in \mathscr{D}$$

Exercise 3.11. Show that $uT \in \mathscr{D}'$.

(Hint: Use $(uf)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(k)} f^{(m-k)}$).

Remark 3.11. We will now motivate for the definition of convolution of a distribution with a test function. For $u, f \in \mathscr{D}$, we know that $u * f \in \mathscr{D}$. Then $T_u, T_{u*f} \in \mathscr{D}'$.

$$T_u(\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\varphi(x)\,dx$$

$$\begin{aligned}
T_{u*f}(\varphi) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (u*f)(x)\,\varphi(x)\,dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) f(x-t)\,dt \right) \varphi(x)\,dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\varphi(x)\,dx \right) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{f}(t-x)\varphi(x)\,dx \right) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t)\ (\widetilde{f} * \varphi)(t)\,dt \\
&= T_u(\widetilde{f} * \varphi)
\end{aligned}$$

Definition 3.17 (Convolution of $u \in \mathscr{D}$ with $T \in \mathscr{D}'$)**.** Let $u \in \mathscr{D}$ and $T \in \mathscr{D}'$. Define $u * T : \mathscr{D} \to \mathbb{F}$ by

$$(u * T)(\varphi) := T(\widetilde{u} * \varphi), \quad \forall \varphi \in \mathscr{D}$$

where $\widetilde{u}(x) = u(-x)$.

Remark 3.12. We will now look at translation of a distribution. Let $x_0 \in \mathbb{R}$ and $f \in L^1_{2\pi}[-\pi, \pi]$. Define

$$(\tau_{x_0} f)(x) = f(x - x_0), \quad \forall x \in \mathbb{R} \quad \text{(translation of an } L^1 \text{ function)}$$

Observe that $T_f,\ T_{\tau_{x_0} f} \in \mathscr{D}'$ (distributions generated by f and translated version of f).

$$T_{\tau_{x_0} f}(\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\tau_{x_0} f)(x)\ \varphi(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - x_0)\ \varphi(x)\, dx$$

Change variables: let $y = x - x_0$,

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, \varphi(y+x_0)\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, (\varphi(x+x_0))\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ (\tau_{-x_0}\varphi)(x)\, dx = T_f\,(\tau_{-x_0}\varphi)$$

for all $\varphi \in \mathscr{D}$. Thus, $T_{\tau_{x_0} f}(\varphi) = T_f\,(\tau_{-x_0}\varphi)\ \ \forall \varphi \in \mathscr{D}$.

Definition 3.18 (Translation of a distribution $T \in \mathscr{D}'$)**.** Let $x_0 \in \mathbb{R}$ and $T \in \mathscr{D}'$. Define translation of T, denoted by $\tau_{x_0}T$ by

$$(\tau_{x_0} T)(\varphi) := T(\tau_{-x_0}\varphi), \quad \forall \varphi \in \mathscr{D}$$

where $\tau_{-x_0}\varphi(x) := \varphi(x + x_0)$.

Exercise 3.12. Check that the translation of a distribution is a distribution, i.e., $\tau_{x_0}T \in \mathscr{D}'$.

Definition 3.19 (Distributional convergence)**.** Let (T_n) be a sequence of distributions, i.e. $(T_n) \subseteq \mathscr{D}'$. We say that $T_n \to T$ in $\mathscr{D}'$ as $n \to \infty$ if $T_n(\varphi) \to T(\varphi),\ \forall \varphi \in \mathscr{D}$.

Theorem 3.2. *The following maps are continuous.*

1. $\sim : \mathscr{D}' \to \mathscr{D}', \quad T \mapsto \widetilde{T}$
2. $- : \mathscr{D}' \to \mathscr{D}', \quad T \mapsto \overline{T}$
3. $\forall x_0 \in \mathbb{R},\ \tau_{x_0} : \mathscr{D}' \to \mathscr{D}', \quad T \mapsto \tau_{x_0} T$
4. $\forall u \in \mathscr{D},\ u : \mathscr{D}' \to \mathscr{D}', \quad T \mapsto uT$
5. $\forall u \in \mathscr{D}, \quad * : \mathscr{D}' \to \mathscr{D}', \quad T \mapsto u * T$
6. $\forall m \in \mathbb{N}, \quad \frac{d^m}{dx^m} : \mathscr{D}' \to \mathscr{D}', \quad T \mapsto T^{(m)}$

Proof. 1. Let $(T_n) \subset \mathscr{D}'$, $T \in \mathscr{D}'$ such that $T_n \to T$ in $\mathscr{D}'$. We claim: $\widetilde{T_n} \to \widetilde{T}$ in $\mathscr{D}'$. For,

$$(\widetilde{T_n} - \widetilde{T})(\varphi) = \left(\widetilde{T_n - T}\right)(\varphi) = (T_n - T)(\widetilde{\varphi}) = T_n(\widetilde{\varphi}) - T(\widetilde{\varphi}) \to 0$$

as $n \to \infty$, by the continuity of T.

The proof of the rest is left to the reader as an exercise. □

Remark 3.13. Let $T \in \mathscr{D}'$, $u \in \mathscr{D}$, then $u * T = T * u$ (convolution of a test function with a distribution commutes).

Recall:

$$\mathscr{D} = C^{\infty}_{2\pi}[-\pi,\pi] \subset C^{(m)}_{2\pi}[-\pi,\pi] \subset C^{1}_{2\pi}[-\pi,\pi] \subset C_{2\pi}[-\pi,\pi] \subset L^{\infty}_{2\pi}[-\pi,\pi] \subset L^{p}_{2\pi}[-\pi,\pi] \subset L^{1}_{2\pi}[-\pi,\pi] \subsetneq \mathscr{D}'.$$

Theorem 3.3. *Let $T : \mathscr{D} \to \mathbb{F}$ be a linear functional. Then $T \in \mathscr{D}'$ if and only if there exists $N \in \mathbb{N}$ and $c > 0$ such that*

$$|T(\varphi)| \leq c \sum_{0 \leq k \leq N} \left\|\varphi^{(k)}\right\|_{\infty}, \quad \forall \varphi \in \mathscr{D} \tag{\dagger}$$

Proof. ($\implies$ Proof by contradiction). Assume that T is a distribution, i.e. $T \in \mathscr{D}'$. Suppose for each $N > 0$, there exists $\varphi_N \in \mathscr{D}$ such that

$$|T(\varphi_N)| \geq N \sum_{0 \leq k \leq N} \left\|\varphi_N^{(k)}\right\|_{\infty} \qquad (*)$$

$$\text{Let } \psi_N = \frac{1}{N}\left(\sum_{0 \leq k \leq N} \left\|\varphi_N^{(k)}\right\|_{\infty}\right)^{-1} \varphi_N, \quad \forall N \in \mathbb{N}$$

For all $m \in \mathbb{N}$,

$$\psi_N^{(m)} = \frac{1}{N}\left(\sum_{0 \leq k \leq N} \left\|\varphi_N^{(k)}\right\|_{\infty}\right)^{-1} \varphi_N^{(m)}$$

We can choose N large enough so that $\left(\sum_{0 \leq k \leq N} \left\|\varphi_N^{(k)}\right\|_{\infty}\right)^{-1} \varphi_N^{(m)} < 1$, as $\sum_{0 \leq k \leq N} \left\|\varphi_N^{(k)}\right\|_{\infty} > \varphi_N^{(m)}$.

$$\implies \forall m \in \mathbb{N}, \quad \psi_N^{(m)} \xrightarrow{\text{uniformly}} 0 \text{ as } N \to \infty \implies \psi_N \to 0 \text{ in } \mathscr{D},$$

by definition of convergence in $\mathscr{D}$ (Definition 3.1).

$$\implies T(\psi_N) \to 0 \text{ as } N \to \infty \quad (\because T \in \mathscr{D}', \text{ so } T \text{ is continuous})$$

But from $(*)$, we have $|T(\psi_N)| \geq 1 \ \forall N$ (contradiction).

$\therefore \exists N \in \mathbb{N}$ and $c > 0$ such that

$$|T(\varphi)| \leq c \sum_{0 \leq k \leq N} \left\|\varphi^{(k)}\right\|_{\infty}, \quad \forall \varphi \in \mathscr{D}.$$

$(\impliedby)$ Suppose $\exists N \in \mathbb{N}$ and $c > 0$ such that $|T(\varphi)| \leq c \sum_{0 \leq k \leq N} \left\|\varphi^{(k)}\right\|_{\infty}, \quad \forall \varphi \in \mathscr{D}$. We have to show that T is continuous, so that $T \in \mathscr{D}'$.
Let (φ_n) be a sequence in $\mathscr{D}$ such that $\varphi_n \to 0$ as $n \to \infty$ in $\mathscr{D}$ (i.e., $\varphi_n^{(k)} \to 0$ uniformly $\forall k \geq 0$).
Then $\varphi_n^{(k)}(x) \to 0 \ \forall x, \ \forall k \geq 0$.
$\implies \sup_x \{\varphi_n^{(k)}(x)\} = \left\|\varphi_n^{(k)}\right\|_{\infty} \to 0$ as $n \to \infty, \ \forall k \geq 0$.

Hence, $T(\varphi_n) \leq c \sum_{0 \leq k \leq N} \left\|\varphi^{(k)}\right\|_{\infty} \to 0$ as $n \to \infty$. Thus, T is a continuous linear functional $\implies T \in \mathscr{D}'$.

$\square$

Definition 3.20 (Order of a distribution)**.** Let $T \in \mathscr{D}'$. The least $N \in \mathbb{N}$ satisfying $(\dagger)$ in Theorem 3.3 is called the *order* of T and we write $O(T) = N$.

Remark 3.14. If $T \in \mathscr{D}'$, then T has a finite order.

3.3 Distributional Fourier Coefficients and Fourier Transform

Definition 3.21 (Fourier coefficient of $T \in \mathscr{D}'$)**.** Let $T \in \mathscr{D}'$ and $n \in \mathbb{Z}$. The n-th Fourier coefficient of is T is defined by

$$\widehat{T}(n) = T(e^{-inx})$$

This definition is motivated by the fact that for $u \in \mathscr{D}$,

$$\widehat{u}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-inx} \, dx = T_u(e^{-inx})$$

,

Definition 3.22 (Fourier series of $T \in \mathscr{D}'$)**.** The Fourier series of T is defined by

$$T \sim \sum_{n=-\infty}^{\infty} \widehat{T}(n) e^{inx}$$

Definition 3.23 (Sequence of polynomial growth)**.** Let $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{F}$. We say that (a_n) is a sequence of polynomial growth if $\exists N \in \mathbb{N}$ and $c > 0$ such that $|a_n| \leq c(1+n^2)^{N/2}$ for all $n \in \mathbb{N}$

Definition 3.24 (Space of sequences of polynomial growth)**.** We define $S'(\mathbb{Z}) = \{(a_n)_{n\in\mathbb{Z}} : (a_n)$ is a sequence of polynomial growth$\}$ as the space of sequences of polynomial growth.

Recall from Definition 3.5:

$$\begin{aligned} S(\mathbb{Z}) &= \text{The space of all rapidly decreasing sequences (Schwartz sequence space)} \\ S'(\mathbb{Z}) &= \text{ The space of all sequences of polynomial growth} \end{aligned}$$

The following exercise establishes a relation between the space of rapidly decreasing sequences $S(\mathbb{Z})$, and the space of sequences of polynomial growth $S'(\mathbb{Z})$.

Exercise 3.13. Show that $S'(\mathbb{Z})$ is the dual of $S(\mathbb{Z})$ (i.e., $(S(\mathbb{Z}))' \cong S'(\mathbb{Z})$) via the dual action: if $(b_n) \in S'(\mathbb{Z}), (a_n) \in S(\mathbb{Z})$,

$$\langle (b_n), (a_n) \rangle := \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

Hint: Define $\eta : S'(\mathbb{Z}) \to (S(\mathbb{Z}))'$ by $a \mapsto \eta(a)$, where $\eta(a) : S(\mathbb{Z}) \to \mathbb{F}$ is defined by $\eta(a = (a_n))(b_n) = \sum_{n=-\infty}^{\infty} b_n \overline{a_n}$, $\forall (a_n) \in S(\mathbb{Z})$. Then show that η is a topological isomorphism.

Lemma 3.9 (Fourier coefficients of a distribution have polynomial growth)**.** *Let $T \in \mathscr{D}'$. Then $\left(\widehat{T}(n)\right)_{n\in\mathbb{Z}} \in S'(\mathbb{Z})$.*

Proof. Given that $T \in \mathscr{D}'$, so by Theorem 3.3, $\exists N \in \mathbb{N}$ and $c > 0$ such that

$$|T(\varphi)| \leq c \sum_{0\leq k\leq N} \|\varphi^{(k)}\|_\infty, \ \forall \varphi \in \mathscr{D}.$$

Let $n \in \mathbb{Z}$, then

$$|\widehat{T}(n)| = |T(e^{-inx})| \leq c \sum_{0 \leq k \leq N} \left\| (-in)^k \; e^{-inx} \right\|_\infty = c \sum_{0 \leq k \leq N} |n|^k \leq \widetilde{c}(1 + |n|)^N, \; \forall n \in \mathbb{Z}$$

(where $\widetilde{c} > 0$ is some constant). So,

$$|\widehat{T}(n)| \leq \widetilde{c} \sum_{0 \leq k \leq N} |n|^k \leq \widetilde{\widetilde{c}}(1 + n^2)^{N/2}, \forall n \in \mathbb{Z}.$$

Hence, $\left(\widehat{T}(n)\right)_{n \in \mathbb{Z}}$ is a sequence of polynomial growth, following Definition 3.23. $\square$

Remark 3.15 (Riemann-Lebesgue lemma may not hold for distributions). Recall that the Riemann-Lebesgue lemma says that if $f \in L^1_{2\pi}[-\pi, \pi]$, then $|\widehat{f}(n)| \to 0$ as $|n| \to \infty$. However, this result **may not be** true for distributions, i.e. if $T \in \mathscr{D}'$, then we cannot say that $\widehat{T}(n) \to 0$ as $|n| \to \infty$.

For example, Let $T = \delta \in \mathscr{D}'$ (Dirac distribution). Then $\widehat{T}(n) = \delta(e^{-inx}) = 1, \forall n \in \mathbb{Z}$. Clearly, $|\widehat{T}(n)| \not\to 0$ as $n \to \infty$.

Exercise 3.14. There exist distributions (that are generated by functions) for which the Riemann-Lebesgue lemma holds good. Let $f \in L^1_{2\pi}[-\pi, \pi]$, then for $T_f \in \mathscr{D}'$, show that $\widehat{T_f}(n) = \widehat{f}(n), \forall n \in \mathbb{Z}$. Hence, $|\widehat{f}(n)| = |\widehat{T_f}(n)| \to 0$ as $n \to \infty$.

Theorem 3.4. *Let $T \in \mathscr{D}'$. Then the Fourier series of T converges to T in the sense of distributions, that is,*

$$T \overset{\mathscr{D}'}{=} \sum_{n=-\infty}^{\infty} \widehat{T}(n) e^{inx}$$

In other words, $\forall \varphi \in \mathscr{D}$,

$$\langle T, \varphi \rangle = \lim_{N \to \infty} \left\langle \sum_{|n| \leq N} \widehat{T}(n) e^{inx}, \varphi \right\rangle$$

Notation: Let $T \in \mathscr{D}'$, $\varphi \in \mathscr{D}$. Write $\langle T, \varphi \rangle := T(\varphi)$.

