Decoding Functional Analysis

Compiled by:

Rajdeep Pathak

Sonali Saha

Instructor: Prof. Venku Naidu D

Indian Institute of Technology Hyderabad

Preface

Functional analysis studies linear spaces provided with suitable topological structures and (continuous) linear transformations between such spaces. This document, *Decoding Functional Analysis*, is compiled as part of our collective efforts to understand and present the core concepts of functional analysis. The following chapters explore essential topics such as normed linear spaces, Banach spaces, Hilbert spaces, and more, structured to provide clarity and deeper insight into this field of mathematics. This material can be used as a reference for a first course in functional analysis. It lays a foundation to several advanced courses, like Banach space theory, Banach algebra, Von Neumann algebra, operator theory, and so on. It is assumed that the reader holds the basic knowledge of linear algebra, real analysis, and complex analysis.

We have taken utmost care to minimze the number of errors, and the presence of any error is completely unintentional. Please feel free to email any suggestions or feedback to pathak.rajdeep@alumni.iith.ac.in or
sahasonali@alumni.iith.ac.in.

The reader is strongly encouraged to attempt the exercises on the way. As Prof. James R. Munkres, in the second edition of his book titled "Topology" has said, "Working problems is a crucial part of learning mathematics. No one can learn (topology) by merely poring over the definitions, theorems, and the examples that are worked out in the text. One must work part of it out for oneself. To provide that opportunity is the purpose of the exercises."

We hope that this compilation proves useful for students, scholars, and anyone interested in gaining a solid foundation in functional analysis. The document was compiled when the authors were final-year M.Sc. students in the Department of Mathematics at IIT Hyderabad. Special thanks to Prof. Venku Naidu D for his continuous support and guidance throughout the course, and for kindly giving a finishing touch to the draft of this document. The content of this document is based on his class notes taught under MA5020: Functional Analysis in Fall 2024 semester at IIT Hyderabad.

Version: 2
Date: August 20, 2025

Notation

We use the notation followed by any standard textbook.

1. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: The set of natural numbers, integers, rationals, reals, and complex numbers respectively

2. $\mathbb{F}$: A field (here, either $\mathbb{R}$ or $\mathbb{C}$, unless mentioned otherwise)

3. $\mathbb{F}^n$: $\mathbb{R}^n$ or $\mathbb{C}^n$, the n-dimensional space

4. $\mathcal{P}[a, b]$: The set of all $\mathbb{F}$-valued polynomials over $[a, b]$

5. $C[a, b]$: The set of all $\mathbb{F}$-valued continuous functions $f : [a, b] \to \mathbb{F}$ over $[a, b]$

6. $C^k[a, b]$: The set of all $\mathbb{F}$-valued k-times continuously differentiable functions over $[a, b]$

Contents

1 Preliminaries

Lemma 1.1 (Zorn's lemma). *Let $(\mathcal{P}, \leq)$ be a partially ordered non-empty set (POSet). If every chain (total ordered set) is bounded above in $(\mathcal{P}, \leq)$, then $\mathcal{P}$ has a maximal element.*

1.1 Vector Spaces and Linear Transformations

Definition 1.1 (Linear Transformations). Let $(V, +, .)$ be a vector space over $\mathbb{F}$. A map $T : V \to W$ is called a linear transformation if the followings are satisfied:

1. $T(x + y) = T(x) + T(y)\ \forall x, y \in V$
2. $T(\alpha x) = \alpha T(x)\ \forall x \in V, \forall \alpha \in \mathbb{F}$

Remark 1.1. Every linear map sends the zero vector to the zero vector. If a map between vector spaces fails to do so, it cannot be classified as a linear transformation. Verifying this property is the first step in determining whether a map is linear.

Exercise 1.1. Answer the following:

1. Let $T : \mathbb{R} \to \mathbb{R}$ be a function such that $T(\alpha x) = \alpha T(x)$ holds $\forall \alpha \in \mathbb{R}$. Then show that $T(x + y) = Tx + Ty$.
2. Let $T : \mathbb{R} \to \mathbb{R}$ be a function such that $T(x + y) = T(x) + T(y)\ \forall x, y \in \mathbb{R}$. Does this imply that $T(\alpha x) = \alpha T(x)\ \forall \alpha \in \mathbb{R}$?
3. Let $T : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $T(x + y) = T(x) + T(y)\ \forall x, y \in \mathbb{R}$. Then show that T is a linear map.
4. Let $T : \mathbb{R} \to \mathbb{R}$ be a function such that $T(x + y) = T(x) + T(y)\ \forall x, y \in \mathbb{R}$. Assume that T is continuous at $x_0 \in \mathbb{R}$. Then show that T is a linear map.

Definition 1.2 (Null space and Range space). Let V and W be two vector spaces over $\mathbb{F}$, and let $T : V \to W$ be a linear transformation. Define the null space of T as

$$N(T) := \{x \in V : T(x) = 0\}\ (= T^{-1}(\{0\}))$$

and the range space of T as

$$R(T) := \{Tx : x \in V\}\ (= T(V)).$$

Clearly, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

Theorem 1.1. *Let V and W be two vector spaces such that $dim(V) = dim(W) = n$, and $T : V \to W$ be a linear transformation. Then the following are equivalent:*

1. *T is one-one.*

2. *T is onto.*

3. *T is a bijection.*

4. *T takes linearly independent set in V to a linearly independent set in W.*

Remark 1.2. If V and W are not finite-dimensional vector spaces, then the above theorem may not hold. We shall see some examples later.

Theorem 1.2. *Let V and W be vector spaces over $\mathbb{F}$. Let $\mathscr{B} = \{e_1, \dots, e_n\}$ be a basis for V and $y_1, \dots, y_n \in W$. Then there exists a unique linear transformation $T : V \to W$ such that $T(e_i) = y_i \; \forall 1 \leq i \leq n$.*

Exercise 1.2. Let $x(\neq 0) \in V$ and $y \in W$. Then show that there exists a linear transformation $T : V \to W$ such that $T(x) = y$. (In fact, we can construct a bunch of (different) linear transforms to do this job.)

Lemma 1.2. *The set of linear transformations from V to W is denoted as $L(V, W)$, and it is a vector space over the underlying field $\mathbb{F}$ of V and W, with the following operations:*

$$(T + S)(x) := T(x) + S(x), \;\; (\alpha T)(x) := \alpha T(x), x \in X, \;\; \alpha \in \mathbb{F}.$$

Definition 1.3 (Linear functional)**.** Let V be a vector space over $\mathbb{F}$. A map $f : V \to \mathbb{F}$ is called linear functional if $f(\alpha v + w) = \alpha f(v) + f(w), \; \forall v, w \in V$ and $\forall \alpha \in \mathbb{F}$.

Example 1.1 (Linear functionals)**.**

1. Define $f : \mathbb{F}^3 \to \mathbb{F}$ by $f((x, y, z)) = x + y \; \forall (x, y, z) \in \mathbb{F}^3$. Then (verify that) f is a linear functional on $\mathbb{F}^3$.

2. Consider the space $C[0, 1]$ of all continuous functions on $[0, 1]$. Then verify that the map $\phi : C[0, 1] \to \mathbb{F}$, defined by

$$\phi(f) = \int_0^1 f(x) \; dx$$

is a linear functional.

Remark 1.3. Note that $L(V, \mathbb{F})$ is a vector space over $\mathbb{F}$. It is denoted by V' - called algebraic dual of V.

1.2 Inner Product Spaces

Definition 1.4 (Inner product)**.** Let X be a vector space over $\mathbb{F}$. A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ is called an inner product on V if the followings are satisfied:

1. $\langle x, x\rangle \geq 0,\ \forall x \in V$

2. $\langle x, x\rangle = 0 \iff x = 0$

3. $\langle x, y\rangle = \overline{\langle y, x\rangle},\ \forall x, y \in X$

4. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ and $\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle,\ \forall \alpha \in \mathbb{F},\ x, y \in X$

5. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,\ \forall x, y, z \in X$

Definition 1.5 (Inner product space)**.** A vector space X over a field $\mathbb{F}$ is said to be an inner product space if there exists an inner product $\langle \cdot, \cdot\rangle$ on X, and is denoted by $(X, \langle \cdot, \cdot\rangle)$.

Remark 1.4. $\langle \cdot, \cdot\rangle$ is linear with respect to the first variable and anti-linear with respect to the second variable.

Exercise 1.3. Show that $\langle x, 0\rangle = \langle 0, x\rangle = 0\ \forall x \in X$.

Example 1.2 (Inner product spaces)**.** The following are some examples of inner product spaces.

1. $\mathbb{R}$ is a vector space over $\mathbb{R}$. Define $\langle \cdot, \cdot\rangle$ as

$$\langle x, y\rangle = x \cdot y,\ \forall x, y \in \mathbb{R}$$

   Verify that $(\mathbb{R}, \langle \cdot, \cdot\rangle)$ is an inner product space.

2. $\mathbb{C}$ is a vector space over $\mathbb{C}$. Define $\langle \cdot, \cdot\rangle$ as

$$\langle z_1, z_2\rangle = z_1 \cdot \overline{z_2},\ \forall z_1, z_2 \in \mathbb{C}$$

   Verify that $(\mathbb{C}, \langle \cdot, \cdot\rangle)$ is an inner product space.

3. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Define

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i$$

   Verify that $(\mathbb{R}^n, \langle \cdot, \cdot\rangle)$ is an inner product space.

4. $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. Let $\tau = (\tau_1, \tau_2, \ldots \tau_n) \in \mathbb{C}^n$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{C}^n$. Define

$$\langle \tau, \omega\rangle = \sum_{i=1}^{n} \tau_i \overline{\omega_i}$$

   Verify that $(\mathbb{C}^n, \langle \cdot, \cdot\rangle)$ is an inner product space.

5. Let $\mathcal{P}[a,b]$ be the collection of all $\mathbb{F}$-valued polynomials on $[a,b] \subseteq \mathbb{R}$. Let $p, q \in \mathcal{P}[a,b]$. Define
$$\langle p, q\rangle = \int_a^b p(x)\overline{q(x)}\ dx$$
Verify that $(\mathcal{P}[a,b], \langle\cdot,\cdot\rangle)$ is an inner product space.

More on inner product spaces is explored in Section 8 (Hilbert Spaces).

1.3 Metric Spaces

Definition 1.6 (Metric)**.** Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a metric on X if $\forall x, y, z \in X$ the followings are hold:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ (Symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Definition 1.7 (Metric space)**.** Let X be a non-empty set and d be a metric on X. The pair (X, d) is called a metric space.

Example 1.3 (Metric spaces)**.** The following are some examples of a metric space.

1. $(\mathbb{R}, d) = (\mathbb{R}, |\cdot|)$ is a metric space, where $d(x, y) = |x - y|,\ \forall x, y \in \mathbb{R}$.
2. $(\mathbb{R}^n, d_E)$ is a metric space, where d_E is the Euclidean metric defined by $d_E(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2},\ \forall \mathbf{x} = (x_1, x_2, \ldots, x_n),\ \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.
3. Let $X \neq \phi$. Let $b(X) =$ the collection of all $\mathbb{F} -$ valued bounded functions on X. Let $f, g \in b(X)$ and
$$d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$
Note that the existence of a supremum is guaranteed by the least upper bound property of $\mathbb{R}$, as $f - g$ is bounded above. Check that d_∞ is a metric.

Definition 1.8. Let $X \neq \phi$. The discrete metric d on X is defined as
$$d = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} .$$

Definition 1.9 (Convergent sequence)**.** Suppose (X, d) is a metric space and (x_n) be a sequence in X. For $x \in X$, we say that (x_n) converges to x if for given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon,\ \forall n > N$.

Remark 1.5. In this case, we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$ and read it as $\{x_n\}$ converges to x in X as $n \to \infty$.

Definition 1.10 (Cauchy sequence)**.** Suppose (X, d) is a metric space and (x_n) is a sequence in X. Then (x_n) is called a Cauchy sequence in X if for each $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon \ \forall n, m > N$.

Remark 1.6. Every convergent sequence is Cauchy, but a Cauchy sequence need not be convergent. Consider $(\mathbb{Q}, |\cdot|)$ and $x_n = \sum_{k=0}^{n} \frac{1}{k!}$. Check that (x_n) is Cauchy but does not converge in $\mathbb{Q}$. In fact, $x_n \to e$ in $\mathbb{R}$ as $n \to \infty$.

Lemma 1.3. *Every Cauchy sequence is bounded.*

Proof. Suppose (X, d) is a metric space and (x_n) is a Cauchy sequence in X. For $\varepsilon = 1$, we have that $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$ $d(x_n, x_m) < 1$. Using the triangle inequality, we have for any $m \in \mathbb{N}$,

$$d(x_m, 0) \leq d(x_N, 0) + d(x_m, x_N) \leq d(x_N, 0) + 1$$

□

Definition 1.11 (Complete metric space)**.** A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 1.4. $(\mathbb{R}, |\cdot|)$ *is a complete metric space.*

Proof. We shall prove this by showing that an arbitrary Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$. Suppose (x_n) is a Cauchy sequence in $\mathbb{R}$. We know by lemma 1.3 that Cauchy sequences are bounded. By Bolzano-Weierstrass Theorem, we know that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. Thus, there is a subsequence $\{x_{n_j}\}$ such that x_{n_j} converges to x in $\mathbb{R}$. This means that

$$\exists N_1 \in \mathbb{N} \text{ such that } \forall j \geq N_1,\ |x_{n_j} - x| < \frac{\varepsilon}{2}$$

The limit of a subsequence is also the limit of the Cauchy sequence, because

$$\forall \varepsilon > 0,\ \exists N_2 \in \mathbb{N} \text{ such that } \forall m, n \geq N_2,\ |x_m - x_n| < \frac{\varepsilon}{2}$$

Now, $\forall \varepsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ such that $\forall n, n_j \geq N_2, \forall j \geq N_1$, we have

$$|x_{n_j} - x| \leq |x_n - x_{n_j}| + |x_{n_j} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, an arbitrary Cauchy sequence is convergent in $\mathbb{R}$, and so $\mathbb{R}$ is complete. □

Exercise 1.4. Show that $\mathbb{R}^n$ is a complete metric space. (**Hint:** Use the completeness of $\mathbb{R}$)

Exercise 1.5. Show that a discrete metric space is complete.

Proposition 1.1. *Let (X, d) be a discrete metric space and (x_n) be a sequence in X. Then (x_n) is convergent in X if and only if $\exists N \in \mathbb{N}$ such that $x_n = x_m$, $\forall n, m > N$. In other words, the only convergent sequences in a discrete metric space are the eventually constant sequences.*

Proposition 1.2. *Let (X, d) be a metric space, (x_n) be a Cauchy sequence in X, and (x_{n_k}) be a convergent subsequence of (x_n) in X such that $x_{n_k} \to x$ in X. Then $x_n \to x$ in X as $n \to \infty$.*

Definition 1.12 (Ball in a metric space)**.** Let (X, d) be a metric space and $x \in X$. For $r > 0$,

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

is said to be a ball (neighbourhood) around x of radius r.

Definition 1.13 (Open set)**.** Let (X, d) be a metric space and $E \subseteq X$. E is said to be open in X if for each $x \in E$, $\exists r = r_x > 0$ such that $B(x, r) \subseteq E$.

Exercise 1.6. Show that the ball $B(x, r)$ is open in X.

Definition 1.14 (Limit point)**.** Let (X, d) be a metric space, and $E \subseteq X$. A point $x \in X$ is said to be a limit point of E if $B(x, r) \setminus \{x\} \cap E \neq \phi$, $\forall r > 0$. That is, x is a limit point of E if every deleted neighborhood of x in X intersects E.

Definition 1.15 (Closed set)**.** Let (X, d) be a metric space, and $G \subseteq X$. G is said to be closed in X if its complement $X \setminus G$ is open in X.
An equivalent definition: A set G is closed in X if it contains all its limit points.

Definition 1.16 (Closure of a set)**.** Given a subset $G \subset X$, the closure of it, denoted by $\overline{G}$, is defined by the smallest closed set that contains G. Equivalently, $\overline{G} := G \cup G'$, where G' is the set of all limit points of G.

Remark 1.7. Let (X, d) be a metric space. X and ϕ are open as well as closed in X.

Remark 1.8. In a discrete metric space, each subset is both open and closed.

Lemma 1.5. *Let (X, d) be a metric space. Then*

1. *Arbitrary union of open sets is open.*

2. *Finite union of closed sets is closed.*

3. *Finite intersection of open sets is open.*

4. *Arbitrary intersection of closed sets is closed.*

Definition 1.17 (Continuous function)**.** Let $f : (X, d_X) \to (Y, d_Y)$ be a function and $x \in X$. f is said to be continuous at $x \in X$ if $\forall \varepsilon > 0, \exists\, \delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta,\ \forall y \in X$.

Proposition 1.3. *Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is a function. Then the following are equivalent:*

1. *f is continuous.*

2. *$f(x_n) \to f(x)$ for every sequence (x_n) in X with $x_n \to x$ in X.*

3. *$f^{-1}(U)$ is open in X whenever U is open in Y.*

4. *$f^{-1}(V)$ is closed in X whenever V is closed in Y.*

Definition 1.18 (Separable metric space)**.** A metric space (X, d) is separable if $\exists$ a countable set $E \subset X$ such that E is dense in X, i.e. $\overline{E} = X$.

Example: $(\mathbb{R}, |\cdot|)$ is separable because $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

Also, observe that the discrete metric space X is separable if and only if X is countable set. Here, a countable set means that it is either a finite set or a countably infinite set. Another interesting example of a separable space is $(C[a, b], d_\infty)$ and the proof can be seen as an application of the Weierstrass approximation theorem (the set of all polynomials with rational coefficients becomes a countable dense set).

Proposition 1.4. *Let K be a compact Hausdorff space, and $C(K)$ denote the set of all $\mathbb{F}$-valued continuous functions defined on K. Then $(C(K), d_\infty)$ is a complete metric space, where*

$$d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in K\},\ \forall f, g \in C(K)$$

Theorem 1.3 (Arzella-Ascoli)**.** *Let K be a compact Hausdorff space and $E \subseteq C(K)$. Then E is compact set in $C(K)$ if and only if E is closed, bounded, and E is an equicontinuous family.*

2 Normed Linear Spaces

Definition 2.1 (Norm)**.** Let V be a vector space over $\mathbb{F}$. A function $\|\cdot\| : V \to [0, \infty)$ $(x \mapsto \|x\|)$ is called a norm on V if :

1. $\|x\| = 0 \iff x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Definition 2.2 (Normed linear space)**.** A vector space V over a field $\mathbb{F}$, equipped with a norm $\|\cdot\|$ is said to be a normed linear space over $\mathbb{F}$. It is denoted by the ordered pair $(V, \|\cdot\|)$.

Example 2.1 (Some examples of normed linear spaces)**.** Consider the following examples of normed linear spaces.

1. Let $V = \mathbb{F}^n$. Define
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$
and
$$\|x\|_\infty = \max\{|x_i| : 1 \leq i \leq n\}$$
where $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$. $(\mathbb{F}^n, \|\cdot\|_1)$ and $(\mathbb{F}^n, \|\cdot\|_\infty)$ are normed linear spaces. To see this, we need to verify that both functions satisfy the properties mentioned in definition 2.1.

Proof that $(\mathbb{F}^n, \|\cdot\|_1)$ *is a normed linear space*:

- Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$. Then $\|\mathbf{x}\|_1 = 0 \iff \sum_{i=1}^n |x_i| = 0 \iff |x_i| = 0 \; \forall i = \{1, \ldots, n\} \iff \mathbf{x} = (x_i)_{i=1}^n = 0$
- Let $\alpha \in \mathbb{F}$. Note that $\alpha\mathbf{x} = \alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots \alpha x_n)$. Then $\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^n |\alpha x_i| = \sum_{i=1}^n |\alpha||x_i| = |\alpha| \sum_{i=1}^n |x_i| = |\alpha| \, \|\mathbf{x}\|_1$
- Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$. Note that $\mathbf{x}+\mathbf{y} = (x_1, \ldots, x_n)+(y_1, \ldots y_n) = (x_1+y_1, \ldots, x_n+y_n)$. Then,
$$\|\mathbf{x}+\mathbf{y}\|_1 = \sum_{i=1}^{n} |x_i+y_i| \leq \sum_{i=1}^{n} |x_i|+\sum_{i=1}^{n} |y_i| \text{ (follows from the triangle inequality in } \mathbb{F})$$
$$= \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1.$$

Proof that $(\mathbb{F}^n, \|\cdot\|_\infty)$ *is a normed linear space*: Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$.

- $\mathbf{x} = 0 \iff x_i = 0 \; \forall \; i \in [n] \iff \max_{i=1}^n(|x_i|) = 0 \iff \|\mathbf{x}\|_\infty = 0$

- For any $c \in \mathbb{F}$, $\|c\mathbf{x}\|_\infty = \max(|cx_1|, \ldots, |cx_n|) = \max(|c||x_1|, \ldots, |c||x_n|) = |c| \max(|x_1|, \ldots, |x_n|) = |c| \; \|\mathbf{x}\|_\infty$
- **Claim:** For any $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots y_n) \in \mathbb{F}^n$,

$$\|\mathbf{x} + \mathbf{y}\|_\infty \leq \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$$

Indeed,

$$\begin{aligned}
&|x_i + y_i| \leq |x_i| + |y_i| \; \forall \; i \in [n] \\
\implies &|x_i + y_i| \leq \max_{i=1}^{n}(|x_i|) + \max_{i=1}^{n}(|y_i|) \\
\implies \max_{i=1}^{n}(&|x_i + y_i|) \leq \max_{i=1}^{n}(|x_i|) + \max_{i=1}^{n}(|y_i|)
\end{aligned}$$

Now,

$$\|\mathbf{x}+\mathbf{y}\|_\infty = \max(|x_1+y_1|, \ldots, |x_n+y_n|) \leq \max(|x_1|, \ldots, |x_n|) + \max(|y_1|, \ldots, |y_n|)$$

$$= \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$$

(the inequality is by virtue of what we have shown above).

2. Let X be a non-empty set. Define $b(X) = \{f : X \to \mathbb{F} \mid f \text{ is a bounded function}\}$. Check that $(b(X), +, \cdot)$ is a vector space over $\mathbb{F}$ (where $+$ and $\cdot$ denote the usual point-wise addition and the usual scalar multiplication of functions). For $f \in b(X)$, define:

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

Exercise 2.1. Show that $(b(X), \|\cdot\|_\infty)$ is a normed linear space.

3. **(Sequence spaces)** Define $\mathcal{S}(\mathbb{N}) = \{(x_n) : (x_n) \text{ is a sequence in } \mathbb{F}\}$. For $(x_n), (y_n) \in \mathcal{S}(\mathbb{N})$, define

$$\begin{aligned}(x_n) + (y_n) &= (x_n + y_n)\\ \alpha(x_n) &= (\alpha x_n)\ \forall (x_n), (y_n) \in \mathcal{S}(\mathbb{N}),\ \forall \alpha \in \mathbb{F}\end{aligned}$$

Exercise 2.2. Check that $(\mathcal{S}(\mathbb{N}), +, \cdot)$ is a vector space over $\mathbb{F}$.

- Define

$$\ell^1(\mathbb{N}) = \{(x_n) \in \mathcal{S}(\mathbb{N}) : \sum_{n=1}^{\infty} |x_n| < \infty\}$$

Check that $\ell^1(\mathbb{N})$ is a subspace of $\mathcal{S}(\mathbb{N})$. For $(x_n) \in \ell^1(\mathbb{N})$, define

$$\|(x_n)\|_1 = \sum_{n=1}^{\infty} |x_n|$$

Exercise 2.3. Check that $(\ell^1(\mathbb{N}), \|\cdot\|_1)$ is a normed linear space.

- Define

$$\ell^\infty(\mathbb{N}) = \{(x_n) \in \mathcal{S}(\mathbb{N}) : (x_n) \text{ is a bounded sequence}\}$$

For $(x_n) \in \ell^\infty(\mathbb{N})$, define

$$\|(x_n)\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$$

Exercise 2.4. Check that $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ is a normed linear space.

4. Let $\mathcal{R}[a, b]$ be the space of all Riemann-integrable functions over $[a, b]$. Let $f \in \mathcal{R}[a, b]$. Define

$$\|f\|_1 = \int_a^b |f(t)| dt$$

Observe that $\|\cdot\|_1$ as defined above does not form a norm on $\mathcal{R}[a, b]$. To see this, take

$$f(x) = \begin{cases} 0, & x \in [a, b) \\ 1, & x = b \end{cases}$$

Clearly, we have $f(x) \neq 0$ for $x \in [0, 1]$, but $\|f\|_1 = \int_a^b |f(t)| dt = 0$. Hence it is not a norm on $\mathcal{R}[a, b]$.

Let $f, g \in \mathcal{R}[a, b]$. Define a relation $\sim$ as $f \sim g$ if and only if $\int_a^b |f(x) - g(x)| dx = 0$.

Exercise 2.5. Check that $\sim$ is an equivalence relation on $\mathcal{R}[a, b]$.

Let $\tilde{\mathcal{R}}[a,b] = \{[f] : f \in \mathcal{R}[a,b]\}$ be the collection of the corresponding equivalence classes. Let $[f],[g] \in \tilde{\mathcal{R}}[a,b]$ and $\alpha \in \mathbb{F}$. Define

$$[f+g] = [f]+[g]$$
$$[\alpha f] = \alpha[f]$$

Check that $\mathcal{R}[a,b]$ is a vector space over $\mathbb{F}$. For $[f] \in \tilde{\mathcal{R}}[a,b]$, define

$$\|[f]\|_1 = \int_a^b |f(t)|dt$$

Exercise 2.6. Show that $\|\cdot\|_1$ is well-defined and forms a norm on $\tilde{\mathcal{R}}[a,b]$.

5. Let $C[a,b]$ denote the set of all continuous $\mathbb{F}$-valued functions over $[a,b]$. Let $f \in C[a,b]$. Define the following norms on $C[a,b]$:

$$\|f\|_1 = \int_a^b |f(x)|dx$$
$$\|f\|_p = \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}},\ 1 < p < \infty$$
$$\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$$

Check that $(C[a,b], \|\cdot\|_p)$ is a normed linear space for $1 \leq p \leq \infty$.

Now we will try to ask a very natural question. Does there exist a norm on a given vector space? The answer is 'Yes', and it is an application of Zorn's lemma.

Remark 2.1. There exists a norm on every vector space.

Exercise 2.7 (Proof of Remark 2.1)**.** Let V be a vector space over $\mathbb{F}$ and $\mathscr{B}$ be a Hamel basis of V (existence of $\mathscr{B}$ is guaranteed by Zorn's lemma). let $x \in V$. Then $\exists\, x_1, x_2, \ldots, x_n \in \mathscr{B}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

Define

$$\|x\| = \sum_{i=1}^{n} |\alpha_i|$$

Check that $(V, \|\cdot\|)$ is a normed linear space.

Definition 2.3 (Conjugate exponent)**.** Let $p \in (1,\infty)$. A number $q \in (1,\infty)$ is called the conjugate exponent of p if $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.1 (Young's inequality)**.** *Let $p, q \in (1, \infty)$ such that p and q are conjugate exponents of each other. Let $a, b \in (0, \infty)$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.1}$$

Proof. Fix $b \in (0, \infty)$. Define $f : (0, \infty) \to \mathbb{R}$ by

$$f(t) = \frac{t^p}{p} + \frac{t^q}{q} - tb \ \forall t \in (0, \infty) \ldots\ldots (*)$$

We know that f is differentiable. So, $f'(x) = t^{p-1} - b \ \forall t \in (0, \infty)$
Observe that $f'(t) \geq 0 \iff t^{p-1} - b \geq 0 \iff t \geq b^{\frac{1}{p-1}}$
and $f'(t) \leq 0 \iff t^{p-1} - b \leq 0 \iff t \leq b^{\frac{1}{p-1}} \implies$ f is increasing on $(b^{\frac{1}{p-1}}, \infty)$, decreasing on $(0, b^{\frac{1}{p-1}})$ and $f(b^{\frac{1}{p-1}}) = 0$ (verify).
Observe that f has a global minimum at $t = b^{\frac{1}{p-1}}$ and $f(b^{\frac{1}{p-1}}) = 0 \implies f(t) \geq 0 \ \forall t \in (0, \infty)$.

$$\implies tb \leq \frac{t^p}{p} + \frac{t^q}{q} \text{ [from (*)]}$$

So for any $a \in (0, \infty)$, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

□

Lemma 2.2 (Hölder's inequality)**.** *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (with the understanding that if $p = 1$, then $q = \infty$ and vice versa).*

1. *Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$. Then*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \begin{cases} \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}, & 1 < p, q < \infty \\ \left(\sum_{i=1}^{n} |a_i|\right) \max\{|b_i| : i \in [1:n]\}, & p = 1, \ q = \infty \end{cases} \tag{2.2}$$

2. *Let $a = (a_n), b = (b_n)$. Then*

$$\left|\sum_{i=1}^{\infty} a_i b_i\right| \leq \begin{cases} \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |b_i|^q\right)^{\frac{1}{q}}, & 1 < p, q < \infty \\ \left(\sum_{i=1}^{\infty} |a_i|\right) \sup_{i=1}^{\infty}\{|b_i|\}, & p = 1, \ q = \infty \end{cases} \tag{2.3}$$

Lemma 2.3 (Minkowski's inequality)**.** *Let $p \in (1, \infty)$. Let $(x_n), (y_n) \in \mathcal{S}(\mathbb{N})$. Then*

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}} \tag{2.4}$$

Exercise 2.8. Prove Hölder's inequality and Minkowski's inequality. (For the proofs, see this and this).

Lemma 2.4. *(Hölder's inequality in $C[a,b]$). Suppose $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^b |x(t)y(t)|dt \leq \|x\|_p \|y\|_q \quad \forall x, y \in C[a,b]$$

Proof. Note that by example 2.1 (5), $\|x\|_p = \left(\int_a^b |x(t)|^p dt\right)^{\frac{1}{p}}$ for $1 < p < \infty$.
Clearly, the given inequality holds if either $\|x\|_p = 0$ or $\|y\|_q = 0$, since

$$\|x\|_p = 0 \implies \int_a^b |x(t)|^p dt = 0 \implies x(t) = 0 \ (\because x \text{ is continuous})$$

Suppose $\|x\|_p, \ \|y\|_q \neq 0$.
For $a, b \in (0, \infty)$, by Young's inequality, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

$$\text{Take } a = \frac{|x(t)|}{\|x\|_p} \text{ and } b = \frac{|y(t)|}{\|y\|_q}$$

Then

$$\begin{aligned}
\frac{|x(t)y(t)|}{\|x\|_p \|y\|_q} &\leq \frac{|x(t)|^p}{p\|x\|_p^p} + \frac{|y(t)|^q}{q\|y\|_q^q} \\
\implies \int_a^b \frac{|x(t)y(t)|}{\|x\|_p \|y\|_q} &\leq \int_a^b \frac{|x(t)|^p}{p\|x\|_p^p} dt + \int_a^b \frac{|y(t)|^q}{q\|y\|_q^q} dt \\
&= \frac{1}{p} + \frac{1}{q} = 1 \\
\therefore \int_a^b |x(t)y(t)|dt &\leq \|x\|_p \|y\|_q
\end{aligned}$$

□

Lemma 2.5. *(Minkowski's inequality in $C[a,b]$). Suppose $1 < p < \infty$. Then*

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \forall x, y \in C[a,b].$$

Proof.

$$\begin{aligned}
\|x+y\|_p^p = \int_a^b |x(t)+y(t)|^p dt &= \int_a^b |x(t)+y(t)||x(t)+y(t)|^{p-1} dt \\
&\leq \int_a^b (|x(t)| + |y(t)|)|x(t)+y(t)|^{p-1} dt \\
&= \int_a^b |x(t)||x(t)+y(t)|^{p-1} dt + \int_a^b |y(t)||x(t)+y(t)|^{p-1} dt \\
\implies \|x+y\|_p^p &\leq (\|x\|_p + \|y\|_p) \|x+y\|_p^{p-1} \quad \text{(by Hölder's inequality in } C[a,b])
\end{aligned}$$

Multiplying both sides with $\frac{\|x+y\|_p}{\|x+y\|_p^p}$, we have

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

□

In the above, we only considered $p \geq 1$. What about $0 < p < 1$? There is an issue with this range of p. See below for example.

Proposition 2.1. *For* $0 < p < 1$, $(\mathbb{R}^2, \|\cdot\|_p)$ *is not a normed linear space.*

Proof. Assume the contrary. Let $x = (x_1, x_2) \in \mathbb{R}^2$. We know, $\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$. Consider $(1,0)$ and $(0,1)$ in $\mathbb{R}^2$. By the triangle inequality,

$$\|(1,0)+(0,1)\|_p = \|(1,1)\|_p = 2^{\frac{1}{p}} \leq \|(1,0)\|_p + \|(0,1)\|_p = 1+1 = 2$$

$\implies 2^{\frac{1}{p}} \leq 2$, for $0 < p < 1$. Which is a contradiction. Hence, the proposition holds. □

Remark 2.2. Observe that $d(x,y) = ||x-y||_p^p$ is a metric.

2.1 Sequence Spaces

We have already seen two sequence spaces $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ in example 2.1 (example 3, Page 12). We define some more sequence spaces and see their characteristics in this section.

1. Let $p \in (1, \infty)$. Define

$$\ell^p(\mathbb{N}) = \{(x_n) \in \mathcal{S}(\mathbb{N}) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

Claim: ℓ^p is a vector space.
Indeed, if $(x_n), (y_n) \in \ell^p(\mathbb{N})$, which means that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ and $\sum_{n=1}^{\infty} |y_n|^p < \infty$, Then

$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq 2^p \left(\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p\right) < \infty$$

Hence, $(x_n + y_n) \in \ell^p(\mathbb{N})$ ($\ell^p(\mathbb{N})$ is closed under addition. Similarly, it can be checked for scalar multiplication).

Let $(x_n) \in \ell^p(\mathbb{N})$. Define

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$

Then

- $\|(x_n)\|_p = 0 \iff (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} = 0 \iff x_n = 0\ \forall n \iff (x_n) = 0$
- $\|\alpha(x_n)\|_p = (\sum_{n=1}^{\infty} |\alpha \cdot x_n|^p)^{\frac{1}{p}} = |\alpha| (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} = |\alpha| \|(x_n)\|_p$
- Using Minkowski's inequality,

$$\|(x_n) + (y_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

$$= \|(x_n)\|_p + \|(y_n)\|_p$$

Remark 2.3. Thus, $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ is a normed linear space for $1 \leq p \leq \infty$.

Lemma 2.6. *Let $p, q \in [1, \infty)$ be such that $p < q$. Then:*

(a) $\|(x_n)\|_p \geq \|(x_n)\|_q\ \forall (x_n) \in \ell^p(\mathbb{N})$

(b) $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$

Proof. (a) We know, $\|(x_n)\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}, \|(x_n)\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$, and $\|(x_n)\|_q = (\sum_{n=1}^{\infty} |x_n|^q)^{\frac{1}{q}}$.

Let $\mathbf{x} = (x_n) \in \ell^p(\mathbb{N})$.

$$\begin{aligned}
|x_n| &\leq \sum_{n=1}^{\infty} |x_n|\ \forall n \in \mathbb{N} \\
\implies |x_n|^p &\leq \sum_{n=1}^{\infty} |x_n|^p\ \forall n \in \mathbb{N} \\
\implies (|x_n|^p)^{\frac{1}{p}} &\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}\ \forall n \in \mathbb{N} \\
\implies |x_n| &\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}\ \forall n \in \mathbb{N} \\
\implies \sup_{n\in\mathbb{N}}\{|x_n|\} &\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \\
\implies \|(x_n)\|_\infty &\leq \|(x_n)\|_p \\
\implies \|\mathbf{x}\|_\infty &\leq \|\mathbf{x}\|_p \ldots\ldots (*)
\end{aligned}$$

Now,

$$\|\mathbf{x}\|_q^q = \sum_{n=1}^{\infty} |x_n|^q = \sum_{n=1}^{\infty} |x_n|^{q-p}|x_n|^p \leq \|\mathbf{x}\|_\infty^{q-p} \sum_{n=1}^{\infty} |x_n|^p \leq \|\mathbf{x}\|_p^{q-p} \sum_{n=1}^{\infty} |x_n|^p \text{ [from (*)]}$$

$$= \|\mathbf{x}\|_p^{q-p} \|\mathbf{x}\|_p^p = \|(x_n)\|_p^q$$

Hence, $\|(x_n)\|_q \leq \|(x_n)\|_q$ (raising both sides to the power $\frac{1}{q}$).

(b) Let $(x_n) \in \ell^p(\mathbb{N})$. We show that $(x_n) \in \ell^q(\mathbb{N})$ as well, so that $\ell^p(\mathbb{N}) \subseteq \ell^q(\mathbb{N})$. Then we prove that $\ell^p(\mathbb{N}) \neq \ell^q(\mathbb{N})$ by getting an element in ℓ^q which is not in ℓ^p.

As $(x_n) \in \ell^p(\mathbb{N})$, $\sum_{n=1}^{\infty} |x_n|^p < \infty \implies (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty \implies \|(x_n)\|_p \leq \infty$. Using (a), we have $\|(x_n)\|_q \leq \|(x_n)\|_p < \infty \implies \|(x_n)\|_q < \infty \implies (x_n) \in \ell^q(\mathbb{N})$.

Thus, $\ell^p(\mathbb{N}) \subseteq \ell^q(\mathbb{N})$.

Now, as $p < q$, let r be such that $p < r < q$ (we can get such an r due to the density property of $\mathbb{R}$).
Claim: The sequence $\left(\frac{1}{n^{1/r}}\right)_{n\in\mathbb{N}} \in \ell^q(\mathbb{N})$ but $\notin \ell^p(\mathbb{N})$.
(Recall that by the p-test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$).
We have

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^{1/r}}\right|^q = \sum_{n=1}^{\infty} \frac{1}{n^{q/r}} < \infty$$

as $\frac{q}{r} > 1$. Thus, $(x_n) \in \ell^q(\mathbb{N}$. However,

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^{1/r}}\right|^p = \sum_{n=1}^{\infty} \frac{1}{n^{p/r}} \to \infty, \text{ as } \frac{p}{r} < 1$$

Thus, $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$.

□

Remark 2.4. The finite dimensional normed linear space $(\mathbb{F}^n, \|\cdot\|_p)$ is often regarded as $\ell^p(n)$.

2. Define

$$c_{00}(\mathbb{N}) = \{x = (x_n) \in \mathcal{S}(\mathbb{N}) : \exists N := N_x \in \mathbb{N} \text{ such that } x_n = 0\ \forall n \geq N\}.$$

$c_{00}(\mathbb{N})$ is the set of all eventually zero sequences.

Exercise 2.9. Check that $(c_{00}(\mathbb{N}), \|\cdot\|_p)$ is a normed linear space for $1 \leq p \leq \infty$.

3. Define

$$c_0(\mathbb{N}) = \{(x_n) \in \mathcal{S}(\mathbb{N}) : x_n \to 0 \text{ as } n \to \infty\}$$

$c_0(\mathbb{N})$ is the set of all sequences converging to 0.
For $(x_n) \in c_0(\mathbb{N})$, define $\|(x_n)\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$.

Exercise 2.10. Check that $(c_0(\mathbb{N}), \|\cdot\|_\infty)$ is a normed linear space.

4. Define
$$c(\mathbb{N}) = \{(x_n) \in \mathcal{S}(\mathbb{N}) : (x_n) \text{ is convergent}\}$$

Exercise 2.11. Show that $(c(\mathbb{N}), \|\cdot\|_\infty)$ is a normed linear space.

Lemma 2.7. $c_{00}(\mathbb{N}) \subset \ell^p(\mathbb{N}) \subset c_0(\mathbb{N}) \subset c(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ *for* $1 \le p < \infty$.

Proof. **Claim 1:** $c_{00}(\mathbb{N}) \subsetneq \ell^p(\mathbb{N})$ for $1 \le p < \infty$.
Let $(x_n) \in c_{00}(\mathbb{N})$. Then $\exists N \in \mathbb{N}$ such that $x_n = 0\ \forall n \ge N$.

$$\implies \sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{N-1} |x_n|^p + \underbrace{\sum_{n=N}^{\infty} |x_n|^p}_{=0} = \sum_{n=1}^{N-1} |x_n|^p < \infty$$

$\implies (x_n) \in \ell^p(\mathbb{N}) \implies c_{00}(\mathbb{N}) \subset \ell^p(\mathbb{N})$.
Now, consider $(x_n) = \left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$. We know by the p-test that $\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^p$ is an absolutely convergent series for $p \ge 1$.
Hence, $\sum_{n=1}^{\infty} |x_n|^p < \infty \implies (x_n) \in \ell^p(\mathbb{N})$. However, the sequence (x_n) is not eventually zero, so $(x_n) \notin c_{00}(\mathbb{N})$.
Thus, we have $c_{00}(\mathbb{N}) \subsetneq \ell^p(\mathbb{N})$.

Claim 2: $\ell^p(\mathbb{N}) \subsetneq c_0(\mathbb{N})$
Let $(x_n) \in \ell^p(\mathbb{N})$. Then $\sum_{n=1}^{\infty} |x_n|^p < \infty$, or the sequence $(|x_n|^p)_{n\in\mathbb{N}}$ converges to 0, i.e. $|x_n|^p \to 0$ as $n \to \infty$.
As $p > 1$, we have $x_n \to 0$ as $n \to \infty \implies (x_n) \in c_0(\mathbb{N}) \implies \ell^p(\mathbb{N}) \subset c_0(\mathbb{N})$.
Now, consider the sequence $\left(\frac{1}{n^{\frac{1}{p}}}\right)_{n\in\mathbb{N}}$. Clearly, $\left(\frac{1}{n^{\frac{1}{p}}}\right)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$, since $\frac{1}{n} \to 0$ as $n \to \infty$.
However, the series $\sum_{n=1}^{\infty} \left|\frac{1}{n^{1/p}}\right|^p = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right|$ is divergent, for $p \in [1, \infty)$.
Thus, $\left(\frac{1}{n^{\frac{1}{p}}}\right)_{n\in\mathbb{N}} \notin \ell^p(\mathbb{N}) \implies \ell^p(\mathbb{N}) \subsetneq c_0(\mathbb{N})$.

Claim 3: $c_0(\mathbb{N}) \subsetneq c(\mathbb{N})$.
Any sequence that converges to 0 is also clearly convergent, so $c_0(\mathbb{N}) \subset c(\mathbb{N})$.
Consider $(x_n) = \left(1 + \frac{1}{n}\right)_{n\in\mathbb{N}}$. Clearly, we have that $x_n \to 1$ as $n \to \infty$, so that $(x_n) \in c(\mathbb{N})$ but $(x_n) \notin c_0(\mathbb{N})$.
Thus, $c_0(\mathbb{N}) \subsetneq c(\mathbb{N})$.

Claim 4: $c(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N})$.
Let $(x_n) \in c(\mathbb{N})$. Then (x_n) is a convergent sequence. We know that a convergent sequence is also bounded. So, $(x_n) \in \ell^\infty(\mathbb{N})$.
Consider the sequence $(x_n) = ((-1)^n)_{n\in\mathbb{N}}$. As it has a finite range, it is bounded. But being oscillatory, it does not converge. Hence, $(x_n) \in \ell^\infty(\mathbb{N}) \setminus c(\mathbb{N})$.
Thus, $c(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N})$. □

Lemma 2.8. *$c_{00}(\mathbb{N})$, $\ell^p(\mathbb{N})$, $c_0(\mathbb{N})$ and $c(\mathbb{N})$ are infinite dimensional normed linear spaces.*

Proof. For $i \in \mathbb{N}$, let $e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the 1 appears only in the i^{th} position. As these elements are eventually zero sequences, we have $e_i \in c_{00}(\mathbb{N})\ \forall i \in \mathbb{N}$.
Claim: The set $\{e_i : i \in \mathbb{N}\}$ is linearly independent.
Let $\{e_{i_1}, \ldots, e_{i_k}\}$ be an arbitrary finite subset of $\{e_i : i \in \mathbb{N}\}$ and $i \in \mathbb{N}$. Let $\alpha_j \in \mathbb{F}$ for $1 \leq j \leq k$ and $\alpha_1 e_1 + \ldots + \alpha_n e_n = 0$.

$$\implies (0, 0, \ldots, \alpha_1, 0, \ldots, \alpha_k, 0, \ldots) = 0$$

where each α_j appears at the i_k^{th} position. Hence, $\alpha_j = 0\ \forall 1 \leq j \leq k$, which implies that our claim is true.
As we have obtained an infinite set which is linearly independent in $c_{00}(\mathbb{N})$, we have that $c_{00}(\mathbb{N})$ is infinite dimensional. Further, by Lemma 2.7, we conclude that $\ell^p(\mathbb{N})$, $c_0(\mathbb{N})$ and $c(\mathbb{N})$ are all infinite dimensional normed linear spaces.

□

2.2 Metric on Normed Linear Spaces

Recall the definition of metric and metric spaces from section 1.3 (Metric Spaces).
Let $(V, \|\cdot\|)$ be a normed linear space. Define $d : V \times V \to [0, \infty)$ by

$$d(x, y) = \|x - y\|\ \forall x, y \in V$$

Lemma 2.9. *d is a metric on V. (This metric d on V is called a metric on X induced by norm on X.)*

Proof. The proof requires verification of the properties of a metric by using the properties of the norm $\|\cdot\|$.

- $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$
- $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$
- Let $x, y, z \in V$. Then $d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$. (The triangle inequality for d follows from the triangle inequality for $\|\cdot\|$).

Thus, d is a metric and (V, d) is a metric space. □

Hence, we have proved that:

Theorem 2.1. *Every normed linear space is a metric space (i.e., every norm induces a metric).*

Remark 2.5. The converse of theorem 2.1 is not true in general - i.e., given a metric d, it need not be induced by a norm. We prove this by showing that the discrete metric on $\mathbb{R}$ cannot be induced by any norm. The discrete metric is given by

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Assume, for the sake of contradiction, that $\|\cdot\|$ is a norm that induces d. Then $\|x\| = \|x-0\| = d(x,0)$. We have $d(1,0) = \|1-0\| = \|1\| = 1$. But then for the norm of 2, we must have $\|2\| = |2| \times \|1\| = 2$. However, $\|2\| = d(2,0) = 1$, a contradiction. ■

Remark 2.6. If $(X, \|\cdot\|)$ is a normed linear space and d be a metric induced by the norm $\|\cdot\|$, then $d : X \times X \to [0,\infty)$ is not a bounded function. Indeed, assuming that $X \neq \{0\}$, if for some $x \neq 0 \in X$ we have $\|x\| = r$, then $\forall n \in \mathbb{N}, \|nx - 0\| = \|nx\| = |n|\|x\| = nr \to \infty$ as $n \to \infty$. Hence, $\|\cdot\|$ is not bounded.

Remark 2.7. No bounded metric on a vector space can be induced by a norm on vector space.

Definition 2.4 (Balls in a normed linear space)**.** Let $(X, \|\cdot\|)$ be a normed linear space and d be the metric on X induced by $\|\cdot\|$. Let $a \in X,\ r > 0$. Then

$$B(a,r) = \{x \in X : d(x,a) < r\} = \{x \in X : \|x - a\| < r\}$$

$$\overline{B}(a,r) = \{x \in X : d(x,a) \leq r\} = \{x \in X : \|x - a\| \leq r\}$$

Definition 2.5 (Convex set)**.** Let X be a set and $A \subseteq X$. We say that A is a convex set if $\forall x, y \in A,\ \forall \lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in A$.

Example 2.2 (Convex sets)**.** Let $(X, \|\cdot\|)$ be a normed linear space. Then

1. $\{x\}$ is a convex set for $x \in X$.
2. Any subspace of X is a convex set.
3. Lines in X are convex sets $(\{x + ty : t \in \mathbb{F},\ x, y \in X\})$

Lemma 2.10. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the balls $B(x,r)$ and $\overline{B}(x,r)$ are convex sets.*

Proof. Recall $B(x,r) = \{y \in X : \|x - y\| < r\}$. Let $x_1, x_2 \in B(x,r)$ and $\lambda \in [0,1]$. Then $\|x_1 - x\| < r$ and $\|x_2 - x\| < r$.

Claim: $\lambda x_1 + (1-\lambda)x_2 \in B(x,r)$

$$\begin{aligned}\|\lambda x_1 + (1-\lambda)x_2 - x\| &= \|\lambda(x_1 - x) + (1-\lambda)(x_2 - x)\| \\ &\leq \|\lambda(x_1 - x)\| + \|(1-\lambda)(x_2 - x)\| \\ &= \lambda\|x_1 - x\| + (1-\lambda)\|x_2 - x\| \\ &< \lambda r + (1-\lambda)r = r\end{aligned}$$

Hence, the distance between $\lambda x_1+(1-\lambda)x_2$ and x is less than $r \implies \lambda x_1+(1-\lambda)x_2 \in B(x,r)$.

To prove that $\overline{B}(x,r)$ is convex, simply replace the $<$ signs with $\leq$. □

Lemma 2.11. *Let $r > 0$. Then $\overline{B(x,r)} = \overline{B}(x,r)$ for $x \in X$ if X is a normed linear space. However, the result is not true if X is a general metric space.*

Proof. By definition, the closure $\overline{B(x,r)}$ is the smallest closed set containing $B(x,r)$, i.e., the intersection of all closed sets containing $B(x,r)$. And, $\overline{B}(x,r)$ is a set containing $B(x,r)$. So, $\overline{B(x,r)} \subset \overline{B}(x,r)$ holds.

Claim: The other inclusion does not hold for a general metric space X. Consider the discrete metric space (X,d) with more than one element. The open balls are

$$B(x,r) = \{y \in X : d(x,y) < r\} = \begin{cases} \{x\}, & r \leq 1 \\ X, & r > 1 \end{cases}$$

Take $r = 1$. Then $\overline{B(x,1)} = \overline{\{x\}} = \{x\}$ (since singletons are closed). However, $\overline{B}(x,1) = \{y \in X : d(x,y) \leq 1\} = X$. So, $\overline{B(x,r)} \neq \overline{B}(x,r)$.

Claim: If X is a normed linear space, then the other inclusion is true, so that $\overline{B(x,r)} = \overline{B}(x,r)$.
We show that $\overline{B}(x,r) \subseteq \overline{B(x,r)}$. Let $y \in \overline{B}(x,r)$. Then $\|x-y\| \leq r$.
If $\|x-y\| < r$, then $y \in B(x,r) \subset \overline{B(x,r)} \implies y \in \overline{B(x,r)}$
Suppose $\|x-y\| = r$. We want to show that y is a limit point of $B(x,r)$: or for $\varepsilon > 0$, $B(y,\varepsilon) \cap B(x,r) \neq \phi$. For $n \in \mathbb{N}$,let $y_n = B\left(y, \frac{1}{n}\right) \cap B(x,r)$. Such a y_n exists because we can choose $t \in [0,1)$ to be arbitrarily close to 1, and $(1-t)x + ty \in B(x,r)$ due to convexity (lemma 2.10). Clearly, $y_n \to y$ as $n \to \infty \implies y$ is a limit point of $B(x,r) \implies y \in \overline{B(x,r)} \implies \overline{B}(x,r) \subseteq \overline{B(x,r)}$. □

Lemma 2.12. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the norm function $\|\cdot\| : (X,d) \to \mathbb{F}$ defined by $x \to \|x\|$ is uniformly continuous.*

Proof. (**Recall:** $f : (Y, d_Y) \to (Z, d_Z)$ is said to be uniformly continuous if for each $\varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $d_Z(f(x), f(y)) < \varepsilon$ whenever $d_Y(x, y) < \delta$).
Let $x, y \in X$. Write $x = x - y + y$ and $y = y - x + x$

$$\implies \|x\| \leq \|x - y\| + \|y\| \text{ and } \|y\| \leq \|x - y\| + \|x\|$$

Subtracting, we have $|\|x\| - \|y\|| \leq \|x - y\|$.
Hence, taking $\delta_\varepsilon = \varepsilon$, we have

$$\|x - y\| < \delta \implies |\|x\| - \|y\|| < \varepsilon$$

□

The following maps play very crucial role in a normed linear space.

Lemma 2.13. *Let $(X, \|\cdot\|)$ be a normed linear space over F. Let $x_0 \in X$, $\alpha \in \mathbb{F} \setminus \{0\}$. Define*

$$T_{x_0} : X \to X \text{ by } T_{x_0}(x) = x + x_0 \ \forall x \in X \text{ (Translation)}$$

$$D_\alpha : X \to X \text{ by } D_\alpha(X) = \alpha x \ \forall x \in X \text{ (Dilation)}$$

Then T_{x_0} and D_α are homeomorphisms (bijective and bicontinuous maps).

Proof. Exercise. **Hint:**

$$\|T_{x_0}(x) - T_{x_0}(y)\| = \|x - y\|$$
$$\|D_\alpha(x) - D_\alpha(y)\| = |\alpha| \|x - y\|$$

□

Lemma 2.14. *Let $(X, \|\cdot\|)$ be a normed linear space and $a \in X$, $r > 0$. Then*

1. $B(a, r) = a + rB(0, 1)$
2. $\overline{B}(a, r) = a + r\overline{B}(0, 1)$

Proof. 1. Let $x \in B(a, r)$. Then

$$\|x - a\| < r \implies \frac{\|x - a\|}{r} < 1 \implies \left\| \frac{x - a}{r} \right\| < 1$$

$\implies \frac{x-a}{r} \in B(0, 1) \implies \exists y \in B(0, 1)$ such that $y = \frac{x-a}{r} \implies x = a + ry \implies x \in a + rB(0, 1)$
$\implies B(a, r) \subseteq a + rB(0, 1)$.
For the reverse inclusion, simply follow the reverse direction.

2. *Exercise.*

□

Remark 2.8. Let $x \in X \setminus \{0\}$. Then $\frac{x}{2\|x\|} \in B(0,1)$. Similarly, if $x \in X \setminus \{0\}$, then $\frac{x}{\|x\|} \in \overline{B}(0,1)$.

Lemma 2.15. *Let $(X, \|\cdot\|)$ be a normed linear space and Y be a subspace of X with $Y^\circ \neq \phi$. Then $Y = X$.*

Proof. Given that $Y \subseteq X$ and $Y^\circ \neq \phi$ (interior of Y is non-empty). This means that $\exists y_o \in Y$ and $\exists r > 0$ such that $B(y_0, r) \subseteq Y$.
Let $x \in X \setminus \{0\}$. Then $y_0 + \frac{rx}{2\|x\|} \in B(y_0, r) \subseteq Y$.
$\implies y_0 + \frac{rx}{2\|x\|} \in Y$
$\implies \frac{rx}{2\|x\|} \in Y$ (Since $y_0 \in Y$ and Y is a subspace of X)
$\implies x \in Y$ (since $\frac{r}{2\|x\|}$ is a scalar and Y is a subspace)
$\implies X \subseteq Y$
But also $Y \subseteq X$, so that $X = Y$.

□

Corollary 2.16. *Let Y be a subspace of a normed linear space $(X, \|\cdot\|)$. Then Y is open $\iff Y = X$. In other words, there are no proper open subspaces of a normed linear space X.*

Proof. $(\implies)$ Y is open in $X \implies Y^\circ \neq \phi$. By lemma 2.15, we have $Y = X$.
$(\impliedby)$ We know that X is open in X. Thus, $Y = X \implies Y$ is open in X. □

Definition 2.6 (Distance between two subsets)**.** Let $(X, \|\cdot\|)$ be a normed linear space. Let $A \subseteq X$ and $B \subseteq X$. Define

$$d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$$

Exercise 2.12. Let $(X, \|\cdot\|)$ be a normed linear space and $A \subseteq X$. Define $\eta_A : X \to \mathbb{R}$ by $\eta_A(x) = d(x, A) = \inf\{\|x - y\| : y \in A\}\ \forall x \in X$. Show that η_A is uniformly continuous.

Lemma 2.17. *Let $(X, \|\cdot\|)$ be a normed linear space and Y be a subset of X. Let $x \in X$. Then*

1. $x \in \overline{Y} \iff d(x, Y) = 0$

2. *If Y is a closed subspace of X and $x \in X \setminus Y$, then $d(x, Y) > 0$*

Proof. 1. $(\implies)$ Let $x \in \overline{Y}$. Then for any $\varepsilon > 0, \exists$ an open ball $B_\varepsilon(x)$ such that $B_\varepsilon(x) \cap Y \neq \phi$ (by definition). Let $y_\varepsilon \in B_\varepsilon(x) \cap Y$. Then

$$d(x, Y) \leq d(x, y_\varepsilon) < \varepsilon \text{ (since } d(x, Y) \text{ is the infimum of all such } y_\varepsilon\text{'s by definition)}$$

As $\varepsilon > 0$ is arbitrary, we have $d(x, Y) = 0$.

($\Longleftarrow$) Conversely, let $d(x, Y) = 0$. Then for any $\varepsilon > 0, \exists\, y_\varepsilon \in Y$ such that $d(x, y_\varepsilon) < \varepsilon$. This means that $B_\varepsilon \cap Y \neq \phi$. As any open ball around x would contain $B_\varepsilon(x)$ for some $\varepsilon > 0$, it follows that $x \in \overline{Y}$.

2. Given that Y is a closed subspace of X. This means that $X \setminus Y$ is open in X. $x \in X \setminus Y$ is an interior point of X. Thus, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset X \setminus Y$.
   $\Longrightarrow$ $B_\varepsilon(x) \cap Y = \phi$ for some $\varepsilon > 0$
   $\Longrightarrow$ $x \notin \overline{Y}$
   $\Longrightarrow$ $d(x, Y) > 0$ by contrapositive of 1.

$\square$

Lemma 2.18. *Let Y be a subspace of $(X, \|\cdot\|)$ and $\alpha \in \mathbb{F} \setminus \{0\}, x \in X$. Then $d(\alpha x, Y) = |\alpha| d(x, Y)$.*

Proof.

$$\begin{aligned} d(\alpha x, Y) &= \inf\{\|\alpha x - y\| : y \in Y\} \\ &= \inf\left\{\left\|\alpha\left(x - \frac{y}{\alpha}\right)\right\| : y \in Y\right\} \\ &= |\alpha| \inf\left\{\left\|x - \frac{y}{\alpha}\right\| : y \in Y\right\} \\ &= |\alpha| \inf\{\|x - y\| : y \in Y\} \ \left(\text{As } Y \text{ is a subspace and } \alpha \in \mathbb{F}, \text{ we can replace } \frac{y}{\alpha} \text{ with } y\right) \\ &= |\alpha| d(x, Y) \end{aligned}$$

$\square$

Remark 2.9. Recall the definition of convergence of a sequence in a metric space from definition 1.9.

Definition 2.7 (Convergence of a sequence in a normed linear space)**.** Let $(X, \|\cdot\|)$ be a normed linear space and (x_n) be a sequence in X. We say that $x_n \to x \in X$ if for given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon \ \forall n > N$.

Lemma 2.19. *Let $(X, \|\cdot\|)$ be a normed linear space and Y be a closed subspace of X and Z be a finite dimensional subspace of X. Then $Y + Z$ is a closed subspace of X.*

Proof. Without loss of generality, assume that $dim(Z) = 1$. Let $Z = span\{x\} = \{\alpha x : \alpha \in \mathbb{F}\}$ and $x \in X \setminus Y$. We show that $Y + Z$ is a closed subspace of X.

$$Y + Z = \{y + z : y \in Y,\ z \in Z\} = \{y + \alpha x, y \in Y,\ \alpha \in \mathbb{F}\}$$

We know that a set is closed if it contains all its limit points. So, let $(y_n + \alpha_n x)_{n\in\mathbb{N}}$ be a sequence in $Y+Z$ such that $y_n+\alpha_n x \to x_0 \in X$ as $n \to \infty$. We have to show that $x_0 \in Y+Z$.

$y_n + \alpha_n x \to x_0 \implies$ for each $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|y_n + \alpha_n x - x_0\| < \varepsilon \; \forall n \geq N$
As convergence $\implies$ the sequence is Cauchy,

$$\|(y_n + \alpha_n x) - (y_m + \alpha_m x)\| < 2\varepsilon \; \forall n, m \geq N$$
$$\|(y_n - y_m) + (\alpha_n - \alpha_m x)\| < 2\varepsilon \; \forall n, m \geq N$$

That is, $(y_n - y_m) - (\alpha_n - \alpha_m x) \to 0$ as $n \to \infty$
Observe that as $d((\alpha_n - \alpha_m)x, Y)$ is the infimum, we have

$$d((\alpha_n - \alpha_m)x, Y) \leq \|(y_n - y_m) + (\alpha_n - \alpha_m x)\|$$

$\implies d((\alpha_n - \alpha_m)x, Y) \to 0$ as $n, m \to \infty$
$\implies |\alpha_n - \alpha_m| d(x, Y) \to 0$ as $n, m \to \infty$ (using lemma 2.18 above).
$\implies |\alpha_n - \alpha_m| \to 0$ as $n, m \to \infty$ [since $d(x, Y) > 0$ because of $x \in X \setminus Y$, by lemma 2.17 (2)]
$\implies (\alpha_n)$ is a Cauchy sequence in $\mathbb{F}$, which is complete (since $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$)
$\implies \exists$ unique $\alpha \in \mathbb{F}$ such that $\alpha_n \to \alpha$ as $n \to \infty$.

Therefore, we have:

$$y_n + \alpha_n x \to x_0$$
$$\alpha_n \to \alpha$$
$$\implies \alpha_n x \to \alpha x \text{ as } n \to \infty$$
$$\implies y_n = y_n + \alpha_n x - \alpha_n x \to x_0 - \alpha x \text{ as } n \to \infty$$
$$\implies y_n \to x_0 - \alpha x$$

$\implies x_0 - \alpha x \in Y$ (since Y is closed, and (y_n) converges to $x_0 - \alpha x$.
$\implies \exists y \in Y$ such that $x_0 = y + \alpha x \implies x_0 \in Y + Z$.

As $(y_n + \alpha_n x)$ was an arbitrary convergent sequence in $Y + Z$ whose limit lies in $Y + Z$, we have that every convergent sequence in $Y + Z$ has its limit in $Y + Z$. Hence, $Y + Z$ is a closed subspace of X. □

Remark 2.10. In the above Lemma, the finite dimensionality condition on Z can not be relaxed in general. Can you think of an example?

Corollary 2.20. *Every finite dimensional subspace of a normed linear space is a closed subspace.*

Lemma 2.21. *Let $(X, +, \cdot, \|\cdot\|)$ be a normed linear space. Then the maps $+ : X \times X \to X$ given by $(x, y) \mapsto x + y$ and $\cdot : \mathbb{F} \times X \to X$ given by $(\alpha, x) \mapsto \alpha x$ are continuous.*

Proof. 1. **Claim:** $+$ is continuous.
Let $\{(x_n, y_n)\} \in X \times X$ be a sequence such that $(x_n, y_n) \to (x, y)$ as $n \to \infty$ in $X \times X$.
$\implies x_n \to x$ and $y_n \to y$ in X and Y respectively, as $n \to \infty$
Then $\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$ as $n \to \infty$
Thus, $x_n + y_n \to x + y$ as $n \to \infty$
or $+(x_n, y_n) \to +(x, y) = x + y$ as $n \to \infty$
So, $+$ is continuous by the sequential criterion.