Proof. Observe that,

$$\begin{aligned}
\left\langle \sum_{|n|\le N} \widehat{T}(n)e^{inx}, \varphi \right\rangle &= \sum_{|n|\le N} \widehat{T}(n)\langle e^{inx}, \varphi\rangle \\
&= \sum_{|n|\le N} \widehat{T}(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\varphi(x)\ dx \\
&= \sum_{|n|\le N} \widehat{T}(n)\ \widehat{\varphi}(-n) \\
&= \sum_{|n|\le N} T(e^{-inx})\ \widehat{\varphi}(-n) \\
&= T\left(\sum_{|n|\le N} \widehat{\varphi}(-n)\ e^{-inx}\right) \\
&= T\left(\sum_{|n|\le N} \widehat{\varphi}(n)\ e^{inx}\right)
\end{aligned}$$

$$= \left\langle T, \sum_{|n|\le N} \widehat{\varphi}(n)e^{inx} \right\rangle \to \langle T, \varphi\rangle \text{ (as T is continuous)}$$

Recall from Exercise 3.7 that for $\varphi \in \mathscr{D}$, we have $\varphi(x) \overset{\mathscr{D}}{=} \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)e^{inx} \overset{\mathscr{D}}{=} \lim_{N\to\infty}\sum_{|n|\le N} \widehat{\varphi}(n)e^{inx}$. So,

$$\lim_{N\to\infty} \left\langle \sum_{|n|\le N} \widehat{T}(n)e^{inx}, \varphi \right\rangle = \langle T, \varphi\rangle, \quad \forall \varphi \in \mathscr{D}.$$

□

Remark 3.16. If $f \in L^1_{2\pi}[-\pi, \pi]$,

$$T_f := f \overset{\mathscr{D}'}{=} \sum_{n=-\infty}^{\infty} \widehat{f}(n)\ e^{inx}$$

(Fourier series converges in distribution sense).

Lemma 3.10. *Let $(f_n) \subseteq L^p_{2\pi}[-\pi, \pi]$, $f \in L^p_{2\pi}[-\pi, \pi]$ $(1 \le p \le \infty)$. If $f_n \to f$ in $L^p_{2\pi}[-\pi, \pi]$, then $T_{f_n} \to T_f$ in $\mathscr{D}'$ as $n \to \infty$.*

Proof. Let $\varphi \in \mathscr{D}$. Then,

$$|(T_{f_n} - T_f)(\varphi)| = |T_{f_n - f}(\varphi)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (f_n(x) - f(x))\ \varphi(x)dx\right| \le \|f_n - f\|_p \|\varphi\|_q \to 0, \text{ as } n \to \infty.$$

Hence, $T_{f_n} \to T_f$ in $\mathscr{D}'$. □

Lemma 3.11. *Let $T \in \mathscr{D}'$, $m \in \mathbb{N}$. Then*

1. $T^{(m)} \in \mathscr{D}'$
2. $T^{(m)} \overset{\mathscr{D}'}{=} \sum_{n=-\infty}^{\infty} (in)^m \, \widehat{T}(n) \, e^{inx}$

Example 3.3 (Fourier series of certain distributions)**.** Recall the Dirac distribution $\delta := T_\delta$ defined as $\delta(\varphi) = \varphi(0) \ \forall \varphi \in \mathscr{D}$, and the distribution δ' that cannot be generated by a measure, defined by $\delta'(\varphi) = -\varphi'(0), \ \forall \varphi \in \mathscr{D}$.

1. Let $\delta \in \mathscr{D}'$. Then $\forall n \in \mathbb{Z}$,

$$\widehat{\delta}(n) = \delta(e^{-inx}) = 1$$

So,

$$\delta \overset{\mathscr{D}'}{=} \sum_{n=-\infty}^{\infty} 1 \cdot e^{inx} \qquad \text{(Fourier series of } \delta\text{)}$$

2. Let $\delta' \in \mathscr{D}'$. Then $\forall n \in \mathbb{Z}$,

$$\widehat{\delta}'(n) = \delta'(e^{-inx}) = -(-in) \; e^{-in(0)} = in$$

So,

$$\delta' \overset{\mathscr{D}'}{=} \sum_{n=-\infty}^{\infty} (in) \; e^{inx} \qquad \text{(Fourier series of } \delta'\text{)}$$

3. Let $m \in \mathbb{N}$. Then $\delta^{(m)} \in \mathscr{D}'$,

$$\delta^{(m)}(\varphi) = (-1)^m \varphi^{(m)}(0), \quad \forall \varphi \in \mathscr{D}$$

Let $n \in \mathbb{Z}$. Then,

$$\widehat{\delta^{(m)}}(n) = \delta^{(m)}(e^{-inx}) = (-1)^m (-in)^m \; e^{in(0)} = (in)^m$$

So,

$$\delta^{(m)} \overset{\mathscr{D}'}{=} \sum_{n=-\infty}^{\infty} (in)^m \; e^{inx}$$

Recall from Lemma 3.9 that the Fourier coefficients of distributions have polynomial growth, i.e. for $T \in \mathscr{D}'$, $\widehat{T}(n) \in S'(\mathbb{Z}) = \{(a_n)_{n \in \mathbb{Z}} : \exists N \in \mathbb{N}, c > 0 \text{ s.t. } |a_n| \leq c(1 +$

$n^2)^{N/2},\ \forall n \in \mathbb{N}\}$. In the following theorem, we will see that given a sequence (a_n) of polynomial growth, there exists a periodic distribution whose Fourier coefficients coincide with (a_n).

Theorem 3.5. *Let $(a_n)_{n\in\mathbb{Z}} \in S'(\mathbb{Z})$. Then $\exists T \in \mathscr{D}'$ such that $\widehat{T}(n) = a_n,\ \forall n \in \mathbb{Z}$.*

Proof. Given $(a_n) \in S'(\mathbb{Z})$, so $\exists N \in \mathbb{N}$ and $c > 0$ such that

$$|a_n| \leq c(1+n^2)^N,\ \forall n \in \mathbb{N}.$$

Define $b_n := a_n(1+n^2)^{-N-1},\ \forall n \in \mathbb{Z}$.

Observe that $(b_n) \in \ell^2(\mathbb{Z})$, since $\sum |b_n|^2 < \infty$ (converges). (In fact, $(b_n) \in \ell^1(\mathbb{Z})$ as $\sum |b_n| \leq \sum \frac{1}{1+n^2} < \infty$).

Then there exists $f \in L^2_{2\pi}[-\pi,\pi]$ such that $\widehat{f}(n) = b_n,\ \forall n \in \mathbb{Z} \ldots (*)$. This is because $\wedge : L^2_{2\pi}[-\pi,\pi] \to \ell^2(\mathbb{Z})\ (f \mapsto \widehat{f}(n))$ is a linear isometrical isomorphism.
Consider T_f, we know $T_f \in \mathscr{D}'$.

Recall:

$$\widehat{f^{(k)}}(n) = (-in)^k \widehat{f}(n), \text{ or } \widehat{f''}(n) = -n^2 \widehat{f}(n)$$

$$\implies \widehat{f}(n) - \widehat{f''}(n) = (1+n^2)\widehat{f}(n)$$

$$\implies \widehat{\left(I - \frac{d^2}{dx^2}\right) f}(n) = (1+n^2)\widehat{f}(n)$$

Define:

$$T := \left(I - \frac{d^2}{dx^2}\right)^{N+1} T_f = \sum_{l=0}^{N+1} \binom{N+1}{l} \left(\frac{d^2}{dx^2}\right)^l T_f(-1)^l = \sum_{l=0}^{N} \binom{N+1}{l} (-1)^l\ T_f^{2l}$$

Clearly, $T \in \mathscr{D}'$, being a finite linear combination of distributions (elements in $\mathscr{D}'$). Observe:

$$\widehat{T}(n) = T(e^{-inx}) = \left(\left(I - \frac{d^2}{dx^2}\right)^{N+1} T_f\right)(e^{-inx}) = T_f\left(\left(I - \frac{d^2}{dx^2}\right)^{N+1}(e^{-inx})\right)$$

But $(I - \frac{d^2}{dx^2})^{N+1}(e^{-inx}) = (1+n^2)^{N+1}e^{-inx}$. So,

$$= T_f((1+n^2)^{N+1}e^{-inx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(1+n^2)^{N+1}e^{-inx}dx = (1+n^2)^{N+1}\widehat{f}(n),\ \forall n \in \mathbb{Z}$$

But from $(*)$, $\widehat{f}(n) = b_n$. Hence,

$$\widehat{T}(n) = (1+n^2)^{N+1}\ b_n = a_n\ \forall n \in \mathbb{Z}$$

Thus, $\exists T \in \mathscr{D}'$ such that $\widehat{T}(n) = a_n$ for all $n \in \mathbb{Z}$.

□

Corollary 3.1. *Let $T \in \mathscr{D}'$. Then $\exists N \in \mathbb{N}$ and $f \in L^2_{2\pi}[-\pi,\pi]$ such that*

$$T = \left(I - \frac{d^2}{dx^2}\right)^{N+1} T_f$$

Definition 3.25 (Distributional Fourier Transform)**.** The distributional Fourier transform is a map $\wedge : \mathscr{D}' \to S'(\mathbb{Z})$ defined by

$$\wedge(T) = \left(\widehat{T}(n)\right)_{n\in\mathbb{Z}}, \quad \forall T \in \mathscr{D}'.$$

Theorem 3.6. *$\wedge : \mathscr{D}' \to S'(\mathbb{Z})$ is a topological isomorphism (for the weak* topology), i.e., $\wedge$ is bijective and bicontinuous.*

Lemma 3.12. *Let $u \in \mathscr{D}$, $T \in \mathscr{D}'$, and $x_0 \in \mathbb{R}$. Then,*

1. $\widehat{\tau_{x_0}T}(n) = e^{-inx_0}\ \widehat{T}(n),\ \forall n \in \mathbb{Z}$
2. $\widehat{u * T}(n) = \widehat{u}(n)\ \widehat{T}(n),\ \forall n \in \mathbb{Z}$
3. $\widehat{\widetilde{T}}(n) = \widehat{T}(-n),\ \forall n \in \mathbb{Z}$
4. $\widehat{\overline{T}}(n) = \overline{\widehat{T}(-n)},\ \forall n \in \mathbb{Z}$
5. $\widehat{T^{(m)}}(n) = (-in)^m\ \widehat{T}(n)$
6. $\widehat{T^{(m)}}(n) = (-in)^m\ \widehat{T}(n),\ \forall n \in \mathbb{Z}$

Proof. 1.

$$\widehat{\tau_{x_0}T}(n) = (\tau_{x_0}T)(e^{-inx}) = T(e^{-in(x+x_0)}) = e^{-inx_0}\ T(e^{-inx}) = e^{-inx_0}\ \widehat{T}(n)$$

The rest of the proof is left to the reader as an exercise. □

3.4 Convolutions of Sequences and Distributions

Definition 3.26 (Convolution operator on sequences in $\ell^1(\mathbb{Z})$)**.** Let $a = (a_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, $b = (b_n)_{n\in\mathbb{Z}} \in l^1(\mathbb{Z})$. We define the convolution of a and b, denoted by $a * b$, by

$$(a * b)(n) := \sum_{l=-\infty}^{\infty} a_{n-l}\, b_l, \quad n \in \mathbb{Z}$$

(if the sum converges). Note that this is the n^{th} term of the convolution, as the convolution is a sequence itself.

Lemma 3.13. *Let $a = (a_n)$, $b = (b_n) \in \ell^1(\mathbb{Z})$. Then $a * b \in \ell^1(\mathbb{Z})$.*

Proof.

$$\sum_{n=-\infty}^{\infty} |(a*b)(n)| = \sum_{n=-\infty}^{\infty} \left| \sum_{l=-\infty}^{\infty} a_{n-l}\ b_l \right| \le \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |a_{n-l}||b_l| \text{ (by triangle inequality)}$$

$$= \sum_{l=-\infty}^{\infty} |b_l| \sum_{n=-\infty}^{\infty} |a_{n-l}| \text{ (by Tonelli's theorem)} = \|b\|_1 \|a\|_1$$

Therefore, $\|a * b\|_1 \le \|a\|_1 \|b\|_1$.

So, $* : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ is a well-defined map: $(a, b) \mapsto a * b$.

□

Lemma 3.14. *If we convolve a rapidly decreasing sequence $(a_n) \in S(\mathbb{Z})$ with a sequence of polynomial growth $(b_n) \in S'(\mathbb{Z})$, then the convolution is rapidly decreasing. That is, let $a = (a_n) \in S(\mathbb{Z})$, $b = (b_n) \in S'(\mathbb{Z})$. Then $a * b \in S(\mathbb{Z})$.*

The standard Convolution Theorem says that convolutions go to products under the Fourier transform. For test functions in $\mathscr{D}$, the converse is also true: the Fourier coefficients of product of two test functions coincide with the convolution of the Fourier coefficients of the test functions:

Lemma 3.15. *Let $u, v \in \mathscr{D}$. Then $uv \in \mathscr{D}$ and $\widehat{uv}(n) = (\widehat{u} * \widehat{v})(n)$ for all $n \in \mathbb{Z}$.*

Proof. Given that $u, v \in \mathscr{D}$, then $uv \in \mathscr{D}$. Recall,

$$u(x) = \sum_{n=-\infty}^{\infty} \widehat{u}(n)\ e^{inx}$$

$$v(x) = \sum_{m=-\infty}^{\infty} \widehat{v}(m)\ e^{imx},\ x \in [-\pi, \pi]$$

Then

$$(uv)(x) = u(x)v(x) = \left(\sum_{n=-\infty}^{\infty} \widehat{u}(n)e^{inx}\right)\left(\sum_{m=-\infty}^{\infty} \widehat{v}(m)e^{imx}\right) \text{ (Cauchy product)}$$

$$= \sum_{l=-\infty}^{\infty} (\widehat{u} * \widehat{v})(l)\ e^{ilx}$$

Also,

$$(uv)(x) = \sum_{l=-\infty}^{\infty} \widehat{uv}(l)\ e^{ilx}$$

$$\implies \widehat{uv}(n) = (\widehat{u} * \widehat{v})(n) \quad \forall n \in \mathbb{Z}$$

□

We have a similar result for convolution of a test function with a distribution:

Lemma 3.16. *Let $u \in \mathscr{D}$, $T \in \mathscr{D}'$. Then*

$$\widehat{uT}(n) = (\widehat{u} * \widehat{T})(n),\ \forall n \in \mathbb{Z}$$

Proof.

$$\begin{aligned}
\widehat{uT}(n) = uT(e^{-inx}) &= T(ue^{-inx}) \\
&= T\left(\sum_{m=-\infty}^{\infty} \widehat{u}(m)\ e^{imx}\ e^{-inx}\right) \\
&= T\left(\sum_{m=-\infty}^{\infty} \widehat{u}(m)\ e^{i(m-n)x}\right) \\
&= T\left(\lim_{N\to\infty} \sum_{|m|\le N} \widehat{u}(m)\ e^{i(m-n)x}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} T\left(\sum_{|m|\le N} \widehat{u}(m)\ e^{i(m-n)x}\right)\\
&= \lim_{N\to\infty} \left(\sum_{|m|\le N} \widehat{u}(m)\ T\left(e^{i(m-n)x}\right)\right)\\
&= \lim_{N\to\infty} \left(\sum_{|m|\le N} \widehat{u}(m)\widehat{T}(n-m)\right)\\
&= \sum_{m=-\infty}^{\infty} \widehat{u}(m)\ \widehat{T}(n-m)\\
&= \widehat{u} * \widehat{T}(n)
\end{aligned}
$$

□

Remark 3.17 (Product of two distributions). Let $T, S \in \mathscr{D}'$. Note that $\widehat{T}$ and $\widehat{S}$ both have polynomial growth. We cannot define TS in such a way that $TS \in \mathscr{D}'$ and $(\widehat{TS})(n) = (\widehat{T} * \widehat{S})(n)$ for all $n \in \mathbb{Z}$ in general.

Recall: If $u \in \mathscr{D}$, $T \in \mathscr{D}'$, then $(u * T)(\varphi) = T(\widetilde{u} * \varphi),\ \forall \varphi \in \mathscr{D}$.

Observe:

$$(\widehat{u * T})(n) = (u * T)(e^{-inx}) = T(\widetilde{u} * e^{-inx}) = \underbrace{\overline{\widehat{u}(-n)}}_{\text{(rapid decay)}} \cdot \underbrace{\widehat{T}(n)}_{\text{(polynomial growth)}},\ n \in \mathbb{Z}.$$

Hence, $\left((\widehat{u * T})(n)\right)_n \in S(\mathbb{Z})$ (is a rapidly decreasing sequence).
Define

$$f(x) = \sum_{n=-\infty}^{\infty} (\widehat{u * T})(n)\ e^{inx}, \quad x \in [-\pi, \pi]$$

Since $(\widehat{u * T})(n) \in S(\mathbb{Z})$, we can see that $f \in \mathscr{D} \implies T_f \in \mathscr{D}'$ (as the Fourier coefficients are rapidly decreasing, the corresponding function is a test function and it generates a periodic distribution).