2. **Claim:** $\cdot$ is continuous.
Let $\{(\alpha_n, x_n)\} \in \mathbb{F} \times X$ be a sequence such that $(\alpha_n, x_n) \to (\alpha, x)$ as $n \to \infty$
Then $\alpha_n \to \alpha$ in $\mathbb{F}$ and $x_n \to x$ in X as $n \to \infty$
Then $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \leq |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\| \to 0$ as $n \to \infty$
Or $\alpha_n x_n \to \alpha x$ as $n \to \infty$

□

Definition 2.8 (Topological Vector Space)**.** Let $(X, +, \cdot)$ be a vector space over $\mathbb{F}$. It is called a topological vector space over $\mathbb{F}$ if there exists a topology τ on X such that the maps $+$ and $\cdot$ as defined in lemma 2.21 are continuous.

Example 2.3. Every normed linear space is a topological vector space. There are some examples of topological vector spaces that are not normed linear spaces! This discussion is not in our scope. The reader may refer to Functional Analysis by Walter Rudin.

2.3 Equivalence of Norms

Definition 2.9. Let $(X, +, .)$ be a vector space over $\mathbb{F}$ and $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on X. We say that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent if there exists constants $c_1 > 0,\ c_2 > 0$ such that

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a \ \forall x \in X$$

Lemma 2.22. *Let $(X, +, .)$ be a vector space over $\mathbb{F}$. Let $\mathcal{N} = \{\|\cdot\| : \|\cdot\| \text{ is a norm on } X\}$. Let $\|\cdot\|_a,\ \|\cdot\|_b \in \mathcal{N}$. We say that $\|\cdot\|_a \sim \|\cdot\|_b$ iff $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent. Then $\sim$ is an equivalence relation on $\mathcal{N}$.*

Proof. We have that $\|\cdot\|_a \sim \|\cdot\|_b$ iff $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, i.e. $\|\cdot\|_a \sim \|\cdot\|_b$ iff $\exists$ constants $c_1 > 0,\ c_2 > 0$ such that

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\| \ x\|_a \ \forall x \in X$$

1. (Reflexive) Take $c_1 = c_2 = 1$. Then

$$c_1\|x\|_a \leq \|x\|_a \leq c_2\| \ x\|_a \ \forall x \in X$$

2. (Symmetric) Let $\|\cdot\|_a \sim \|\cdot\|_b$. Then $\exists$ constants $c_1 > 0,\ c_2 > 0$ such that $c_1\|x\|_a \leq \|x\|_b \leq c_2\| \ x\|_a \ \forall x \in X$. Then

$$\frac{1}{c_2}\|x\|_b \leq \|x\|_a \leq \frac{1}{c_1}\|x\|_b$$

   and clearly we have $\frac{1}{c_1}, \frac{1}{c_2} > 0$. Hence, $\|\cdot\|_b \sim \|\cdot\|_a$.

3. (Transitive) Let $\|\cdot\|_a \sim \|\cdot\|_b$ and $\|\cdot\|_b \sim \|\cdot\|_c$. Then $\exists c_1, c_2, c_3, c_4 > 0$ such that $c_1\|x\|_a \leq \|x\|_b \leq c_2\| \ x\|_a$ and $c_3\|x\|_b \leq \|x\|_c \leq c_4\| \ x\|_b \ \forall x \in X$.

   $\implies c_1\|x\|_a \leq \|x\|_b \leq \frac{1}{c_3}\|x\|_c \implies c_1c_3\|x\|_a \leq \|x\|_c$.

   Also, $\|x\|_c \leq c_4\|x\|_b \leq c_4c_2\|x\|_a \implies \|x\|_c \leq c_4c_2\|x\|_a$
   $\therefore c_1c_3\|x\|_a \leq \|x\|_c \leq c_4c_2\|x\|_a$

   As $c_1c_3,\ c_4c_2 > 0$, we have $\|\cdot\|_a \sim \|\cdot\|_c$.

□

Theorem 2.2. *Any two norms on a finite dimensional vector space over $\mathbb{F}$ are equivalent.*

Proof. Let $(X, +, .)$ be a finite dimensional vector space over $\mathbb{F}$ and $\|\cdot\|_a,\ \|\cdot\|_b$ be two norms on X.

Claim: $\exists$ constants $c_1 > 0,\ c_2 > 0$ such that $c_1\|x\|_a \leq \|x\|_b \leq c_2\| \ x\|_a \ \forall x \in X$.
Let $\mathscr{B} = \{e_1, \ \ldots \ , e_n\}$ be a basis for X. Let $x \in X$. Then x can be written as $x = \sum_{i=1}^n c_ie_i,\ c_i \in \mathbb{F}$. Define:

$$\|x\| = \max\{|c_i| : 1 \leq i \leq n\}$$

Exercise 2.13. Check that $\|\cdot\|$ is a norm on X.

We claim that $\|\cdot\|_a$ and $\|\cdot\|$ are equivalent, so that in a similar way, we will have that $\|\cdot\|_b$ and $\|\cdot\|$ are equivalent, and then using transitivity (lemma 2.22), we can conclude that

$\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent.

$$\|x\|_a = \left\|\sum_{i=1}^{n} c_i e_i\right\|_a \leq \sum_{i=1}^{n} \|c_i e_i\|_a = \sum_{i=1}^{n} |c_i| \|e_i\|_a \leq \|x\| \sum_{i=1}^{n} \|e_i\|_a$$

(since $\|x\| = \max_i |c_i|$ by definition).
$\implies \exists c = \sum_{i=1}^{n} \|e_i\|_a$ such that $\|x\|_a \leq c\|x\| \; \forall x \in X$

Again,

$$\|x\|_a = \left\|\sum_{i=1}^{n} c_i e_i\right\|_a = \left\|c_i e_i + \sum_{j=1,\, j\neq i}^{n} c_j e_j\right\|_a$$

Let

$$Y_i = span\{e_j : j = \{1, 2, \ldots, n\} \setminus \{i\}\}$$

As Y is finite dimensional, by corollary 2.20, Y is a closed subspace of X. Then

$$\begin{aligned}
\|x\|_a = \left\|\sum_{i=1}^{n} c_i e_i\right\|_a &= \left\|c_i e_i + \sum_{j=1,\, j\neq i}^{n} c_j e_j\right\|_a \\
&\geq d(c_i e_i, Y_i) \; \forall 1 \leq i \leq n \text{ (since } d(c_i e_i, Y_i) \text{ is the infimum of these quantities, by definiti} \\
&= |c_i| d(e_i, Y_i) \; \forall i \\
&\geq |c_i| \alpha \; \forall i \text{ (where } \alpha = \min\{d(e_i, Y_i) : 1 \leq i \leq n\}) \\
&\geq \alpha \max\{|c_i| : 1 \leq i \leq n\} = \alpha \|x\| \; \forall x \in X
\end{aligned}$$

Therefore, $\exists \alpha, c > 0$ such that

$$\alpha\|x\| \leq \|x\|_a \leq c\|x\| \; \forall x \in X$$

So, $\|\cdot\| \sim \|\cdot\|_a$. Similarly, $\|\cdot\| \sim \|\cdot\|_b$. By transitivity, $\|\cdot\|_a \sim \|\cdot\|_b$ □

Remark 2.11. If the underlying vector space X is infinite dimensional, theorem 2.2 need not be true in general (i.e. there might exist two non-equivalent norms on X). We show this by constructing two norms on an infinite dimensional space X that are not equivalent.

Let $\mathscr{B}$ be a Hamel basis for X (existence is guaranteed by Zorn's lemma). For $x \in X$, write $x = \sum_{i=1}^{n} c_i x_i$, $x_i \in \mathscr{B}$. Define

$$\|x\| = \sum_{i=1}^{n} |c_i|$$

From exercise 2.7, we know that $(X, \|\cdot\|)$ is a normed linear space. Note that $\|x\| = 1$ if $x \in \mathscr{B}$. (Because if $x \in \mathscr{B}$, then $x = 1.x + \sum_{i=2}^{n} c_i x_i$, where $c_1 = 1$ and all other c_i's are 0)

$\ldots\ldots(*)$

Let $\{x_i : i \in \mathbb{N}\} \subseteq \mathscr{B}$ (countably infinite subset of $\mathscr{B}$). Let $f : X \to \mathbb{F}$ be a linear functional such that $f(x_i) = i \; \forall i \in \mathbb{N}$. Let $x \in X$. Define

$$\|x\|_f = \|x\| + |f(x)|$$

.

Exercise 2.14. Verify that $\|\cdot\|_f$ is a norm on X.
(*Hint:* Use the properties of f being a linear functional, i.e. $f(x+y) = f(x) + f(y)$ and $f(cx) = cf(x)$ for $x, y \in X,\ c \in \mathbb{F}$).

We claim that $\|\cdot\|$ and $\|\cdot\|_f$ are not equivalent. Clearly, we have $\|x\| \leq \|x\| + |f(x)| = \|x\|_f \; \forall x \in X$.

Claim: $\nexists$ any $\alpha > 0$ such that $\|x\|_f \leq \alpha\|x\| \; \forall x \in X$.
Suppose $\exists$ an $\alpha > 0$ such that $\|x\|_f \leq \alpha\|x\| \; \forall x \in X$. Then

$$\begin{aligned}
&\|x_i\|_f \leq \alpha\|x_i\| \; \forall i \in \mathbb{N} \; (x_i \in X \text{ are basis elements}) \\
\implies &\|x_i\| + |f(x_i)| \leq \alpha\|x_i\| \; \forall i \in \mathbb{N} \\
&\implies 1 + i \leq \alpha \; \forall i \in \mathbb{N} \;\; [\because \|x_i\| = 1 \text{ by } (*) \text{ and } f(x_i) = i \; \forall i \in \mathbb{N}]
\end{aligned}$$

But this means that α is an upper bound for $\mathbb{N}$, which is not possible. Hence, we arrive at a contradiction, which means that our claim is true. Thus, the two norms $\|\cdot\|$ and $\|\cdot\|_f$ on X are not equivalent.

Example 2.4. Let $1 \leq p \leq q \leq \infty$. Consider the vector space $(\mathbb{F}^n, +, .)$ over $\mathbb{F}$. By theorem 2.2, we know that $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on $\mathbb{F}^n$. Let us explicitly calculate the constants c_1 and c_2 such that

$$c_1 \|x\|_p \leq \|x\|_q \leq c_2 \|x\|_p \; \forall x \in \mathbb{F}^n$$

As $p \leq q$, we have $\|x\|_q \leq \|x\|_p \; \forall x \in \mathbb{F}^n$ (see lemma 2.6 and remark 2.4), which means that $c_2 = 1$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{F}$. Then

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \implies \|x\|_p^p = \sum_{i=1}^{n} |x_i|^p \cdot 1$$

Let $r, r' \in (1, \infty)$ be such that $\frac{1}{r} + \frac{1}{r'} = 1$. Using Hölder's inequality, we have

$$\begin{aligned} \|x\|_p^p &= \sum_{i=1}^{n} |x_i|^p \cdot 1 \\ &\leq \left(\sum_{i=1}^{n} 1^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} |x_i|^{pr'}\right)^{\frac{1}{r'}} \\ &= n^{\frac{1}{r}} \left(\sum_{i=1}^{n} |x_i|^{pr'}\right)^{\frac{1}{r'}} \end{aligned}$$

Choose $r' \in (1, \infty)$ such that $pr' = q$. Then

$$\frac{1}{r} = 1 - \frac{1}{r'} \implies r = \frac{q}{q-p}$$

Hence, we have

$$\begin{aligned} & \|x\|_p^p \leq n^{\frac{1}{r}} \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{q}{qr'}} \\ \implies & \|x\|_p^p \leq n^{\frac{1}{r}} \|x\|_q^{\frac{q}{r'}} \\ \implies & \|x\|_p \leq n^{\frac{1}{rp}} \|x\|_q^{\frac{q}{pq}} \\ \implies & \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \ \forall x \in \mathbb{F}^n \end{aligned}$$

Thus, we have

$$n^{\frac{1}{q}-\frac{1}{p}} \|x\|_p \leq \|x\|_q \leq \|x\|_p \quad \blacksquare$$

Definition 2.10 (Normed subspace). Let $(X, \|\cdot\|_X)$ be a normed linear space. A subspace Y of X is called a normed subspace if $(Y, \|\cdot\|_Y)$ is a normed linear space, where

$$\|x - y\|_Y := \|x - y\|_X \ \forall x, y \in Y$$

Exercise 2.15. Find optimal constants $c_1, \ c_2, \ c_3 > 0$ such that:

1. For $q > 1$, $\|x\|_q \leq \|x\|_1 \leq c_1 \|x\|_q \ \forall x \in \mathbb{F}^n$
2. For $1 \leq q < \infty$, $c_2 \|x\|_q \leq \|x\|_\infty \leq c_3 \|x\|_q \ \forall x \in \mathbb{F}^n$

Lemma 2.23. *Let $1 < p < q < \infty$. We know (lemma 2.6) that $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$ and $\|x\|_q \leq \|x\|_p \ \forall x \in \ell^p(\mathbb{N})$. Also, $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ and $(\ell^p(\mathbb{N}), \|\cdot\|_q)$ are normed linear spaces. $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent norms on $\ell^p(\mathbb{N})$.*

Proof. We know that $\|x\|_q \leq \|x\|_p \ \forall x \in \ell^p(\mathbb{N})$
Claim: $\nexists c > 0$ such that $\|x\|_p \leq c \|x\|_q \ \forall x \in \ell^p(\mathbb{N})$

Assume for the sake of contradiction that $\exists c > 0$ such that $\|x\|_p \leq c\,\|x\|_q$ $\forall x \in \ell^p(\mathbb{N}) \ldots\ldots (*)$
Let $p < r < q$. Then the sequence $\left(\frac{1}{n^{1/r}}\right)$ is in $\ell^q(\mathbb{N})$, but not in $\ell^p(\mathbb{N})$ (see the proof of lemma 2.6 (b)).
Let $x = \left(\frac{1}{n^{1/r}}\right) \in \ell^q(\mathbb{N}) \setminus \ell^p(\mathbb{N})$.
Let $(x_n) = \left(1, \frac{1}{2^{1/r}}, \frac{1}{3^{1/r}}, \ldots, \frac{1}{n^{1/r}}, 0, 0, \ldots\right)$ $\forall n \in \mathbb{N}$

Note that $(x_n) \in c_{00}(\mathbb{N}) \subset \ell^p(\mathbb{N}) \implies (x_n) \in \ell^p(\mathbb{N})$ $\forall n \in \mathbb{N}$.
(Observe that $x_1 = (1, 0, 0, \ldots)$, $x_2 = (1, \frac{1}{2^{1/r}}, 0, 0, \ldots), \ldots$)

From (*), we have that

$$\|x_n\|_p \leq c\|x_n\|_q \;\forall n \in \mathbb{N}$$
$$\implies \left(\sum_{k=1}^{n} \frac{1}{k^{p/r}}\right)^{\frac{1}{p}} \leq c\left(\sum_{k=1}^{n} \frac{1}{k^{q/r}}\right)^{\frac{1}{q}} \;\forall n \in \mathbb{N}$$
$$\implies \left(\sum_{k=1}^{n} \frac{1}{k^{p/r}}\right)^{\frac{1}{p}} \leq c\left(\sum_{k=1}^{\infty} \frac{1}{k^{q/r}}\right)^{\frac{1}{q}} \;\forall n \in \mathbb{N}$$

However, the LHS diverges as $n \to \infty$ ($\because \frac{p}{r} < 1$) and the RHS converges (as $\frac{q}{r} > 1$), so we arrive at a contradiction. Hence, our claim is true. $\square$

Lemma 2.24. *Let $(X, \|\cdot\|_a)$ and $(X, \|\cdot\|_b)$ be normed linear spaces and $\|\cdot\|_a, \|\cdot\|_b$ be equivalent norms. Let (x_n) be a sequence in X. Then we have the following:*

1. *$x_n \to x$ in $(X, \|\cdot\|_a)$ iff $x_n \to x$ in $(X, \|\cdot\|_b)$.*
2. *(x_n) is Cauchy in $(X, \|\cdot\|_a)$ iff (x_n) is Cauchy in $(X, \|\cdot\|_b)$.*
3. *$K \subseteq X$ is compact in $(X, \|\cdot\|_a)$ iff $K \subseteq X$ is compact in $(X, \|\cdot\|_b)$.*

Moreover, the identity map $f : (X, \|\cdot\|_a) \to (X, \|\cdot\|_b)$ given by $x \mapsto x$ is a homeomorphism.

Proof. Given that the two norms are equivalent $\implies$ $\exists c_1, c_2 > 0$ such that $c_1\,\|x\|_b \leq \|x\|_a \leq c_2\,\|x\|_b$ $\forall x \in X$.

1. Suppose $x_n \to x$ in $(X, \|\cdot\|_a)$. Then for given $\varepsilon' > 0$, $\exists N \in \mathbb{N}$ such that

$$\|x_n - x\|_a < \varepsilon' \;\forall n > N \text{ (see definition 2.7)}$$
$$\implies c_1\|x_n - x\|_b \leq \|x_n - x\|_a < \varepsilon' \;\forall n > N$$
$$\implies \|x_n - x\|_b < \frac{\varepsilon'}{c_1} = \varepsilon \;\forall n > N$$
$$\implies x_n \to x \text{ in } (X, \|\cdot\|_b)$$

Conversely, suppose $x_n \to x$ in $(X, \|\cdot\|_b)$. Let $\varepsilon' > 0$ be given. Then $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} \|x_n - x\|_b &< \varepsilon' \ \forall n > N \\ \implies \frac{1}{c_2}\|x_n - x\|_a \leq \|x_n - x\|_b &< \varepsilon' \ \forall n > N \\ \implies \|x_n - x\|_a &< c_2\varepsilon' = \varepsilon \ \forall n > N \\ \implies x_n &\to x \text{ in } (X, \|\cdot\|_a) \end{aligned}$$

2. Let (x_n) be Cauchy in $(X, \|\cdot\|_a)$. Let $\varepsilon' > 0$ be given. Then $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} \|x_m - x_n\|_a &< \varepsilon' \ \forall m, n > N \\ \implies c_1\|x_m - x_n\|_b < \|x_m - x_n\|_a &< \varepsilon' \ \forall m, n > N \\ \implies \|x_m - x_n\|_b &< \frac{\varepsilon'}{c_1} = \varepsilon \ \forall m, n > N \end{aligned}$$

$\implies$ (x_n) is Cauchy in $(X, \|\cdot\|_b)$

Conversely, let (x_n) be Cauchy in $(X, \|\cdot\|_a)$. Proceeding similarly, it is easy to show that (x_n) is also Cauchy in $(X, \|\cdot\|_b)$.

3. We know that a subset K of a metric space X is compact if and only if K is sequentially compact, i.e. every sequence in K has a convergent subsequence.
Let $K \subseteq X$ be compact in $(X, \|\cdot\|_a)$. Then K is sequentially compact. Let $(x_n) \subset K$ be an arbitrary sequence and (x_{n_k}) be a subsequence of (x_n) that converges to x in K with respect to $\|\cdot\|_a$. But by (1), if $x_{n_k} \to x$ w.r.t $\|\cdot\|_a$, then $x_{n_k} \to x$ w.r.t $\|\cdot\|_b$ as well.
$\implies$ Every sequence in K has a convergent subsequence w.r.t $\|\cdot\|_b$.
$\implies$ K is sequentially compact in $(X, \|\cdot\|_b)$ $\implies$ K is compact in $(X, \|\cdot\|_b)$.

Conversely, suppose that $K \subseteq X$ is compact in $(X, \|\cdot\|_b)$. Then following a similar argument, one can show that K is also compact in $(X, \|\cdot\|_a)$.

□

2.4 Riesz Lemma and Best Approximation Property

Lemma 2.25 (Riesz Lemma)**.** *Let $(X, \|\cdot\|)$ be a normed linear space and Y be a proper closed subspace of X. Let $r \in (0, 1)$. Then $\exists\ x_r \in X$ with $\|x_r\| = 1$ such that $d(x_r, Y) > r$, where d is the metric induced by the norm $\|\cdot\|$.*

Proof. As Y is a proper subspace, let $x_0 \in X \setminus Y$ and $r \in (0, 1)$. Let $d = dist(x_o, Y)$. By lemma 2.17 (2), we have that $d > 0$. Also, as $0 < r < 1$, we have $\frac{d}{r} > d$.

Choose $y \in Y$ such that

$$d \leq \|x_0 - y\| < \frac{d}{r}$$

Let $x_r = \frac{x_0 - y}{\|x_0 - y\|}$
As $x_0 \in X \setminus Y$, we have $x_r \in X \setminus Y$ and $\|x_r\| = 1$. Thus,

$$d(x_r, Y) = \frac{1}{\|x_0 - y\|} d(x_0 - y, Y) = \frac{1}{\|x_0 - y\|} d(x_0, Y) \text{ (as } y \in Y) = \frac{d}{\|x_0 - y\|} > \frac{d}{d/r} = r$$

□

Proposition 2.2. *Riesz Lemma may not hold for $r = 1$.*

Proof. The proof is taken from [2]. Consider $X = \{x \in C[0,1] : x(0) = 0\}$ with $\|\cdot\|_\infty$
Let $Y = \{x \in X : \int_0^1 x(t)dt = 0\}$
Verify that Y is a proper closed subspace of X.

Suppose Riesz Lemma holds for $r = 1$. Then $\exists x \in X$ such that $\|x\|_\infty = 1$ and $dist(x, Y) = 1 \ldots\ldots (*)$.

Claim 1: $\left|\int_0^1 x(t)dt\right| < 1$

$$\left|\int_0^1 x(t)dt\right| \leq \int_0^1 |x(t)|\, dt \leq \|x\|_\infty = 1$$

$\implies \left|\int_0^1 x(t)dt\right| \leq 1$. But if $\left|\int_0^1 x(t)dt\right| = 1$, then $\int_0^1 (1 - |x(t)|)dt = 0$, which means that $|x(t)| = 1\ \forall t \in [0,1]$, which contradicts that $x(0) = 0$. Thus, $\left|\int_0^1 x(t)dt\right| < 1$.

Claim 2: $|\int_0^1 u(t)dt| \leq \left|\int_0^1 x(t)dt\right|\ \forall u \in X$ with $\|u\|_\infty = 1$.
If $u \in Y$, then $\int_0^1 u(t)dt = 0 \implies$ claim 2 holds.
Suppose $u \in X \setminus Y$ and $\|u\|_\infty = 1$. Let

$$\tilde{u} = x - \frac{\int_0^1 x(t)dt}{\int_0^1 u(t)dt}\, u$$

As $\int_0^1 \tilde{u}(t)dt = \int_0^1 x(t)dt - \frac{\int_0^1 x(t)dt}{\int_0^1 u(t)dt} \int_0^1 u(t)dt = 0$,
$\implies \tilde{u} \in Y$, and

$$\|x - \tilde{u}\|_\infty = \left|\frac{\int_0^1 x(t)dt}{\int_0^1 u(t)dt}\right|$$

By assumption, $\|x - \tilde{u}\| \geq dist(x, Y) = 1$
$\implies \left|\int_0^1 u(t)dt\right| \leq \left|\int_0^1 x(t)dt\right|$

But claims 1 and 2 leads to a contradiction to $(*)$, because $\forall \varepsilon > 0$, we can find $u_\varepsilon \in X$ with $\|u_\varepsilon\|_\infty = 1$ and $\left|\int_0^1 u(t)dt\right| > 1 - \varepsilon$. □

Definition 2.11 (Best approximation property)**.** Let X be a normed linear space and X_0 be a subspace of X. For $x \in X$, $x_0 \in X_0$ is the **best approximation** of x if $\|x - x_0\| = dist(x, X_0)$. A subspace X_0 is said to have the **best approximation property** if every $x \in X$ has a best approximation in X_0.

Theorem 2.3. *Suppose X_0 is a proper closed subspace of a normed linear space X and X_0 has the Best approximation property. Then $\exists\, x \in X$ with $\|x\| = 1$ and $dist(x, X_0) = 1$.*

Proof. Choose $x \in X \setminus X_0$ and let $x_0 \in X_0$ be the best approximation of x, i.e., $\|x - x_0\| = dist(x, X_0)$. Let

$$\tilde{x} = \frac{x - x_0}{\|x - x_0\|}$$

Then $\|\tilde{x}\| = 1$ and

$$\begin{aligned} dist(\tilde{x}, X_0) &= dist\left(\frac{x - x_0}{\|x - x_0\|}, X_0\right) \\ &= \frac{1}{\|x - x_0\|} dist\,(x - x_0, X_0) \\ &= \frac{1}{\|x - x_0\|} dist\,(x, X_0) \;\; [\text{as } x_0 \in X_0] \\ &= \frac{\|x - x_0\|}{\|x - x_0\|} = 1 \end{aligned}$$

□

Definition 2.12 (Bounded set in a normed linear space)**.** Let $(X, \|\cdot\|)$ be a normed linear space and $A \subseteq X$. Then A is bounded in X if $\exists M > 0$ such that $\|x\| \leq M \; \forall x \in A$, i.e., $A \subseteq \overline{B}(0, M)$.

Lemma 2.26. *Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space over $\mathbb{F}$ and $\mathscr{B} = \{e_1, \ldots, e_n\}$ be a basis for X. Define $T : X \to (\mathbb{F}^n, \|\cdot\|_2)$, by $x = \sum_{i=1}^n c_i e_i \mapsto (c_1, \ldots, c_n)$, where $\{e_1, \ldots, e_n\}$ is a basis for X and $c_1, \ldots, c_n \in \mathbb{F}$. T is a well-defined linear and bijective map. Also, $\exists \alpha_1, \alpha_2 > 0$ such that $\alpha_1 \|Tx\|_2 \leq \|x\| \leq \alpha_2 \|Tx\|_2 \; \forall x \in X$. In other words, T is bijective and bicontinuous (T and T^{-1} are both continuous, and hence a homeomorphism).*

Proof. It is easy to check that T is well-defined, linear, and bijective. For the last part, let $x \in X$. Define $\|x\|_* = \|Tx\|_2$. Clearly (verify), $\|\cdot\|_*$ is a norm in X. Hence, we have two norms $\|\cdot\|_*$ and $\|\cdot\|$ on a finite dimensional normed linear space X. By theorem 2.2, they are equivalent. Thus, $\exists \alpha_1, \alpha_2 > 0$ such that

$$\begin{aligned} \alpha_1 \|x\|_* \leq \|x\| \leq \alpha_2 \|x\|_* \; \forall x \in X \\ \implies \alpha_1 \|Tx\|_2 \leq \|x\| \leq \alpha_2 \|x\|_* \; \forall x \in X \end{aligned}$$

The last part means that T and T^{-1} are both a bounded linear transformations, and are hence continuous (see remark 5.2 in Bounded Linear Transformations). $\square$

Theorem 2.4 (Heine-Borel Theorem). *Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space over $\mathbb{F}$. Let $K \subseteq X$. Then K is compact in X if and only if K is closed and bounded in X.*

Proof. ($\Longrightarrow$) Suppose K is compact in X. We prove that K is closed and bounded. Assume that K is not bounded. Then $\forall n \in \mathbb{N}$, $\exists x_n \in K$ such that $\|x_n\| > n$. This means that there is a sequence (x_n) in K which has no convergent subsequence. This contradicts that K is compact (since compactness implies sequential compactness). Thus, K is bounded.

Now, we will show that K is closed by showing that $X \setminus K$ is open. Let $p \in X \setminus K$. For each $x \in K$, consider $\delta_x = \frac{1}{2}\|p - x\|$.
Clearly, $\{B_{\delta_x}(x)\}_{x \in K}$ forms an open cover for K. As K is compact, $\exists x_1, \ldots, x_n$ such that $K \subseteq \bigcup_{i=1}^{n} B_{\delta_{x_i}}(x_i)$.
Consider $V = \bigcap_{i=1}^{n} B_{\delta_{x_i}}(p)$. Clearly, V is an open set (being finite intersection of open sets) containing p (because each ball in the intersection contains p). Further, $V \subset X \setminus K$ (As if $y \in V \cap K$, then $y \in B_{\delta_{x_i}}(p)$ $\forall i$ and $y \in B_{\delta_{x_j}}(x_j)$ for some j, which implies that $\|p - x_j\| < 2\delta_{x_j}$, contradicting $\delta_{x_j} = \frac{1}{2}\|p - x_j\|$).
Hence, we have V, an open set containing $p \in X \setminus K$, which is strictly inside $X \setminus K$. This means that p is an interior point of $X \setminus K$. As p is arbitrary, we have that every point in $X \setminus K$ is an interior point. Thus, $X \setminus K$ is open $\implies$ K is closed in X.

($\Longleftarrow$) Assume that K is closed and bounded in X. Consider $T : X \to \mathbb{F}^n$, defined as in lemma 2.26. T is a well-defined linear, bijective, and continuous (bounded) map. Thus, $K = T^{-1}(T(K)) \implies T(K)$ is closed and bounded in $\mathbb{F}^n$. By Heine-Borel theorem in $\mathbb{F}^n$ (which you might have studied in a first course of metric spaces or real analysis), $T(K)$ is compact in $\mathbb{F}^n$. Then $T^{-1}(T(K)) = K$ is compact in X, as T^{-1} is continuous. $\square$

Theorem 2.5 (Finite dimensional subspaces of a normed linear space has the best approximation property). *If X_0 is a finite dimensional subspace of a normed linear space X, then $\forall x \in X$, $\exists x_0 \in X_0$ such that $\|x - x_0\| = dist(x, X_0)$.*

Proof. Let $x \in X$. Fix $u_0 \in X_0$ and consider

$$S = \{u \in X_0 : \|x - u\| \leq \|x - u_0\|\}$$

Consider the map $f : S \to \mathbb{R}$ given by $u \mapsto \|x - u\|$.
Essentially, $S = \overline{B}(u, \|x - u_0\|)$, so S is closed. Also, S is a bounded subset of a finite dimensional space X_0. By Heine-Borel Theorem, S is compact.

Claim: f is continuous.
Take any point $s \in S$ such that $\|u - s\| < \delta$. Then

$$\|f(u) - f(s)\| = |\|x - u\| - \|x - s\|| \leq \|x - u - x + s\| = \|u - s\| < \delta$$

As S is compact and f is continuous, $\exists$ some $x_0 \in S$ such that $f(x_0) = \inf\{f(u) : u \in S\}$ (infimum is attained).
$\implies \|x - x_0\| = \inf\{\|x - u\| : u \in S\}$
But $S \subset X_0$. So, we have $\|x - x_0\| = \inf\{\|x - u\| : u \in X_0\} = dist(x, X_0)$ $\square$

Definition 2.13 (Unit sphere in a normed linear space)**.** Let $(X, \|\cdot\|)$ be a normed linear space. The unit sphere is defined as

$$S_X = \{x \in X : \|x\| = 1\}$$

Theorem 2.6 (A characterization of finite dimensional normed linear spaces)**.** *Let* $(X, \|\cdot\|)$ *be a normed linear space. Then the following are equivalent:*

1. *X is finite dimensional.*

2. *The unit sphere $S_X := \{x \in X : ||x|| = 1\}$ is compact in X.*

Proof. [(1) $\implies$ (2)]
Let X be a finite dimensional normed linear space. We show that S_X is closed and bounded. Clearly,

$$X \setminus S_X = \{x \in X : \|x\| < 1\} \cup \{x \in X : \|x\| > 1\} = B(0,1) \cup \left(\overline{B}(0,1)\right)^c$$

where $\left(\overline{B}(0,1)\right)^c = X \setminus \overline{B}(0,1)$. Clearly, $B(0,1)$ is open in X and $\overline{B}(0,1)$ is closed in $X \implies \left(\overline{B}(0,1)\right)^c$ is also open in X. So, $X \setminus S_X$ is open, being a union of two open sets, which means that S_X is closed.
Clearly, for any $x \in S_X$, $\|x\| \leq 1 \implies S_X$ is bounded.
As X is finite dimensional, S_X is compact in X by Heine-Borel Theorem.