Observe,

$$\widehat{T_f}(n) = \widehat{u * T}(n),\ \forall n \in \mathbb{Z} \implies T_f = u * T.$$

Remark 3.18. Hence, for $u \in \mathscr{D}$ and $T \in \mathscr{D}'$, $u * T \in \mathscr{D}'$ (identified as a distribution), and $u * T \in \mathscr{D}$ (identified as a test function).

Definition 3.27 (Convolution of two distributions)**.** Let $T, S \in \mathscr{D}'$. We define the convolution of T and S by

$$(T * S)(\varphi) = S(\widetilde{T} * \varphi), \quad \forall \varphi \in \mathscr{D}$$

.

This definition is valid because from our above discussion, we have that $\varphi \in \mathscr{D}$ and $\widetilde{T} \in \mathscr{D}' \implies \widetilde{T} * \varphi \in \mathscr{D}$, and so it is a valid input to $S \in \mathscr{D}'$.

Exercise 3.15. Show that for $T, S \in \mathscr{D}'$, $T * S \in \mathscr{D}'$ (i.e. the convolution of two distributions is also a distribution), and

$$\left(\widehat{T * S}\right)(n) = \widehat{T}(n)\ \widehat{S}(n),\ \forall n \in \mathbb{Z}.$$

4 Non-Periodic Fourier Analysis

This chapter will deal with the analysis of non-periodic functions. We will specifically study the Fourier transforms of functions in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Recall that,

$$L^p(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \text{ measurable} : \int_{\mathbb{R}} |f(x)|^p dx < \infty \right\} \; (1 \le p < \infty)$$
$$L^\infty(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : \exists M_f > 0 \text{ such that } |f(x)| \le M_f \text{ a.e. on } \mathbb{R} \}$$

Keep in mind that the Fourier coefficients of a function $f \in L^1_{2\pi}[-\pi,\pi]$, denoted by $\widehat{f}(n)$, was a sequence indexed by $n \in \mathbb{Z}$. For non-periodic functions $g \in L^1(\mathbb{R})$, the Fourier coefficients will be denoted by $\widehat{g}(\xi),\ \xi \in \mathbb{R}$, which are themselves functions on $\mathbb{R}$.

We will skip the proofs of some theorems and lemmas in this chapter because they can be proved in a similar way as their periodic counterparts. The reader is, however, encouraged to prove all the unproved statements that they encounter.

Remark 4.1.

1. Let $1 \le p < q \le \infty$, then $L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R})$ and $L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R})$.
   This differs from the space of 2π-periodic L^p functions, which we call $L^p_{2\pi}[-\pi,\pi]$, because in those spaces we had that $L^1_{2\pi}[-\pi,\pi]$ was the largest space, and $L^p_{2\pi}[-\pi,\pi] \subset L^q_{2\pi}[-\pi,\pi]$ for $p > q$. Therefore, in case of periodic functions, defining a notion for a function in $L^1_{2\pi}[-\pi,\pi]$ was enough - because all other $L^p_{2\pi}$-functions were already inside $L^1_{2\pi}[-\pi,\pi]$. However, for non-periodic L^p functions, this will not be the case - as we shall see how to define L^1 and L^2 Fourier transforms differently.

2. Let $1 \le p \le r \le q \le \infty$, then $L^p \cap L^q \subseteq L^r \subseteq L^p + L^q$.

Remark 4.2. Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)| dx < \infty$.

1. Then for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\int_A |f(x)| dx < \varepsilon$ whenever A is measurable set with $m(A) < \delta$ (where m is the Lebesgue measure).

2. By Dominated Convergence Theorem or DCT, we can see that

$$\lim_{R \to \infty} \int_{-R}^{R} |f(x)| dx = \int_{-\infty}^{\infty} |f(x)| dx$$

Reason: Define $A_N = [-N, N]$, $N \in \mathbb{N}$.

Define $g_N(x) = |f(x)|\ \chi_{A_N}(x)$, $x \in \mathbb{R}$.

$g_N(x) \to |f(x)|$ pointwise almost everywhere on $\mathbb{R}$, and $|g_N(x)| \le |f(x)|$ for all $N \in \mathbb{N}$, $f \in L^1(\mathbb{R})$.

4.1 L^1-Fourier Transform

Definition 4.1 (L^1-Fourier Transform)**.** Let $f \in L^1(\mathbb{R})$. The L^1-Fourier transform of f, denoted by $\widehat{f}$ or $\mathscr{F}f$, is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\ e^{-2\pi i x\xi}\, dx, \quad \xi \in \mathbb{R}.$$

Note that we will also view the L^1-Fourier transform as a linear transformation from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, as in Theorem 4.4.

Definition 4.2 (Inverse L^1-Fourier Transform)**.** The inverse L^1-Fourier transform of f, denoted by $\mathscr{F}^{-1}f$, is defined as

$$\mathscr{F}^{-1}f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\ e^{2\pi i x\xi}\, d\xi, \quad x \in \mathbb{R}.$$

Lemma 4.1. *Let $f \in L^1(\mathbb{R})$. Then:*

1. $|\widehat{f}(\xi)| \le \|f\|_1, \quad \forall \xi \in \mathbb{R}$
2. $\widehat{f} \in L^\infty(\mathbb{R})$ *and* $\left\|\widehat{f}\right\|_\infty \le \|f\|_1$
3. $\widehat{f} \in C(\mathbb{R})$ *(in fact, $\widehat{f} \in C_0(\mathbb{R})$, i.e. $\widehat{f}$ is continuous and vanishes at infinity).*

Proof. 1. Note that $|e^{2\pi i x\xi}| = 1$.

$$|\widehat{f}(\xi)| = \left|\int_{\mathbb{R}} f(x)\ e^{-2\pi i x\xi}\, dx\right| \le \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1,\ \forall \xi \in \mathbb{R}$$

2.

$$|\widehat{f}(\xi)| \le \|f\|_1\ \ \forall \xi \in \mathbb{R} \implies \sup_{\xi} |\widehat{f}(\xi)| = \left\|\widehat{f}\right\|_\infty \le \|f\|_1$$

3. Let $(\xi_n) \subseteq \mathbb{R}$ such that $\xi_n \to \xi \in \mathbb{R}$ as $n \to \infty$.

Claim: $\widehat{f}(\xi_n) \to \widehat{f}(\xi)$ as $n \to \infty$, so that $\widehat{f}$ is continuous. Indeed,

$$\widehat{f}(\xi_n) - \widehat{f}(\xi) = \int_{\mathbb{R}} f(x) \left(e^{-2\pi i x \xi_n} - e^{-2\pi i x \xi}\right) dx$$

Let $g_n(x) = f(x)(e^{-2\pi i x \xi_n} - e^{-2\pi i x \xi})$.

Clearly, $|g_n(x)| \to 0$ as $n \to \infty$, and $|g_n(x)| \leq 2|f(x)|$, $f \in L^1(\mathbb{R})$ (or (g_n) has an L^1-bound for all n).

Hence, by Dominated Convergence Theorem or DCT, we have,

$$\widehat{f}(\xi_n) - \widehat{f}(\xi) = \int_{\mathbb{R}} g_n(x)\ dx = \int_{\mathbb{R}} f(x) \left(e^{-2\pi i x \xi_n} - e^{-2\pi i x \xi}\right) dx \to 0 \text{ as } n \to \infty.$$

So, $\widehat{f}$ is continuous on $\mathbb{R}$.

□

Lemma 4.2 (Riemann-Lebesgue lemma)**.** *Let* $f \in L^1(\mathbb{R})$*. Then* $\widehat{f}(\xi) \to 0$ *as* $|\xi| \to \infty$*.*

Proof. We first prove the result for real-valued characteristic functions on $[a, b]$, and then use the fact that characteristic functions are dense in $L^1(\mathbb{R})$.

Let $f(x) = \chi_{[a,b]}(x)$, $x \in \mathbb{R}$, where $a, b \in \mathbb{R},\ a < b$. Then,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx = \int_a^b e^{-2\pi i x \xi}\, dx = \frac{e^{-2\pi i b \xi} - e^{-2\pi i a \xi}}{-2\pi i \xi},\ \forall \xi \in \mathbb{R}$$

(for $\xi = 0$, understand the above expression in the sense of limits). Hence,

$$|\widehat{f}(\xi)| \leq \frac{c}{|\xi|}\ \forall |\xi| > 1, \implies |\widehat{f}(\xi)| \to 0 \text{ as } |\xi| \to \infty.$$

Recall that characteristic functions are dense in $L^1(\mathbb{R})$. That is, $\text{span}\{\chi_{[a,b]} : a, b \in \mathbb{R},\ a \leq b\} \subseteq L^1(\mathbb{R})$.

Let $g \in \text{span}\{\chi_{[a,b]} : a, b \in \mathbb{R}\}$ (i.e., g is a finite linear combination of such χ's). Clearly, $|\widehat{g}(\xi)| \to 0$ as $|\xi| \to \infty$ from what we have just proved above.

Let $f \in L^1(\mathbb{R})$, then $\exists (g_n) \in \text{span}\{\chi_{[a,b]}\}$ s.t. $g_n \xrightarrow{L^1} f$ (density).

Write $\widehat{f}(\xi) = \widehat{f}(\xi) - \widehat{g_n}(\xi) + \widehat{g_n}(\xi),\ n \in \mathbb{N}$. Then,

$$\begin{aligned} |\widehat{f}(\xi)| &\leq |\widehat{f}(\xi) - \widehat{g_n}(\xi)| + |\widehat{g_n}(\xi)| \\ &= \left|\widehat{f - g_n}(\xi)\right| + |\widehat{g_n}(\xi)| \\ &\leq \|f - g_n\|_1 + |\widehat{g_n}(\xi)| \end{aligned}$$

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ and $M > 0$ such that:

$$\|f - g_N\|_1 < \varepsilon/2 \quad \forall n \geq N, \text{ and } |\widehat{g_N}(\xi)| < \varepsilon/2,\ \forall |\xi| \geq M.$$

$$\implies |\widehat{f}(\xi)| < \varepsilon,\ \forall |\xi| \geq M \implies |\widehat{f}(\xi)| \to 0 \text{ as } |\xi| \to \infty.$$

$\square$

Lemma 4.3 (Translations and Modulations under Fourier Transform). *Let $f \in L^1(\mathbb{R})$, $a, b \in \mathbb{R}$. Then,*

$$\begin{aligned} &\widehat{T_a f}(\xi) = M_a \widehat{f}(\xi)\ \forall \xi \in \mathbb{R},\ \textit{where } T_a f(x) = f(x - a)\ \forall x \in \mathbb{R}. \\ &\widehat{M_b f}(\xi) = T_{-b} \widehat{f}(\xi),\ \textit{where } M_b f(x) = e^{-2\pi i b x} f(x)\ \forall x \in \mathbb{R}. \end{aligned}$$

That is, translations go to modulations and modulations go to translations under the Fourier transform. In other words, translating a function or signal in the original (spatial or temporal) domain by a factor of a, modulates (or shifts the phase) of the function in the Fourier (frequency) domain by the same factor a.

Proof.

$$\widehat{T_a f}(\xi) = \int_{\mathbb{R}} f(x - a)\ e^{-2\pi i x \xi} dx$$

Let $y = x - a$:

$$= \int_{\mathbb{R}} f(y) e^{-2\pi i (y+a)\xi}\ dy = e^{-2\pi i a \xi} \int_{\mathbb{R}} f(y) e^{-2\pi i y \xi}\ dy = e^{-2\pi i a \xi}\ \widehat{f}(\xi) = M_a \widehat{f}(\xi)$$

And,

$$\widehat{M_b f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i b x} f(x) e^{-2\pi i x \xi} dx = \int_{\mathbb{R}} f(x) e^{-2\pi i x (\xi + b)} dx = \widehat{f}(\xi + b) = T_{-b} \widehat{f}(\xi)$$

□

Definition 4.3 (Convolution)**.** Let $f, g \in L^1(\mathbb{R})$. The convolution $f * g$ of f and g is defined by

$$(f * g)(\xi) = \int_{\mathbb{R}} f(x)\ g(\xi - x)\, dx \quad \forall \xi \in \mathbb{R}.$$

Lemma 4.4. *Let $f, g \in L^1(\mathbb{R})$. Then:*

1. $f * g \in L^1(\mathbb{R})$, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, *and* $f * g = g * f$ *(convolution is commutative).*
2. ***Convolution theorem:*** *convolution goes to product under the Fourier transform, i.e.* $\widehat{f * g}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi) \quad \forall \xi \in \mathbb{R}$
3. *Let* $f, f' \in L^1(\mathbb{R}) \cap C_C(\mathbb{R})$ *(compactly supported continuous* L^1 *functions), then* $f * g$ *is differentiable and* $(f * g)' = f' * g$.

Proof. The proofs of 1 and 2 are similar as in case of periodic functions, and are left to the reader as an exercise.

3. Let $\xi \in \mathbb{R}$.

$$\begin{aligned} \frac{(f * g)(\xi + h) - (f * g)(\xi)}{h} &= \frac{1}{h} \int_{\mathbb{R}} f(x) \left[g(\xi + h - x) - g(\xi - x)\right]\ dx \\ &= \frac{1}{h} \int_{\mathbb{R}} \left(f(\xi + h - x) - f(\xi - x)\right)\ g(x) dx \\ &= \int_{\mathbb{R}} \frac{f(\xi + h - x) - f(\xi - x)}{h}\ g(x)\ dx \end{aligned}$$

Define $\varphi_h(x) = \frac{f(\xi+h-x) - f(\xi - x)}{h} g(x),\ \forall x \in \mathbb{R}$.
Observe that $\varphi_h(x) \to f'(\xi - x)\ g(x)$ almost everywhere on $\mathbb{R}$ as $h \to 0$.

By the Mean Value Theorem,

$$|f(\xi + h - x) - f(\xi - x)| \leq |f'(\xi - x + \delta_{x,h})||h|$$

Thus,

$$\left| \frac{f(\xi + h - x) - f(\xi - x)}{h} \right| \leq M|g(x)| \in L^1(\mathbb{R})$$

By Dominated Convergence Theorem or DCT,

$$\int_{\mathbb{R}} \varphi_h(x)dx \to \int_{\mathbb{R}} f'(\xi - x)\ g(x)\ dx \text{ as } h \to 0$$

Hence,

$$(f * g)' = f' * g.$$

$\square$

Exercise 4.1. Prove the following.

1. Let $f \in C_C(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then $f * g \in C(\mathbb{R})$.
2. Let $f \in C_C^{(n)}(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then $(f * g)^{(m)} = f^{(m)} * g$ for $0 \leq m \leq n$.
3. $(f * g)^{(n)} \in C(\mathbb{R})$.
4. Let $f, g \in C_c^{(k)}(\mathbb{R})$. Then $f * g \in C_c^{(k)}(\mathbb{R})$.

Definition 4.4 (Dilation of a function)**.** Let $f, g \in L^1(\mathbb{R})$ and $\delta > 0$. Define the dilation operator D_δ by

$$D_\delta f(x) = \frac{1}{\delta} f\left(\frac{x}{\delta}\right),\ x \in \mathbb{R}$$

Lemma 4.5 (Fourier transform of dilation)**.**

$$\widehat{D_\delta f}(\xi) = \widehat{f}(\delta\xi) = \frac{1}{\delta} D_{1/\delta}\widehat{f}(\xi), \quad \xi \in \mathbb{R}$$

Lemma 4.6. *Let $f(x) = e^{-\pi x^2}$. Then $\widehat{f}(\xi) = f(\xi)$ for all $\xi \in \mathbb{R}$. That is, the Fourier coefficients of the Gaussian function coincide with itself.*

Proof.

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\ e^{-2\pi i x\xi}\ dx = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x\xi}\ dx$$

Note that the Gaussian integral formula is:

$$\int_{\mathbb{R}} e^{-(ax^2 + ibx)} dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$$

In our case, taking $a = \pi$, $b = 2\pi\xi$, we have:

$$= \int_{\mathbb{R}} e^{-(\pi x^2 + 2\pi i x\xi)}\ dx$$

Thus,

$$\widehat{f}(\xi) = e^{\frac{4\pi^2 i^2 \xi^2}{4\pi}} = e^{-\pi\xi^2}$$

And since $f(\xi) = e^{-\pi\xi^2}$, so $\widehat{f}(\xi) = f(\xi)\ \forall \xi \in \mathbb{R}$. □

Remark 4.3. If $P(x)$ is any polynomial, $P(x)\ e^{-\pi x^2} \in L^1(\mathbb{R})$, because:

$$\int_{\mathbb{R}} |P(x)|\ e^{-\pi x^2} dx = \int_{\mathbb{R}} \underbrace{P(x)\ e^{-\frac{\pi x^2}{2}}}_{\leq M \text{ (bounded)}}\ \underbrace{e^{-\frac{\pi x^2}{2}}}_{\in L^1(\mathbb{R})} < \infty$$

Note that $P(x)\ e^{-\frac{\pi x^2}{2}} \leq M$ because exponentially decaying Gaussian kills the growth of any polynomial.

Corollary 4.1. *Let* $\delta > 0$ *and* $f(x) = e^{-\pi\delta^2 x^2}$. *Then*

$$\widehat{f}(\xi) = \frac{1}{\delta} e^{-\pi\xi^2/\delta^2}, \quad \xi \in \mathbb{R}.$$

Proof.

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi\delta^2 x^2} e^{-2\pi i x\xi} dx = \frac{1}{\delta}\int_{\mathbb{R}} e^{-\pi x^2} e^{2\pi i x\xi/\delta} dx = \frac{1}{\delta} e^{-\pi\xi^2/\delta^2}.$$

□

Theorem 4.1 (Multiplication formula)**.** *Let* $f, g \in L^1(\mathbb{R})$. *Then*

$$\int_{\mathbb{R}} \widehat{f}(x) g(x)\ dx = \int_{\mathbb{R}} f(x)\widehat{g}(x)\ dx$$

Proof. First, observe that both the integrals are valid. The left integral is valid because $\widehat{f}$ is bounded and $g \in L^1(\mathbb{R})$. The right integral is valid since $f \in L^1(\mathbb{R})$ and $\widehat{g}$ is bounded (as $g \in L^1$). Now,

$$\begin{aligned}\int_{\mathbb{R}} \widehat{f}(x) g(x) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi) e^{-2\pi i\xi x} d\xi\right) g(x)\ dx &= \int_{\mathbb{R}} f(\xi) \left(\int_{\mathbb{R}} e^{-2\pi i\xi x} g(x)\ dx\right) d\xi \\ &\text{(by Fubini's theorem)} \\ &= \int_{\mathbb{R}} f(\xi)\widehat{g}(\xi) d\xi\end{aligned}$$

□

Definition 4.5 (Approximation to the identity)**.** A sequence $(\varphi_n) \subseteq L^1(\mathbb{R})$ is called an approximation to the identity if:

1. $\int_{\mathbb{R}} \varphi_n(x)dx = 1$ for all $n \in \mathbb{N}$
2. $\exists M > 0$ such that $\int_{\mathbb{R}} |\varphi_n(x)|dx \leq M$ for all $n \in \mathbb{N}$
3. $\forall \delta > 0,\ \int_{|x|>\delta} |\varphi_n(x)|dx \to 0$ as $n \to \infty$

Example 4.1. Let $\varphi \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(x)dx = 1$. Define the dilation of φ as $D_\delta\varphi(x) = \frac{1}{\delta}\varphi\left(\frac{x}{\delta}\right)$, $x \in \mathbb{R}$. Then $\{D_\delta : \delta > 0\}$ is an Approximation to the identity. Let us verify the three properties to show this.

1. $\int_{\mathbb{R}} D_\delta\varphi(x)dx = \int_{\mathbb{R}} \frac{1}{\delta}\varphi\left(\frac{x}{\delta}\right) dx = \int_{\mathbb{R}} \varphi(y)\ dy = 1$ for all $\delta > 0$.
2. $\int_{\mathbb{R}} |D_\delta\varphi(x)|dx = \int_{\mathbb{R}} |\varphi(x)|dx < \infty,\ \forall\delta > 0$.
3. $$\text{Let } \epsilon > 0,\ \int_{|x|>\epsilon} |D_\delta\varphi(x)|\ dx = \int_{|x|>\epsilon} \left|\frac{1}{\delta}\varphi\left(\frac{x}{\delta}\right)\right| dx = \int_{|y|>\epsilon/\delta} |\varphi(y)|\ dy \to 0 \text{ as } \delta \to 0^+.$$

Lemma 4.7. *Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R})$, $g \in L^1(\mathbb{R})$. Then $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \leq \|f\|_p \|g\|_1$.*

Theorem 4.2. *Let $1 \leq p < \infty$ and $(\varphi_n) \subseteq L^1(\mathbb{R})$ be an Approximation to the identity. Then,*

1. *For every $f \in L^p(\mathbb{R})$, $f * \varphi_n \to f$ in $L^p(\mathbb{R})$ as $n \to \infty$.*
2. *For $f \in C_C(\mathbb{R})$, $f * \varphi_n \to f$ uniformly as $n \to \infty$.*

Theorem 4.3. *$C_C^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $1 \leq p < \infty$ (with respect to L^p norm). $C_C^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ (with respect to sup-norm).*

Proof. Let $\varphi \in C_C^\infty(\mathbb{R})$, such that $\int_{\mathbb{R}} \varphi(x)\, dx = 1$. Let $f \in L^p(\mathbb{R})$. By Theorem **??**, we have:

$$f * D_\delta\varphi \to f \text{ in } L^p(\mathbb{R}) \text{ as } \delta \to 0, \text{ and } f * D_\delta\varphi \in L^p(\mathbb{R}) \cap C^\infty(\mathbb{R}).$$

Select $\psi \in C_C^\infty(\mathbb{R})$ such that $|\psi(x)| \leq 1$ and $\psi(x) = 1\ \forall x \in [-1, 1]$. Let $\epsilon > 0$, and define

$$F_\epsilon(x) = \psi(\epsilon x) \cdot f * D_\delta\varphi(x),\ x \in \mathbb{R}$$

Note that ψ is a cutoff function, and $f * D_\delta\varphi$ is in $C^\infty(\mathbb{R})$. Clearly, we have $F_\epsilon \in C_C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$.

Claim: $F_\epsilon \to f$ in $L^p(\mathbb{R})$ as $\epsilon \to 0$. For,

$$\begin{aligned}
\|F_\epsilon - f\|_p &= \left(\int_{\mathbb{R}} |F_\epsilon(x) - f(x)|^p dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}} |\psi(\epsilon x) \cdot f * D_\delta\varphi(x) - f(x)|^p \ dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}} |\psi(\epsilon x) \ f * D_\delta\varphi(x) - \psi(\epsilon x) \ f(x) + \psi(\epsilon x) \ f(x) - f(x)|^p \, dx\right)^{1/p} \\
&\le \left(\int_{\mathbb{R}} |\psi(\epsilon x) \ f * D_\delta\varphi(x) - \psi(\epsilon x) \ f(x)|^p \ dx\right)^{1/p} + \left(\int_{\mathbb{R}} |\psi(\epsilon x) \ f(x) - f(x)|^p \ dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}} |\psi(\epsilon x)|^p \ |f * D_\delta\varphi(x) - f(x)|^p \ dx\right)^{1/p} + \left(\int_{\mathbb{R}} |f(x)|^p \ |1 - \psi(\epsilon x)|^p \ dx\right)^{1/p} \\
&\le \left(\int_{\mathbb{R}} |f * D_\delta\varphi(x) - f(x)|^p \ dx\right)^{1/p} + \left(\int_{|x|>1/\epsilon} |f(x)|^p \ dx\right)^{1/p} \\
&= \|f * D_\delta\phi - f\|_p + \left(\int_{|x|>1/\epsilon} |f(x)|^p \ dx\right)^{1/p} \to 0 \text{ as } \delta \to 0, \ \epsilon \to 0
\end{aligned}$$

$\therefore C_C^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $1 \le p < \infty$, with respect to L^p norm. □

Lemma 4.8 (Inversion formula)**.** *Let* $f \in L^1(\mathbb{R})$ *and* $\widehat{f} \in L^1(\mathbb{R})$ *(in general, if* $f \in L^1(\mathbb{R})$*, then* $\widehat{f} \in C_0(\mathbb{R})$*, not necessarily* $L^1(\mathbb{R})$*). Then*

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) \ e^{2\pi i x\xi} \ d\xi, \ \textit{almost everywhere on } \mathbb{R}$$

Proof. Given $f, \widehat{f} \in L^1(\mathbb{R})$. Set $h(y) = e^{2\pi i y x} e^{-\pi\delta^2 y^2}$ for $y \in \mathbb{R}$.

Observe that,

$$\int_{\mathbb{R}} \widehat{f}(\xi) \ h(\xi) \ d\xi = \int_{\mathbb{R}} \widehat{f}(\xi) \ e^{2\pi i x\xi} \ e^{-\pi\delta^2\xi^2} d\xi$$

and,

$$\widehat{h}(\xi) = \int_{\mathbb{R}} h(y) e^{-2\pi i y\xi} dy = \int_{\mathbb{R}} e^{2\pi i y x} e^{-\pi\delta^2 y^2} e^{-2\pi i y\xi} dy = \int_{\mathbb{R}} e^{-\pi\delta^2 y^2} e^{2\pi i y(x-\xi)} dy$$

By the Gaussian formula,

$$= \frac{1}{\delta} e^{-\pi(x-\xi)^2/\delta^2}$$

Thus,

$$\widehat{h}(\xi) = D_\delta g_1(x-\xi), \text{ where } g_1(x) = e^{-\pi x^2}, \ x \in \mathbb{R}.$$

So,

$$\int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, e^{-\pi\delta^2\xi^2}\, d\xi = \int_{\mathbb{R}} f(\xi)\, \widehat{h}(\xi)\, d\xi \text{ (using Multiplication formula)} = \int_{\mathbb{R}} f(\xi)\, D_\delta g_1(x-\xi)\, d\xi$$

Now, as $\int_{\mathbb{R}} g_1(x) = \int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$ and $\{D_\delta g_1 : \delta > 0\}$ is an Approximation to the identity, we have

$$f * D_\delta g_1 \to f \text{ in } L^1(\mathbb{R}) \text{ as } \delta \to 0^+.$$

Recall: If a sequence $f_n \to f$ in $L^p(\mathbb{R})$, then it does not necessarily mean that $f_n \to f$ pointwise. All we can say is that, $\exists$ a subsequence (f_{n_k}) such that $f_{n_k} \to f$ pointwise almost everywhere (the subsequence of (f_n) may be (f_n) itself!)

$$\implies \exists\ (\delta_n)_{n=1}^{\infty} \text{ such that } f * D_{\delta_n} g_1 = \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, e^{-\pi\delta_n^2\xi^2}\, d\xi \to f(x) \text{ pointwise a.e. on } \mathbb{R} \ldots (1)$$

Also, by Dominated Convergence Theorem or DCT, we have

$$\int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, e^{-\pi\delta_n^2\xi^2}\, d\xi \to \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi \text{ pointwise a.e. as } \delta_n \to 0^+ \ldots (2)$$

From (1) and (2), it follows that

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi, \text{ almost everywhere on } \mathbb{R}$$

□

Corollary 4.2. *Let $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Then $f(x) = 0$ almost everywhere on $\mathbb{R}$. That is, a function whose Fourier coefficients are 0 is itself 0 almost everywhere.*

Proof. Observe that $f,\ \widehat{f} \in L^1(\mathbb{R})$. By Inversion formula,

$$f(x) = \int_{\mathbb{R}} 0 \cdot e^{2\pi i x\xi}\, d\xi = 0 \text{ almost everywhere on } \mathbb{R}.$$

$\square$

Theorem 4.4 (L^1-Fourier Transform)**.** *The L^1-Fourier transform is a linear transformation $\wedge : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ such that $\wedge$ is a bounded linear transformation with $\|\wedge\| = 1$. Moreover, $\wedge$ is one-to-one.*

Proof. We know (Lemma 4.1) that $|\widehat{f}(\xi)| \leq \|f\|_1$, $\forall f \in L^1(\mathbb{R})$.

$\implies \left\|\widehat{f}\right\|_\infty \leq \|f\|_1$ $\forall f \in L^1(\mathbb{R})$, so that $\wedge$ is a bounded linear transformation with $\|\wedge\| \leq 1$.

Recall from Lemma 4.6 that if $g(x) = e^{-\pi x^2}$, $x \in \mathbb{R}$, then $\widehat{g}(\xi) = e^{-\pi\xi^2}$, $\xi \in \mathbb{R}$. Hence, $\|\widehat{g}\|_\infty = 1$ and $\|g\|_1 = 1 \implies \exists$ a function $g \in L^1(\mathbb{R})$ such that $\|\wedge(g)\|_\infty = \|\widehat{g}\|_\infty = 1$. Hence, $\|\wedge\| = 1$. $\square$

Remark 4.4. A natural question to ask here is whether $\wedge$ is onto. The answer is no. Can you think why?

4.2 The Schwartz Space

Definition 4.6 (Schwartz Space $\mathcal{S}(\mathbb{R})$)**.** The Schwartz space (or Schwartz class), denoted by $\mathcal{S}(\mathbb{R})$, is a function space that contains infinitely differentiable functions that decay to zero faster than any polynomial, along with all their derivatives. We call such functions as 'rapidly decreasing functions.'

$$\mathcal{S}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \in C^\infty(\mathbb{R}),\ \forall\, n, m \in \mathbb{N} \cup \{0\},\ \exists c_{n,m} > 0 \text{ s.t. } \left| x^n \frac{d^m f(x)}{dx^m} \right| \leq c_{n,m}\ \forall x \in \mathbb{R} \right\}$$

Remark 4.5. $(\mathcal{S}(\mathbb{R}), +)$ is a vector space over $\mathbb{C}$.

Example 4.2.

1. $e^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$
2. $P(x)\ e^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$ for any polynomial $P(x)$
3. $\frac{1}{1+x^2} \notin \mathcal{S}(\mathbb{R})$ (why?)
4. $C_C^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$

Lemma 4.9.