[(2) $\implies$ (1)]
Suppose that S_X is compact in X. Let X be an infinite dimensional normed linear space. Let $\{x_1, \ldots, x_n\}$ be a linearly independent set in X. Define $X_n = span\{x_i : 1 \leq i \leq n\}$
Clearly, X_n is a finite dimensional proper subspace of X.
Moreover, $X_n \subsetneq X_{n+1}$, and X_n is in fact a proper closed subspace of X_{n+1} $\forall n \in \mathbb{N}$.

By Riesz Lemma, for each $n \in \mathbb{N}$, $\exists x_{n+1} \in X_{n+1}$ such that $\|x_{n+1}\| = 1$ and $d(x_{n+1}, X_n) > \frac{1}{2}$. Observe:

1. $d(x_{n+1}, y) > \frac{1}{2} \; \forall y \in X_n$
2. $d(x_{n+1}, x_i) > \frac{1}{2} \; \forall 1 \leq i \leq n \; (\because x_i \in X_n \; \forall i)$
3. $d(x_n, x_m) > \frac{1}{2} \; \forall n \neq m$
4. $\|x_n\| = 1 \; \forall n \in \mathbb{N}$

Thus, (x_n) is a sequence in S_X such that $\|x_n - x_m\| > \frac{1}{2} \; \forall n \neq m$
$\implies$ No subsequence of (x_n) is convergent $\implies$ S_X is not sequentially compact, which contradicts that S_X is compact. Thus, X must be finite dimensional.

□

Remark 2.12. Let $(X, \|\cdot\|)$ be a normed linear space. Then

1. $\overline{B}(0, 1)$ is compact in X if and only if X is finite dimensional.
2. Let $x_0 \in X, r > 0$. Then $\overline{B}(x_0, r)$ is compact in X if and only if X is finite dimensional.

Corollary 2.7. *Let $(X, \|\cdot\|)$ be an infinite dimensional normed linear space. Then S_X and $\overline{B}(x, r)$ for $x \in X$ and $r > 0$ are non-compact sets in X.*

This is the major drawback of infinite dimensional normed linear spaces, namely the "good sets" (above balls) are non-compact sets. But there is a remedy for this if the space is an infinite dimensional normed linear space. This is not within our scope. Interested readers can see Banach–Alaoglu Theorem. Below, we present some examples of infinite dimensional normed linear spaces X and show that S_X is not compact.

1. Consider $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ for $1 \leq p \leq \infty$. Then

   $$S_{\ell^p} = \{x \in \ell^p(\mathbb{N}) : \|x\|_p = 1\}$$

   We know, $e_i \in S_{\ell^p} \; \forall i \in \mathbb{N}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots)$, where 1 occurs in the i^{th} position. Clearly, $\|e_i\| = 1 \; \forall i$.
   Then $\|e_i - e_j\|_p = \|(0, \ldots, 1, \ldots, -1, 0, \ldots)\|$ (where the 1 occurs at the i^{th} position and -1 at the j^{th} position).
   $\implies \|e_i - e_j\| = (1+1)^{\frac{1}{p}} = 2^{\frac{1}{p}} \; \forall i \neq j$
   $\implies$ (e_i) has no convergent subsequence $\implies$ S_{ℓ^p} is not compact.

2. Consider $(C[0, 1], \|\cdot\|_\infty)$, the space of all $\mathbb{F}$-valued continuous function over $[0, 1]$. Then

   $$S_{C[0,1]} = \{f \in C[0, 1] : \|f\|_\infty = 1\}$$

To show that S_X is not compact, we must construct a sequence (f_n) in $S_{C[0,1]}$ which has no convergent subsequence.
Let $f_n(x) = x^n \; \forall x \in [0,1]$
Observe that $\|f_n(x)\|_\infty = \sup_{x\in[0,1]}\{|f_n(x)|\} = 1 \; \forall n \in \mathbb{N}$
We show that no subsequence of (f_n) is Cauchy (and hence, no subsequence is convergent).

$$\|f_{2n} - f_n\| = \sup_{x\in[0,1]}(x^n - x^{2n}) = \sup_{x\in[0,1]}(x^n - (x^n)^2) = \sup_{t\in[0,1]}(t - t^2) = \frac{1}{4}$$

As (f_n) is monotonic, $k \geq 2n \implies \|f_k - f_n\| \geq \frac{1}{4}$
Hence, a subsequence of (f_n) is not Cauchy. This is because for any given k_0, we can find $k' > k_0$ such th at $n_{k'} > 2n_{k_0}$ and $\|f_{n'_k} - f_{n_{k_0}}\| \geq \frac{1}{4}$.

3 Banach Spaces

Recall (from Real Analysis):

- Let (a_n) be a sequence in $\mathbb{F}$. The formal sum $\sum_{n=1}^{\infty} a_n$ is called a *formal series.*
- Let $S_m = \sum_{k=1}^{m} a_k,\ m \in \mathbb{N}$. Then S_m is called the m^{th} partial sum.
- We say that the series $\sum_{n=1}^{\infty} a_n$ is *convergent* if $\exists\ a \in \mathbb{F}$ such that the sequence of partial sums converges to a, i.e. $\lim_{m\to\infty} S_m = a$.
- We say that $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is a convergent series in $\mathbb{R}$.
- If a series is absolutely convergent, then it is convergent. The converse, however, need not be true.

Definition 3.1 (Banach Space)**.** Let $(X, \|\cdot\|)$ be a normed linear space. We say that $(X, \|\cdot\|)$ is a Banach space if every Cauchy sequence in X is convergent in X. In other words, a complete normed linear space is called a Banach space.

Example 3.1 (Some Banach spaces)**.** The following are some examples of Banach spaces.

1. $(\mathbb{F}^n, \|\cdot\|_p)$ for $1 \leq p \leq \infty$ is a Banach space.

   Proof. **Case 1:** $1 \leq p < \infty$
   Then $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, where $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$
   Let (x_m) be a Cauchy sequence in $\mathbb{F}^n$. We show that (x_m) is convergent in $\mathbb{F}^n$.

   Each term of the sequence (x_m) is an n-tuple:

$$
\begin{aligned}
x_1 &= \left(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}\right)\\
x_2 &= \left(x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n)}\right)\\
&\cdots\\
x_m &= \left(x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(n)}\right),\ m \in \mathbb{N}
\end{aligned}
$$

   (x_m) is Cauchy $\implies$ for each $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$
\begin{aligned}
\|x_l - x_m\|_p &< \varepsilon\ \forall l, m > N\\
\implies \left(\sum_{i=1}^{n} \left|x_l^{(i)} - x_m^{(i)}\right|^p\right)^{\frac{1}{p}} &< \varepsilon\ \forall l, m > N\\
\implies \sum_{i=1}^{n} \left|x_l^{(i)} - x_m^{(i)}\right|^p &< \varepsilon^p\ \forall l, m > N\\
\implies \text{ for each } i, \text{ we have } \left|x_l^{(i)} - x_m^{(i)}\right| &< \varepsilon\ \forall l, m > N
\end{aligned}
$$

$\implies$ for each i, the sequence $\left(x_l^{(i)}\right)_{l=1}^{\infty}$ is Cauchy in $\mathbb{F}$.
As $\mathbb{F}$ is complete, for each i, $\exists x_i \in \mathbb{F}$ such that $x_l^{(i)} \to x_i$ as $l \to \infty$
In other words, for each i, given $\varepsilon > 0$, $\exists N_i \in \mathbb{N}$ such that $\left|x_l^{(i)} - x_i\right| < \varepsilon \ \forall l > N_i \ldots\ldots (*)$
Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$

Claim: $x_m \to x$ as $m \to \infty$
Let $\varepsilon > 0$ be given. Then

$$\|x_m - x\|_p = \left(\sum_{i=1}^{n} \left|x_m^{(i)} - x_i\right|^p\right)^{\frac{1}{p}} < n^{\frac{1}{p}}\varepsilon \ \forall m > N = \max\{N_1, \ldots, N_n\} \text{ [by (*)]}$$

$\implies$ (x_m) is convergent in $\mathbb{F}^n$ with respect to $\|\cdot\|_p$.

Case 2: $p = \infty$
For $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$, $\|x\|_\infty = \max\{|x_i| : 1 \le i \le n\}$
Let (x_m) as above be a Cauchy sequence in $(\mathbb{F}^n, \|\cdot\|_\infty)$.
$\implies$ Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} \|x_l - x_m\|_\infty &< \varepsilon \ \forall l, m > N \\ \implies \max_{i=1}^{n}\left\{\left|x_l^{(i)} - x_m^{(i)}\right|\right\} &< \varepsilon \ \forall l, m > N \\ \implies \forall i, \text{ we have } \left|x_l^{(i)} - x_m^{(i)}\right| &< \varepsilon \ \forall l, m > N \end{aligned}$$

$\implies$ for each i, $\left(x_l^{(i)}\right)_{l=1}^{\infty}$ is Cauchy in $\mathbb{F}$.
As $\mathbb{F}$ is complete, for each i, $\exists\ x_i \in \mathbb{F}$ such that $x_l^{(i)} \to x_i$ as $l \to \infty$
$\implies$ Given $\varepsilon > 0$, $\exists N_i \in \mathbb{N}$ such that $\left|x_l^{(i)} - x_i\right| < \varepsilon \ \forall l > N_i \ldots\ldots (**)$

Let $x = (x_1, \ldots, x_n)$. Clearly, $x \in \mathbb{F}^n$.
Claim: $x_m \to x$ as $m \to \infty$
For given $\varepsilon > 0$, we have $\|x_m - x\|_\infty = \max\left\{\left|x_m^{(i)} - x_i\right| : 1 \le i \le n\right\} < \varepsilon \ \forall m > N = \max_{i=1}^{n}\{N_i\}$ (using (**))
Thus, (x_m) converges to x in $(\mathbb{F}^n, \|\cdot\|_\infty)$.

$\therefore$ $(\mathbb{F}^n, \|\cdot\|_p)$ is a Banach space for $1 \le p \le \infty$. □

2. $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ is a Banach space for $1 \le p \le \infty$.

Proof. **Case 1:** $1 < p < \infty$

Let (x_n) be a Cauchy sequence in $\ell^p(\mathbb{N})$. Each term of the sequence is an infinite tuple:

$$\begin{aligned}
x_1 &= (x_{11}, x_{12}, \ldots, x_{1n}, \ldots) \\
x_2 &= (x_{21}, x_{22}, \ldots, x_{2n}, \ldots) \\
&\ldots \\
x_m &= (x_{m1}, x_{m2}, \ldots, x_{mn}, \ldots) \in \ell^p(\mathbb{N}) \\
&\ldots
\end{aligned}$$

Let $\varepsilon > 0$ be given. Then $\exists N \in \mathbb{N}$ such that

$$\begin{aligned}
\|x_n - x_m\|_p &< \varepsilon \; \forall n, m > N \\
\sum_{k=1}^{\infty} |x_{nk} - x_{mk}|^p &< \varepsilon^p \; \forall n, m > N \ldots\ldots (*) \\
\implies \text{ For each } k \in \mathbb{N}, |x_{nk} - x_{mk}| &< \varepsilon^p \; \forall n, m > N \\
\implies \text{ For each } k \in \mathbb{N}, |x_{nk} - x_{mk}| &< \varepsilon \; \forall n, m > N
\end{aligned}$$

$\implies$ For each $k \in \mathbb{N}, (x_{mk})_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$.
$\implies$ For each $k \in \mathbb{N}$, $\exists$ unique $y_k \in \mathbb{F}$ such that $x_{mk} \to y_k$ as $m \to \infty \ldots\ldots (**)$

Let $y = (y_1, y_2, \ldots, y_m, \ldots)$
Claim: $x_n \to y$ as $n \to \infty$ w.r.t. $\|\cdot\|_p$ and $y \in \ell^p(\mathbb{N})$.
From $(*)$, we have $\sum_{k=1}^{\infty} |x_{nk} - x_{mk}|^p < \varepsilon^p \; \forall n, m > N$
$\implies$ For each $L \in \mathbb{N}$, we have

$$\sum_{k=1}^{L} |x_{nk} - x_{mk}|^p < \varepsilon^p \; \forall n, m > N$$

Now letting $m \to \infty$, we have

$$\begin{aligned}
&\sum_{k=1}^{L} |x_{nk} - y_k|^p < \varepsilon^p \; \forall n > N, \; \forall L \in \mathbb{N} \text{ (Using } (**), \text{ that } x_{mk} \to y_k \text{ as } m \to \infty) \\
\implies &\sum_{k=1}^{\infty} |x_{nk} - y_k|^p < \varepsilon^p \; \forall n > N \\
&\implies \|x_n - y\|_p < \varepsilon \; \forall n > N \\
&\qquad \implies x_n \to y \;\; \text{w.r.t } p-\text{norm}
\end{aligned}$$

Now, we show that $y \in \ell^p(\mathbb{N})$. Write $y = y - x_N + x_N$ for some $N \in \mathbb{N}$. Then $\|y\|_p \leq \|y - x_N\|_p + \|x_N\|_p < \infty$ (since both terms are finite. $\|y - x_N\|_p$ is finite as

$x_N \to y$ w.r.t $\|\cdot\|_p$, and $\|x_N\|_p$ is finite because $x_N \in \ell^p(\mathbb{N})$).

Thus, we have $x_n \to y \in \ell^p(\mathbb{N})$.

Case 2: $p = \infty$

Recall that $\ell^\infty(\mathbb{N}) = \{(x_n) : (x_n) \text{ is bounded}\}$. Let (x_n) as above be a Cauchy sequence in $\ell^\infty(\mathbb{N})$. Let $\varepsilon > 0$ be given. Then $\exists N \in \mathbb{N}$ such that $\|x_n - x_m\|_\infty < \varepsilon \; \forall m, n \in \mathbb{N}$

$$\implies \sup\{|x_{nk} - x_{mk}| : k \in \mathbb{N}\} < \varepsilon \; \forall n, m > N \ldots\ldots (*)$$
$$\implies \text{ for each } k \in \mathbb{N}, |x_{nk} - x_{mk}| < \varepsilon \; \forall n, m > N$$

$\implies$ For each $k \in \mathbb{N}, (x_{mk})_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$.
$\implies$ For each $k \in \mathbb{N}$, $\exists$ unique $y_k \in \mathbb{F}$ such that $x_{mk} \to y_k$ as $m \to \infty$

Let $y = (y_1, \ldots, y_m, \ldots)$. We claim that $x_n \to y$ w.r.t. $\|\cdot\|_\infty$ and $y \in \ell^\infty(\mathbb{N})$.
From $(*)$, we have

$$\sup\{|x_{nk} - x_{mk}| : k \in \mathbb{N}\} < \varepsilon \; \forall n, m > N$$
$$\text{Letting } m \to \infty, \text{ we have}$$
$$\sup\{|x_{nk} - y_k| : k \in \mathbb{N}\} < \varepsilon \; \forall n > N$$
$$\implies \|x_n - y\|_\infty < \varepsilon \; \forall n > N$$
$$\implies x_n \to y \text{ as } n \to \infty \text{ w.r.t. } \|\cdot\|_\infty$$

To see that $y \in \ell^\infty(\mathbb{N})$, write $y = y - x_N + x_N$ for some $N \in \mathbb{N}$ and observe that $\|y\|_\infty \le \|y - x_N\|_\infty + \|x_N\|_\infty < \infty$.

Thus, $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ is a Banach space for $1 \le p \le \infty$. □

3. Let K be a compact Hausdorff space and define $C(K) = \{f : K \to \mathbb{F} \mid f \text{ is continuous}\}$. We know that $C(K)$ is a vector space over $\mathbb{F}$. Then $(C(K), \|\cdot\|_\infty)$ is a Banach space.

Proof.

Exercise 3.1. Check that $(C(K), \|\cdot\|_\infty)$ is a normed linear space.

Let (f_n) be a Cauchy sequence in $C(K)$. Let $\varepsilon > 0$ be given. Then $\exists N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_\infty < \varepsilon \; \forall n, m > N$$
$$\implies \sup\{|f_n(x) - f_m(x)| : x \in K\} < \varepsilon \; \forall n, m > N$$

$\implies$ For each $x \in K$, the sequence $(f_m(x))_{m=1}^{\infty}$ is Cauchy in $\mathbb{F}$
As $\mathbb{F}$ is complete, for each $x \in K$, $\exists$ unique $f(x) \in \mathbb{F}$ such that $\lim_{n\to\infty} f_n(x) = f(x) \ldots\ldots (*)$

Claim: $f \in C(K)$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$
From $(*)$, we have that for each $\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon \ \forall n > N_\varepsilon$ and $\forall x \in K$
$\implies$ $f_n \to f$ uniformly on K
$\implies$ The limit function f is continuous $\implies$ $f \in C(K)$

Further, we have for each $x \in K$, $|f_n(x) - f_m(x)| < \varepsilon \ \forall n, m > N$. Letting $m \to \infty$,

$$\begin{aligned} &|f_n(x) - f(x)| < \varepsilon \ \forall n > N \ \forall x \in K \\ \implies &\sup_{x\in K}\{|f_n(x) - f(x)|\} < \varepsilon \ \forall n > N \\ \implies &\|f_n - f\|_\infty < \varepsilon \ \forall n > N \end{aligned}$$

□

Proposition 3.1. *Every finite dimensional normed linear spaces are Banach spaces.*

Proof. Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space over $\mathbb{F}$. Let $\mathscr{B} = \{e_1, \ldots, e_n\}$ be a basis for X and $x \in X$. Write $x = \sum_{i=1}^n c_i e_i$, where $c_i \in \mathbb{F}$. Define

$$\|x\|_\infty = \max\{|c_i| : 1 \le i \le n\}$$

Exercise 3.2. Verify that $(X, \|\cdot\|)_\infty$ is a Banach space.

As X is finite dimensional, the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ (arbitrary norm) are equivalent by theorem 2.2. Hence, $(X, \|\cdot\|)$ is Banach. □

Definition 3.2 (Formal series and its convergence)**.** Let $(X, \|\cdot\|)$ be a normed linear space. Let (x_n) be a sequence in X. The formal sum $\sum_{n=1}^\infty x_n$ is called *formal series* in X.

Let $S_m = \sum_{k=1}^m x_k$, $m \in \mathbb{N}$.
We say that $\sum_{n=1}^\infty x_n$ is convergent in X if $\exists\, x \in X$ such that $\lim_{m\to\infty} S_m = x$, i.e.

$$\exists x \in X \text{ such that } \|S_m - x\| \to 0 \text{ as } m \to \infty$$

Definition 3.3 (Absolutely convergent series)**.** Let $\sum_{n=1}^\infty x_n$ be a formal series in $(X, \|\cdot\|)$. We say that $\sum_{n=1}^\infty x_n$ is *absolutely convergent* in X if $\sum_{n=1}^\infty \|x_n\|$ is convergent in $\mathbb{R}$.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Let (x_n) be a Cauchy sequence in X. Suppose there exists a subsequence (x_{n_k}) of (x_n) and $x \in X$ such that x_{n_k} converges to x, i.e. $\lim_{k\to\infty} x_{n_k} = x$. Then $x_n \to x$ as well, i.e. $\lim_{n\to\infty} x_n = x$. In other words, to show that a Cauchy sequence (x_n) in a normed linear space X is convergent, it is enough to show that $\exists$ a subsequence (x_{n_k}) of it which converges in X.*

Proof. Let $\varepsilon > 0$ be given. Then $\exists N_1 \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon \ \forall n, m > N_1$ (as (x_n) is Cauchy)
Also, $\exists N_2 \in \mathbb{N}$ such that $\|x_{n_k} - x\| < \varepsilon \ \forall n_k > N_2$ (as (x_{n_k}) converges to x). Thus,

$$\|x_n - x\| = \|x_n - x_{n_k} + x_{n_k} - x\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - x\| < 2\varepsilon \ \forall n, n_k > \max\{N_1, N_2\}$$

$\implies x_n \to x$ as $n \to \infty$. □

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Then X is a Banach space if and only if every absolutely convergent series in X is convergent in X.*

Proof. ($\implies$) Let $(X, \|\cdot\|)$ be a Banach space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X which is absolutely convergent, i.e. $\sum_{n=1}^{\infty} \|x_n\|$ is convergent in $\mathbb{R} \ldots\ldots (*)$. We prove that $\sum_{n=1}^{\infty} x_n$ is a convergent series in X.

Let (S_m) be the sequence of partial sums of $\sum_{n=1}^{\infty} x_n$ in X.
Claim: (S_m) is Cauchy in X.
Let $\varepsilon > 0$ be given and $m < n$. Then

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| \text{ (by the triangle inequality)}$$

From $(*)$, we have that $\exists N \in \mathbb{N}$ such that $\sum_{k=m+1}^{n} \|x_k\| < \varepsilon \ \forall n > m > N$
Hence, we have

$$\|S_n - S_m\| \leq \sum_{k=m+1}^{n} \|x_k\| < \varepsilon \ \forall n > m > N$$

$\implies (S_m)$ is Cauchy in X.
As X is Banach, (S_m) converges in $X \implies \sum_{n=1}^{\infty} x_n$ converges in X.

($\impliedby$) Conversely, suppose that $(X, \|\cdot\|)$ is a normed linear space and every absolutely convergent series in X is convergent in X. We prove that X is a Banach space.
Let (x_n) is a Cauchy sequence in X.
$\implies$ for each $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon \ \forall n, m > N$

Let $\varepsilon = 1$. Then $\exists N_1 \in \mathbb{N}$ such that $\|x_n - x_m\| < 1 \ \forall n, m > N_1$

In particular, $\|x_n - x_{N_1}\| < 1 \; \forall n > N_1$

Let $\varepsilon = \frac{1}{2}$. Then $\exists N_2 > N_1 \in \mathbb{N}$ such that $\|x_n - x_m\| < \frac{1}{2} \; \forall n, m > N_2$
In particular, $\|x_n - x_{N_2}\| < \frac{1}{2} \; \forall n > N_2$ and $\|x_{N_2} - x_{N_1}\| < 1$

Let $\varepsilon = \frac{1}{2^2}$. Then $\exists N_3 > N_2 \in \mathbb{N}$ such that $\|x_n - x_m\| < \frac{1}{2^2} \; \forall n, m > N_3$
In particular, $\|x_n - x_{N_3}\| < \frac{1}{2^2} \; \forall n > N_3$ and $\|x_{N_3} - x_{N_2}\| < \frac{1}{2}$

At the $(k+1)^{th}$ stage, let $\varepsilon = \frac{1}{2^k}$. Then $\exists N_{k+1} > N_k \in \mathbb{N}$ such that $\|x_n - x_m\| < \frac{1}{2^k} \; \forall n, m > N_{k+1}$
In particular, $\left\|x_n - x_{N_{k+1}}\right\| < \frac{1}{2^k} \; \forall n > N_{k+1}$ and $\left\|x_{N_{k+1}} - x_{N_k}\right\| < \frac{1}{2^{k-1}}$

Now, observe that (x_{N_k}) is a subsequence of (x_n) such that

$$\left\|x_{N_{k+1}} - x_{N_k}\right\| < \frac{1}{2^{k-1}}$$

We have

$$\sum_{k=1}^{\infty} \left\|x_{N_{k+1}} - x_{N_k}\right\| \leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} < \infty$$

$\implies \sum_{k=1}^{\infty} \left(x_{N_{k+1}} - x_{N_k}\right)$ is absolutely convergent.
$\implies$ By hypothesis, $\sum_{k=1}^{\infty} (x_{N_{k+1}} - x_{N_k})$ is a convergent series in X.
$\implies (S_m) = \left(\sum_{k=1}^{m} (x_{N_{k+1}} - x_{N_k})\right)$ is a convergent sequence in X.
$\implies (S_m) = \left(x_{N_{m+1}} - x_{N_m}\right)_{m=1}^{\infty}$ is a convergent sequence in X.

$\implies (x_{N_k})$ is a convergent subsequence of (x_n) in X.
$\implies \exists x \in X$ such that $\lim_{k \to \infty} x_{N_k} = x$
$\implies$ By lemma 3.1, $x_n \to x \in X$ as $n \to \infty$
As (x_n) is an arbitrary Cauchy sequence that is convergent, we have that every Cauchy sequence in X converges in X, i.e. X is a Banach space.

□

Exercise 3.3. Recall from example 2.1 (5), the two norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $C[a,b]$. Show that $(C[a,b], \|\cdot\|_\infty)$ is a Banach space, while $(C[a,b], \|\cdot\|_1)$ is not a Banach space. In fact, $(C[a,b], \|\cdot\|_p)$ is not Banach for $1 \leq p < \infty$, where the norm is defined as

$$\|f\|_p = \left(\int_a^b |f(t)|^p\right)^{\frac{1}{p}} \quad \forall f \in C[a,b]$$

Example 3.2. Let $C^1[a,b]$ denote the set of all continuously differentiable $\mathbb{F}$-valued functions over $[a, b]$. Define $\|\cdot\|_*$ as

$$\|x\|_* = \|x\|_\infty + \|x'\|_\infty \ \forall x \in C^1[a, b]$$

$\|\cdot\|_*$ is called the C^1-norm. Then

1. $(C^1[a, b], \|\cdot\|_*)$ is a Banach space.
2. $(C^1[a, b], \|\cdot\|_\infty)$ is not a Banach space.

Proof. We first prove that $\|\cdot\|_*$ is a norm on $C^1[a, b]$.

- For $x \in C^1[a, b]$,

$$\|x\|_* = 0 \iff \|x\|_\infty + \|x'\|_\infty = 0 \iff \|x\|_\infty = 0 \text{ and } \|x'\|_\infty = 0 \iff \|x\|_\infty = 0 \iff x \equiv 0$$

- For $\alpha \in \mathbb{F}$,

$$\|\alpha x\|_* = \|\alpha x\|_\infty + \|\alpha x'\|_\infty = |\alpha| \, \|x\|_\infty + |\alpha| \, \|x'\|_\infty = |\alpha|(\|x\|_\infty + \|x'\|_\infty) = |\alpha| \, \|x\|_*$$

- Let $x, y \in C^1[a, b]$. Then

$$\begin{aligned} \|x + y\|_* &= \|x + y\|_\infty + \|x' + y'\|_\infty \\ &\leq (\|x\|_\infty + \|y\|_\infty) + (\|x'\|_\infty + \|y'\|_\infty) \\ &= (\|x\|_\infty + \|x'\|_\infty) + (\|y\|_\infty + \|y'\|_\infty) \\ &= \|x\|_* + \|y\|_* \end{aligned}$$

Thus, $(C^1[a, b], \|\cdot\|_*)$ is a normed linear space.

1. **Claim:** $(C^1[a, b], \|\cdot\|_*)$ is Banach.
   Let (f_n) be a Cauchy sequence in $(C^1[a, b], \|\cdot\|)$. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} & \|f_n - f_m\|_* < \varepsilon \ \forall n, m > N \\ \implies & \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty < \varepsilon \ \forall n, m > N \\ \implies & \|f_n - f_m\|_\infty < \varepsilon \text{ and } \|f_n' - f_m'\|_\infty < \varepsilon \ \forall n, m > N \end{aligned}$$

   $\implies$ (f_n) and (f_n') are Cauchy sequences in $(C[a, b], \|\cdot\|_\infty)$.
   As $(C[a, b], \|\cdot\|_\infty)$ is Banach (exercise 3.3), $\exists f, g \in C[a, b]$ such that $f_n \to f$ and $f_n' \to g$ uniformly w.r.t. $\|\cdot\|_\infty$

Claim: $f' = g$.
Observe that as $f_n' \to g$ uniformly, we have $\int_a^x f_n'(t)dt \to \int_a^x g(t)dt$ because

$$\left\|\int_a^x f_n'(t)dt - \int_a^x g(t)dt\right\|_\infty \leq \sup_{a\leq x\leq b}\left\{\int_a^x |f_n'(t) - g(t)|dt\right\} \leq \|f_n' - g\|_\infty < \varepsilon$$

By the fundamental theorem of Calculus, we have $\int_a^x f_n'(t)dt = f_n(x) - f_n(a)$
As $f_n \to f$ uniformly,

$$\int_a^x f_n'(t)dt \to \int_a^x g(t)dt \implies f_n(x) - f_n(a) \to \int_a^x g(t)dt \implies f(x) - f(a) = \int_a^x g(t)dt$$

Again, by the fundamental theorem of Calculus, we have that f is the anti-derivative of g. Thus, $f' = g$, or $f_n' \to f'$ uniformly as $n \to \infty$.

Claim: $f_n \to f$ in $(C^1[a,b], \|\cdot\|)$.
Let $\varepsilon > 0$ be given. Then $\exists N \in \mathbb{N}$ such that

$$\begin{aligned}
\|f_n - f\|_\infty < \frac{\varepsilon}{2} \text{ and } \|f_n' - f'\|_\infty < \frac{\varepsilon}{2}\ \forall n > N \\
\implies \|f_n - f\|_\infty + \|f_n' - f'\|_\infty < \varepsilon\ \forall n > N \\
\implies \|f_n - f\|_* < \varepsilon\ \forall n > N
\end{aligned}$$

$\implies f_n \to f$ in $C^1[a,b]$ w.r.t $\|\cdot\|_* \implies (C^1[a,b], \|\cdot\|_*)$ is Banach.

2. We have to show that $(C^1[a,b], \|\cdot\|_\infty)$ is not a Banach space.
Without loss of generality, let $[a,b] \equiv [0,1]$. Consider $f_n(x) = \sqrt{x + \frac{1}{n}}$. Clearly, $f_n \in C^1[0,1]$ because $f_n'(x) = \frac{1}{2\sqrt{x+\frac{1}{n}}}$ is continuous in $[0,1]$.

Claim: $f_n \to \sqrt{x}$ uniformly on $[0,1]$ w.r.t $\|\cdot\|_\infty$.
Note that $\sqrt{x}$ is uniformly continuous on compact sets (in particular, on $[0,1]$).
$\implies$ Given $\varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in [0,1]$ with $|x-y| < \delta$, we have $|\sqrt{x} - \sqrt{y}| < \varepsilon$.

On (say) $[0,2]$, taking $\delta = \frac{1}{N}$, we have

$$\begin{aligned} &\left|\sqrt{x+\frac{1}{N}}-\sqrt{x}\right| < \varepsilon \\ \implies &\left|\sqrt{x+\frac{1}{n}}-\sqrt{x}\right| < \varepsilon \ \forall n > N \\ \implies \sup_{x\in[0,1]} &\left\{\left|\sqrt{x+\frac{1}{n}}-\sqrt{x}\right|\right\} < \varepsilon \ \forall n > N \\ \implies &\left\|\sqrt{x+\frac{1}{n}}-\sqrt{x}\right\|_\infty < \varepsilon \ \forall n > N \end{aligned}$$

As $f_n \to \sqrt{x}$ uniformly, (f_n) is Cauchy in $(C^1[0,1], \|\cdot\|_*)$. However, $\sqrt{x} \notin C^1[0,1]$, as its derivative is not continuous at $x = 0$. Thus, $(C^1[a,b], \|\cdot\|_\infty)$ is not Banach.

□

Exercise 3.4. Let $\alpha \in [a,b]$. Then $C^1[a,b]$ is also Banach space with the norm $\|f\| := |f(\alpha)| + \|f'\|_\infty$. Is this norm and $\|\cdot\|_*$ (defined above) equivalent? Study the same for the spaces $C^k[a,b],\ k \in \mathbb{N}$.

Example 3.3 (Normed linear spaces that are not Banach spaces)**.** Consider the following examples:

1. $\left(c_{00}(\mathbb{N}), \|\cdot\|_p\right)$ for $1 \le p \le \infty$ is not a Banach space.

   Proof. We show this by constructing a Cauchy sequence in $c_{00}(\mathbb{N})$ which does not converge in $c_{00}(\mathbb{N})$.
   We know that the sequence $\left(\frac{1}{n^2}\right)$ is in $\ell^p(\mathbb{N})$ but not in $c_{00}(\mathbb{N})$ (because it is not eventually zero). Let $x = \left(\frac{1}{n^2}\right) \in \ell^p(\mathbb{N}) \setminus c_{00}(\mathbb{N})$.