1. *$\overline{\mathcal{S}(\mathbb{R})} = L^p(\mathbb{R})$, $1 \leq p < \infty$ (i.e. Schwartz class is dense in L^p for $1 \leq p < \infty$).*
2. *$\overline{\mathcal{S}(\mathbb{R})} = C_0(\mathbb{R})$, $p = \infty$*

Lemma 4.10. $\mathcal{S}(\mathbb{R}) \subseteq L^p(\mathbb{R}),\ 1 \leq p \leq \infty.$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$, then $\exists c > 0$ s.t. $|f(x)| \leq \frac{c}{1+x^2}$ $\forall x \in \mathbb{R}$.

$$\implies |f(x)|^p \leq \frac{c^p}{(1+x^2)^p} \leq \frac{c^p}{1+x^2}$$

$$\implies \int_{\mathbb{R}} |f(x)|^p dx \leq c^p \int_{\mathbb{R}} \frac{1}{1+x^2} dx$$

$$\leq c^p \tan^{-1} x$$

$$\implies \left(\int_{\mathbb{R}} |f(x)|^p \ dx \right)^{1/p} = \|f\|_p \leq c^p (\tan^{-1} x)^{1/p}$$

$$\implies f \in L^p(\mathbb{R})$$

□

Lemma 4.11. *The Multiplication formula holds true for Schwartz class functions, i.e. for* $f, g \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \widehat{f}(\xi) g(\xi) \ d\xi = \int_{\mathbb{R}} f(\xi) \widehat{g}(\xi) \ d\xi$$

Lemma 4.12. *Let* $f \in \mathcal{S}(\mathbb{R})$. *Then,*

1.
$$\widehat{\left(\frac{d}{dx} f\right)}(\xi) = 2\pi i \xi \ \widehat{f}(\xi),\ \xi \in \mathbb{R}$$

2.
$$x\widehat{f}(\xi) = -\frac{1}{2\pi i} \frac{\widehat{df}}{d\xi}(\xi)$$

Proof.

$$\widehat{\frac{d}{dx}} f(\xi) = \int_{\mathbb{R}} \left(\frac{df}{dx} \right)(x) \ e^{-2\pi i x \xi} \ dx = -\int_{\mathbb{R}} f(x)(-2\pi i \xi)(e^{-2\pi i x \xi}) dx$$

Thus,

$$\widehat{\frac{d}{dx}} f(\xi) = 2\pi i \xi \ \widehat{f}(\xi),\ \xi \in \mathbb{R}.$$

Secondly,

$$\widehat{xf}(\xi) = \int_{\mathbb{R}} x f(x) \ e^{-2\pi i x \xi} \ dx = \int_{\mathbb{R}} f(x) \ x e^{-2\pi i x \xi} \ dx$$

$$= \frac{1}{-2\pi i} \int_{\mathbb{R}} f(x) \frac{d}{d\xi} (e^{-2\pi i x \xi}) dx$$

By Dominated Convergence Theorem or DCT, we can interchange the differentiation and integration.

$$= -\frac{1}{2\pi}\frac{d}{d\xi}\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\, dx = -\frac{1}{2\pi i}\frac{d}{d\xi}\widehat{f}(\xi),\ \xi \in \mathbb{R}.$$

$\square$

Corollary 4.3. *Let $f \in \mathcal{S}(\mathbb{R})$, $m, n \in \mathbb{N} \cup \{0\}$. Then*

$$\widehat{\frac{d^m}{d\xi^m}} f(\xi) = (2\pi i\xi)^m(-1)^{m+1} f(\xi),\ \xi \in \mathbb{R}.$$

$$\widehat{x^n f}(\xi) = \left(-\frac{1}{2\pi i}\right)^n \frac{d^n}{d\xi^n}\widehat{f}(\xi),\ \xi \in \mathbb{R}.$$

Corollary 4.4. *The Fourier transform is closed under Schwartz class, i.e., if $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$. (Recall that this is not true in general for L^1-functions. If $f \in L^1(\mathbb{R})$, all we can say is $\widehat{f} \in C_0(\mathbb{R})$).*

Proof. Given that $f \in \mathcal{S}(\mathbb{R})$. Let $m, n \in \mathbb{N} \cup \{0\}$. Consider

$$\xi^m \frac{d^n}{d\xi^n}\widehat{f}(\xi) = (-2\pi i)^n\ \xi^m\ \widehat{x^n f}(\xi)$$

$$= (-2\pi i)^n \frac{1}{(2\pi i)^m(-1)^{m+1}}\left(\widehat{\frac{d^m}{dx^m}(x^n f)}\right)(\xi)$$

As $f \in \mathcal{S}(\mathbb{R})$, we know that $\frac{d^m}{dx^m}(x^n f)$ is in $L^1(\mathbb{R})$, and hence bounded.

$\therefore \exists c_{n,m} > 0$ such that

$$\left|\xi^m \frac{d^n}{d\xi^n}\widehat{f}(\xi)\right| \leq c_{n,m} \quad \forall \xi \in \mathbb{R}$$

Therefore, $\widehat{f} \in \mathcal{S}(\mathbb{R})$. $\square$

Exercise 4.2 (Alternative definitions of the Schwartz class)**.** Show that,

1.

$$\mathcal{S}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \ \Big|\ f \in C^\infty(\mathbb{R}),\ \forall n, m \in \mathbb{N} \cup \{0\}, \left|x^n \frac{d^m}{dx^m} f(x)\right| \to 0 \text{ uniformly on } \mathbb{R}\}$$

2.

$$\begin{aligned}\mathcal{S}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \,\big|\, f \in C^\infty(\mathbb{R}),\ \forall n, m \in \mathbb{N} \cup \{0\},\ \exists c_{n,m} > 0 \\ \text{s.t. } \left|\frac{d^m}{dx^m}(x^n f)(x)\right| \leq c_{n,m},\ \forall x \in \mathbb{R}\}\end{aligned}$$

3.

$$\begin{aligned}\mathcal{S}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C},\ f \in C^\infty(\mathbb{R}),\ \forall n, m \in \mathbb{N} \cup \{0\}, \\ \left|\frac{d^m}{dx^m}(x^n f)(x)\right| \to 0 \text{ uniformly on } \mathbb{R}\}\end{aligned}$$

Lemma 4.13 (The L^2-norm is an isometry on $\mathcal{S}(\mathbb{R})$)**.** *Let $f \in \mathcal{S}(\mathbb{R})$. Then* $\|f\|_2 = \left\|\widehat{f}\right\|_2$.

Proof. Given $f \in \mathcal{S}(\mathbb{R})$.

Let $h(x) = f * \widetilde{\overline{f}}(x)$, where $\widetilde{f}(x) = f(-x)$ for $x \in \mathbb{R}$. Observe that,

$$h(0) = f * \widetilde{\overline{f}}(0) = \int_{\mathbb{R}} f(t)\ \widetilde{\overline{f}}(0-t)\ dt = \int_{\mathbb{R}} f(t)\overline{f}(t)\ dt = \int_{\mathbb{R}} |f(t)|^2 dt = \|f\|_2^2$$

Now,

$$\widehat{h}(\xi) = \widehat{f}(\xi)\ \widehat{\widetilde{\overline{f}}}(\xi) = \widehat{f}(\xi)\ \overline{\widehat{f}(\xi)}\ \text{(verify!)} = |\widehat{f}(\xi)|^2, \quad \xi \in \mathbb{R} \ldots (1)$$

So,

$$\int_{\mathbb{R}} \widehat{h}(\xi) d\xi = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi$$

Recall Inversion formula: If $f, \widehat{f} \in L^1(\mathbb{R})$ then $f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\ e^{2\pi i x \xi}\ d\xi$, almost everywhere on $\mathbb{R}$. Applying the inversion formula for $h(x)$ at $x = 0$,

$$h(0) = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi \ldots (2)$$

From (1) and (2), we have,

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi \implies \|f\|_2 = \left\|\widehat{f}\right\|_2$$

□

Lemma 4.14. *The Fourier transform on Schwartz class, $\wedge : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined by $f \mapsto \widehat{f}$ is a bijective linear transformation.*

Proof. We have seen that $\wedge$ is linear an d clearly, $\wedge$ is one-to-one, being an isometry (Lemma 4.13) (for the proof of the fact that a linear isometry is one-one, see Lemma 7.2 in Decoding Functional Analysis).

Onto: Let $f \in \mathcal{S}(\mathbb{R})$. Let $g(x) := \widehat{f}(-x)$, $x \in \mathbb{R}$. Then $g \in \mathcal{S}(\mathbb{R})$. We show that the Fourier transform of g is f. Observe that,

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi = \int_{\mathbb{R}} \widehat{f}(-\xi) e^{-2\pi i x \xi} d\xi = \int_{\mathbb{R}} g(\xi) e^{-2\pi i x \xi} d\xi = \widehat{g}(x)$$

Thus, $f = \widehat{g}$, $g \in \mathcal{S}(\mathbb{R})$.

$\implies$ $\wedge$ is onto on $\mathcal{S}(\mathbb{R})$. $\square$

4.3 L^2-Fourier Transform

Let $f \in L^2(\mathbb{R})$. As $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ (Lemma 4.9), there exists $(f_n) \subseteq \mathcal{S}(\mathbb{R})$ such that $f_n \to f$ in $L^2(\mathbb{R})$. Observe that,

$$\left\| \widehat{f_n} - \widehat{f_m} \right\|_2 = \|f_n - f_m\|_2 \to 0 \text{ as } n,\ m \to \infty$$

So, $(\widehat{f_n})$ is a Cauchy sequence in $L^2(\mathbb{R})$. Hence, $\exists g \in L^2(\mathbb{R})$ such that $\widehat{f_m} \to g$ as $m \to \infty$.

Define $\widehat{f} := g \overset{L^2}{=} \lim_{n\to\infty} \widehat{f_n}$, i.e.

$$\widehat{f} \overset{L^2}{=} \lim_{n\to\infty} \widehat{f_n}$$

Well-definedness: Let (h_n) be another sequence in $L^2(\mathbb{R})$ such that $h_n \to f$ in $L^2(\mathbb{R})$. Then (h_n) is a Cauchy sequence in $L^2(\mathbb{R})$, so $\exists h \in L^2(\mathbb{R})$ such that

$$h \overset{L^2}{=} \lim_{n\to\infty} \widehat{h_n}$$

Claim: $h = g$. For,

$$
\begin{aligned}
\|h-g\|_2 &= \left\| \lim_{n\to\infty} \widehat{h_n} - \lim_{n\to\infty} \widehat{f_n} \right\|_2 \\
&= \lim_{n\to\infty} \left\| \widehat{h_n} - \widehat{f_n} \right\|_2 \ \ (\|\cdot\|_2 \text{ is continuous}) \\
&= \lim_{n\to\infty} \left\| \widehat{h_n - f_n} \right\|_2 \\
&= \lim_{n\to\infty} \|h_n - f_n\|_2 \ \ (\text{since } h_n, f_n \in \mathcal{S}(\mathbb{R}),\ \|\cdot\|_2 \text{ is an isometry}) \\
&= \lim_{n\to\infty} \|h_n - f + f - f_n\|_2 \to 0 \ (\text{as } h_n \to f,\ f_n \to f) \\
\implies \|h-g\|_2 &= 0 \\
\implies h &= g
\end{aligned}
$$

Remark 4.6. Instead of taking the sequence (f_n) in $\mathcal{S}(\mathbb{R})$, we can use any dense class $\mathscr{D}$ in $L^1 \cap L^2$. We shall now formally define the L^2-Fourier transform, in line with our above discussion.

Definition 4.7. Let $f \in L^2(\mathbb{R})$. Define the L^2-Fourier transform of f as

$$\widehat{f} \overset{L^2}{=} \lim_{n\to\infty} \widehat{f_n}$$

where (f_n) is any sequence in $\mathcal{S}(\mathbb{R})$ such that $f_n \to f$ in $L^2(\mathbb{R})$ as $n \to \infty$.
In other words, for a function $f \in L^2(\mathbb{R})$, the L^2-Fourier transform, denoted by $\mathscr{F}_2$, is defined as

$$\widehat{f} = \mathscr{F}_2 f \overset{L^2}{=} \lim_{n\to\infty} \mathscr{F}_1 f_n$$

where $\mathscr{F}_1$ denotes the L^1-Fourier transform, which acts on the sequence $(f_n) \subseteq \mathcal{S}(\mathbb{R})$ (recall that $\mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R})$ from Lemma 4.10).

Remark 4.7. The L^2-Fourier transform of a function $f \in L^2(\mathbb{R})$ is thus defined as the L^2-limit of the L^1-Fourier transform of a sequence of Schwartz class functions (f_n) that converges to f in $L^2(\mathbb{R})$. Note that such a sequence $(f_n) \subseteq \mathcal{S}(\mathbb{R})$ always exists because $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ (Lemma 4.9). We write

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx \overset{L^2}{=} \lim_{N\to\infty} \int_{\mathbb{R}} f(x)\, \chi_{[-N,N]}(x)\, e^{2\pi i x\xi}\, dx, \quad \forall \xi \in \mathbb{R}.$$

$\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. If $b_n \to f$ in L^2, then

$$\widehat{f}(\xi) = \lim_{n\to\infty} \widehat{b}_n(\xi), \quad \widehat{b}_n(\xi) = \int_{\mathbb{R}} b_n(x)e^{-2\pi i x\xi}dx$$

for all $\xi \in \mathbb{R}$.

We write

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx, \quad \forall \xi \in \mathbb{R}$$

and

$$\widehat{f}^{L^2(\mathbb{R})} = \lim_{N\to\infty} \int_{-N}^{N} f(x)\chi_{[-N,N]}(x)e^{-2\pi i x\xi}dx$$

Theorem 4.5 (Plancherel's Theorem)**.** *The L^2-Fourier transform $\wedge : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by $f \mapsto \widehat{f}$ is a linear isometry and is onto ($\wedge$ is a unitary operator).*

Proof. We know that $\|f\|_2 = \left\|\widehat{f}\right\|_2$, $\forall f \in \mathcal{S}(\mathbb{R})$ (Lemma 4.13). As $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ (Lemma 4.9), $\wedge$ is an isometry on $L^2(\mathbb{R})$ (isometry on the dense class). $\implies \wedge : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry.

As $\wedge$ is an isometry, we can see that $\overline{\wedge(L^2(\mathbb{R}))} = \wedge(L^2(\mathbb{R}))$ (Range of $\wedge$ is closed in L^2).

Also, $\mathcal{S}(\mathbb{R}) \subseteq \wedge(L^2(\mathbb{R}))$ (since $\wedge : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is onto by Lemma 4.14).
And, $\overline{\mathcal{S}(\mathbb{R})} = L^2(\mathbb{R})$.
$\implies \wedge(L^2(\mathbb{R})) = L^2(\mathbb{R}) \implies \wedge$ is onto. □

Remark 4.8.
1. $\wedge : L^1(\mathbb{R}) \to C_0(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ is a bounded linear transformation.
2. $\wedge : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear transformation.

Exercise 4.3. Show that for functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the L^1 and L^2-Fourier transforms coincide. That is, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$\widehat{f}^{L^1} = \widehat{f}^{L^2}, \text{ or } \mathscr{F}_1(f) = \mathscr{F}_2(f)$$

4.4 L^p-Fourier Transform

Remark 4.9. By Interpolation Theorem, $\wedge : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ is a bounded linear transformation, where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{2}, \quad 0 < \theta < 1.$$

First, we define L^p Fourier transform for $1 < p < 2$ (we have already seen $p = 1, 2$).

Definition 4.8 (L^p-Fourier Transform)**.** Let $f \in L^p(\mathbb{R})$ $(1 < p < 2)$. Let $(f_n) \subseteq S(\mathbb{R})$ such that $f_n \to f$ in L^p. From Remark 4.9, $\exists c_p > 0$, such that $\left\| \widehat{f_n} \right\|_q \leq c_p \left\| f_n \right\|_p$, $\forall f \in L^p(\mathbb{R})$ (where q is the conjugate index).

So, $\left(\widehat{f_n}\right)_n$ (L^1-Fourier transforms, as $f_n \in \mathcal{S}(\mathbb{R})$) is Cauchy in $L^q(\mathbb{R})$.
$\implies \exists g \in L^q(\mathbb{R})$ such that $\widehat{f_n} \to g$ in L^q as $n \to \infty$.

Define the L^p-transform of f as:

$$\widehat{f} := g \overset{L^q}{=} \lim_{n \to \infty} \widehat{f_n}$$

Remark 4.10. Here are some observations.

1. $\wedge : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is bijective (Lemma 4.14)
2. $\forall f \in \mathcal{S}(\mathbb{R})$, $\widehat{\widehat{f}}(x) = f(-x)$, $x \in \mathbb{R}$
   $\forall f \in \mathcal{S}(\mathbb{R}),\ \mathscr{F}^{-1}\left(\widehat{f}\right)(x) = f(x),\ x \in \mathbb{R}$
3. $\forall f \in \mathcal{S}(\mathbb{R})$, $\widehat{\widehat{\widehat{\widehat{f}}}}(x) = f(x)$, $x \in \mathbb{R}$
4. $\wedge^4 = I$ $(\wedge : L^2(\mathbb{R}) \to L^2(\mathbb{R}))$
5. $\sigma(\wedge) = \{\pm 1,\ \pm i\}$ (spectrum of $\wedge$ or eigenvalues($\wedge$))

Theorem 4.6 (Parseval's theorem)**.** *For $f, g \in L^2(\mathbb{R})$,*

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$$

Note:

$$\|f\|_2^2 = \left\|\widehat{f}\right\|_2^2, \quad \implies \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx \right|^2 d\xi$$

Let us now find out the Fourier coefficients for some common non-periodic functions.

Example 4.3. Let $a > 0$ and $f(x) = e^{-a|x|}$, $x \in \mathbb{R}$. Then,

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{-a|x|}\ e^{-2\pi i x\xi}\ dx \\ &= \int_0^\infty e^{-ax}e^{-2\pi i x\xi}dx + \int_{-\infty}^0 e^{ax}e^{-2\pi i x\xi}dx \\ &= \int_0^\infty e^{-x(a+2\pi i\xi)}dx + \int_0^\infty e^{-x(a-2\pi i\xi)}dx \qquad (*)\end{aligned}$$

These integrals might seem trivial to solve, but don't miss the fact that they are complex integrals. Let us first find the general answer for $\int_0^\infty e^{-zx}\ dx$ for $z \in H^+ = \{a + ib : a, b \in \mathbb{R}, a > 0\}$ (right side of the complex plane), and then we shall revisit and solve $(*)$.

Let $a > 0$. We know $\int_0^\infty e^{-ax}dx = \frac{1}{a}$ for $a \in \mathbb{R} \ldots (**)$.
Define $h_1(z) = \frac{1}{z}$, $\forall z \in H^+$.
Define $h_2(z) = \int_0^\infty e^{-zx}dx$, $z \in H^+$.
Clearly, h_1 is analytic on H^+, and by Dominated Convergence Theorem or DCT, one can verify that so is h_2.
From $(**)$, we have $h_1(a) = h_2(a)$ for $a > 0 \in \mathbb{R}$.

By the Uniqueness Theorem (of complex analysis), we thus have $h_1(z) = h_2(z)$ for $z \in H^+$ (if two functions analytic on a domain coincide on the entire real line, they must coincide on the entire domain). Hence,

$$\int_0^\infty e^{-zx}dx = \frac{1}{z}, \quad \forall z \in H^+$$

Coming back to $(*)$, we have

$$\begin{aligned}\widehat{f}(\xi) &= \int_0^\infty e^{-x(a+2\pi i\xi)}dx + \int_0^\infty e^{-x(a-2\pi i\xi)}dx \\ &= \left[-\frac{e^{-x(a+2\pi i\xi)}}{a+2\pi i\xi}\right]_0^\infty - \left[\frac{e^{-x(a-2\pi i\xi)}}{a-2\pi i\xi}\right]_0^\infty \\ &= \frac{1}{a+2\pi i\xi} + \frac{1}{a-2\pi i\xi} = \frac{2a}{a^2+4\pi^2\xi^2},\ \xi \in \mathbb{R}\end{aligned}$$

$$\boxed{\therefore \text{For } f(x) = e^{-a|x|},\ \widehat{f}(\xi) = \frac{2a}{a^2+4\pi^2\xi^2},\ \xi \in \mathbb{R}}$$

Example 4.4. Let $f(x) = e^{-a|x|}$, $g(x) = e^{-b|x|}$, $a > 0$, $b > 0$. Then

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

(using Parseval's theorem)

$$\begin{aligned}
\implies \int_{\mathbb{R}} e^{-(a+b)|x|}\ dx &= \int_{-\infty}^{\infty} \frac{2a}{a^2+4\pi^2\xi^2}\cdot\frac{2b}{b^2+4\pi^2\xi^2}d\xi \\
\implies \frac{2}{a+b} &= 4ab\int_{\mathbb{R}} \frac{d\xi}{(a^2+4\pi^2\xi^2)(b^2+4\pi^2\xi^2)} \\
\implies \frac{1}{2ab(a+b)} &= \int_{-\infty}^{\infty} \frac{d\xi}{(a^2+4\pi^2\xi^2)(b^2+4\pi^2\xi^2)}
\end{aligned}$$

Therefore, the expression for the general integral:

$$\boxed{\int_{\mathbb{R}} \frac{d\xi}{(a^2+\xi^2)(b^2+\xi^2)} = \frac{\pi}{ab(a+b)},\ \ a,b>0,\ \ \xi\in\mathbb{R}}$$

Example 4.5. Suppose the question is to evaluate the integral

$$\int_{-\infty}^{\infty}\left|\frac{\sin\xi}{\xi}\right|^2\ d\xi$$

Let $f(x) = \chi_{[-1,1]}(x)$, $x\in\mathbb{R}$. Then,

$$\widehat{f}(\xi) = \int_{-1}^{1} e^{-2\pi i x\xi}\ dx = \left.\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\right|_{-1}^{1} = \frac{e^{-2\pi i\xi}-e^{2\pi i\xi}}{-2\pi i\xi} = \frac{e^{-2\pi i\xi}-e^{2\pi i\xi}}{-2\pi i\xi} = \frac{\sin 2\pi\xi}{\pi\xi},\ \ \xi\in\mathbb{R}$$

Therefore,

$$\int_{-\infty}^{\infty}\left|\widehat{f}(\xi)\right|^2 d\xi = \int_{-\infty}^{\infty}\left|\frac{\sin(2\pi\xi)}{\pi\xi}\right|^2 d\xi = \int_{-\infty}^{\infty}|\chi_{[-1,1]}(x)|^2dx = \int_{-1}^{1}dx \text{ (by Plancherel's Theorem)} = 2$$

Hence, by change of variable $\xi\mapsto 2\pi\xi$,

$$\boxed{\implies \int_{-\infty}^{\infty}\left|\frac{\sin\xi}{\xi}\right|^2\ d\xi = 4\pi}$$

Remark 4.11. If $f(x) = \chi_{(-a,a)}(x)$ and $g(x) = \chi_{(-b,b)}(x)$, $a, b > 0$, then:

$$\int_{\mathbb{R}} \frac{\sin(2\pi a\xi)}{\pi\xi} \frac{\sin(2\pi b\xi)}{\pi\xi} d\xi = 2\min\{a, b\}$$

We have seen that the L^p-Fourier transform namely $\wedge : L^p(\mathbb{R}) \to L^q(\mathbb{R})$, is a bounded linear transform for $1 < p < 2$, where $\frac{1}{p} + \frac{1}{q} = 1$ (q is the conjugate exponent of p). A very natural question to ask is, whether we can define the Fourier transform $\wedge$ for $f \in L^p(\mathbb{R})$, $p > 2$, such that it is a bounded linear transformation. The answer is no, which we shall answer in the next few steps.

Question: For $p > 2$, can we say $\wedge : (\mathcal{S}(\mathbb{R}), \|\cdot\|_p) \to (\mathcal{S}(\mathbb{R}))$ is a bounded linear transformation for some q? The answer is *no*. The following lemma shows that if $\wedge$ is indeed bounded, then p and q must be conjugate exponents.

Lemma 4.15. *Let $p > 2$. Assume $\wedge : (\mathcal{S}(\mathbb{R}), \|\cdot\|_p) \to (\mathcal{S}(\mathbb{R}), \|\cdot\|_q)$ is bounded linear transformation. Then $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Given that $\|\widehat{f}\|_q \leq c_{p,q}\|f\|_p$ for all $f \in \mathcal{S}(\mathbb{R}) \ldots (*)$

For $a > 0$, define $f_a(x) = e^{-\pi a^2 x^2}$, $x \in \mathbb{R}$

Clearly, $f_a \in \mathcal{S}(\mathbb{R})$ and $\widehat{f_a}(\xi) = \frac{1}{a}e^{-\pi^2\xi^2/a^2}$, $\xi \in \mathbb{R}$.

From $(*)$, we have

$$\|f_a\|_p = \left(\int_{\mathbb{R}} e^{-p\pi a^2 x^2} dx\right)^{1/p} = \left(e^{-\pi x^2}\frac{dx}{a\sqrt{p}}\right)^{1/p} = \frac{1}{a^{1/p}\, p^{1/2p}}$$

Similarly, we have

$$\left\|\widehat{f_a}\right\|_q = \frac{1}{a}\frac{a^{1/q}}{q^{1/2q}}$$

$$\implies \frac{1}{a}\frac{a^{1/q}}{q^{1/2q}} \leq c_{p,q}\frac{1}{a^{1/p}\, p^{1/2p}} \;\forall a > 0$$

$$\implies a^{\frac{1}{p}+\frac{1}{q}-1} \leq c_{p,q}\frac{q^{1/2q}}{p^{1/2p}}$$

$$\implies \frac{1}{p} + \frac{1}{q} - 1 = 0$$

$$\implies \frac{1}{p} + \frac{1}{q} = 1$$

□

Lemma 4.16. *Let $p > 2$. The Fourier transform $\wedge : (\mathcal{S}(\mathbb{R}), \|\cdot\|_p) \to (\mathcal{S}(\mathbb{R}), \|\cdot\|_q)$ is not a bounded linear transformation: i.e.,*

$$\nexists\, c_p > 0 \text{ such that } \|\widehat{f}\|_q \leq c_p \|f\|_p, \ \ \forall f \in \mathcal{S}(\mathbb{R}).$$

Proof. We provide a sketch of the proof and leave the readers to verify the details. Assume the contrary: $\exists c_p > 0$ s.t. $\|\widehat{f}\|_q \leq c_p \|f\|_p$ for all $f \in \mathcal{S}(\mathbb{R}) \ldots (*)$.

Let $a > 0$, $b \in \mathbb{R}$. Define $f_{a,b}(x) = e^{-\pi(a+ib)x^2}$, $x \in \mathbb{R}$.

1. Verify that $f_{a,b} \in S(\mathbb{R})$ and $\widehat{f_{a,b}}(\xi) = \frac{1}{\sqrt{a+ib}} e^{-\pi^2 \xi^2/(a+ib)}$, $\xi \in \mathbb{R}$.
2. Compute $\|f_{a,b}\|_p$, $\left\|\widehat{f_{a,b}}\right\|_q$ and substitute in $(**)$ to get a contradiction.

□

Remark 4.12. In light of the above discussion, the Fourier transform $\wedge$ cannot be defined for all $f \in L^p(\mathbb{R})$ $(p > 2)$ such that $\wedge : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a bounded linear transformation.

4.5 Poisson Summation Formula

Theorem 4.7 (Poisson Summation Formula)**.** *Let $f \in L^2(\mathbb{R})$ such that $|f(x)| \leq \frac{c}{1+x^2}$, $\forall x \in \mathbb{R}$ and $|\widehat{f}(\xi)| \leq \frac{\widetilde{c}}{1+\xi^2}$, $\forall \xi \in \mathbb{R}$. Then,*

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \widehat{f}\left(\frac{n}{2\pi}\right)$$

Proof. Given that $f \in L^2(\mathbb{R})$ and $|f(x)| \leq \frac{c}{1+x^2}$, $|\widehat{f}(\xi)| \leq \frac{\widetilde{c}}{1+\xi^2}$. Define

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + 2\pi n), \quad x \in \mathbb{R}.$$

Observe that

$$F(x + 2\pi) = F(x), \quad \forall x \in \mathbb{R}$$

and

$$\int_{-\pi}^{\pi} |F(x)|^2 dx = \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} f(x + 2\pi n) \right|^2 dx \overset{how?}{=} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} |f(x + 2\pi n)|^2 dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{(2n-1)\pi}^{(2n+1)\pi} |f(x)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 dx < \infty$$

Now, observe that

$$\begin{aligned}
\widehat{F}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-inx}\ dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(x + 2\pi n) e^{-inx}\ dx \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} f(x + 2\pi n) e^{-inx}\ dx \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{(2n-1)\pi}^{(2n+1)\pi} f(x) e^{-inx}\ dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-inx}\ dx \\
&= \frac{1}{2\pi} \widehat{f}\left(\frac{n}{2\pi}\right) \\
\implies \widehat{F}(n) &= \frac{1}{2\pi} \widehat{f}\left(\frac{n}{2\pi}\right)
\end{aligned}$$

Let $\sum_{n=-\infty}^{\infty} \widehat{F}(n)\ e^{inx}$, $x \in \mathbb{R}$, be the Fourier series of F. Then,

$$\sum_{n=-\infty}^{\infty} |\widehat{F}(n)| = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left| \widehat{f}\left(\frac{n}{2\pi}\right) \right| < \infty$$

So, $F(x)$ is uniformly the Fourier series $\sum_{n=-\infty}^{\infty} \widehat{F}(n)\ e^{inx},\ x \in \mathbb{R}$.

$$\implies \sum_{n=-\infty}^{\infty} f(x + 2\pi n) = \sum_{n=-\infty}^{\infty} \widehat{F}(n)\ e^{inx}, \forall x \in \mathbb{R}$$

Putting $x = 0$, we have:

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \sum_{n=-\infty}^{\infty} \widehat{F}(n) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \widehat{f}\left(\frac{n}{2\pi}\right)$$

$\square$

The Poisson summation formula helps us in solving certain problems, such as simplifying the expressions of certain series. Let us illustrate this through some examples.

Example 4.6. Let $\delta > 0$. Take $f_\delta(x) = e^{-\pi\delta x^2}$, $x \in \mathbb{R}$. Then (see Corollary 4.1),

$$\widehat{f_\delta}(\xi) = \frac{1}{\sqrt{\delta}} e^{-\pi\xi^2/\delta}, \quad \xi \in \mathbb{R}$$

By Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} e^{-\delta\ 4n^2\pi^3} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{\delta}} e^{-\pi n^2/(4\pi^2\delta)} = \frac{1}{2\pi\sqrt{\delta}} \sum_{n=-\infty}^{\infty} e^{-n^2/(4\pi\delta)}$$

Example 4.7. $a > 0$, $f(x) = e^{-a|x|}$, $x \in \mathbb{R}$. Recall from Example 4.3 that,

$$\widehat{f}(\xi) = \frac{2a}{a^2 + 4\pi^2\xi^2}, \quad \xi \in \mathbb{R}$$

By Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{2a}{a^2 + \frac{4\pi^2 n^2}{4\pi^2}} = \frac{a}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2}$$

$$\implies \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{a} \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

$$\implies \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{a} + 2\sum_{n=1}^{\infty} e^{-2\pi a n}$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{2a} - \frac{1}{2a^2} + \sum_{n=1}^{\infty} e^{-2\pi a n}$$

Example 4.8. Let

$$g(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\widehat{g}(\xi) = \int_{-1}^{1} (1 - |x|) e^{-2\pi i x \xi} dx$$

Due to symmetry, this becomes

$$= 2\int_{0}^{1} (1 - x)\cos(2\pi x \xi)\ dx$$

Integrating by parts, we have:

$$= 2\left[(1-x)\frac{\sin(2\pi x\xi)}{2\pi\xi} - \frac{\cos(2\pi x\xi)}{4\pi^2\xi^2}\right]_0^1$$

$$= 2\left[-\frac{\cos(2\pi\xi)}{4\pi^2\xi^2} + \frac{1}{4\pi^2\xi^2}\right] = \frac{1}{2\pi^2\xi^2}\cdot 2\sin^2(\pi\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2$$

(using $\cos(2\theta) = 1 - 2\sin^2(\theta)$).

By Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} g(2\pi n) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \widehat{g}\left(\frac{n}{2\pi}\right)$$

$$\implies g(0) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\frac{\sin^2\left(\frac{\pi n}{2\pi}\right)}{\frac{\pi^2 n^2}{2\pi^2}}\right)$$

Note that in LHS, only $g(0) = 1$ survives, because the other terms in the summation, that are $g(2\pi),\ g(4\pi), \ldots$ and $g(-2\pi),\ g(-4\pi), \ldots$ vanish by definition. So,

$$\implies 1 = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{\sin^2\left(\frac{n}{2}\right)}{n^2}$$

$$\implies \sum_{n=-\infty}^{\infty}\frac{\sin^2(n/2)}{n^2} = \frac{\pi}{2}$$

So,

$$\frac{1}{4} + 2\sum_{n=1}^{\infty}\frac{\sin^2(n/2)}{n^2} = \frac{\pi}{2}$$

$$\boxed{\implies \sum_{n=1}^{\infty}\frac{\sin^2(n/2)}{n^2} = \frac{\pi}{4} - \frac{1}{8}}$$

Recall: If $f \in S(\mathbb{R})$, then $\widehat{f} \in S(\mathbb{R})$ (Corollary 4.4).

4.6 Some Important Theorems

In this section, we shall cover a special case of Benedick's theorem, Heisenberg's Uncertainty Principle, and the Paley-Wiener Theorem.