   Let $x_n = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \dots, \frac{1}{n^2}, 0, 0, \dots\right) \in c_{00}(\mathbb{N}),\ n \in \mathbb{N}$

   Claim: (x_n) is Cauchy in $c_{00}(\mathbb{N})$.
   Let $n > m$. Then

   $$\|x_n - x_m\|_p^p = \sum_{k=m+1}^{n} \frac{1}{k^{2p}} \le \sum_{k=m+1}^{\infty} \frac{1}{k^{2p}} < \infty$$

   $\implies$ (x_n) is Cauchy.
   Assume for the sake of contradiction that $c_{00}(\mathbb{N})$ is Banach. Then $\exists y = (y_1, \dots, y_N, 0, 0, \dots) \in$

$c_{00}(\mathbb{N})$ such that $x_n \to y$ as $n \to \infty \ldots\ldots (*)$
Observe that:

$$
\begin{aligned}
\|x_n - y\|_p^p &= \sum_{k=1}^{\infty} \left| \frac{1}{k^2} - y_k \right|^p \\
&= \sum_{k=1}^{N} \left| \frac{1}{k^2} - y_k \right|^p + \sum_{k=N+1}^{\infty} \frac{1}{k^{2p}} \\
&\geq \sum_{k=N+1}^{\infty} \frac{1}{k^{2p}} \\
&\geq \frac{1}{(N+1)^{2p}} \ \forall n \geq N \\
\implies \|x_n - y\|_p &\geq \frac{1}{(N+1)^2} \ \forall n \geq N
\end{aligned}
$$

which is a contradiction to $(*)$.
Hence, $\nexists\, y \in c_{00}(\mathbb{N})$ such that $x_n \to y$ as $n \to \infty$
$\implies$ (x_n) is a Cauchy sequence in $c_{00}(\mathbb{N})$ which does not converge in $c_{00}(\mathbb{N})$. Thus, $\left(c_{00}(\mathbb{N}), \|\cdot\|_p\right)$ for $1 \leq p \leq \infty$ is not Banach. □

2. Consider $\mathcal{P}[a,b] = \{p : [a,b] \to \mathbb{F} \mid p$ is a polynomial over $\mathbb{F}\}$. Check that $(\mathcal{P}[a,b], +, \cdot)$ is a vector space over $\mathbb{F}$.
Let $1 \leq p \leq \infty$ and $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathcal{P}[a,b]$ $(x \in [a,b])$. Define

$$
\|f\|_p = \begin{cases} \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max\{|a_i| : 0 \leq i \leq n\}, & p = \infty \end{cases}
$$

Then $(\mathcal{P}[a,b], \|\cdot\|_p)$ is not a Banach space.

Remark 3.1. Note that $\mathscr{B} = \{1, x, x^2, \ldots, x^n, \ldots\} \subseteq \mathcal{P}[a,b]$ is a countable basis for $\mathcal{P}[a,b]$. Thus, by theorem 3.3 that follows, we can directly say that $(\mathcal{P}[a,b], \|\cdot\|_p)$ is not a Banach space.

3. $\exists$ a family of sequence spaces which are non-Banach spaces.

Exercise 3.5. Let $p < q$. We know that $\ell^p(\mathbb{N}) \subsetneq \ell^q(\mathbb{N})$. Show that $(\ell^p(\mathbb{N}), \|\cdot\|_q)$ is a normed linear space, which is not a Banach space.

Definition 3.4 (Nowhere dense set)**.** Let (X,d) be a metric space and Y be a subset of X. We say that Y is nowhere dense in X if the interior of closure of Y is empty, i.e. if $\left(\overline{Y}\right)^o = \phi$.

Theorem 3.2 (Baire's Category Theorem)**.** *A complete metric space cannot be written as a countable union of nowhere dense sets.*

Definition 3.5 (Denumerable set)**.** A set A is called a denumerable set if A is countably infinite.

Theorem 3.3. *A Banach space cannot have a denumerable basis. In other words, a Banach space cannot have a countably infinite basis.*

Proof. Let $(X, \|\cdot\|)$ be a Banach space and the dimension of X be countably infinite ($dim(X) = \aleph_o$).
Let $\mathscr{B} = \{x_1, \ldots, x_n, \ldots\}$ be a basis for X.
Let $X_n = span\{x_i : 1 \leq i \leq n\}$
Clearly, X_n is a proper closed subspace of X $\forall n \in \mathbb{N}$

Observe that:

1. $\left(\overline{X_n}\right)^o = (X_n)^o = \phi$ (by the contrapositive of lemma 2.15, as X_n is a proper closed subspace)
2. $X_n \subset X_{n+1}$ $(x_{n+1} \in X_{n+1} \setminus X_n)$
3. $X = \bigcup_{n=1}^{\infty} X_n$

$\implies$ X is a countable union of nowhere dense sets in X, which is a contradiction to Baire's Category Theorem since X is complete. Thus, X cannot have a countably infinite basis (or $dim(X)$ cannot be $\aleph_o$). □

Remark 3.2. The spaces in example 3.3 (1) and (2) have denumerable basis, so they are not Banach with any norm. In other words, $\nexists$ any norm $\|\cdot\|$ on $c_{00}(\mathbb{N})$ and $\mathcal{P}[a, b]$ such that they are Banach spaces.

4 Separability

Definition 4.1 (Dense set)**.** Let (X, d) be a metric space and $A \subseteq X$. We say that A is dense in X if $\overline{A} = X$, i.e. $\forall x \in X,\ \forall r > 0, B(x, r) \cap A \neq \phi$.

Lemma 4.1. *Let (X, d) be a metric space and $A \subseteq B \subseteq X$. Then*

1. $B^o = \phi \iff \overline{B^c} = X$.
2. *If $\overline{A} = X$, then $\overline{B} = X$.*
3. *If $\overline{B} = X$ and $B \subseteq \overline{A}$, then $\overline{A} = X$.*

Proof. 1. $(\implies)$ Let $B^o = \phi$. We show that $\overline{B^c} = X$. Clearly, $\overline{B^c} \subseteq X$.
Let $x \in X$. Then either $x \in B$ or $x \in X \setminus B = B^c$.
If $x \in B^c$, then $x \in \overline{B^c}$ ($\because B^c \subset \overline{B^c}$), and we are done.
If $x \in B$, then we know that x is not an interior point of B. This means that $\forall r > 0,\ B(x, r) \cap B^c \neq \phi$
$\implies$ x is a limit point of B^c
$\implies$ $x \in \overline{B^c}$
Thus, $\overline{B^c} = X$.

$(\impliedby)$ Let $\overline{B^c} = \overline{X \setminus B} = X$. Let $x \in B,$ and $\exists r > 0$ such that $B(x, r) \subset B \ldots\ldots (*)$
But $x \in B \implies x \in X$ ($\because B \subset X$) $\implies x \in \overline{X \setminus B}$. This means that either $x \in X \setminus B$, or x is a limit point of $X \setminus B$.
If $x \in X \setminus B$, then $B(x, r) \subset B$ (our assumption $(*)$)is not possible.
Hence, x is a limit point of $X \setminus B$. Then ever deleted neighborhood of x intersects $X \setminus B \implies B(x, r) \subset B$ is again not possible.

Hence, $(*)$ is not possible $\implies$ no point in B is its interior point $\implies B^o = \phi$.

2. Let $\overline{A} = X$. Clearly, we have $\overline{B} \subseteq X$.
Let $x \in X$. Then $x \in \overline{A}$. So, either $x \in A$ or x is a limit point of A.
If $x \in A$, then $x \in B \subseteq \overline{B} \implies x \in \overline{B}$ ($\because A \subseteq B \subseteq \overline{B}$)
Otherwise, x is a limit point of A. Then $\exists$ some $y(\neq x) \in B(x, r) \cap A\ \forall r > 0$
$\implies y \in B(x, r),\ y \in A \implies y \in B$ ($\because A \subseteq B$)
$\implies y \in B(x, r) \cap B\ \forall r > 0 \implies x$ is a limit point of B.
$\implies x \in \overline{B}$. Thus, $\overline{B} = X$.

3. Exercise.

□

Definition 4.2 (Separability)**.** Let (X, d) be a metric space. We say that X is separable if there exists a countable set A such that A is dense in X, i.e. $\overline{A} = X$.

Example 4.1 (Separable spaces)**.** Consider the following examples of separable spaces.

1. $(\mathbb{R}, |\cdot|)$ is a separable metric space.

   Proof. We show that $\mathbb{Q} \subset \mathbb{R}$ is a countable dense set in $\mathbb{R}$. We know (prove!) that $\mathbb{Q}$ is countable. Let $a, b \in \mathbb{R}, a < b$. Then $b - a > 0$. By the Archimedean property, $\exists n \in \mathbb{N}$ such that $n(b-a) > 1 \implies na + 1 < nb$.
   Let $m = [na]$. Then $na < m + 1 \leq na + 1 < nb$
   $\implies a < \frac{m+1}{n} \leq nb$
   $\implies \frac{m+1}{n} \in (a, b)$
   $\implies$ Between any two real numbers, we can find a rational. Thus, $\mathbb{Q}$ is dense in $\mathbb{R}$. □

2. $(\mathbb{R}^n, \|\cdot\|_2)$ is a separable normed linear space.

   Proof. We show that $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$. We have to show that $B(x, \varepsilon) \cap \mathbb{Q}^n \neq \phi$.
   Let $y = (y_1, \ldots, y_n) \in \mathbb{Q}^n$ such that $|x_i - y_i| < \frac{\varepsilon}{\sqrt{n}}\ \forall 1 \leq i \leq n$ (because of the density property of $\mathbb{Q}$ in $\mathbb{R}$).
   Consider
   $$\|x-y\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} < \left(\sum_{i=1}^{n} \frac{\varepsilon^2}{n}\right) = \frac{n^{1/2}}{n^{1/2}}\varepsilon = \varepsilon$$
   $\implies \|x - y\|_2 = \varepsilon \implies y \in B(x, \varepsilon) \cap \mathbb{Q}^n$ □

3. $(\mathbb{C}^n, \|\cdot\|_2)$ is separable. $(\mathbb{Q} + i\mathbb{Q})^n$ is countable and dense in $\mathbb{C}^n$.

4. $(\mathbb{F}^n, \|\cdot\|_2)$ is separable.

Lemma 4.2 (Continuous image of a separable space is separable)**.** *Let $(X, d_1), (Y, d_2)$ be two metric spaces. Let $f : X \to Y$ be a continuous function. If X is separable, then $(f(X), d_2)$ is separable.*

Proof. Let X be separable and $f : X \to Y$ be continuous. Clearly, $f : X \to f(X)$ is onto and continuous. Let A be countable an dense in X. Then $f(A)$ is also countable, f being onto.
Claim: $f(A)$ is dense in $f(X)$.
We show that $f(A) \cap V \neq \phi\ \forall$ non-empty open sets $V \subset f(X)$. Indeed, if $V(\neq \phi) \subset f(X)$ is an open set, then $\phi \neq f^{-1}(V) \subset X$ is open in X as f is continuous.
$\implies \exists a \in A$ such that $a \in f^{-1}(V)$ [as A is dense in X]
$\implies f(a) \in V \implies f(a) \in f(A) \cap V$
$\implies f(A)$ is countable and dense in $f(X)$. Thus, $f(X)$ is separable. □

Proposition 4.1. *Any finite dimensional normed linear space is separable.*

Proof. Let $(X, \|\cdot\|)$ be a given normed linear space with $dim\ X = n$. Consider the map $T : X \to \mathbb{F}^n$ as defined in lemma 2.26. T is a linear homeomorphism. We know that $(\mathbb{F}^n, \|\cdot\|_2)$ is separable. So, using lemma 4.2, $X = T^{-1}(\mathbb{F}^n)$ is separable since T^{-1} is continuous. □

Proposition 4.2. *$\ell^p(\mathbb{N})$ is separable for* $1 \leq p < \infty$.

Proof. Define $A = C_{00}^{\mathbb{Q}} = \{(x_n) \in C_{00} : x_n \in \mathbb{Q} + i\mathbb{Q},\ \forall n \in \mathbb{N}\}$. Clearly, A is a countable subset of $\ell^p(\mathbb{N})$. We will show that A is dense in $\ell^p(\mathbb{N})$.

Claim 1: $c_{00}(\mathbb{N})$ is dense in $\ell^p(\mathbb{N})$.
For any $\mathbf{x} = (x_n) \in \ell^p(\mathbb{N})$, we show that $\exists$ a $\mathbf{y} = (y_n) \in c_{00}(\mathbb{N})$ such that $y_n \to \mathbf{x}$ in ℓ^p-norm. Let $\mathbf{x} = (x_1, x_2, \ldots) \in \ell^p(\mathbb{N})$. Then we have that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Hence, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |x_n| < \infty$.
Define $\mathbf{y} \in c_{00}(\mathbb{N})$ by

$$(y_n) = \begin{cases} x_n, & n \leq N \\ 0, & n > N \end{cases}$$

That is, $\mathbf{y} = (x_1, x_2, \ldots, x_N, 0, 0, \ldots) \in c_{00}(\mathbb{N})$

Clearly, $\|\mathbf{x} - \mathbf{y}\|_p = \left(\sum_{n=N+1}^{\infty} |x_n|^p\right) < \varepsilon$. That is, $\mathbf{y} \in B(\mathbf{x}, \varepsilon) \cap c_{00}(\mathbb{N})$. Hence, $\overline{c_{00}(\mathbb{N})} = \ell^p(\mathbb{N})$.

However, $c_{00}(\mathbb{N})$ is uncountable.
Claim 2: A is countable and dense in $\ell^p(\mathbb{N})$.
Choose $z_1, \ldots, z_N \in \mathbb{Q} + i\mathbb{Q}$ such that $\forall i \in [1 : N], |z_i - y_i| < \frac{\varepsilon}{N^{1/p}}$ (since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{F}$).
Set $\mathbf{z} = (z_1, \ldots, z_N, 0, 0, \ldots) \in A$. We have

$$\|\mathbf{z} - \mathbf{y}\|_p = \left(\sum_{n=1}^{N} |z_i - y_i|^p\right)^{\frac{1}{p}} < \left(\frac{\varepsilon^p}{N} N\right)^{\frac{1}{p}} = \varepsilon$$

Hence, $\|\mathbf{x} - \mathbf{y}\|_p < \varepsilon$ and $\|\mathbf{z} - \mathbf{y}\|_p < \varepsilon \implies \|\mathbf{x} - \mathbf{z}\|_p < 2\varepsilon$
$\implies \mathbf{z} \in B(\mathbf{x}, 2\varepsilon) \cap A \neq \phi$
Thus, $\overline{A} = \ell^p(\mathbb{N})$. □

Theorem 4.1. *Let (X, d) be a metric space and $A = \{x_i : i \in \mathcal{I}\}$ be a subset of X, where $\mathcal{I}$ is an uncountable index set. Suppose $\exists r > 0$ such that*

$$d(x_i, x_j) \geq r\ \forall i, j \in \mathcal{I},\ i \neq j$$

Then (X, d) is non-separable. In other words, if X has an uncountable subset A and the *distance between each distinct elements in A is greater than a fixed positive number r, then X is non-separable.*

Proof. Assume that $\exists r > 0$ such that $d(x_i, x_j) \geq r \; \forall i, j \in \mathcal{I}, \; i \neq j \ldots\ldots (*)$
Let Y be a dense subset of X, i.e. $Y \subset X$ with $\overline{Y} = X$. We claim that Y is not countable.

Let $\mathcal{C} = \{B\left(x_i, \frac{r}{2}\right) : x_i \in A\}$. Clearly, $\mathcal{C}$ is uncountable because the balls are disjoint: $B\left(x_i, \frac{r}{2}\right) \cap B\left(x_j, \frac{r}{2}\right) = \phi \; \forall i, j \in \mathcal{I}, \; i \neq j$ by $(*)$.

But since Y is dense in X, we have $Y \cap B\left(x_i, \frac{r}{2}\right) \neq \phi \; \forall i \in \mathcal{I}$
$\implies$ Y intersects uncountably many disjoint balls $\implies$ Y is uncountable.

Thus, $\nexists$ any countable dense subset of X, whence X cannot be separable. □

Proposition 4.3. $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ *is not separable.*

Proof. We know that

$$A = \{(x_n) \in \ell^\infty(\mathbb{N}) : x_n = 0 \text{ or } 1 \; \forall n \in \mathbb{N}\}$$

is an uncountable subset of $\ell^\infty(\mathbb{N})$.
If $x, y \in A$ with $x \neq y$, then $\|x - y\|_\infty = \sup_i\{|x_i - y_i|\} = 1$. Hence, it follows by theorem 4.1 that $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ is not separable. □

Theorem 4.2 (Weierstrass Approximation Theorem)**.** *Let $f \in C[a, b]$ and $\varepsilon > 0$. Then $\exists$ a polynomial $p : [a, b] \to \mathbb{F}$ such that $\|p - f\|_\infty < \varepsilon$. That is, every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated as closely as desired by a polynomial function.*

Proposition 4.4. $(C[a, b], \|\cdot\|_\infty)$ *is separable.*

Proof. Let $\mathcal{P}[a, b] = \{p : [a, b] \to \mathbb{F} \mid p \text{ is a polynomial with coefficients in } \mathbb{F}\}$. Define

$$A = \mathcal{P}^{\mathbb{Q}+i\mathbb{Q}}[a, b] = \left\{ p(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{P}[a, b] : a_i \in \mathbb{Q} + i\mathbb{Q} \; \forall 1 \leq i \leq n, \; n \in \mathbb{N} \right\}$$

Claim: A is countable and $\overline{A} = C[a, b]$.
Observe that $A \simeq \bigcup_{n=1}^{\infty} (\mathbb{Q} + i\mathbb{Q})^n$. Hence, A is countable, being a countable union of countable sets.

Let $f \in C[a, b]$ and $\varepsilon > 0$. By Weierstrass Approximation Theorem, $\exists p(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{P}[a, b]$ such that $p \in B(f, \varepsilon)$. We need a $q(x) = \sum_{i=0}^{n} b_i x^i \in A$ (i.e., $b_i \in \mathbb{Q} + i\mathbb{Q} \; \forall i = [1 : n]$)

such that $\|p-q\|_\infty < \varepsilon$. Observe that

$$\begin{aligned}\|p-q\|_\infty &= \sup\left\{\left|\sum_{i=0}^{n}(a_i-b_i)x^i\right| : a \le x \le b\right\} \\ &\le \sup\left\{\sum_{i=0}^{n}|(a_i-b_i)|\left|x^i\right| : a \le x \le b\right\} \\ &\le M\left(\sum_{i=0}^{n}|a_i-b_i|\right)\end{aligned}$$

where $M = \max\{1, |x|, |x|^2, \ldots, |x|^n : x \in [a,b]\}$
Choose $b_i \in \mathbb{Q} + i\mathbb{Q}$ $(0 \le i \le n)$ such that

$$|a_i - b_i| < \frac{\varepsilon}{M(n+1)}$$

Let $q(x) = \sum_{i=0}^{n} b_i x^i$, $x \in [a,b]$. Clearly, $q \in A = \mathcal{P}^{\mathbb{Q}+i\mathbb{Q}}[a,b]$.
We have

$$\|p-q\|_\infty \le M\left(\sum_{i=0}^{n}|a_i-b_i|\right) < \frac{M(n+1)\varepsilon}{M(n+1)} = \varepsilon$$

Thus, $p \in B(f,\varepsilon)$ and $q \in B(p,\varepsilon) \implies q \in B(f, 2\varepsilon)$
$\implies B(f,2\varepsilon) \cap A \neq \phi$
$\implies \overline{A} = C[a,b] \implies C[a,b]$ is separable.

□

Theorem 4.3. *Let K be a compact Hausdorff space. Then $(C(K), \|\cdot\|_\infty)$ is a separable normed linear space if and only if K is separable. (Proof is not within the scope of these notes.)*

Proposition 4.5. *$(c_0(\mathbb{N}), \|\cdot\|_\infty)$ is separable.*

Proof. Recall from Sequence Spaces (section 2.1, page 18) that $C_0(N) = \{(x_n) : x_n \to 0 \text{ as } n \to \infty\}$. Define

$$A = \{(x_n) \in c_{00}(\mathbb{N}) : x_n \in \mathbb{Q} + i\mathbb{Q}\} \subseteq c_0(\mathbb{N})$$

Observe that

$$A = \bigcup_{k=1}^{\infty}\{(x_1, \ldots, x_k, 0, 0, \ldots) \in c_{00}(\mathbb{N}) : x_i \in \mathbb{Q} + i\mathbb{Q}\ \forall i\}$$

Hence, A is countable, being a countable union of countable sets. We will show that A is dense in C_0 w.r.t $\|\cdot\|_\infty$ by showing that $\forall \mathbf{y} = (y_n) \in c_0(\mathbb{N})$, $\exists \mathbf{x} = (x_n) \in A$ such that $\mathbf{x} \in B(\mathbf{y}, \varepsilon)$ for a given $\varepsilon > 0$.

Let $\mathbf{y} = (y_n) \in c_0(\mathbb{N})$. That is, $y_n \to 0$ as $n \to \infty$. That is, for given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|y_n\|_\infty < \infty\ \forall n > N$.
$\implies \sup\{|y_n| : n \in \mathbb{N}\} < \varepsilon\ \forall n > \mathbb{N}$
$\implies |y_n| < \varepsilon\ \forall n > N \ldots\ldots (*)$

As $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{F}$, for all $n \in \{1, 2, \ldots, N\}$, $\exists x_n \in \mathbb{Q} + i\mathbb{Q}$ such that $|x_n - y_n| < \varepsilon \ldots\ldots (**)$
Define $\mathbf{x} = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$. Clearly, $\mathbf{x} \in A$.

$$\begin{aligned}\|\mathbf{x} - \mathbf{y}\|_\infty &= \sup_{n \in \mathbb{N}}\{|x_n - y_n|\} \\ &= \sup\{|x_1 - y_1|, \ldots, |x_N - y_N|, |y_n| : n > N\} \\ &< \varepsilon \quad [\text{by } (*) \text{ and } (**)]\end{aligned}$$

$\implies \mathbf{x} \in B(\mathbf{y}, \varepsilon) \implies B(\mathbf{y}, \varepsilon) \cap A \neq \phi \implies \overline{A} = c_0(\mathbb{N})$. □

Proposition 4.6. $(c(\mathbb{N}), \|\cdot\|_\infty)$ *is separable.*

Proof. Recall from Sequence Spaces (section 2.1, page 18) that $c(\mathbb{N}) = \{(x_n) \in \mathcal{S}(\mathbb{N}) : (x_n) \text{ is convergent}\}$. Define

$$A = \bigcup_p \bigcup_{k=1}^{\infty} \{(x_1, x_2, \ldots, x_k, p, p, \ldots) : x_i, p \in \mathbb{Q} + i\mathbb{Q}\ \forall i\}$$

A is a union of eventually constant sequences, where each term of the sequence is in $\mathbb{Q}+i\mathbb{Q}$. Thus, A is a countable union of countable sets, and hence countable.

Claim: A is dense in $c(\mathbb{N})$ w.r.t $\|\cdot\|_\infty$.
Let $\mathbf{y} = (y_n) \in c(\mathbb{N})$. Then $\exists x \in \mathbb{F}$ such that $y_n \to x$ as $n \to \infty$.
$\implies$ For given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} \|y_n - x\|_\infty &< \frac{\varepsilon}{2}\ \forall n > N \\ \implies \sup\{|y_n - x|\} &< \frac{\varepsilon}{2}\ \forall n > N \\ \implies |y_n - x| &< \frac{\varepsilon}{2}\ \forall n > N \end{aligned}$$

Also, $\exists m \in \mathbb{Q} + i\mathbb{Q}$ such that $|m - x| < \frac{\varepsilon}{2}$, by the density of $\mathbb{Q} + i\mathbb{Q}$ in $\mathbb{F}$. Hence,

$$|y_n - m| \leq |y_n - x| + |x - m| < \varepsilon\ \forall n > N \ldots\ldots (*)$$

For each y_n $(n \leq N)$, $\exists x_n \in \mathbb{Q} + i\mathbb{Q}$ such that $|y_n - x_n| < \varepsilon$, $1 \leq n \leq N \ldots\ldots (**)$

Define $\alpha = (x_1, x_2, \ldots, x_N, m, m, \ldots)$. Clearly, $\alpha \in A$.

$$\begin{aligned}\|\mathbf{y} - \alpha\|_\infty &= \sup_{n \in \mathbb{N}}\{|y_n - \alpha_n|\} \\ &= \sup\{|y_1 - x_1|, \ldots, |y_N - x_N|, |y_n - m| : n > N\} \\ &< \varepsilon \ \text{ [by } (*) \text{ and } (**)]\end{aligned}$$

Thus, $\alpha \in B(\mathbf{y}, \varepsilon) \cap A \implies \overline{A} = c(\mathbb{N})$. $\square$

Proposition 4.7. *$c_{00}(\mathbb{N})$ is a proper dense subspace of $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$. Also, $c_{00}(\mathbb{N})$ is dense in $c_0(\mathbb{N})$ but not dense in $\ell^\infty(\mathbb{N})$ with respect to the sup norm.*

Proof. Let $1 \leq p < \infty$. From lemma 2.7, we know that $c_{00}(\mathbb{N}) \subsetneq \ell^p(\mathbb{N})$.
Let $\mathbf{x} = (x_n) \in \ell^p(\mathbb{N})$. Then $\sum_{n=1}^{\infty} |x_n|^p < \infty$.
That is, for given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |x_n|^p < \varepsilon$.
Let $\mathbf{y} = (x_1, x_2, \ldots, x_N, 0, 0, \ldots) \in c_{00}(\mathbb{N})$.

$$\|\mathbf{x} - \mathbf{y}\|_p = \sum_{n=N+1}^{\infty} |x_n|^p < \varepsilon$$

$\implies \mathbf{y} \in B(\mathbf{x}, \varepsilon) \cap c_{00}(\mathbb{N})$
$\implies \overline{c_{00}(\mathbb{N})} = \ell^p(\mathbb{N}) \implies c_{00}(\mathbb{N})$ is dense in $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$.

Now, consider the case for $p = \infty$. We show that $c_{00}(\mathbb{N})$ is dense in $c_0(\mathbb{N})$ w.r.t. $\|\cdot\|_\infty$, and $c_0(\mathbb{N}) \neq \ell^\infty(\mathbb{N})$, so that $(c_{00}(\mathbb{N}), \|\cdot\|_\infty)$ is not dense in $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$.
Let $\mathbf{x} = (x_n) \in c_0(\mathbb{N})$, i.e. $x_n \to 0$ as $n \to \infty$.
That is, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned}&\|x_n\|_\infty < \varepsilon \ \forall n > N \\ \implies &\sup\{|x_n|\} < \varepsilon \ \forall n > N \\ \implies &|x_n| < \varepsilon \ \forall n > N \ldots\ldots (*)\end{aligned}$$

Let $\mathbf{y} = (x_1, x_2, \ldots, x_N, 0, 0, \ldots) \in c_{00}(\mathbb{N})$. Then

$$\|\mathbf{x} - \mathbf{y}\|_\infty = \sup\{0, |x_{N+1}|, |x_{N+2}|, \ldots\} < \varepsilon \ \text{ [by } (*)]$$

Thus, $\mathbf{y} \in B(\mathbf{x}, \varepsilon) \cap c_{00}(\mathbb{N}) \implies \overline{c_{00}(\mathbb{N})} = c_0(\mathbb{N})$.

Now, $c_0(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N})$, because the sequence $(1, 1, \ldots) \in \ell^\infty(\mathbb{N})$, but $\notin c_0(\mathbb{N})$. Hence, $\overline{c_{00}(\mathbb{N})} = c_0(\mathbb{N}) \neq \ell^\infty(\mathbb{N})$. $\square$

Lemma 4.3. *(The space of direct sum of normed linear spaces) Let $(X_i, \|\cdot\|_i)$ be a family* *of non-zero normed linear spaces indexed by a set $\mathcal{I}$. Let $X := \left(\sum_i^{\oplus} X_i,\ \|\cdot\|\right)$ be a normed linear space, where $\|x\| := \sum_i \|x_i\|_i$. Then X is non-separable space iff $\mathcal{I}$ is uncountable or one of the $(X_i, ||\cdot||_i)$ is non-separable. Moreover, X is a Banach space iff each X_i is a Banach space.*

Now that the warm-up for the course is complete, let's dive into the heart of this journey — the world of Functional Analysis!

5 Bounded Linear Transformations

Recall the definition of a bounded set from definition 2.12.

Example 5.1 (Bounded sets)**.** Let $(X, \|\cdot\|)$ be a normed linear space. Then:

1. $B(x, r)$ (see definition 2.4) is a bounded set.

2. S_X (see definition 2.13) is a bounded set.

3. Every Cauchy and convergent sequence in X is a bounded set in X.

Definition 5.1 (Continuity, Uniform continuity, and Lipschitz continuity)**.** Let (X, d_1) and (Y, d_2) be two metric spaces. Let $f : X \to Y$ be a function. We say that

1. f is continuous at $x_0 \in X$ iff for each $\varepsilon > 0$, $\exists \delta = \delta(x_0, \varepsilon) > 0$ such that $d_2(f(x), f(x_0)) < \varepsilon$ whenever $d_1(x, x_0) < \delta$.

2. f is continuous on X if f is continuous at each point $x \in X$.

3. f is uniformly continuous if for each $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $d_2(f(x), f(y)) < \varepsilon$ whenever $d_1(x, y) < \delta \ \forall x, y \in X$.

4. f is Lipschitz continuous if $\exists k > 0$ such that $d_2(f(x), f(y)) \leq k\, d_1(x, y)\ x, y \in X$.

Remark 5.1. Lipschitz continuity $\implies$ uniform continuity $\implies$ continuity $\implies$ continuity at a single point x_0, but the converse need not be true.

Definition 5.2 (The space of all linear transformations)**.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces over $\mathbb{F}$. Define

$$L(X, Y) = \{T : X \to Y \mid T \text{ is a linear transformation}\}$$

Let $T, S \in L(X, Y)$, $\alpha \in \mathbb{F}$. Define

$$(T + S)(x) = T(x) + S(x)\ \forall x \in X$$
$$(\alpha T)(x) = \alpha T(x)\ \forall x \in X$$

Claim: $T + S,\ \alpha T \in L(X, Y)$
For $x, y \in X$ and $c \in \mathbb{F}$, $(T+S)(cx+y) = T(cx+y)+S(cx+y) = T(cx)+T(y)+S(cx)+S(y) = c(T(x) + S(x)) + T(y) + S(y) = c(T + S)(x) + (T + S)(y)$.
And for $\alpha \in \mathbb{F}$, $\forall x, y \in X$ and $c \in \mathbb{F}$, $(\alpha T)(cx + y) = \alpha \cdot T(cx + y) = \alpha(cT(x) + T(y)) = \alpha cT(x) + \alpha T(y) = c(\alpha(T(x) + T(y))) = c\alpha T(x + y) = c \cdot (\alpha T)(x + y)$.

Further, $L(X, Y)$ is a vector space over $\mathbb{F}$.

Theorem 5.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces over $\mathbb{F}$. Let $T : X \to Y$ be a linear transformation. Then the following are equivalent:*

1. *T is continuous at $x = 0$.*
2. *T is continuous on X.*
3. *T is uniformly continuous.*
4. *$\exists\, k > 0$ such that $\|Tx\|_Y \leq k\,\|x\|_X$ $\forall x \in X$.*
5. *T takes bounded sets in X to bounded sets in Y, i.e. if A is bounded in X, then $T(A) = \{T(a) : a \in A\}$ is a bounded set in Y.*

Proof. $[\mathbf{(1)} \iff \mathbf{(3)}]$

$(\impliedby)$ (3) $\implies$ (1) is trivial, by remark 5.1.

$(\implies)$ Assume that T is continuous at $x = 0$, i.e., for each $\varepsilon > 0$, $\exists\delta > 0$ such that $d_Y(T(x), T(0)) < \varepsilon$ whenever $d_X(x, 0) < \delta$. That is,

$$\|T(x)\|_Y < \varepsilon \text{ whenever } \|x\|_X < \delta \ldots\ldots (*)$$

Let $y \in X$. Note that $x - y \in X$ as X is a normed linear space. As $(*)$ holds $\forall x \in X$, it must hold for $x - y$. Hence,

$$\begin{aligned} &\|T(x-y)\|_Y < \varepsilon \text{ whenever } \|x-y\|_X < \delta \\ \implies &\|Tx - Ty\|_Y < \varepsilon \text{ whenever } \|x-y\|_X < \delta \end{aligned}$$

But as $y \in X$ was arbitrary, we have that

$$\|Tx - Ty\|_Y < \varepsilon \text{ whenever } \|x-y\|_X < \delta \;\forall x, y \in X$$

This means that T is uniformly continuous on X.

$[\mathbf{(1)} \iff \mathbf{(2)}]$

$(\impliedby)$ (2) $\implies$ (1) is clear by remark 5.1.

$(\implies)$ We have shown that if (1) holds, then T is uniformly continuous. Hence, using remark 5.1 again, we have that T is continuous.

$[\mathbf{(1)} \iff \mathbf{(4)}]$

$(\impliedby)$ (4) $\implies$ (1) is trivial by remark 5.1.

$(\implies)$ Assume that T is continuous at $x = 0$. That is,

$$\|T(x)\|_Y < \varepsilon \text{ whenever } \|x\|_X < \delta \ldots\ldots (*)$$

Let $x_0 \in X \setminus \{0\}$. Then $\frac{\delta x_0}{2\|x_0\|_X} \in X$ (since $\frac{\delta}{2\|x_0\|_X} \in \mathbb{F}$) and

$$\left\| \frac{\delta x_0}{2\left\|x_0\right\|_X} \right\| = \frac{\delta}{2\left\|x_0\right\|_X} \left\|x_0\right\|_X = \frac{\delta}{2} < \delta$$

From $(*)$, we have

$$\begin{aligned} \left\| T\left(\frac{\delta x_0}{2\left\|x_0\right\|_X}\right) \right\|_Y &< \varepsilon \\ \implies \left\| \frac{\delta}{2\left\|x_0\right\|_X} T(x_0) \right\|_Y &< \varepsilon \\ \implies \frac{\delta}{2\left\|x_0\right\|_X} \left\|T(x_0)\right\|_Y &< \varepsilon \\ \implies \left\|T(x_0)\right\|_Y &< \frac{2\varepsilon}{\delta} \left\|x_0\right\|_X \\ \implies \left\|Tx\right\|_Y &< k\ \left\|x\right\|_X \ \forall x \in X, \text{ where } k = \frac{2\varepsilon}{\delta} \end{aligned}$$

[(4) $\iff$ (5)]
($\implies$) Given, $\exists k > 0$ such that $\|Tx\|_Y \leq k\ \|x\|_X\ \forall x \in X$.
Let A be a bounded set in X. We show that $T(A) = \{T(a) : a \in A\}$ is a bounded set in Y.
$\exists M > 0$ such that $\|a\|_X \leq M\ \forall a \in A$, since A is bounded.
Then $\|Ta\|_Y \leq k\ \|a\|_X \leq kM$
$\implies \exists N = kM > 0$ such that $\|Ta\|_Y \leq N\ \forall a \in A$
$\implies T(A)$ is bounded in Y.

($\impliedby$) (Proof by contradiction) Suppose that T takes bounded sets in X to bounded sets in Y, and $\nexists\ k > 0$ such that $\|Tx\|_Y \leq k\ \|x\|_X\ \forall x \in X$.
$\implies$ For each $n \in \mathbb{N}$, $\exists x_n \in X \setminus \{0\}$ such that

$$\|Tx_n\|_Y > n \left\|x_n\right\|_X$$
$$\left\| T\left(\frac{x_n}{\left\|x_n\right\|_X}\right) \right\|_Y > n\ \forall n \in \mathbb{N} \ldots\ldots (*)$$

Let $A = \left\{ \frac{x_n}{\|x_n\|_X} : n \in \mathbb{N} \right\}$. Clearly, A is a bounded set in X.
$\implies T(A)$ is bounded in Y (by hypothesis).
$\implies \exists M > 0$ such that

$$\left\| T\left(\frac{x_n}{\left\|x_n\right\|_X}\right) \right\|_Y \leq M\ \forall n \in \mathbb{N}$$

which is a contradiction to $(*)$. Thus, $\exists k > 0$ such that $\|Tx\|_Y \leq k\ \|x\|_X\ \forall x \in X$.

□

Definition 5.3 (Bounded linear transformation)**.** A linear transformation $T : (X, \|\cdot\|_X) \to$

$(Y, \|\cdot\|_Y)$ is called a bounded linear transformation if $\exists\, k > 0$ such that

$$\|Tx\|_Y \leq k\ \|x\|_X\ \forall x \in X$$

Remark 5.2. A bounded linear transformation is also continuous, and a continuous linear transformation is bounded. This follows from theorem 5.1 [(2) $\iff$ (4)].

Remark 5.3. A non-zero bounded linear transformation is not a bounded function.

Example 5.2 (Bounded linear transformations)**.** The following are some examples of bounded linear transforms.

1. Let $(X, \|\cdot\|)$ be a normed linear space. Define the **zero transformation** as $\mathcal{O} : X \to X$ by $\mathcal{O}(x) = 0\ \forall x \in X$. Clearly, $\mathcal{O}$ is a linear map.
   Observe that $\|\mathcal{O}(x)\| = \|0\| = 0 \leq \|x\|\ \forall x \in X$.
   $\implies \mathcal{O}$ is a bounded linear transformation (as definition 5.3 is satisfied with $k = 1$).

2. Let $(X, \|\cdot\|)$ be a normed linear space. Define the **identity transformation** as $\mathcal{I} : X \to X$ by $\mathcal{I}(x) = x\ \forall x \in X$.
   Observe that $\|\mathcal{I}(x)\| = \|x\|\ \forall x \in X$.
   $\implies \mathcal{I}$ is a bounded linear transformation (as definition 5.3 is satisfied for any $k \geq 1$).

3. For $n > m$, define $T : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ by

   $$T\left((x_1, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_n)\right) = (x_1, \ldots, x_m)\ \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$$

   Clearly, T is a linear transformation. Observe that

   $$\begin{aligned}\|T((x_1, \ldots, x_n))\|_2 &= \|(x_1, \ldots, x_m)\|_2 \\ &= \left(\sum_{i=1}^{m} |x_i|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}\quad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n \\ &= \|(x_1, \ldots, x_n)\|_2\ \forall (x_1, \ldots, x_n) \in \mathbb{R}^n\end{aligned}$$

   Hence, T is a bounded linear transformation.

4. Let $1 \leq p < q \leq \infty$. Define $T : (\mathbb{F}^n, \|\cdot\|_p) \to (\mathbb{F}^n, \|\cdot\|_q)$ by $T((x_1, \ldots, x_n)) = (x_1, \ldots, x_n)\ \forall (x_1, \ldots, x_n) \in \mathbb{F}^n$. Then

   $$\|Tx\|_q = \|x\|_q \leq \|x\|_p\ \forall x = (x_1, \ldots, x_n) \in \mathbb{F}^n$$

$\implies T$ is a bounded linear transformation.
Similarly, one can show that $T : (\ell^p(\mathbb{N}), \|\cdot\|_p) \to (\ell^p(\mathbb{N}), \|\cdot\|_q)$ given by $(x_n) \mapsto (x_n)\ \forall (x_n) \in \ell^p(\mathbb{N})$ is a bounded linear transformation.

5. Let $1 \leq p < q \leq \infty$. Define $T : (\ell^p(\mathbb{N}), \|\cdot\|_q) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)$ by $T((x_n)) = (x_n)\ \forall (x_n) \in \ell^p(\mathbb{N})$. We show that T is **not** a bounded linear transformation.

   Clearly, T is linear. We know from lemma 2.23 ($\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent) that $\nexists k > 0$ such that $\|(x_n)\|_p \leq k\ \|(x_n)\|_q$
   That is, $\nexists k > 0$ such that $\|T((x_n))\|_p \leq k\ \|(x_n)\|_q\ \forall (x_n) \in \ell^p(\mathbb{N})$.
   Hence, T is not a bounded linear transformation.

6. Let $a = (a_n) \in \ell^\infty(\mathbb{N})$ and $1 \leq p \leq \infty$. Define the **multiplication operator** $M_a : (\ell^p(\mathbb{N}), \|\cdot\|_p) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)$ by $M_a((x_n)) = (a_n x_n)\ \forall (x_n) \in \ell^p(\mathbb{N})$.
   Clearly, M_a is a linear transformation. We show that M_a is bounded.

   Case 1: $1 \leq p < \infty$

$$\begin{aligned}
\|M_a((x_n))\|_p &= \|(a_n x_n)\|_p \\
&= \left(\sum_{n=1}^{\infty} |a_n x_n|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{\infty} K^p |x_n|^p\right)^{\frac{1}{p}} \quad (\text{where } K = \|a\|_\infty\,, \text{ so that } |a_n| \leq K\ \forall n) \\
&= K\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \\
&= K\,\|(x_n)\|_p
\end{aligned}$$

   Thus, $\|M_a((x_n))\|_p \leq \|a\|_\infty \|(x_n)\|_p\ \forall (x_n) \in \ell^p(\mathbb{N})$.

   Case 2: $p = \infty$

$$\begin{aligned}
\|M_a((x_n))\|_\infty &= \|(a_n x_n)\|_\infty \\
&= \sup_{n\in\mathbb{N}}\{|a_n x_n|\} \\
&= \sup_{n\in\mathbb{N}}\{|a_n||x_n|\} \\
&\le \sup_{n\in\mathbb{N}}\{\|a\|_\infty |x_n|\} \\
&= \|a\|_\infty \sup_{n\in\mathbb{N}}\{|x_n|\} \\
&= \|a\|_\infty \|(x_n)\|_\infty
\end{aligned}$$

7. Let $1 \le p \le \infty$. Define the **right-shift operator** by $R : (\ell^p(\mathbb{N}), \|\cdot\|_p) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)$ by $R((x_1, x_2, \dots)) = (0, x_1, x_2, \dots)\ \forall (x_n) \in \ell^p(\mathbb{N})$.
   Clearly, R is a linear transformation. Observe that

   $$\|R((x_n))\|_p = \|(0, x_1, x_2, \dots)\|_p = \|(x_n)\|_p \ \forall (x_n) \in \ell^p(\mathbb{N})$$

   Thus, R is a bounded linear transformation.

   Remark 5.4. The right-shift operator R is one-one but not onto.

8. Let $1 \le p \le \infty$. Define the **left-shift operator** by $L : (\ell^p(\mathbb{N}), \|\cdot\|_p) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)$ by $L((x_1, x_2, \dots)) = (x_2, x_3, \dots)\ \forall (x_n) \in \ell^p(\mathbb{N})$.
   Clearly, L is a linear transformation. Observe:

   $$\|L((x_n))\|_p = \|(x_2, x_3, \dots)\|_p \le \|(x_n)\|_p \ \forall (x_n) \in \ell^p(\mathbb{N})$$

   Hence, L is bounded.

   Remark 5.5. The left-shift operator L is onto, but not one-to-one.

9. Let $x_0 \in [a, b]$. Define $T_{x_0} : (C[a, b], \|\cdot\|_\infty) \to (\mathbb{F}, |\cdot|)$ by $T_{x_0}(f) = f(x_0)\ \forall f \in C[a, b]$. Check that T_{x_0} is a linear transformation. Observe that

   $$\|T_{x_0}(f)\| = |f(x_0)| \le \|f\|_\infty = \sup_{x\in[a,b]}\{|f(x)|\}\ \forall f \in C[a, b]$$

   Thus, T_{x_0} is a bounded linear transformation, as definition 5.3 is satisfied with $k = 1$.

10. Define $T : (C[a, b], \|\cdot\|_\infty) \to (C[a, b], \|\cdot\|_\infty)$ by

    $$T(f(x)) = \int_a^x f(t)dt\ \forall f \in C[a, b]$$

Clearly, T is a linear transformation. Observe that

$$\begin{aligned}
|T(f(x))| \leq \int_a^x |f(t)|dt \leq \int_a^b |f(t)|dt \ \forall x \in [a,b] \\
\implies \sup_{x\in[a,b]} |Tf(x)| \leq \int_a^b |f(t)|dt \ \forall f \in C[a,b] \\
\leq \|f\|_\infty \int_a^b 1dt \ (\because |f(t)| \leq \|f\|_\infty) \\
\leq \|f\|_\infty (b-a) \\
\implies \|Tf\|_\infty \leq (b-a)\|f\|_\infty \ \forall f \in C[a,b]
\end{aligned}$$

So, T is bounded as definition 5.3 is satisfied with $k = b-a$.

11. Let $1 \leq p \leq \infty$ and $N \in \mathbb{N}$. Define $T : \ell^p(\mathbb{N}) \to \mathbb{F}$ by $T((x_n)) = x_N \ \forall (x_n) \in \ell^p(\mathbb{N})$. Check that T is a linear transformation. Observe that

$$\|T((x_n))\| = |T((x_n))| = |x_N| \leq 1 \cdot \|(x_n)\|_p \ \forall (x_n) \in \ell^p(\mathbb{N})$$

$\implies$ T is a bounded linear transformation.

Definition 5.4 (Space of all bounded linear transformations)**.** Let X and Y be normed linear spaces. Define

$$\mathcal{B}(X,Y) = \{T \in L(X,Y) : T \text{ is a bounded linear transformation}\}$$

where $L(X,Y)$ is the set of all linear transformations from X to Y (see definition 5.2). Also,

$$\mathcal{B}(X) = \mathcal{B}(X,X) = \{T : X \to X \mid T \text{ is a bounded linear operator}\}$$

Remark 5.6. $\mathcal{B}(X,Y) \neq \phi$, as $\mathcal{O} \in \mathcal{B}(X,Y)$ (see example 5.2 (1)).

Definition 5.5 (Operator norm)**.** Let $T \in \mathcal{B}(X,Y)$, i.e. T is a bounded linear transformation from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$. The operator norm of T is defined as

$$\|T\| = \sup\{\|T(x)\|_Y : x \in S_X\} = \sup\{\|T(x)\|_Y : x \in X,\ \|x\|_X = 1\}$$

Often times, the operator norm is denoted by $\|\cdot\|_{op}$.

Remark 5.7. For the sake of simplicity, we shall simply write $\|\cdot\|$ to represent the norms when the presence of X and Y in the subscript is contextually clear. Hence, the operator norm can be defined as $\|T\| = \sup\{\|T(x)\| : x \in S_X\}$.

Lemma 5.1. *If $T \in \mathcal{B}(X,Y)$, then*

$$\|Tx\|_Y \leq \|T\|\,\|x\|_X \;\; \forall x \in X$$

Following remark 5.7, we have that if T is a bounded linear transformation from X to Y, then

$$\|Tx\| \leq \|T\|\|x\| \; \forall x \in X$$

Proof. By definition, $\|T\| = \sup\{\|Tx\| : x \in S_X\}$.

$$\begin{aligned}
&\implies \|Tx\| \leq \|T\| \text{ (the supremum) } \forall x \in S_X \\
&\implies \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \|T\| \; \forall x \in X \; \left(\because \left\| \frac{x}{\|x\|} \right\| = 1, \implies \frac{x}{\|x\|} \in S_X \; \forall x \in X \right) \\
&\implies \frac{1}{\|x\|}\|Tx\| \leq \|T\| \; \forall x \in X \\
&\implies \|Tx\| \leq \|T\|\|x\| \; \forall x \in X
\end{aligned}$$

□

Proposition 5.1. *Let $T, S \in \mathcal{B}(X,Y)$, $\alpha \in \mathbb{F}$. Define*

$$\begin{aligned}
(T+S)(x) &= T(x) + S(x) \; \forall x \in X \\
(\alpha T)(x) &= \alpha \cdot T(x) \;\; \forall x \in X
\end{aligned}$$

Then $T+S,\ \alpha T \in \mathcal{B}(X,Y)$

Proof.

$$\begin{aligned}
\|(T+S)(x)\| = \|T(x) + S(x)\| &\leq \|Tx\| + \|Sx\| \\
&\leq \|T\|\|x\| + \|S\|\|x\| \text{ (by lemma 5.1) } \forall x \in X \\
&= (\|T\| + \|S\|)\|x\| \; \forall x \in X
\end{aligned}$$

Thus, $T+S \in \mathcal{B}(X,Y)$, as definition 5.3 is satisfied with $k = \|T\| + \|S\|$.

$$\begin{aligned}
\|(\alpha T)(x)\| = \|\alpha T(x)\| &= |\alpha|\|Tx\| \\
&\leq |\alpha|\|T\|\|x\| \text{ (by lemma 5.1) } \forall x \in X
\end{aligned}$$

$\implies \alpha T \in \mathcal{B}(X,Y)$ as definition 5.3 is satisfied with $k = |\alpha|\|T\|$. □

Exercise 5.1. Check that $(\mathcal{B}(X,Y), +, \cdot)$ (with addition $+$ and scalar multiplication $\cdot$ as defined above in proposition 5.1) is a vector space.

Lemma 5.2 (The operator norm is a norm on $\mathcal{B}(X,Y)$). *Define*

$$\|\cdot\| : \mathcal{B}(X,Y) \to [0,\infty) \text{ by}$$
$$T \mapsto \|T\| = \sup\{\|Tx\| : x \in S_X\},\ \forall T \in \mathcal{B}(X,Y)$$

Then the operator norm $\|\cdot\|$ *forms a norm on* $\mathcal{B}(X,Y)$*, i.e.* $(\mathcal{B}(X,Y), \|\cdot\|)$ *is a normed linear space.*

Proof. We know that $\|\cdot\|$ is a well-defined map.

1. **Claim:** $\|T\| = 0 \iff T \equiv 0$
   $(\Longleftarrow)$ If $T \equiv 0$, then $T(x) = 0\ \forall x \in X$
   $\implies \|Tx\| = 0\ \forall x \in X$
   $\implies \|T\| = \sup\{\|Tx\| : x \in S_X\} = 0$

   $(\Longrightarrow)$ If $\|T\| = \sup\{\|Tx\|_Y : x \in S_X\} = 0$,
   $\implies \|Tx\|_Y = 0\ \forall x \in S_X$
   $\implies Tx = 0\ \forall x \in S_X$ (as $\|\cdot\|_Y$ is a norm on Y)
   $\implies T\left(\frac{x}{\|x\|}\right) = 0\ \forall x \in X$
   $\implies Tx = 0\ \forall x \in X$ (as X is a normed linear space and $\frac{1}{\|x\|}$ is a scalar)
   $\implies T \equiv 0$

2. Let $\alpha \in \mathbb{F}$. Then

$$\begin{aligned}
\|\alpha T\| &= \sup\{\|(\alpha T)(x)\| : x \in S_X\} \\
&= \sup\{\|\alpha \cdot T(x)\| : x \in S_X\} \\
&= \sup\{|\alpha|\|Tx\| : x \in S_X\} \\
&= |\alpha| \sup\{\|Tx\| : x \in S_X\} \\
&= |\alpha|\|T\|
\end{aligned}$$

3. Let $T, S \in \mathcal{B}(X,Y)$. We know from proposition 5.1 that $T + S \in \mathcal{B}(X+Y)$. Then

$$\begin{aligned}
\|T+S\| &= \sup\{\|(T+S)(x)\| : x \in S_X\} \\
&= \sup\{\|Tx + Sx\| : x \in S_X\} \\
&\leq \sup\{\|Tx\| + \|Sx\| : x \in S_X\} \\
&\leq \sup\{\|Tx\| : x \in S_X\} + \sup\{\|Sx\| : x \in S_X\} \\
&= \|T\| + \|S\|
\end{aligned}$$

□

Example 5.3 (Calculating operator norms)**.** Let $T \in \mathcal{B}(X, Y)$. Then $\|T\| = \sup\{\|Tx\| : x \in S_X\}$.

There are two ways to calculate $\|T\|$:

First, try to find an optimal upper bound M for $\|T\|$, i.e. $\|T\| \leq M$.

- Then, if possible, choose $x_0 \in S_X$ such that $\|Tx_0\|_Y = M$, i.e. the upper bound M is attained at $x_0 \in S_X$. Then $\|T\| = M$.
- If such an $x_0 \in S_X$ does not exist, then the supremum is not attained. Then find a sequence $(x_n) \subseteq S_X$ (i.e., a sequence (x_n) in X with $\|x_n\|_X = 1\ \forall n \in \mathbb{N}$) such that $\|Tx_n\|_Y \to M$ as $n \to \infty$. Then $\|T\| = \lim_{n\to\infty} \|Tx_n\| = M$.

Let us calculate some operator norms now. Some of the bounded linear transformations below are already defined in example 5.2 (page 65).

1. Let $(X, \|\cdot\|)$ be a normed linear space. Define $\mathcal{O} : X \to X$ by $\mathcal{O}(x) = 0;\ \forall x \in X$. We have seen that $\mathcal{O}$ (the zero transformation) is bounded. Now,

$$\begin{aligned}
\|\mathcal{O}(x)\| = \|0\| = 0 &\leq \frac{1}{n}\|x\|\ \forall n \in \mathbb{N} \\
\implies \frac{1}{\|x\|}\|\mathcal{O}(x)\| &\leq \frac{1}{n}\ \forall n \in \mathbb{N} \\
\implies \left\|\mathcal{O}\left(\frac{x}{\|x\|}\right)\right\| &\leq \frac{1}{n}\ \forall n \in \mathbb{N},\ \forall x \in X \setminus \{0\} \\
\implies \|\mathcal{O}(x)\| &\leq \frac{1}{n}\ \forall n \in \mathbb{N},\ \forall x \in S_X \\
\implies \sup\{\|\mathcal{O}(x)\| : x \in S_X\} &\leq \frac{1}{n}\ \forall n \in \mathbb{N} \\
\implies \|\mathcal{O}\| &\leq \frac{1}{n}\ \forall n \in \mathbb{N} \\
\implies \|\mathcal{O}\| &= 0
\end{aligned}$$

2. Define $\mathcal{I} : X \to X$ by $\mathcal{I}(x) = x\ \forall x \in X$. We know that $\mathcal{I} \in \mathcal{B}(X, Y)$. Now,

$$\begin{aligned}
\|\mathcal{I}(x)\| &= \|x\|\ \forall x \in X \\
\implies \left\|\mathcal{I}\left(\frac{x}{\|x\|}\right)\right\| &= 1\ \forall x \in X \setminus \{0\} \\
\implies \|\mathcal{I}(x)\| &= 1\ \forall x \in S_X \\
\implies \sup\{\|\mathcal{I}(x)\| : x \in S_X\} &= 1 \\
\implies \|\mathcal{I}\| &= 1
\end{aligned}$$

3. Define the right-shift operator as $R : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ by

$$R((x_1, x_2, \dots)) = (0, x_1, x_2, \dots), \forall (x_1, x_2, \dots) \in \ell^p(\mathbb{N}).$$

We know that $R \in \mathcal{B}(\ell^p(\mathbb{N}))$. Now for $x = (x_1, x_2, \ldots) \in \ell^p(\mathbb{N})$,

$$\begin{aligned}
\|Rx\|_p &= \|x\|_p \ \forall x \in \ell^p(\mathbb{N}) \\
\implies \left\| R\left(\frac{x}{\|x\|}\right) \right\|_p &= 1 \ \forall x \in X \setminus \{0\} \\
\implies \|Rx\|_p &= 1 \ \forall x \in S_X \\
\implies \sup\{\|Rx\|_p : x \in S_X\} &= 1 \\
\implies \|R\| &= 1
\end{aligned}$$

4. Define the left-shift operator as

$$L : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}) \text{ by } L((x_1, x_2, \ldots)) = (x_2, x_3 \ldots) \ \forall (x_1, x_2, \ldots) \in \ell^p(\mathbb{N}).$$

We know that $T \in \mathcal{B}(\ell^p(\mathbb{N}))$ and

$$\begin{aligned}
\|L(x)\|_p &\leq \|x\|_p \ \forall x = (x_n) \in \ell^p(\mathbb{N}) \\
\left\| L\left(\frac{x}{\|x\|}\right) \right\|_p &\leq 1 \ \forall x \in \ell^p(\mathbb{N}) \setminus \{0\} \\
\implies \|Lx\|_p &\leq 1 \ \forall x \in S_{\ell^p(\mathbb{N})} \\
\implies \sup\{\|Lx\|_p : x \in S_{\ell^p(\mathbb{N})}\} &\leq 1 \\
\implies \|L\| &\leq 1
\end{aligned}$$

Note that we have obtained an upper bound 1 for $\|L\|$. Now, we will try to find an element $(x_n) \in S_{\ell^p(\mathbb{N})}$ such that the upper bound 1 is attained, i.e. $\|L(x_n)\|_p = 1$.

Observe that $e_2 = (0, 1, 0, 0, \ldots) \in \ell^p(\mathbb{N})$ and $\|e_2\|_p = 1$. So, $e_2 \in S_{\ell^p(\mathbb{N})}$. Note that

$$\|L(e_2)\|_p = \|L((0, 1, 0, 0, \ldots))\|_p = \|(1, 0, 0, \ldots)\|_p = \|e_1\|_p$$

Thus, $\|L(e_2)\|_p = \|e_1\|_p = 1$
$\implies \|L\| = 1$.

5. Let $a = (a_n) \in \ell^\infty(\mathbb{N})$ and $1 \leq p \leq \infty$. Define the multiplication operator

$$M_a : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}) \text{ by } M_a((x_n)) = (a_n x_n) \ \forall (x_n) \in \ell^p(\mathbb{N}).$$

We know that $M_a \in \mathcal{B}(\ell^p(\mathbb{N}))$. Let $x = (x_n) \in \ell^p(\mathbb{N})$. Now,

$$\|M_a(x)\|_p = \|(a_n x_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n x_n|^p\right)^{\frac{1}{p}} \leq \|(a_n)\|_\infty \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} = \|a\|_\infty \|x\|_p$$

$$\begin{aligned} &\implies \|M_a(x)\|_p \leq \|a\|_\infty \|x\|_p \ \forall x \in \ell^p(\mathbb{N}) \\ &\implies \left\| M_a\left(\frac{x}{\|x\|}\right) \right\|_p \leq \|a\|_\infty \ \forall x \in \ell^p(\mathbb{N}) \\ &\implies \|M_a(x)\|_p \leq \|a\|_\infty \ \forall x \in S_{\ell^p(\mathbb{N})} \\ &\implies \|M_a\| \leq \|a\|_\infty \ldots\ldots (*) \end{aligned}$$

Claim: $\|M_a\| = \|a\|_\infty$

Observe that $e_i = (0, \ldots, 0, 1, 0, \ldots) \in \ell^p(\mathbb{N})$ (where the 1 is in the i^{th} position), and $\|e_i\|_p = 1 \ \forall i \in \mathbb{N}$. Hence, $e_i \in S_{\ell^p(\mathbb{N})} \ \forall i \in \mathbb{N}$ and $M_a(e_i) = (0, \ldots, 0, a_i, 0, \ldots)$, where a_i appears in the i^{th} position. Thus,

$$\begin{aligned} &\|M_a(e_i)\|_p = |a_i| \\ &\implies |a_i| \leq \|M_a\| \ \forall i \in \mathbb{N} \text{ (since } \|M_a\| \text{ is the supremum)} \\ &\implies \sup_{i \in \mathbb{N}} |a_i| \leq \|M_a\| \\ &\implies \|(a_n)\|_\infty = \|a\|_\infty \leq \|M_a\| \ldots\ldots (**) \end{aligned}$$

From $(*)$ and $(**)$, it follows that $\|M_a\| = \|a\|_\infty$.

6. Fix $t_0 \in [a, b]$. Define $f : (C[a, b], \|\cdot\|_\infty) \to (\mathbb{F}, |\cdot|)$ by $f(x) = x(t_0) \ \forall x \in C[a, b]$. We know that f is bounded, because

$$\begin{aligned} &|f(x)| = |x(t_0)| \leq \|x\|_\infty \ \forall x \in C[a, b] \ (\implies f \text{ is bounded}) \\ &\implies \left| f\left(\frac{x}{\|x\|}\right) \right| \leq 1 \ \forall x \in C[a, b] \\ &\implies \|f\| = 1 \end{aligned}$$

Claim: $\|f\| = 1$

Let $x(t) = 1 \ \forall t \in [a, b]$. Then $\|x\|_\infty = 1 \implies x \in S_{C[a,b]}$. Also, $|f(x)| = |x(t_0)| = 1$. Hence, $\|f\| = 1$.

7. Define $f : (C[a, b], \|\cdot\|_\infty) \to (\mathbb{F}, |\cdot|)$ by $f(x) = \int_a^b x(t)dt$. We know that f is bounded, because

$$|f(x)| = \left| \int_a^b x(t)dt \right| \leq \int_a^b |x(t)|dt \leq \|x\|_\infty \int_a^b dt = (b - a) \|x\|_\infty \ \forall x \in C[a, b]$$

Now,

$$\begin{aligned} |f(x)| &\leq (b-a)\,\|x\|_\infty \;\forall x \in C[a,b] \\ \implies \left| f\left(\frac{x}{\|x\|}\right)\right| &\leq b-a \;\forall x \in C[a,b] \\ \implies |f(x)| &\leq b-a \;\forall x \in S_{C[a,b]} \\ \implies \|f\| &\leq b-a \end{aligned}$$

Claim: $\|f\| = b-a$
Let $x(t) = 1 \;\forall t \in [a,b]$
Then $\|x\|_\infty = 1$, or $x \in S_{C[a,b]}$. Also, $|f(x)| = \left|\int_a^b 1dt\right| = b-a$
Thus, $\|f\| = b-a$.

8. Define $T : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ by

$$(T(x))(t) = \int_a^t x(s)ds \;\forall x \in C[a,b],\; \forall t \in [a,b]$$

Check that T is a linear transformation.

$$|(T(x))(t)| = \left|\int_a^t x(s)ds\right| \leq \int_a^t |x(s)|ds \leq \|x\|_\infty (b-a) \;\forall x \in C[a,b],\; \forall t \in [a,b]$$

$\implies$ T is a bounded linear transformation. Further,

$$\begin{aligned} |(T(x))(t)| &\leq (b-a)\,\|x\|_\infty \;\forall x \in C[a,b],\; \forall t \in [a,b] \\ \implies \|T(x)\|_\infty &\leq (b-a)\,\|x\|_\infty \;\forall x \in C[a,b] \\ \implies \|T(x)\|_\infty &\leq b-a \;\forall x \in S_{C[a,b]} \\ \implies \|T\| &\leq b-a \end{aligned}$$

Claim: $\|T\| = b-a$
Let $x(t) = 1 \;\forall t \in [a,b]$. Then $\|x\|_\infty = 1$ and $(T(x))(t) = t-a \implies \|Tx\|_\infty = b-a$
$\implies \|T\| = b-a$.