Lemma 4.17 (Special case of Benedick's Theorem)**.** *A nonzero continuous function and its Fourier transform cannot be both compactly supported. That is, if* $f,\ \widehat{f} \in C_C^\infty(\mathbb{R})$*, then*

$f = 0$. *In other words, if* $f(\neq 0) \in C_C^\infty(\mathbb{R})$*, then* $\widehat{f} \notin C_C^\infty(\mathbb{R})$*.*

Proof. Let $f \in C_C^\infty(\mathbb{R})$ and support$f \subseteq [-a, a]$, $a > 0$. Further, let $\widehat{f} \in C_C^\infty(\mathbb{R})$ with support$\widehat{f} \subseteq [-M, M]$. Then,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx = \int_{-a}^{a} f(x)e^{-2\pi i x\xi}dx, \quad \xi \in \mathbb{R} \ldots (*)$$

Define:

$$F(z) = \int_{-a}^{a} f(x)e^{-2\pi i x z}\, dx, \ z \in \mathbb{C}.$$

By Dominated Convergence Theorem or DCT, one can show that F is continuous on $\mathbb{C}$. Let γ be a simple closed contour in $\mathbb{C}$. Then

$$\int_\gamma F(z)dz = \int_\gamma \int_{-a}^{a} f(x)e^{-2\pi i x z}\, dx\, dz = \int_{-a}^{a} f(x) \left(\int_\gamma e^{-2\pi i x z} dz \right) dx$$

But $\int_\gamma e^{-2\pi i x z}dz = 0$ (by Cauchy's Theorem, because the integrand is an analytic function and γ is a simple closed contour).
$\implies \int_\gamma F(z)dz = 0$ for all closed contours γ. Hence, by Morera's theorem, F is analytic in $\mathbb{C}$.

From $(*)$, $F(\xi) = \widehat{f}(\xi)$, $\forall \xi \in \mathbb{R}$.
$\implies F(\xi) = 0$, $\forall |\xi| > M$ (since sup $\widehat{f} \subseteq [-M, M]$).
Then $F(z) = 0$ for all $z \in \mathbb{C}$ (by Uniqueness theorem).
Thus, $\widehat{f}(\xi) = 0\ \forall \xi \in \mathbb{R}$
$\implies f(x) = 0$ on $\mathbb{R}$ (by Corollary 4.2).

□

Theorem 4.8 (Heisenberg's Uncertainty Principle)**.** *A function and its Fourier transform cannot both be sharply localized. Let* $f \in L^2(\mathbb{R})$ *and* $|f(x)| \leq \frac{c}{1+x^2}$*,* $x \in \mathbb{R}$*. Then,*

$$\int_{\mathbb{R}} x^2 |f(x)|^2\, dx \cdot \int_{\mathbb{R}} \xi^2 \left| \widehat{f}(\xi) \right|^2\, d\xi \geq c\ (> 0)$$

Moreover, equality holds if $f(x) = \tilde{c}e^{-\pi x^2}$ *(Gaussian).*

Proof. Consider

$$\left| \int_{\mathbb{R}} x \frac{d}{dx} |f(x)|^2\, dx \right| = \left| \int_{\mathbb{R}} x \frac{d}{dx} (f(x)\overline{f(x)})\, dx \right|$$

$$= \left| \int_{\mathbb{R}} x \left[f'(x)\overline{f(x)} + f(x)\overline{f'(x)} \right] \, dx \right|$$

$$\leq 2 \int_{\mathbb{R}} |x||f'(x)||f(x)|dx$$

$$\leq 2 \left(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |f'(x)|^2 dx \right)^{1/2} \text{ (by Cauchy-Schwarz)}$$

$$= 2 \left(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \left| \widehat{f'}(\xi) \right|^2 d\xi \right)^{1/2}$$

$$= 2 \left(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |2\pi i \xi|^2 \left| \widehat{f'}(\xi) \right|^2 d\xi \right)^{1/2}$$

$$= 4\pi \left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \xi^2 \left| \widehat{f}(\xi) \right|^2 d\xi \right)^{1/2} \ldots (1)$$

Observe that,

$$\int_{\mathbb{R}} x \frac{d}{dx} |f(x)|^2 dx = - \int_{\mathbb{R}} |f(x)|^2 dx \ldots (2)$$

(integration by parts; boundary terms = 0 due to decay)

From (1) and (2):

$$\int_{\mathbb{R}} |f(x)|^2 dx \leq 4\pi \left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \xi^2 \left| \widehat{f}(\xi) \right|^2 d\xi \right)^{1/2}$$

$$\implies \frac{1}{16\pi^2} \|f\|_2^4 \leq \left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx \right) \left(\int_{\mathbb{R}} \xi^2 \left| \widehat{f}(\xi) \right|^2 d\xi \right)$$

(the terms cannot be both made small together).

□

Definition 4.9 (Type of a function)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. We say that f is of type T if $\exists c > 0$ such that $|f(z)| \leq ce^{T|z|}$, $\forall z \in \mathbb{C}$. If $T < 0$, then $f \equiv 0$; the only analytic function vanishing at end points is 0.

Theorem 4.9. *Let $f \in L^2(\mathbb{R})$, such that* support$(f) \subset [-T, T]$, $T > 0$. *Then the Fourier transform of f, $\widehat{f}$, is the restriction of an entire function of type $2\pi T$. That is, $\exists$ an entire function $F : \mathbb{C} \to \mathbb{C}$ such that*

$$|F(z)| \leq ce^{2\pi T|z|} \quad \forall z \in \mathbb{C}, \text{ such that } F(\xi) = \widehat{f}(\xi), \quad \xi \in \mathbb{R}.$$

Proof. Given $f \in L^2(\mathbb{R})$, $\operatorname{supp}(f) \subset [-T,T]$, $T>0$. Recall that

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx = \int_{-T}^{T} f(x)e^{-2\pi i x\xi}dx,\ \xi \in \mathbb{R}$$

Define

$$F(z) = \int_{-T}^{T} f(x)e^{-2\pi i x z}dx, \quad z \in \mathbb{C}$$

One can show that F is entire (by Cauchy-Schwarz inequality, Morera's theorem, and DCT). Observe,

$$|F(z)| \leq \int_{-T}^{T} |f(x)| \cdot |e^{-2\pi i x z}|\ dx$$

Let $z = a+ib$,

$$= \int_{-T}^{T} |f(x)|e^{-2\pi i x(a+ib)}\ dx = \int_{-T}^{T} |f(x)|e^{-2\pi x b}dx$$

$$\leq e^{2\pi T|b|}\int_{-T}^{T} |f(x)|dx \leq c\ e^{2\pi T|b|} \leq c\ e^{2\pi T|z|},\ \forall z \in \mathbb{C}$$

$\implies \exists c>0$ such that $|F(z)| \leq ce^{2\pi T|z|}$, $\forall z \in \mathbb{C}$.

Thus, F is an entire function of type $2\pi T$ and $F(\xi) = \widehat{f}(\xi)$ for $\xi \in \mathbb{R}$. $\square$

Theorem 4.10 (Payley-Wiener Theorem)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function of type $2\pi T$ $(T>0)$. Then $\exists f \in L^2(\mathbb{R})$ such that*

$$\operatorname{supp} f \subseteq [-T,T], \qquad F(\xi) = \widehat{f}(\xi),\ \forall \xi \in \mathbb{R}$$

(converse of the previous theorem).

Example 4.9. Let $a, b \in \mathbb{R}$.

1. Construct $f \in C_C^\infty(\mathbb{R})$ such that $f(x) = 0$ for $x < a$, $b < x$.
2. Construct $f \in C^\infty(\mathbb{R})$ such that $f(x) = 0$ for $x \leq a$ and $f(x) = 1$ for $x \geq b$, and f is increasing.
3. Let $\delta > 0$. Construct $f \in C_C^\infty(\mathbb{R})$ such that $f(x) = 1$ for $a+\delta < x < b-\delta$.

Answers:

1. $f(x) = \begin{cases} e^{-\left(\frac{1}{x-a} - \frac{1}{b-x}\right)}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$

   $\operatorname{supp} f = [a,b]$

2.
$$F(x) = \frac{\int_a^x f(t)\,dt}{\int_a^b f(t)\,dt}$$

Clearly, $F \in C^\infty(\mathbb{R})$, $F(x) = 0$ if $x \le a$, $F(x) = 1$ if $x \ge b$.

3. Exercise.

Result: Let K be a compact set in $\mathbb{R}^n$. Let U be an open set in $\mathbb{R}^n$ with $\overline{U}$ compact and $K \subseteq U$. Then $\exists\, \varphi \in C_C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ for all $x \in K$, $\operatorname{supp}\varphi \subset U$.

5 Non-periodic Distribution Theory

Notation: We will call functions in $\mathscr{D}(\mathbb{R}) = C_C^\infty(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ as test functions. The definitions for convergence in both the spaces will be different, as we shall see next.

Definition 5.1 (Convergence in $\mathscr{D}(\mathbb{R}) = C_C^\infty(\mathbb{R})$)**.** Let $(\varphi_n) \subset \mathscr{D}(\mathbb{R})$. We say that $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R})$ if $\exists$ a compact set $K \subseteq \mathbb{R}$ s.t. $\operatorname{supp}\varphi_n \subseteq K$ and $\forall m \in \mathbb{N} \cup \{0\}$,

$$\left| \frac{d^m}{dx^m} \varphi_n(x) \right| \to 0 \text{ uniformly on } K \text{ (on } \mathbb{R})$$

Definition 5.2 (Convergence in $\mathcal{S}(\mathbb{R})$))**.** Let $(\varphi_k) \subseteq \mathcal{S}(\mathbb{R})$. We say that $\varphi_k \to 0$ in $\mathcal{S}(\mathbb{R})$ if $\forall n, m \in \mathbb{N} \cup \{0\}$,
$$\left| x^n \frac{d^m}{dx^m} \varphi_k(x) \right| \to 0 \text{ uniformly on } \mathbb{R} \text{ as } k \to \infty$$

Remark 5.1 (Metric on $\mathscr{D}(\mathbb{R})$). Write $\mathbb{R} = \bigcup_{n=0}^\infty K_n$, where $K_n \subseteq \mathbb{R}$ are compact. Let $m, n \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathscr{D}(\mathbb{R})$. Define

$$\|\varphi\|_{m,n} = \sum_{k=0}^m \left\| \varphi^{(k)}|_{K_n} \right\|_\infty \text{ (restriction of } \varphi^{(k)} \text{ to } K_n)$$

Then $\|\cdot\|_{m,n}$ is a pseudo-norm on $\mathscr{D}(\mathbb{R})$. ($\|\varphi\|_{m,n} = 0 \implies \varphi^{(k)} = 0$ for all k only on K_n, not on $\mathbb{R}$, and so the metric condition fails.)

Let $\|\cdot\|_N$ be the family of pseudo-norms on $\mathscr{D}(\mathbb{R})$ as above. Let $\varphi, \psi \in \mathscr{D}(\mathbb{R})$. Define

$$d(\varphi, \psi) = \sum_{N=0}^\infty \frac{1}{2^N} \frac{\|\varphi - \psi\|_N}{1 + \|\varphi - \psi\|_N}$$

Check that $(\mathscr{D}(\mathbb{R}), d)$ is a complete metric space.

Remark 5.2 (Metric on $\mathcal{S}(\mathbb{R})$). Let $N \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{S}(\mathbb{R})$. Define

$$\|\varphi\|_N = \sum_{0 \leq m+n \leq N} \left\| x^m \frac{d^n}{dx^n} \varphi(x) \right\|_\infty$$

Check that $\|\cdot\|_N$ is a norm on $\mathcal{S}(\mathbb{R})$.

Let $\varphi, \psi \in \mathcal{S}(\mathbb{R})$. Define

$$d(\varphi, \psi) = \sum_{N=0}^{\infty} \frac{1}{2^N} \frac{\|\varphi - \psi\|_N}{1 + \|\varphi - \psi\|_N}$$

Show that $(\mathcal{S}(\mathbb{R}), d)$ is a complete metric space.

Exercise 5.1. Show that the identity map $Id : \mathscr{D}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ $(\mathscr{D}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}))$ defined by $f \mapsto f$ is continuous.

Hint: Take $(\varphi_n) \subset \mathscr{D}(\mathbb{R})$ such that $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R})$ (by Definition 5.1), and show that $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R})$ (using Definition 5.2).

Exercise 5.2. Show that the class of compactly supported infinitely differentiable functions on $\mathbb{R}$ is dense in the Schwartz class, i.e., $\overline{\mathscr{D}(\mathbb{R})} = \mathcal{S}(\mathbb{R})$.

Hint: Let $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\varphi(x) = 1$ for $-1 \leq x \leq 1$. Let $m \in \mathbb{N}$, define $\varphi_m(x) = \varphi(x/m)$ for all $x \in \mathbb{R}$.

Let $f \in \mathcal{S}(\mathbb{R})$. Show that $\varphi_m f \to f$ in $\mathcal{S}(\mathbb{R})$.

Theorem 5.1. *Let $x, y \in \mathbb{R}$ and $\varphi \in \mathscr{D}(\mathbb{R})$. Then the following $\mathscr{D}(\mathbb{R}) \to \mathscr{D}(\mathbb{R})$ maps are continuous:*

1. $\psi \mapsto \tau_x \psi$ *(translation)*
2. $\psi \mapsto M_y \psi$, *where* $M_y(\varphi)(t) = e^{-2\pi i y t} \varphi(t)$ *(modulation)*
3. $\psi \mapsto \phi\psi$
4. *For* $m \in \mathbb{N}$, $\psi \mapsto \psi^{(m)}$
5. $\psi \mapsto \phi * \psi$

6. $\psi \mapsto \overline{\psi}$

7. $\psi \mapsto \widetilde{\psi}$ *(reflection)*

Definition 5.3 (Non-periodic distribution)**.** A continuous linear functional $T : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ is called a non-periodic distribution.

Definition 5.4 (Space of non-periodic distributions)**.** $\mathscr{D}'(\mathbb{R}) = \{T : \mathscr{D}(\mathbb{R}) \to \mathbb{F} \mid T \text{ is continuous}\}$ is the space of all non-periodic distributions on $\mathbb{R}$.

Definition 5.5 (Locally integrable function)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be a Lebesgue measurable function. We say f is *locally integrable* if

$$\int_K |f(x)|dx < \infty \text{ for every compact } K \subseteq \mathbb{R}.$$

The space of all locally integrable functions is denoted by $L^1_{\text{loc}}(\mathbb{R})$.

Example 5.1 (Locally integrable functions)**.**

1. $C(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R})$ (all continuous functions on $\mathbb{R}$ are locally integrable).

2. $L^p(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R})$, $1 \leq p < \infty$.
   Reason: Let K (compact) $\subseteq \mathbb{R}$ and $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Then

$$\int_K |f(x)|dx = \int_K |f(x)|\cdot 1dx \leq \left(\int_K |f(x)|^p dx\right)^{1/p} \left(\int_K 1^{p'} dx\right)^{1/p'} = \|f\|_p \; m(K)^{1/p'} < \infty$$

   (by Hölder's inequality, with $1/p + 1/p' = 1$ and m the Lebesgue measure).

Example 5.2 (Non-periodic distributions)**.** Let us see some examples of non-periodic distributions.

1. **Non-periodic distribution generated by locally integrable function:**
   Let $f \in L^1_{\text{loc}}(\mathbb{R})$. Define $T_f : \mathscr{D}(\mathbb{R}) \to \mathbb{C}$ by

$$T_f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx, \quad \forall \varphi \in \mathscr{D}$$

   Clearly, T_f is a linear functional. To prove that T_f is a distribution, we have to prove its continuity.

To this end, let $(\varphi_n) \subset \mathscr{D}(\mathbb{R})$ with $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R})$. That is, $\exists$ a compact set $K \subseteq \mathbb{R}$ such that $\operatorname{supp} \varphi_m \subseteq K$ and

$$\left| \frac{d^n}{dx^n} \varphi_m(x) \right| \to 0 \text{ uniformly on } \mathbb{R} \text{ as } m \to \infty.$$

Claim: $T_f(\varphi_m) \to 0$ as $m \to \infty$. For,

$$\begin{aligned} |T_f(\varphi_m)| = \left| \int_{\mathbb{R}} f(x)\varphi_m(x)dx \right| &= \left| \int_K f(x)\varphi_m(x)dx \right| \\ &\leq \int_K |f(x)||\varphi_m(x)|dx \\ &\leq \|\varphi_m\|_\infty \int_K |f(x)|dx \to 0 \text{ as } m \to \infty. \end{aligned}$$

$\implies T_f \in \mathscr{D}'(\mathbb{R})$.