9. Let $C^1[0,1]$ be the set of all $\mathbb{F}$-valued continuously differentiable functions over $[0,1]$ and $\|\cdot\|_*$ be the C^1-norm (see example 3.2). Define $T : (C^1[0,1], \|\cdot\|_*) \to (C[0,1], \|\cdot\|_\infty)$ by $Tf = f' \;\forall f \in C^1[0,1]$. Check that T is a linear transformation. T is bounded, because

$$\|Tf\|_\infty = \|f'\|_\infty \leq \|f\|_\infty + \|f'\|_\infty = \|f\|_* \;\forall f \in C^1[0,1]$$

Also,

$$
\begin{aligned}
\|Tf\|_\infty &\leq \|f\|_* \ \forall f \in C^1[0,1] \\
\implies \left\| T\left(\frac{f}{\|f\|}\right) \right\| &\leq 1 \ \forall f \in C^1[0,1] \\
\implies \|T(f)\| &\leq 1 \ \forall f \in S_{C^1[0,1]} \\
\implies \|T\| &\leq 1 \ (\|T\| = \sup\{\|Tf\|_\infty : \|f\|_* = 1\})
\end{aligned}
$$

Claim: $\|T\| = 1$
By remark 5.8 that follows, $\nexists$ any $f \in C^1[0,1]$ such that $\|f\|_* = 1$ and $\|Tf\|_\infty = 1$. So the supremum 1 is not attained by any $f \in S_{C^1[0,1]}$.
Let $x_n(t) = \frac{t^n}{n+1} \ \forall t \in [0,1]$
Observe that

$$\|x_n\|_* = \|x_n\|_\infty + \|x_n'\|_\infty = \frac{1}{n+1} + \frac{n}{n+1} = 1$$

$\implies x_n \in S_{C^1[0,1]}$, and

$$\|T(x_n)\|_\infty = \|x_n'\|_\infty = \frac{n}{n+1} \to 1 \text{ as } n \to \infty$$

Thus, $\|T\| = 1$.

Remark 5.8. Define $T : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ by $(T(x))(t) = \int_a^t x(s)ds \ \forall x \in C[a,b]$, $\forall t \in [a,b]$. Then $\nexists$ any $f \in C^1[0,1]$ such that $\|f\|_* = 1$ and $\|Tf\|_\infty = 1$.

Proof. Let $\exists\ f \in C^1[0,1]$ such that $\|f\|_* = 1$ and $\|Tf\|_\infty = \|f'\|_\infty = 1$.

$$
\begin{aligned}
&\implies \|f\|_\infty + \|f'\|_\infty = 1 \\
&\implies \|f\|_\infty = 0 \ (\text{as } \|f'\|_\infty = 1) \\
&\implies f \equiv 0 \\
&\implies f' = 0,
\end{aligned}
$$

which is a contradiction to $\|f'\|_\infty = 1$. □

Proposition 5.2 (Some equivalent definitions of the operator norm)**.** *Let* $T \in \mathcal{B}(X,Y)$*.*

Then

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : x \in S_X,\ \textit{i.e. } \|x\| = 1\} \textit{ (our definition)}\\ &= \inf\{k > 0 : \|Tx\| \le k\|x\|\ \forall x \in X\}\\ &= \sup\{\|Tx\| : x \in B(0,1),\ \textit{i.e. } \|x\| < 1\}\\ &= \sup\{\|Tx\| : x \in \overline{B}(0,1),\ \textit{i.e. } \|x\| \le 1\}\end{aligned}$$

Proof. If $X = \{0\}$, then there is nothing to prove. Let $X \neq \{0\}$. Let

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| = 1\}\\ \alpha &= \inf\{k > 0 : \|Tx\| \le k\|x\|\ \forall x \in X\}\\ \beta &= \sup\{\|Tx\| : \|x\| \le 1\}\\ \gamma &= \sup\{\|Tx\| : \|x\| < 1\}\end{aligned}$$

We know that $\{\|Tx\| : \|x\| \le 1\}$ is a bigger set than $\{\|Tx\| : \|x\| = 1\}$ and $\{\|Tx\| : \|x\| < 1\}$. Thus,

$$\|T\| \le \beta \text{ and } \gamma \le \beta \ldots\ldots (*)$$

As T is bounded, from lemma 5.1, we know that $\|Tx\| \le \|T\|\|x\|\ \forall x \in X$. So, by definition, we have $\alpha \le \|T\|$ (since α is the infimum of all such $k > 0$ and $k = \|T\|$ is one such possibility).

Consider $x \neq 0 \in X$ and $0 < r < 1$. As T is linear,

$$\|T(x)\| = \left\|T\left(\frac{rx}{\|x\|}\right)\right\|\frac{\|x\|}{r} \le \sup\{\|T(z)\| : z \in X, \|z\| = r\}\frac{\|x\|}{r}$$

As $r < 1$,

$$\begin{aligned}\|Tx\| < \sup\{\|Tz\| : z \in X, \|z\| < 1\}\frac{\|x\|}{r} &= \gamma\frac{\|x\|}{r}\\ \implies \|Tx\| &\le \frac{\gamma}{r}\|x\|\end{aligned}$$

Letting $r \to 1$, we have $\|Tx\| \le \gamma\|x\|\ \forall x \in X$, and so $\alpha \le \gamma$ (since again, α is the infimum of such γ's).

Claim: $\beta \le \alpha$

Consider $k > 0$ such that $\|Tx\| \le k\|x\|\ \forall x \in X$ (such a $k > 0$ exists as T is bounded). Taking supremum over all $x \in X$ with $\|x\| \le 1$, we have $\beta \le k$. Since α is the infimum of all such k's, we have $\beta \leq \alpha$.

$$\therefore \beta \leq \alpha \leq \min\{\|T\|, \gamma\} \leq \beta \text{ (using } (*))$$

Hence, $\|T\| = \alpha = \beta = \gamma$. $\square$

Recall that the space of all bounded linear transformations from X to Y, equipped with the operator norm, is a normed linear space. In other words, $(\mathcal{B}(X,Y), \|\cdot\|)$ is a normed linear space.

Theorem 5.2. *$(\mathcal{B}(X,Y), \|\cdot\|)$ is a Banach space if Y is a Banach space.*

Proof. Given that Y is a Banach space. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X,Y)$.

$$\implies \text{ for each } \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \|T_n - T_m\| < \varepsilon\ \forall n,m > N$$
$$\implies \text{ for each } \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \sup_{x \in S_X} \{\|T_n(x) - T_m(x)\|\} < \varepsilon\ \forall n,m > N$$
$$\implies \|T_n(x) - T_m(x)\| < \varepsilon\ \forall n,m > N,\ \forall x \in S_X$$

$\implies$ for each $x \in S_X$, $\{T_n(x)\}_{n=1}^{\infty}$ is Cauchy in Y.
As Y is Banach, for each $x \in X$, $\{T_n(x)\}_{n=1}^{\infty}$ is convergent in Y.
Define

$$T : X \to Y \text{ by } T(x) = \lim_{n\to\infty} T_n(x)\ \forall x \in X$$

Clearly, $\lim_{n\to\infty} T_n(x)$ exists $\forall x \in X$. Also, T is linear because for $x_1, x_2 \in X$ and $\alpha \in \mathbb{F}$,

$$\begin{aligned} T(\alpha x_1 + x_2) &= \lim_{n\to\infty} T_n(\alpha x_1 + x_2) \\ &= \lim_{n\to\infty} \alpha T_n(x_1) + T_n(x_2) \text{ (as } (T_n) \in \mathcal{B}(X,Y)) \\ &= \alpha \lim_{n\to\infty} T_n(x_1) + \lim_{n\to\infty} T_n(x_2) \\ &= \alpha T(x_1) + T(x_2) \end{aligned}$$

Claim: $T \in \mathcal{B}(X,Y)$ and $T_n \to T$ as $n \to \infty$.
We know that given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|T_n(x) - T_m(x)\| < \varepsilon\ \forall n,m > N,\ \forall x \in S_X$
Taking $m \to \infty$, we have

$$\|T_n(x) - T(x)\| < \varepsilon\ \forall n > N, \forall x \in S_X$$
$$\implies \sup_{x \in S_X} \{\|T_n(x) - T_m(x)\|\} < \varepsilon\ \forall n > N$$
$$\implies \|T_n - T\| < \varepsilon\ \forall n > N \ldots\ldots (*)$$
$$\implies T_n - T \text{ is bounded } \forall n > N$$
$$\implies T_n - T \in \mathcal{B}(X,Y)\ \forall n > N \ldots (**)$$

Observe that $T = T - T_{N+1} + T_{N+1}$, where $T - T_{N+1} \in \mathcal{B}(X,Y)$ by $(**)$ and $T_{N+1} \in \mathcal{B}(X,Y)$ by construction. Thus, $T \in \mathcal{B}(X,Y)$, because $\mathcal{B}(X,Y)$ is a vector space.
Further, from $(*)$, it follows that $T_n \to T \in \mathcal{B}(X,Y)$ as $n \to \infty$.
Hence, every Cauchy sequence $(T_n) \subseteq \mathcal{B}(X,Y)$ is convergent in $\mathcal{B}(X,Y)$ if Y is Banach $\implies \mathcal{B}(X,Y)$ is Banach if Y is Banach. □

Remark 5.9. The converse of theorem 5.2 is also true. In other words, $\mathcal{B}(X,Y)$ is Banach if and only if Y is Banach. The proof for the converse is given in theorem 6.11.

Example 5.4. We know from remark 3.2 that $c_{00}(\mathbb{N})$ is not a Banach space with any norm. As a corollary of theorem 5.2, it follows that $\mathcal{B}(\ell^p(\mathbb{N}), c_{00}(\mathbb{N}))$ is not a Banach space for $1 \leq p \leq \infty$.

Definition 5.6 (Bounded linear functional)**.** Recall the definition of a linear functional from definition 1.3. Let $(X, \|\cdot\|)$ be a normed linear space. Then the linear functional $f : X \to \mathbb{F}$ is a bounded linear functional if $\exists k > 0$ such that $|f(x)| \leq k\|x\| \ \forall x \in X$.

Definition 5.7 (Topological and Algebraic dual spaces)**.** Recall the definition of a dual space from definition **??**. Let $(X, \|\cdot\|)$ be a normed linear space.

1. The space of all bounded linear functionals from X is called the **topological dual space** of X. It is denoted by $X^* = \mathcal{B}(X, \mathbb{F})$.

2. The space of all linear functionals from X is called the **algebraic dual space** of X. It is denoted by $X' = L(X, \mathbb{F})$.

Remark 5.10. Hereon, we shall refer to **topological dual spaces** by **dual spaces**, i.e. X^*.

Corollary 5.3 (The dual of a normed linear space is Banach)**.** *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$. Then $(X^*, \|\cdot\|_{op})$ is a Banach space.*

Proof. Note that $X^* = \mathcal{B}(X, \mathbb{F})$. For $f \in X^*$, $f : X \to \mathbb{F}$ is a bounded linear functional, i.e. $\exists k > 0$ such that $|f(x)| \leq k\|x\| \ \forall x \in X$. The operator norm is defined as $\|f\|_{op} = \|f\| = \sup\{|f(x)| : x \in S_X\}$. We know that $\mathbb{F}$ is a Banach space. From theorem 5.2, with $Y = \mathbb{F}$, it follows that $X^* = \mathcal{B}(X, \mathbb{F})$ is a Banach space. □

Theorem 5.4 (Every linear transformation on a finite dimensional normed linear space is bounded (continuous))**.** *Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space and $(Y, \|\cdot\|)$ be any normed linear space. Then $\mathcal{B}(X,Y) = L(X,Y)$.*

Proof. Given that X is finite dimensional. We know that every bounded linear transformation is also a linear transformation, so $\mathcal{B}(X,Y) \subseteq L(X,Y)$.

Claim: $L(X,Y) \subseteq \mathcal{B}(X,Y)$.

Let $dim\ X = n$. Let $T \in L(X,Y)$ and $\{x_1, \dots, x_n\}$ be a basis for X. For $x \in X$, $\exists a_1, \dots, a_n \in \mathbb{F}$ such that $x = \sum_{i=1}^{n} a_i x_i$.
Define $\|x\|_* = \sum_{i=1}^{n} |a_i|$, $x \in X$. From exercise 2.7, we know that $(X, \|\cdot\|_*)$ is a normed linear space.

As X is finite dimensional, the norms $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, i.e., $\exists \alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|x\| \leq \|x\|_* \leq \alpha_2 \|x\| \ \forall x \in X$$

Let $x \in X$, $x = \sum_{i=1}^{n} a_i x_i$. Then

$$\begin{aligned}
T(x) &= T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i) \\
\implies \|T(x)\| &= \left\|\sum_{i=1}^{n} a_i T(x_i)\right\| \leq \sum_{i=1}^{n} |a_i| \|T(x_i)\| \ \forall x \in X \\
\implies \|T(x)\| &\leq M \sum_{i=1}^{n} |a_i|, \text{ where } M = \max_{1 \leq i \leq n} \{\|T(x_i)\|\} \\
&= M \|x\|_* \ \forall x \in X \\
&= M\alpha_2 \|x\| \ \forall x \in X \\
\implies \|Tx\| &\leq M\alpha_2 \|x\| \ \forall x \in X \\
\implies T &\in \mathcal{B}(X,Y) \\
\implies L(X,Y) &\subseteq \mathcal{B}(X,Y)
\end{aligned}$$

Thus, $L(X,Y) = \mathcal{B}(X,Y)$ if X is finite dimensional, or every linear transformation over X is bounded. $\square$

Remark 5.11. Let X and Y be two normed linear spaces. Then $L(X,Y) = \mathcal{B}(X,Y)$ if and only if X is finite dimensional. We prove this in lemma 5.3 that follows, by showing that if X is infinite dimensional and $Y \neq \{0\}$, then $\exists$ a linear transformation from X to Y which is not bounded (i.e., $\mathcal{B}(X,Y) \subsetneq L(X,Y)$).

Corollary 5.5. *If X is a finite dimensional normed linear space, then*

$$X^* = X'$$

That is, the set of all bounded linear functionals coincides with the set of all linear functionals over X.

Lemma 5.3. *Let $(X, \|\cdot\|)$ be any infinite dimensional normed linear space and $(Y, \|\cdot\|)$ be a normed linear space with $dim\ Y \geq 1$. Then $\exists T \in L(X,Y) \setminus \mathcal{B}(X,Y)$, i.e., $\exists$ a linear transformation which is not bounded, whence $\mathcal{B}(X,Y) \subsetneq L(X,Y)$.*

Proof. Let $\mathscr{B}$ be a basis for X. Let $\mathscr{A} = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a countable subset of $\mathscr{B}$. Let $y_0(\neq 0) \in Y$. Let $T : X \to Y$ be a linear transformation defined on the basis as

$$T(x) = \begin{cases} jy_0\|x_j\|, & \text{if } x = x_j \in \mathscr{A} \\ 0, & \text{if } x \in \mathscr{B} \setminus \mathscr{A} \end{cases}$$

Claim: T is not a bounded linear transformation. Observe that

$$\begin{aligned} T(x_j) &= jy_0\|x_j\| \ \forall j \in \mathbb{N} \\ \implies \|T(x_j)\| &= j\|x_j\|\|y_0\| \ \forall j \in \mathbb{N} \\ \implies \left\|T\left(\frac{x_j}{\|x_j\|}\right)\right\| &= j\|y_0\| \ \forall j \in \mathbb{N} \ldots (*) \end{aligned}$$

We know that $\left\|\frac{x_j}{\|x_j\|}\right\| = \frac{\|x_j\|}{\|x_j\|} = 1$, so $\left\|\frac{x_j}{\|x_j\|}\right\| \in S_X \ \forall j \in \mathbb{N}$.
If T is bounded, then $\exists k > 0$ such that $\|Tx\| \leq k \ \forall x \in S_X$. But

$$\begin{aligned} \|Tx\| &\leq k \ \forall x \in S_X \\ \implies j\|y_0\| &\leq k \ \forall j \in \mathbb{N} \text{ (from } (*)) \end{aligned}$$

which is a contradiction, because $\mathbb{N}$ is not bounded above. Thus, T is not bounded, and we have $T \notin \mathcal{B}(X, Y)$. □

Corollary 5.4. *If X is an infinite dimensional normed linear space, then there exists a linear functional $f : X \to \mathbb{F}$ such that f is not bounded.*

Proof. Following the proof of lemma 5.3 with $y_0 = 1 \in \mathbb{F}$, the result follows. □

Corollary 5.5. *If X is an infinite dimensional normed linear space, then*

$$X^* \subsetneq X'$$

That is, the topological dual space of X (set of all bounded linear functionals on X) is strictly contained inside the algebraic dual space (set of all linear functionals on X).

Lemma 5.6. *Let X, Y, Z be normed linear spaces, $T \in \mathcal{B}(X, Y)$, and $S \in \mathcal{B}(Y, Z)$. Then*

1. $S \circ T \in \mathcal{B}(X, Z)$
2. $\|S \circ T\| \leq \|S\|\|T\|$

Proof. 1. Clearly, $S \circ T \in L(X, Z)$. Observe that

$$\begin{aligned} \|(S \circ T)(x)\|_Z = \|S(T(x))\|_Z &\leq \|S\| \, \|Tx\|_Y \ \forall x \in X \\ &\leq \|S\|\|T\| \, \|x\|_X \ \forall x \in X \end{aligned}$$

Thus, $S \circ T \in \mathcal{B}(X, Z)$ because S satisfies definition 5.3 with $k = \|S\|\|T\|$.

2. Observe that

$$\|S \circ T\| = \sup\{\|(S \circ T)(x)\|_Z : x \in S_X\} \leq \sup\{\|S\|\|T\|\,\|x\|_X : x \in S_X\} = \|S\|\|T\|$$

(since $x \in S_X \implies \|x\|_X = 1$).

$\square$

Remark 5.12. If $T \in \mathcal{B}(X)$, then

$$T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n \text{ times}} \in \mathcal{B}(X)\ \forall n \in \mathbb{N}$$

and $\|T^n\| \leq \|T\|^n\ \forall n \in \mathbb{N}$.

Remark 5.13. Let $p(t) = \sum_{j=0}^{n} a_j t^n\ \forall t \in \mathbb{F}$ be a polynomial over $\mathbb{F}$. With the notation of $T^0 = I$, define

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \ldots + a_n T^n = \sum_{j=0}^{n} a_j T^j$$

Observe that $p(T) \in \mathcal{B}(X)$ if $T \in \mathcal{B}(X)$. $p(T)$ is called a **polynomial operator** in variable T.

Remark 5.14. Let $(X, \|\cdot\|)$ be a Banach space and $T \in \mathcal{B}(X)$. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n\ \forall z \in \mathbb{C}$ with radius of convergence ∞.
Consider the formal series $\sum_{n=0}^{\infty} a_n T^n$.
Observe that

$$\sum_{n=0}^{\infty} \|a_n T^n\| \leq \sum_{n=0}^{\infty} |a_n| \|T\|^n < \infty$$

$\implies \sum_{n=0}^{\infty} a_n T^n$ is absolutely convergent.
As $\mathcal{B}(X)$ is Banach, from theorem 3.1, it follows that $\sum_{n=0}^{\infty} a_n T^n$ is a convergent series in $\mathcal{B}(X)$. Hence, $f(T) = \sum_{n=0}^{\infty} a_n T^n$ is a bounded linear transformation on X.

Definition 5.8 (Finite rank linear transformation)**.** Let $T \in \mathcal{B}(X, Y)$. We say that T is a finite rank linear transformation if the range of T, defined as $R(T) = \{Tx : x \in X\}$ is a finite dimensional subspace of Y. In this case, the rank of T is $\rho(T) = dim(R(T))$.

Remark 5.15. Let $T \in L(X, Y)$, where X and Y are normed linear spaces. Suppose that T is a finite rank linear transformation (i.e., $\rho(T) \in \mathbb{N}$. We cannot say that T is bounded. In other words, it is not true in general that $T \in \mathcal{B}(X, Y)$ if T is of finite rank. To see this, recall the linear transformation T that we defined in lemma 5.3. Even when $dim\ Y = 1$ (so that $\rho(T) < \infty$) and X is infinite dimensional, we can obtain a linear transformation that is not bounded. In fact, we have seen such example already.

Lemma 5.7. *Let $T \in L(X,Y)$. Then the null space of T is defined as $N(T) = \{x \in X : Tx = 0\}$. If T is bounded, then $N(T)$ is a closed subspace of X.*

Proof. $T \in \mathcal{B}(X,Y) \implies T$ is continuous (remark 5.2). Hence, inverse image of a closed set in Y is closed in X under T. We know that $\{0\}$ is a closed set in Y, being a singleton. Observe that $N(T) = T^{-1}(\{0\})$, which is closed in X. $\square$

Remark 5.16. The converse of lemma 5.7 is not true for general linear transformations. That is, if for a linear transformation $T \in L(X,Y)$, we have $N(T)$ is closed in X, then we cannot say that T is bounded. Consider the following examples:

1. Define $T : (c_{00}(\mathbb{N}), \|\cdot\|_p) \to (c_{00}(\mathbb{N}), \|\cdot\|_p)$ by

$$T((x_1, x_2, \ldots, x_n, 0, 0, \ldots)) = (x_1, 2x_2, \ldots, nx_n, 0, 0, \ldots)\ \forall x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in c_{00}(\mathbb{N})$$

Clearly, T is a linear transformation, and $T(x) = 0 \iff x = 0 \implies N(T) = \{0\}$. Hence, $N(T)$ is closed in $c_{00}(\mathbb{N})$.
Observe that

$$\begin{aligned} e_n &= (0, \ldots, 0, 1, 0, \ldots), \text{ where 1 appears in the } n^{th} \text{ position} \\ \implies T(e_n) &= (0, \ldots, n, 0, 0, \ldots), \text{ where } n \text{ appears in the } n^{th} \text{ position} \end{aligned}$$

Also, $\|e_n\|_p = 1\ \forall n \in \mathbb{N}$. Then $\|T(e_n)\|_p = n\ \forall n \in \mathbb{N}$.
$\implies T$ takes a bounded set $\{e_n : n \in \mathbb{N}\}$ to an unbounded set $\{ne_n : n \in \mathbb{N}\} \subseteq c_{00}(\mathbb{N})$.
$\implies T$ is not bounded.

2. Define $T : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ by $Tf = f'\ \forall f \in C^1[0,1]$. Clearly, T is a linear transformation, and

$$\begin{aligned} N(T) &= \{f \in C^1[0,1] : f' = 0 \text{ on } [0,1]\} \\ &= \{f \in C^1[0,1] : f'(x) = 0\ \forall x \in [0,1]\} \\ &= \{f \in C^1[0,1] : f \text{ is a constant function}\} \simeq \mathbb{F} \end{aligned}$$

As $dim\ N(T) = 1$, $N(T)$ is a closed subspace of $C^1[0,1]$.
Claim: T is not bounded.
Let $n \in \mathbb{N}$, $f_n(x) = x^n\ \forall x \in [0,1]$. Observe that $\|f_n\|_\infty = \sup_{x\in[0,1]}\{|f_n(x)|\} = 1$.

$$\|Tf_n\|_\infty = \|f_n'\|_\infty = \left\|nx^{n-1}\right\|_\infty = |n| \left\|x^{n-1}\right\|_\infty = n,\ \forall n \in \mathbb{N}$$

Hence, T is not bounded as $\|Tf_n\|_\infty \to \infty$ as $n \to \infty$.

Remark 5.17. Let $(X, \|\cdot\|)$ be a normed linear space and $f : X \to \mathbb{F}$ be a non-zero linear functional. Then

1. f is onto. We know that $f(0) = 0$ as f is linear. Let $x_0 \in X$ such that $f(x_0) \neq 0$ and $\alpha \in \mathbb{F} \setminus \{0\}$. Then
$$f\left(\frac{\alpha x_0}{f(x_0)}\right) = \frac{\alpha}{f(x_0)} f(x_0) = \alpha$$
This means that every $\alpha \in \mathbb{F}$ has a pre-image in X, due to which f is onto.

2. Let $x_0 \in X$ such that $f(x_0) \neq 0$. For each $x \in X$, we have $u = x - \frac{f(x)}{f(x_0)} x_0 \in N(f)$
To see this, observe that
$$f(u) = f\left(x - \frac{f(x)}{f(x_0)} x_0\right) = f(x) - f(x) f\left(\frac{x_0}{f(x_0)}\right) = f(x) - f(x)\frac{f(x_0)}{f(x_0)} = 0$$
Hence, for each $x \in X$, we can write $x = u + \frac{f(x)}{f(x_0)} x_0$ for some $u \in N(f)$.

Remark 5.18. The converse of lemma 5.7 is true for linear functionals. In other words, a linear functional $f : X \to \mathbb{F}$ is bounded (continuous) if and only if its null space $N(f)$ is closed in X (theorem 5.6).

Theorem 5.6. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and $f : X \to \mathbb{F}$ is a non-zero linear functional. Then the following are equivalent:*

1. *f is a continuous linear functional.*
2. *$N(f)$ is closed subspace of X.*
3. *$\overline{N(f)} \neq X$.*

Proof. **(1 $\implies$ 2)**
Since f is continuous, so f^{-1} takes closed set to closed set. And $\{0\}$ is a closed set in $\mathbb{F}$, $f^{-1}(0) = N(f)$ is a closed set in X.

(2 $\implies$ 1)
Assume that f is a linear functional and $N(f)$ is a closed subspace of X.
Without loss of generality, let $f \neq 0$, so $\exists x_0 \in X$ such that $f(x_0) \neq 0$. By remark 5.17, for each $x \in X$, we can write $x = u + \frac{f(x)}{f(x_0)} x_0$ for some $u \in N(f)$.
Now,
$$\begin{aligned} x &= u + \frac{f(x)}{f(x_0)} x_0 \\ \implies d(x, N(f)) &= d\left(u + \frac{f(x)}{f(x_0)} x_0, N(f)\right) \\ &= d\left(\frac{f(x)}{f(x_0)} x_0, N(f)\right) \text{ (Since } u \in N(f)) \\ &= \frac{|f(x)|}{|f(x_0)|} d(x_0, N(f)) \end{aligned}$$

Since $0 \in N(f)$ and $d(x, N(f)) = \inf\{\|x - u\| : u \in N(f)\}$,
$\implies d(x, N(f)) \leq \|x - 0\| = \|x\|$

$$\implies \frac{|f(x)|}{|f(x_0)|} d(x_0, N(f)) \leq \|x\|$$
$$\implies |f(x)| \leq \frac{|f(x_0)|}{d(x_0, N(f))} \|x\|$$

Note that here, $d(x_0, N(f)) \neq 0$ by lemma 2.17 (2), because $x_0 \in X \setminus N(f)$, since $f(x_0) \neq 0$.
$\therefore f \in X^*$, i.e. f is continuous (bounded) linear functional.

(2 $\implies$ 3)
As $N(f)$ is closed subspace of X, we have $\overline{N(f)} = N(f)$. And f is a non-zero functional, so there exists some $x_0 \notin N(f)$. Then,

$$\overline{N(f)} = N(f) \subset X$$

Thus, $\overline{N(f)} \neq X$

(3 $\implies$ 2)
Given, $\overline{N(f)} \neq X$

Claim: $\overline{N(f)} = N(f)$. We will prove this by contradiction.
If possible, let $x_0 \in \overline{N(f)} \setminus N(f)$ such that $f(x_0) \neq 0$.
Let $x \in X$ such that
$$x = x - \frac{f(x)}{f(x_0)} x_0 + \frac{f(x)}{f(x_0)} x_0$$

$y = x - \frac{f(x)}{f(x_0)} x_0$ and $z = \frac{f(x)}{f(x_0)} x_0$ (say)

Then clearly, $y \in N(f) \subseteq \overline{N(f)}$ and $z \in \overline{N(f)} \implies x = y + z \in \overline{N(f)}$. This leads us to say $X = \overline{N(f)}$, which is a contradiction.
Thus, there is no $x_0 \in \overline{N(f)} \setminus N(f) \implies \overline{N(f)} \setminus N(f) = \phi$.
Hence $\overline{N(f)} = N(f)$, so $N(f)$ is a closed subspace of X. □

Lemma 5.8. *Let $(X, \|\cdot\|)$ be a normed linear space. If $f \in X^* \setminus \{0\}$ (i.e. if f is a non-zero linear functional on X), then*

$$\|f\| = \frac{|f(x_0)|}{d(x_0, N(f))}, \text{ where } x_0 \in X \text{ such that } f(x_0) \neq 0$$

Proof. From the proof of theorem 5.6, we have

$$\begin{aligned}
|f(x)| &\leq \frac{|f(x_0)|}{d(x_0, N(f))}\|x\| \ \forall x \in X \\
\implies |f(x)| &\leq \frac{|f(x_0)|}{d(x_0, N(f))} \ \forall x \in S_X \\
\implies \|f\| &\leq \frac{|f(x_0)|}{d(x_0, N(f))} \ldots\ldots (*)
\end{aligned}$$

Observe that

$$\begin{aligned}
|f(x_0)| &= |f(x_0 - u)| \ \forall u \in N(f) \ (\because f(u) = 0) \\
&\leq \|f\|\|x_0 - u\| \ \forall u \in N(f) \\
\implies |f(x_0)| &\leq \|f\| \inf_{u \in N(f)} \{\|x_0 - u\|\} \\
&= \|f\| d(x_0, N(f)) \\
\implies \frac{|f(x_0)|}{d(x_0, N(f))} &\leq \|f\| \ldots\ldots (**)
\end{aligned}$$

From $(*)$ and $(**)$, it follows that $\|f\| = \frac{|f(x_0)|}{d(x_0,N(f))}$.
Note that $d(x_0, N(f)) > 0$ by theorem 2.17 (2), because $f(x_0) \neq 0$ by assumption, so that $x_0 \notin N(f) = \overline{N(f)}$. □

Corollary 5.9. *Let $f \in X^* \setminus \{0\}$ and $x_0 \in X$ such that $f(x_0) = 1$. Then*

$$\|f\| = \frac{1}{d(x_0, N(f))}$$

6 The Five Pillars of Functional Analysis

The five pillars of functional analysis: Hahn-Banach Theorem, Uniform Boundedness Principle, Open Mapping Theorem, Closed Graph Theorem, and Bounded Inverse Theorem are fundamental tools for understanding the interplay between linear transformations and topological structures in normed spaces. They provide essential frameworks for studying continuity, boundedness, and the extension of functionals, forming the backbone for applications in several fields.

6.1 Hahn-Banach Extension Theorems & Applications

The extension lemma is the core principle of the Hahn-Banach Theorem, as it allows the extension of a bounded linear functional defined on a subspace by one dimension at a time. This process preserves the functional norm and is crucial in proving the full Hahn-Banach extension.