Definition 5.6. Let $T \in \mathscr{D}'(\mathbb{R})$. We say T is a function if $\exists f \in L^1_{\text{loc}}(\mathbb{R})$ such that $T = T_f$ (that is, if a distribution is generated by a locally integrable function, we identify the distribution as a function).

Note: $C(\mathbb{R}),\ L^p(\mathbb{R}) \subseteq \mathscr{D}'(\mathbb{R})$ and $L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathscr{D}'(\mathbb{R})$. Hence, if $f \in C(\mathbb{R})$ or $f \in L^p(\mathbb{R})$, $\implies T_f \in \mathscr{D}'(\mathbb{R})$. In other words, all continuous functions and L^p functions generate non-periodic distributions.

2. **Non-periodic distribution generated by measure:**

   Let μ be a regular Borel complex measure on $\mathbb{R}$ (or μ a positive measure) such that $\mu(K) < \infty$ for every compact $K \subseteq \mathbb{R}$. Define

   $$T_\mu(\varphi) = \int_{\mathbb{R}} \varphi(x)\ d\mu(x) \quad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

   Clearly, T_μ is a linear functional on $\mathscr{D}(\mathbb{R})$; Verify that T_μ is continuous, so that $T_\mu \in \mathscr{D}'(\mathbb{R})$.

   Definition 5.7. Let $T \in \mathscr{D}'(\mathbb{R})$. We say T is a measure if $\exists$ a measure μ (as above) such that $T = T_\mu$.

3. **Not all non-periodic distributions can be generated by locally integrable functions:**

Let $x \in \mathbb{R}$. Define $\delta_x : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by

$$\delta_x(\varphi) = \varphi(x), \quad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

Show that $\delta_x \in \mathscr{D}'(\mathbb{R})$. δ_x is called the Dirac measure at x, and $\delta_0 := \delta$ is known as the Dirac measure (at $x = 0$).

We show that $\delta \in \mathscr{D}'(\mathbb{R})$ is not a function, i.e. $\nexists f \in L^1_{\text{loc}}(\mathbb{R})$ such that $\delta = T_f$.

Suppose $\exists f \in L^1_{\text{loc}}(\mathbb{R})$ such that $\delta = T_f$. Then,

$$\delta(\varphi) = T_f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx \qquad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

$$\implies \varphi(0) = \int_{\mathbb{R}} f(x)\varphi(x)dx \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}) \ldots (*)$$

Let $\epsilon > 0$. Select $\varphi_\epsilon \in \mathscr{D}(\mathbb{R})$ such that $\varphi_\epsilon(x) = 1 \ \forall x \in (-\epsilon/2,\ \epsilon/2)$, $|\varphi_\epsilon(x)| \leq 1$, and $\operatorname{supp} \varphi_\epsilon \subseteq (-\epsilon, \epsilon)$.

From (*), we have:

$$\varphi_\epsilon(0) = \int_{\mathbb{R}} f(x)\varphi_\epsilon(x)dx \implies 1 = \int_{-\epsilon}^{\epsilon} f(x)dx$$

$$\implies 1 \leq \int_{-\epsilon}^{\epsilon} |f(x)|dx \to 0 \quad \text{as } \epsilon \to 0$$

which is not possible; contradiction.

4. **Not all non-periodic distributions can be generated by measures:**

Define $\delta' := T : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by

$$T(\varphi) = -\varphi'(0) \qquad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

Check that $T \in \mathscr{D}'(\mathbb{R})$. We show that the above T cannot be generated by a measure, i.e., there does not exist a measure μ (as above) such that $T = T_\mu$.

Suppose $\exists$ a measure such that $T = T_\mu$, then

$$T(\varphi) = T_\mu(\varphi) \ \forall \varphi \in \mathscr{D}(\mathbb{R}) \implies -\varphi'(0) = \int_{\mathbb{R}} \varphi(x)\ d\mu(x) \quad \forall \varphi \in \mathscr{D}(\mathbb{R}) \qquad (*)$$

Let $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\varphi(x) = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, $|\varphi(x)| \leq 1$, $\operatorname{supp}\varphi \subseteq [-1,1]$. Define

$$\psi_k(x) = \sin(kx)\ \varphi(x), \quad x \in \mathbb{R},\ k \in \mathbb{N}$$

Clearly, $\psi_k \in \mathscr{D}(\mathbb{R})\ \forall k \in \mathbb{N}$.

Observe that,

$$\psi_k'(x) = k\cos(kx)\varphi(x) + \sin(kx)\varphi'(x), \quad x \in \mathbb{R}$$

$$\implies \psi_k'(0) = k$$

From $(*)$, we have

$$-\psi_k'(0) = \int_{\mathbb{R}} \psi_k(x) d\mu(x)$$

$$\implies -k = \int_{-1}^{1} \sin(kx)\varphi(x)\ d\mu(x) \leq \int_{-1}^{1} |\sin(kx)||\varphi(x)|\ d\mu(x)$$

$$\leq \int_{-1}^{1} d\mu(x) = \mu([-1,1]) < \infty\ \forall k \in \mathbb{N}$$

This is a contradiction, as $k \to \infty$.

Definition 5.8. Let $x, y \in \mathbb{R}$, $\psi \in \mathscr{D}(\mathbb{R})$, and $T \in \mathscr{D}'(\mathbb{R})$. For the distribution T, we define:

1. **Translation:** $\tau_x T : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $\tau_x T(\varphi) = T(\tau_x\varphi)\ \forall \varphi \in \mathscr{D}(\mathbb{R})$
2. **Modulation:** $M_y T : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $M_y T(\varphi) = T(M_y\varphi)\ \forall \varphi \in \mathscr{D}(\mathbb{R})$
3. **Multiplication:** $\psi T : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $(\psi T)(\varphi) = T(\psi\varphi)\ \forall \varphi \in \mathscr{D}(\mathbb{R})$
4. **Derivatives:** $T^{(m)} : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $T^{(m)}(\varphi) = (-1)^m T(\varphi^{(m)})$ for all $\varphi \in \mathscr{D}(\mathbb{R})$
5. **Convolution:** $\psi * T : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $(\psi * T)(\varphi) = T(\widetilde{\psi} * \varphi)\ \forall \varphi \in \mathscr{D}(\mathbb{R})$
6. **Conjugation:** $\overline{T} : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $\overline{T}(\varphi) = \overline{T(\overline{\varphi})}\ \forall \varphi \in \mathscr{D}(\mathbb{R})$
7. **Reflection:** $\widetilde{T} : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $\widetilde{T}(\varphi) = T(\widetilde{\varphi})$ for all $\varphi \in \mathscr{D}(\mathbb{R})$.

Exercise 5.3. Verify that all the maps in the above definition are distributions.

Definition 5.9. Let Ω be an open set in $\mathbb{R}$. Let $\varphi : \Omega \to \mathbb{F}$ be a $C_C^\infty(\Omega)$ function. Define $\varphi^E : \mathbb{R} \to \mathbb{F}$ by

$$\varphi^E(x) = \begin{cases} \varphi(x), & x \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

Then $\varphi^E \in \mathscr{D}(\mathbb{R})$.

Let Ω be an open set in $\mathbb{R}$ and $T \in \mathscr{D}'(\mathbb{R})$. We say that T is *vanishing on* Ω if $T(\varphi^E) = 0$ for all $\varphi \in C_C^\infty(\Omega)$.

Definition 5.10 (Support of a distribution)**.** Let $T \in \mathscr{D}'(\mathbb{R})$. The *support* of T is the complement of the largest open set in $\mathbb{R}$ on which T vanishes.

Remark 5.3.

1. Product of two distributions cannot be defined.

2. We can define convolution of two distributions if one of them is compactly supported.

6 Problems

The following are a few problems that the reader may attempt to test their understanding.

1. Define Fourier coefficients and Fourier series of $f \in L^1_{2\pi}[-\pi, \pi]$.

2. Define trigonometric polynomial.

3. Define Approximation to the identity.

4. Define the convolution of $f, g \in L^1_{2\pi}[-\pi, \pi]$.

5. Define Dirichlet, Fejer and Poisson kernels.

6. Let $1 \leq p \leq q \leq \infty$. Then show that $L^q_{2\pi}[-\pi, \pi] \subseteq L^p[-\pi, \pi]$. Is it a proper inclusion? Justify your response.

7. Let $1 \leq p \leq q \leq \infty$. Show that $L^p(\mathbb{R})$, $L^q(\mathbb{R})$ spaces are not comparable.

8. Show that trigonometric polynomials are dense in $L^p_{2\pi}[-\pi, \pi]$ if $1 < p < \infty$. Can we say the same for $p = \infty$? Justify your response.

9. State and prove the Riemann-Lebesgue lemma.

10. Show that $F(z) = \int e^{-izx} e^{-x^2} dx$, $z \in \mathbb{C}$, is an entire function.

11. Define Hilbert transform and show that it is not a bounded operator on $L^1_{2\pi}[-\pi, \pi]$.

12. Let $a \in [-\pi, \pi]$. Show that there is a continuous function whose Fourier series diverges at a.

13. Let $\alpha \in (0, 1)$ and f be an α-Hölder 2-periodic function. Show that the Fourier series of f converges to f pointwise.

14. Define convergence in 2π-periodic test function space $C^\infty_{2\pi}[-\pi, \pi]$, and show that the collection of all trigonometric polynomials is dense in it.

15. Define a periodic distribution and the notion of convergence on it. Show that the set of all trigonometric polynomials is dense in the space of periodic distributions.

16. Show that the Dirac distribution is not a function, and give an example of a distribution that is not generated by any measure. Justify your response.

17. Let $f \in L^1_{2\pi}[-\pi, \pi]$. Show that f is infinitely differentiable as a distribution.

18. Define L^1 Fourier transform of $f \in L^1(\mathbb{R})$ and L^2 Fourier transform of $f \in L^2(\mathbb{R})$. Calculate the Fourier transform of $f(x) = \frac{1}{a^2+x^2}$, $x \in \mathbb{R}$, where a is fixed non-zero real number.

19. Define $\mathcal{S}(\mathbb{R})$. Show that

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall n, m \in \mathbb{N} \cup \{0\},\, x^n f^{(m)}(x) \in L^p(\mathbb{R})\}$$

20. Let $f(x) = \chi_{(0,1)}(x)$, $x \in \mathbb{R}$. Calculate $f * f$.

21. Let $a > 0$. Define $f_a(x) = e^{-ax^2}$, $x \in \mathbb{R}$. Compute $f_a * f_a$.

22. Let $c > 0$ and define $f_c(x) = e^{-\pi c x^2}$, $x \in \mathbb{R}$. Compute $\widehat{f}_{a+ib}$ where $a > 0$, $b \in \mathbb{R}$.

23. Show that $C_C^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

24. Let $1 \le p, q \le \infty$. If the Fourier transform

$$\wedge : (\mathcal{S}(\mathbb{R}), \|\cdot\|_p) \to (\mathcal{S}(\mathbb{R}), \|\cdot\|_q)$$

is continuous, then show that $\frac{1}{p} + \frac{1}{q} = 1$.

25. State the Riemann–Lebesgue Lemma for the Fourier transform of a function $f \in L^1(\mathbb{R})$. Is it also true for functions in $L^2(\mathbb{R})$? If yes, prove it; otherwise, justify your answer with an appropriate example.

26. Let $\{a_n\}_{n\in\mathbb{Z}}$ be a doubly infinite sequence such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. Then show that there is a continuous 2π-periodic function f on $\mathbb{R}$ with $\widehat{f}(n) = a_n$ for all $n \in \mathbb{Z}$.

27. Let $\{D_n\}$ be the sequence of Dirichlet kernels. Show that $\{\|D_n\|_1\}_{n=0}^{\infty}$ is unbounded.

28. State and prove Dirichlet's theorem.

29. Let $f \in L^2_{2\pi}[-\pi, \pi]$ and $(a, b) \subseteq [-\pi, \pi]$. Show that

$$\frac{1}{2\pi}\int_a^b f(t)dt = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \widehat{f}(n) \int_a^b e^{int}dt$$

30. Let $f \in L^1_{2\pi}[-\pi, \pi]$. Then show that

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(x) - f\left(x + \frac{\pi}{n}\right)\right] e^{-inx} dx \text{ for } n \in \mathbb{Z} \setminus \{0\}.$$

If f satisfies the Hölder condition of order $\alpha \in (0, 1]$, then there is $c > 0$ such that $|\widehat{f}(n)| \leq \frac{c}{n^\alpha},\ n \in \mathbb{Z} \setminus \{0\}$.

31. Show that the Hilbert transform is bounded on $L^p_{2\pi}[-\pi, \pi]$, $1 < p < \infty$.

32. Let $f(x) = |x|$, $x \in [-\pi, \pi]$. Write the Fourier series of f and show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{8} \text{ and } \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^2}{96}.$$

33. Show that the functions $e^x, e^{|x|}, \frac{1}{1+x^2}$ do not belong to $\mathcal{S}(\mathbb{R})$. Also, show that $f(x) = e^{-\frac{1}{1-x^2}}$ for $|x| < 1$, $f(x) = 0$ otherwise, and $g(x) = e^{-\pi x^{2n}}$ belong to $\mathcal{S}(\mathbb{R})$.

34. Define the Fourier transform on $L^2(\mathbb{R})$. Show that $\|f\|_2 = \left\|\widehat{f}\right\|_2$ for all $f \in \mathcal{S}(\mathbb{R})$. Also show that the $L^2(\mathbb{R})$ Fourier transform is a unitary transform on $L^2(\mathbb{R})$.

35. If $f, \widehat{f} \in C_C^\infty(\mathbb{R})$, then show $f = 0$ on $\mathbb{R}$. Give an example where f is compactly supported and $\widehat{f} \notin C_C^\infty(\mathbb{R})$, yet f is non-zero.

36. Let $f \in L^2(\mathbb{R})$ and $f(x) = 0$ for all $|x| > M$, for some $M > 0$. Show that f is the restriction of an entire function of exponential type. If $f \in C_C^\infty(\mathbb{R})$, then show that $\widehat{f}$ is the restriction of an entire function such that for each $n \in \mathbb{N}$ there is $c_n > 0$ with $|\widehat{f}(y)| \leq c_n \frac{1}{(1+y^2)^n}$, $y \in \mathbb{R}$.

37. Define convergent sequences in $\mathscr{D}(\mathbb{R})$. Let $f \in \mathscr{D}(\mathbb{R})$. Show that the maps $\varphi \mapsto f\varphi$ and $\varphi \mapsto f * \varphi$ are continuous on $\mathscr{D}(\mathbb{R})$.

38. Let $T \in \mathscr{D}'(\mathbb{R})$, $f \in \mathscr{D}(\mathbb{R})$. Then define the convolution of f and T as a function and show that $f * T$ is an infinitely differentiable function. Moreover, show that

$$(f * T)^{(m)} = f^{(m)} * T = f * T^{(m)}, \quad m \in \mathbb{N}.$$

39. Let $m \in \mathbb{N}$. Define $T_m : \mathscr{D} \to \mathbb{F}$ by

$$T_m(\varphi) = (-1)^m \varphi^{(m)}(0) \ \forall \varphi \in \mathscr{D}$$

Show that T_m is a periodic distribution, i.e., $T_m \in \mathscr{D}'$.

40. Let $x \in \mathbb{R}$, $m \in \mathbb{N}$. Define $T : \mathscr{D}(\mathbb{R}) \to \mathbb{F}$ by $T(\varphi) = (-1)^m \varphi^{(m)}(x)$ for all $\varphi \in \mathscr{D}(\mathbb{R})$. Then show that T is a non-periodic distribution, i.e. $T \in \mathscr{D}'(\mathbb{R})$.

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