Let $(\mathbf{X}, \|\cdot\|)$ is normed linear space over $\mathbb{F}$ and $\mathcal{U}$ be a subspace of X. Let $x \in X$, then

$$\mathcal{U} + \mathbb{F}x = \{u + \alpha x : u \in \mathcal{U}, \alpha \in \mathbb{F}\}$$

Theorem 6.1 (Extension Lemma)**.** *Let $(\mathbf{X}, \|\cdot\|)$ is normed linear space over $\mathbb{R}$, $\mathcal{U}$ be a subspace of $\mathbf{X}$ and $x \in \mathbf{X}\backslash\mathcal{U}$. Let $f : \mathcal{U} \to \mathbb{R}$ be a continuous (bounded) real linear functional, then $\exists$ a continuous real linear functional $\tilde{f} : \mathcal{U} + \mathbb{R}x \to \mathbb{R}$ such that $\tilde{f}|_{\mathcal{U}}$=f and $\| \tilde{f} \| = \|f\|$*

Proof. Since $f : \mathcal{U} \to \mathbb{R}$ is a continuous real linear functional, i.e.

$$|f(u)| \leq \|f\| \, \|u\|, \forall u \in \mathcal{U}$$

Let $c \in \mathbb{R}$, define $\tilde{f} : \mathcal{U} + \mathbb{R}x \to \mathbb{R}$ by $\tilde{f}(u + \alpha x) = f(u) + \alpha c$
Clearly $\tilde{f}$ is a real continuous linear functional (since f is continuous real linear functional) and $\tilde{f}|_{\mathcal{U}} = f$. Now,

$$\begin{aligned}
\|f\| &= \sup\left\{\frac{|f(u)|}{\|u\|} : u \in \mathcal{U} \text{ and } \|u\| \neq 0\right\} \\
&\leq \sup\left\{\frac{|f(y)|}{\|y\|} : y \in \mathcal{U} + \mathbb{R}x \text{ and } \|y\| \neq 0\right\} \;\; (\text{Since } \mathcal{U} \subset \mathcal{U} + \mathbb{R}x)
\end{aligned}$$

So, $\|f\| \leq \|\tilde{f}\|$

To show the other inequality, we will try to choose $c \in \mathbb{R}$ such that $\|\tilde{f}\| \leq \|f\|$

i.e. select c $\in \mathbb{R}$ such that$|\tilde{f}(u+\alpha x)| \leq \|\tilde{f}\|\|u+\alpha x\| \leq \|f\|\|u+\alpha x\|$

i.e. select $c \in \mathbb{R}$ such that $|f(u)+\alpha c| \leq \|f\|\|u+\alpha x\|, \forall u \in \mathcal{U}$ and $\alpha \in \mathbb{R}$

i.e. select $c \in \mathbb{R}$ such that $|\alpha\|f(\frac{u}{\alpha})+c\| \leq |\alpha|\|f\|\|\frac{u}{\alpha}+x\|, \forall u \in \mathcal{U}$ and $\alpha \in \mathbb{R}$

i.e. select $c \in \mathbb{R}$ such that $|f(u)+c| \leq \|f\|\|u+x\|, \forall u \in \mathcal{U}$

i.e. select $c \in \mathbb{R}$ such that $-\|f\|\|u+x\| \leq f(u)+c \leq \|f\|\|u+x\|, \forall u \in \mathcal{U}$

i.e. select $c \in \mathbb{R}$ such that $-\|f\|\|u+x\|-f(u) \leq c \leq \|f\|\|u+x\|-f(u), \forall u \in \mathcal{U}$

Subclaim: $\exists c \in \mathbb{R}$ such that $-\|f\|\|u+x\|-f(u) \leq c \leq \|f\|\|v+x\|-f(v), \forall u, v \in \mathcal{U}$
For $u, v \in \mathcal{U}$,

$$\begin{aligned}
-\|f\|\|u+x\|-f(u) &= -\|f\|\|u+x+v-v\|-f(u) \\
&\leq \|f\|(\|v+x\|-\|v-u\|)-f(u) \text{ (using reverse triangle inequality)} \\
&= \|f\|\|v+x\|-\|f\|\|v-u\|-f(u) \\
&= \|f\|\|v+x\|-\|f\|\|v-u\|-f(v-u)-f(v) \\
&= (\|f\|\|v+x\|-f(v)-(\|f\|\|v-u\|-f(v-u)) \\
&\leq \|f\|\|v+x\|-f(v)
\end{aligned}$$

(Since $\|f\|$ is the supremum of $\frac{|f(y)|}{\|y\|}$ such that $\|y\| \neq 0$ hence $\|f\| \geq \frac{f(y)}{\|y\|} \implies -\|f\|\|v-u\|-f(v-u) \geq 0$)

$$\text{Thus, } -\|f\|\|u+x\|-f(u) \leq \|f\|\|v+x\|-f(v), \forall u.v \in \mathcal{U}$$
$$\sup\{-\|f\|\|u+x\|-f(u) : u \in \mathcal{U}\} \leq \inf\{\|f\|\|v+x\|-f(v) : v \in \mathcal{U}\}$$

Let c $\in \mathbb{R}$ such that

$$\sup\{-\|f\|\|u+x\|-f(u) : u \in \mathcal{U}\} \leq c \leq \inf\{\|f\|\|v+x\|-f(v) : v \in \mathcal{U}\}$$

Hence we have proven the sub-claim, and now we have $c \in \mathbb{R}$ such that

$$-\|f\|\|u+x\|-f(u) \leq c \leq \|f\|\|u+x\|-f(u), \forall u \in \mathcal{U}$$

$$\implies \|\tilde{f}\| \leq \|f\|$$

Thus $\|\tilde{f}\| = \|f\|$
So, there exists a continuous (bounded) real linear functional $\tilde{f} : \mathcal{U} + \mathbb{R}x \to \mathbb{R}$ such that $\tilde{f}|_{\mathcal{U}} = f$ and $\|\tilde{f}\| = \|f\|$

$\square$

Theorem 6.2 (Hahn-Banach Extension Theorem). *Let* $(\mathbf{X}, \|\cdot\|)$ *is normed linear space over* $\mathbb{F}$, $\mathcal{U}$ *be a subspace of* $\mathbf{X}$. *Let* $f : \mathcal{U} \to \mathbb{F}$ *be a continuous (bounded) linear functional, then* $\exists$ *a continuous linear functional* $\tilde{f} : \mathcal{U} + \mathbb{F}x \to \mathbb{F}$ *such that* $\tilde{f}|_{\mathcal{U}} = f$ *and* $\|\tilde{f}\| = \|f\|$

Proof. We will divide the proof into two versions, one for $\mathbb{F} = \mathbb{R}$ and another for $\mathbb{F} = \mathbb{C}$.

Real Version: When $\mathbb{F} = \mathbb{R}$
Given that, $\mathcal{U}$ be a subspace of $\mathbf{X}$. Let $f : \mathcal{U} \to \mathbb{R}$ be a continuous (bounded) linear functional.
Let,

$$\mathcal{A} = \{g : W \to \mathbb{R} : g \text{ is cont. linear functional}, \mathcal{U} \subseteq \mathcal{W} \subseteq X, g|_{\mathcal{U}} = f, \|g\| = \|f\|\}$$

Clearly $f \in \mathcal{A}$, so $\mathcal{A} \neq \phi$
Let W_1, W_2 are subspaces of X containing $\mathcal{U}$, g_1, $g_2 \in \mathcal{A}$, where $g_1 : W_1 \to \mathbb{R}$ and $g_2 : W_2 \to \mathbb{R}$. Define a relation in $\mathcal{A}$ by,

$$g_1 \leq g_2 \text{ iff } W_1 \subseteq W_2 \text{ and } g_2|_{W_1} = g_1$$

Clearly, $\mathcal{A}$ is a POset. Let $\mathcal{C}$ be a chain in $\mathcal{A}$, say, $\mathcal{C} = \{f_i \in \mathcal{A} : i \in \mathcal{I}\}$, where $\mathcal{I}$ is some index set.
So, for each $i \in \mathcal{I}, \exists W_i$ such that $f_i : W_i \to \mathbb{R}$ is continuous and linear functional, $f_i|_{\mathcal{U}} = f$ and $\|f_i\| = \|f\|$
Let $W = \cup_{i \in \mathcal{I}} W_i$, since $\mathcal{C}$ is a chain, so the subspaces are in increasing order, and so, W is a subspace of X. $\mathcal{U} \subseteq W$ and for each $i \in \mathcal{I}, W_i \subseteq W$.
Define $g : W \to \mathbb{R}$ by $g(x) := f_i(x)$ if $x \in W_i$
Clearly, (check) g is well defined and linear.
Now, we can observe that, for each $x \in W, \exists i \in \mathcal{I}$ such that $x \in W_i$ and by our definition,

$$\begin{aligned} g(x) &= f_i(x) \\ \implies |g(x)| &= |f_i(x)| \\ &\leq ||f_i|| ||x|| \\ &= ||f||x|| \\ \implies ||g|| = \sup\{|g(x)| : x \in W, ||x|| = 1\} &\leq ||f|| \end{aligned}$$

Thus, g is a Bounded (continuous) linear functional from W to $\mathbb{R}$ and $g|_{\mathcal{U}} = f$,

$$g|_{\mathcal{U}} = f \implies ||g|| \geq ||f||$$

Hence we have, $||g|| = ||f||$
So, $g : W \to \mathbb{R}, g \in \mathcal{A}$ which is the upper bound of the chain $\mathcal{C}$.
By Zorn's lemma, $(\mathcal{A}, \leq)$ has a maximal element, say $\tilde{f}$. i.e. there exists a subspace $\tilde{U}$, $\mathcal{U} \subseteq \tilde{U} \subseteq X$ and
1. $\tilde{f} : \tilde{U} \to \mathbb{R}$ is continuous and linear functional
2. $\tilde{f}|_{\mathcal{U}} =$ f
3. $||\tilde{f}|| = ||f||$

Claim: $\tilde{U} = X$
(proof by contradiction)
If possible let, $\exists x \in X$ such that $x \notin \tilde{U}$, let,

$$\tilde{W} := \tilde{U} + \mathbb{R}x = \{w + \alpha x : w \in \tilde{U}, \alpha \in \mathbb{R}\}$$

Then, obviously $\tilde{W}$ is a subspace of X and $\mathcal{U} \subseteq \tilde{W}$, by Extension Lemma there exists a continuous real linear functional $\hat{f} : \tilde{W} \to \mathbb{R}$ such that

$$\hat{f}|_{\tilde{U}} = \tilde{f} \ , \ \hat{f}|_U = f \text{ and } ||\hat{f}|| = ||\tilde{f}|| = ||f||$$

which leads to $\hat{f} \in \mathcal{A}$ and $\tilde{f} \leq \hat{f}$
But, $\tilde{f}$ is the maximal element of $\mathcal{A}$, so $\tilde{f} = \hat{f}$.
Which implies, domain of $\tilde{f} = \tilde{U} = \tilde{W} =$ domain of $\hat{f}$, i.e. $\tilde{U} = \tilde{W} := \tilde{U} + \mathbb{R}x$

$$\implies x \in \tilde{U}$$
$$\implies \tilde{U} = X$$

Hence, $\tilde{f} : X \to \mathbb{R}$ is continuous (bounded) linear functional in $\mathcal{A}$, that is the full extension of f.

$$\tilde{f}|_U = f \text{ and } ||\tilde{f}|| = ||f||$$

Complex Version: When $\mathbb{F} = \mathbb{C}$
Given that, $\mathcal{U}$ be a subspace of $\mathbf{X}$. Let $f : \mathcal{U} \to \mathbb{C}$ be a continuous (bounded) $\mathbb{C}$ linear functional. Since every complex normed linear space is also real normed linear space, so we can assume $\mathcal{U}$ as a real N.L. space.

Define $f_1 : \mathcal{U} \to \mathbb{R}$ by $f_1(x) = Ref(x), \forall x \in X$. We can easily observe that,

$$|f_1(x)| = |Re(f(x))| \leq |f(x)| \leq \|f\| \; \|x\|$$

which gives that, f_1 is bounded (continuous) linear functional, such that $\|f_1\| \leq \|f\|$.
(Note that, $f(ix) = if(x) = iRef(x) - Imf(x) \implies f_1(ix) = Ref(ix) = -Imf(x)$, since f is linear functional)
Observe that, $f(x) = f_1(x) - if_1(ix), \forall x \in X$

Now, if we treat X as a real normed linear space and $\mathcal{U}$ as a real subspace of X, then we have $f_1 : \mathcal{U} \to \mathbb{R}$ is a real continuous linear functional. Then by the real version of Hahn Banach extension theorem, we have that there exists a continuous linear functional $\tilde{f}_1 : \mathcal{U} \to \mathbb{R}$ such that $\tilde{f}_1|_{\mathcal{U}} = f_1$ and $||\tilde{f}_1|| = ||f_1||$

Now, let us define $\tilde{f} : X \to \mathbb{C}$ by $\tilde{f}(x) = \tilde{f}_1(x) - i\tilde{f}_1(ix)$. Clearly, $\tilde{f}$ is well defined linear map and since $\tilde{f}_1$ is $\mathbb{R}$-linear then $\tilde{f}$ is $\mathbb{R}$-linear functional.
Claim: $\tilde{f}$ is a $\mathbb{C}$-linear functional
It is enough to show that $\tilde{f}(ix) = i\tilde{f}(x)$

$$\begin{aligned}\tilde{f}(ix) &= \tilde{f}_1(ix) - i\tilde{f}_1(i^2x) \\ &= \tilde{f}_1(ix) + i\tilde{f}_1(x) \\ &= i(\tilde{f}_1(x) - i\tilde{f}_1(ix)) \\ &= i\tilde{f}(x)\end{aligned}$$

Hence, $\tilde{f}$ is a $\mathbb{C}$-linear functional and $\tilde{f}|_{\mathcal{U}} = f$
Remaining to show, $\tilde{f}$ is continuous (bounded) linear functional and $||\tilde{f}|| = ||f||$ Since, $\tilde{f}$ is an extension of f then obviously $||f|| \leq ||\tilde{f}||$

For the reverse inequality,

$$\begin{aligned}
|\tilde{f}(x)|^2 &= \tilde{f}(x)\overline{\tilde{f}(x)} = \tilde{f}\left(\overline{\tilde{f}(x)}x\right) \\
&= \tilde{f}_1\left(\overline{\tilde{f}(x)}x\right) \text{ (since in the LHS, we have real value: } |\tilde{f}(x)|^2 \in \mathbb{R}) \\
&\leq \|\tilde{f}_1\| \left\|\overline{\tilde{f}(x)}x\right\| \\
&= \|f_1\| \left\|\overline{\tilde{f}(x)}x\right\| \\
&\leq \|f\| \left\|\overline{\tilde{f}(x)}x\right\| \\
&= \|f\| \, |\tilde{f}(x)| \, \|x\| \\
\implies |\tilde{f}(x)| &\leq \|f\| \, \|x\| \\
\implies \|\tilde{f}\| &\leq \|f\|
\end{aligned}$$

Thus $\tilde{f}$ is bounded linear functional and $\|\tilde{f}\| = \|f\|$, therefore we can conclude that there exists a $\mathbb{C}$-linear functional $\tilde{f} : X \to \mathbb{C}$ such that $\tilde{f}|_{\mathcal{U}} = f$ and $\|\tilde{f}\| = \|f\|$ □

Proposition 6.1. *Hahn-Banach extension of a linear functional need not be unique.*

Proof. Let $X = (\mathbb{F}^2, \|\cdot\|_1)$ and $U = \{x \in X : x_2 = 0\} = \{(x_1, 0) : x_1 \in \mathbb{F}\}$. Define $g : U \to \mathbb{F}$ by

$$g((x_1, 0)) = x_1 \; \forall (x_1, 0) \in U$$

Clearly, g is linear. As $dim\, U < \infty$, g is continuous by theorem 5.4. Then $g \in U^*$. Also,

$$\|g\| = \sup\{|g(u)| : u \in U, \|u\| = 1\} = \sup\{|g((x_1, 0))| : |x_1| = 1\} = \sup\{|x_1| : |x_1| = 1\} = 1$$

$\implies \|g\| = 1 = g(x_0)$, where $x_0 = (1, 0)$ and $U = span\{x_0\}$.
Let $f \in X^*$. Then f is of the form $f(x) = ax_1 + bx_2,\; x = (x_1, x_2) \in X$ for some $a, b \in \mathbb{F}$. Also,

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\} = \sup\{|ax_1 + bx_2| : |x_1| + |x_2| = 1\} = \max\{|a|, |b|\}$$

For the above f to be a Hahn-Banach extension of g, it is necessary and sufficient that $f(x_0) = g(x_0) = 1$ ($\because U = span\{x_0\}$, so it is enough for f and g to coincide on x_0 to coincide on the entire set U) and $\|f\| = \|g\| = 1$.

$\therefore$ For any b with $|b| \leq 1$, $f(x) = x_1 + bx_2,\; x \in X$ is a Hahn-Banach extension of g. □

Corollary 6.3. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and $x \in X \setminus \{0\}$. Then there exists a continuous linear functional $f : X \to \mathbb{F}$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Proof. Let $x \in X \setminus \{0\}$, and $U = span\{x\}$
Define $g : U \to \mathbb{F}$ by $g(\alpha x) = a\|x\|$. Since $dim(U) = 1$, so clearly g is continuous linear

functional.
We can observe that $g(x) = \|x\|$ and

$$\begin{aligned} |g(\alpha x)| &= |\alpha| \ \|x\| \\ &= \|\alpha x\| \\ \implies \|g\| &= 1 \end{aligned}$$

So, by Hahn Banach extension theorem, we can extend g to a continuous linear functional $f : X \to \mathbb{F}$ such that $\|f\| = 1$ and $f(x) = \|x\|$ $\square$

Corollary 6.4. *Let $(X, ||\cdot||)$ be a normed linear space over $\mathbb{F}$. Then $X^* \neq \{0\}$ if $X \neq \{0\}$.*

Proof. It is easy to see from corollary 6.3 that if we have at least one non-zero element $x \in X$, then we can get a bounded (continuous) linear functional from X to $\mathbb{F}$. $\square$

Corollary 6.5. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$. Then X^* separates points in X, i.e. for any $x, y \in X (x \neq y)$, $\exists f \in X^*$ such that $f(x) \neq f(y)$.*

Proof. Let $x, y \in X$ such that $x \neq y$, $z = x - y \in X \setminus \{0\}$ (say). Then by corollary 6.3, there exists a bounded linear functional $f : X \to \mathbb{F}$ such that

$$\begin{aligned} f(z) &= \|z\| \\ \implies f(x-y) &= \|x-y\| \\ \implies f(x) - f(y) &\neq 0 \text{ (since } x \neq y \implies \|x-y\| \neq 0) \\ \implies f(x) &\neq f(y) \end{aligned}$$

So, there exists a $f \in X^*$ which separates x and y. Hence X^* separates X. $\square$

Corollary 6.6. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and $x \in X$, then*

$$\|f\| = \sup\{|f(x)| : f \in X^* \textit{ and } \|f\| = 1\} = \sup\{|f(x)| : f \in S_{X^*}\}$$

Proof. Without loss of generality, let $x \neq 0$. Then

$$\begin{aligned} |f(x)| &\leq \|f\|\|x\| \quad \text{(using lemma 5.1, as } f \text{ is bounded)} \\ \implies |f(x)| &\leq \|x\| \ \forall f \in S_{X^*} \quad (\text{as } f \in S_{X^*} \implies \|f\| = 1) \\ \implies \sup\{|f(x)| : f \in S_{X^*}\} &\leq \|x\| \end{aligned}$$

Since $x \neq 0$, $\exists f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$ by corollary 6.3.
$\implies$ $\|x\|$ is one of the elements in $\{|f(x)| : f \in S_{X^*}\}$
$\implies$ $\|x\| = \sup\{|f(x)| : f \in S_{X^*}\}$

$\square$

Corollary 6.7. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and M be a proper closed subspace of X. Let $x_0 \in X \setminus M$. Then $\exists f \in X^*$ such that $M \subseteq N(f)$, i.e. $f(m) = 0\ \forall m \in M$ and $\|f\| = dist(x_0, M)$.*

Proof. Given that M is a proper closed subspace of X and $x_0 \in X \setminus M$. Let $U = M + \mathbb{F}x_0 = \{m+\alpha x_0 : m \in M,\ \alpha \in \mathbb{F}\}$. Define $g : U \to \mathbb{F}$ by $g(m+\alpha x_0) = \alpha\, dist(x_0, M)^2\ \forall m+\alpha x_0 \in U$.

Claim: g is a linear functional.
Let $u_1 = m_1 + \alpha_1 x_0$ and $u_2 = m_2 + \alpha_2 x_0 \in U$, and $\beta \in \mathbb{F}$. Then

$$\begin{aligned} g(\beta u_1 + u_2) &= g(\beta m_1 + \beta\alpha_1 x_0 + m_2 + \alpha_2 x_0) \\ &= g((\beta m_1 + m_2) + (\beta\alpha_1 + \alpha_2)x_0) \\ &= (\beta\alpha_1 + \alpha_2)\ dist(x_0, M)^2 \\ &= \beta\alpha_1\ dist(x_0, M)^2 + \alpha_2\ dist(x_0, M)^2 \\ &= \beta g(u_1) + g(u_2) \end{aligned}$$

Claim: $M = N(g)$
$N(g) = \{u = m + \alpha x_0 \in U : g(u) = 0\}$
Let $m \in M$, then $g(m) = g(m+0x_0) = 0\cdot dist(x_0, M)^2 = 0 \implies m \in N(g) \implies M \subseteq N(g)$.
Let $u = m + \alpha x_0 \in N(g)$. Then $g(m + \alpha x_0) = \alpha\ dist(x_0, M)^2 = 0$. But $x_0 \in X \setminus M \implies dist(x_0, M) > 0$ (by lemma 2.17 (2)). So, it must be that $\alpha = 0$. Then $u = m \in M \implies N(g) \subseteq M$.
Thus, $N(g) = M$.

As $N(g) = M$ is a closed subspace of U and g is a linear functional, it follows from theorem 5.6 that g is a continuous linear functional. Also, $g(x_0) = dist(x_0, M)^2 = dist(x_0, N(g))^2 \neq 0$. By lemma 5.8, we have

$$\|g\| = \frac{|g(x_0)|}{dist(x_0, N(g))} = \frac{dist(x_0, M)^2}{dist(x_0, M)} = dist(x_0, M)$$

By Hahn-Banach Extension Theorem, $\exists f \in X^*$ such that $f|_U = g$ and $\|f\| = \|g\|$.
$\implies N(g) = M \subseteq N(f)$ and $\|f\| = dist(x_0, M)$. □

Corollary 6.8. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$, M be a proper closed subspace of X and $x_0 \in X$. Then there exists a $f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = dist(x_0, M)$ and $M \subseteq N(f)$.*

Proof. Given that $x_0 \in X \setminus M$ and M is a proper closed subspace of X. Hence $dist(x_0, M) \geq 0$. Let $\tilde{U} = M + \mathbb{F}x_0$, clearly $\tilde{U}$ is a subspace of X. Define $g : \tilde{U} \to \mathbb{F}$ by $g(m + \alpha x_0) = \alpha\ dist(x_0, M)$, where $m \in M$. We can easily verify that g is linear functional and $N(g) = M$.

(Since $m + \alpha x_0 \in N(g) \iff g(m + \alpha x_0) = 0 \iff \alpha\ dist(x_0, M) = 0 \iff \alpha = 0$)

By theorem 5.6, we have that g is continuous linear functional since $N(g) = M$ is closed. Also, $g(x_0) = dist(x_0, M)$. Using lemma 5.8,

$$\begin{aligned} \implies \|g\| &= \frac{|g(x_0)|}{dist(x_0, N(g))} \\ &= \frac{dist(x_0, M)}{dist(x_0, M)} \\ &= 1 \end{aligned}$$

Now, by Hahn-Banach Extension Theorem, $\exists f \in X^*$ such that $f|_{\tilde{U}} = g$, $\|f\| = \|g\| = 1$ and $f(x_0) = g(x_0) = dist(x_0, M)$

Let $x \in M = N(g)$

$$\begin{aligned} g(x) &= 0 \\ \text{and } f|_{\tilde{U}} &= g \\ \implies f(x) = g(x) &= 0 \\ \implies x &\in N(f) \\ M &\subseteq N(f) \end{aligned}$$

$\exists f \in X^*$ such that, $\|f\| = 1$ and $f(x_0) = dist(x_0, M)$ and $M \subseteq N(f)$ □

Exercise 6.1. Let $(X, \|\cdot\|)$ be a normed linear space and $(Y, \|\cdot\|)$ be a Banach space over $\mathbb{F}$. U be a subspace of X such that $\overline{U} = X$, and let $T : U \to Y$ be bounded linear transformation. Then there exists unique bounded linear transformation $\tau : X \to Y$ such that $\tau|_U = T$ and $\|\tau\| = \|T\|$.

Corollary 6.9. *Let $x \in X$. Then $x = 0 \iff f(x) = 0\ \forall f \in X^*$.*

Proof. ($\implies$) $x = 0 \implies f(x) = 0\ \forall f \in X^*$ by the linearity of f.
($\impliedby$) Let $f(x) = 0\ \forall f \in X^*$. From corollary 6.6, we have $\|x\| = \sup\{|f(x)| : f \in S_{X^*}\} = 0$. And, $\|x\| = 0 \implies x = 0$. □

Corollary 6.10. *Let $x \in X$ and W be a subspace of X^* such that $\overline{W} = X^*$. Then $x = 0 \iff f(x) = 0\ \forall x \in W$.*

Proof. ($\implies$) If $x = 0$, then $f(x) = 0\ \forall f \in W$.
($\impliedby$) Assume that $g(x) = 0\ \forall g \in W$. Let $f \in X^*$. Then $\exists\ (g_n) \subseteq W$ such that $g_n \to f$ in X^* as $n \to \infty$.

$\implies \|g_n - f\|_{X^*} \to 0$ as $n \to \infty$

$$\begin{aligned}
&\|g_n - f\|_{X^*} = \sup\{|(g_n - f)(x)| : x \in S_X\} \\
\implies &|(g_n - f)(x)| \leq \|g_n - f\|_{X^*} \to 0 \text{ as } n \to \infty \\
\implies &g_n(x) - f(x) \to 0 \text{ as } n \to \infty \\
\implies &g_n(x) \to f(x) \text{ as } n \to \infty \\
\implies &f(x) = 0 \ \forall f \in X^* \\
\implies &x = 0
\end{aligned}$$

□

Theorem 6.11. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed linear spaces. Then $(\mathcal{B}(X,Y), \|\cdot\|_{op})$ is a Banach space if and only if Y is a Banach space.*

Proof. ($\impliedby$) We have already proved in theorem 5.2 that Y is Banach $\implies$ $\mathcal{B}(X,Y)$ is Banach.

($\implies$) Assume that $\mathcal{B}(X,Y)$ is Banach. Let (y_n) is a Cauchy sequence in Y and $x_0 \in X \setminus \{0\}$ with $\|x_0\| = 1$
From 6.3, $\exists f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = 1$
Let $n \in \mathbb{N}$, define $T_n : X \to Y$ by $T_n(x) = f(x)y_n, \forall x \in X$. Clearly for each $n \in \mathbb{N}$, T_n is linear since f is linear and

$$\|T_n(x)\| = |f(x)| \ \|y_n\| \leq \|f\| \ \|x\| \ \|y_n\|$$

Hence $\forall n \in \mathbb{N}$, T_n is bounded linear functional, and $\|T_n\| \leq \|f\| \ \|y_n\| = \|y_n\|$. Observe that for any $m, n \in \mathbb{N}$

$$\begin{aligned}
\|T_n - T_m\| &= \sup\{\|T_n(x) - T_M(x)\| : \ \|x\| = 1\} \\
&= \sup\{\|f(x)y_n - f(x)y_m\| : \ \|x\| = 1\} \\
&= \sup\{\|f(x) \ \|y_n - y_m\| : \ \|x\| = 1\} \\
&= \|y_n - y_m\| \ \sup\{\|f(x)\| : \ \|x\| = 1\} \\
&= \|y_n - y_m\| \quad (\text{since } \|f\| = 1)
\end{aligned}$$

Which implies (T_n) is a Cauchy sequence in $\mathcal{B}(X,Y)$, since $\mathcal{B}(X,Y)$ is Banach, so there exists a $T \in \mathcal{B}(X,Y)$ such that $T_n \to T$ as $n \to \infty$.

Observe that,

$$\begin{aligned} T_n(x) \to T(x), \text{ as } n \to \infty \text{ in } Y, \ \forall x \in X \\ \text{(in particular for } x_0), \ T_n(x_0) \to T(x_0) \text{ as } n \to \infty \\ f(x_0)y_n \to T(x_0) \text{ as } n \to \infty \\ y_n \to T(x_0) \text{ as } n \to \infty \ \ (\text{since } f(x_0) = 1) \end{aligned}$$

Thus (y_n) is convergent in Y. So Y is a Banach space □

6.2 Uniform Boundedness Theorem

Theorem 6.12 (Uniform Boundedness Theorem)**.** *Let* $(X, \|\cdot\|)$ *be Banach space and* $(Y, \|\cdot\|)$ *be a normed linear space over* $\mathbb{F}$. *For an index set* $\mathcal{I}$, *let* $\{T_\alpha : \alpha \in \mathcal{I}\} \subseteq \mathcal{B}(X, Y)$. *Suppose for each* $x \in X$, $\exists M_x > 0$ *such that* $\|T_\alpha(x)\| \leq M_x$ *for all* $\alpha \in \mathcal{I}$. *Then* $\exists M > 0$ *such that* $\|T_\alpha\| \leq M, \forall \alpha \in \mathcal{I}$. *In other words, if the family* $\{T_\alpha : \alpha \in \mathcal{I}\}$ *is pointwise bounded, then it is uniformly bounded.*

Proof. Let $n \in \mathbb{N}$, define $\mathcal{A}_n = \{x \in X : \|T_\alpha(x)\| \leq n, \forall \alpha \in \mathcal{I}\}$
Clearly $\mathcal{A}_n \neq \phi, 0 \in \mathcal{A}_n, \forall \alpha \in \mathcal{I}$. Observe that,

$$\bigcap_{\alpha \in \mathcal{I}} (\| \cdot \| \circ T_\alpha)^{-1}([-n, n]) = \mathcal{A}_n$$

Hence, $\mathcal{A}_n$ is closed set, since $(\| \cdot \| \circ T_\alpha)$ is continuous functional, so it takes closed set to closed set, and the intersection of closed sets is a closed set.
Also, we have that

$$X = \bigcup_{n=1}^{\infty} \mathcal{A}_n$$

Since, clearly $\bigcup_{n=1}^{\infty} \mathcal{A}_n \subseteq X$
For the other inclusion, let $x \in X$. Then $\exists M_x$ (by our hypothesis) such that $\|T_\alpha(x)\| \leq M_x$ for all $\alpha \in \mathcal{I}$.
By the Archimedean property, $\exists N \in \mathbb{N}$ such that $M_x \leq N$

$$\begin{aligned} &\implies \|T_\alpha(x)\| \leq N \ \forall \alpha \in \mathcal{I} \\ &\implies x \in \{x \in X : \|T_\alpha(x)\| \leq N, \forall \alpha \in \mathcal{I}\} \\ &\implies x \in \mathcal{A}_N \implies X \subseteq \bigcup_{n=1}^{\infty} \mathcal{A}_n \\ &\implies X = \bigcup_{n=1}^{\infty} \mathcal{A}_n \end{aligned}$$

Since X is complete, by Baire's Category Theorem, $\exists N \in \mathbb{N}$ such that $(\mathcal{A}_N)^\circ \neq \phi$
i.e. there exists $x_0 \in X$ and $r \geq 0$ such that $B(x_0, r) \subseteq \mathcal{A}_N$
Now for $x \in X$,

$$x_0 + \frac{rx}{2\|x\|} \in B(x_0, r) \subseteq \mathcal{A}_N$$

$$\implies \left\| T_\alpha \left(x_0 + \frac{rx}{2\|x\|} \right) \right\| \leq N$$

$$\text{By triangle inequality, } \left\| \frac{r}{2\|x\|} T_\alpha(x) \right\| \leq \|T_\alpha(x_0)\| + \left\| T_\alpha \left(x_0 + \frac{rx}{2\|x\|} \right) \right\|$$

$$\leq M_{x_0} + N$$

$$\implies \|T_\alpha(x)\| \leq (M_{x_0} + N) \frac{2\|x\|}{r}$$

$$\implies \|T_\alpha\| \leq M, \text{ where } M = (M_{x_0} + N) \frac{2\|x\|}{r}$$

Hence, $\{T_\alpha\}$ is uniformly bounded. □

Exercise 6.2. If X is not a Banach space, then the above theorem may not be true. Construct a counter example.

6.3 Open Mapping Theorem & Applications

Definition 6.1 (Open map)**.** Let (X, d) an (Y, ρ) be two metric spaces. A function $f : X \to Y$ is called an **open map** if f takes open sets in X to open sets in Y. That is, f is an open map if $f(U)$ is open in Y whenever U is open in X.

Remark 6.1. We know that for a continuous function $f : X \to Y$, the inverse image of an open set in Y is open in X. An open map need not be continuous, and a continuous map need not be an open map. The reader is encouraged to think of some examples.

Exercise 6.3. Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and $f : X \to \mathbb{F}$ be a non-zero linear functional. Then f is an open map.

Lemma 6.1. *Let X and Y be normed linear spaces and $T : X \to Y$ be a non-zero linear open map. Then T is onto.*

Proof. We know that X is open in X. As T is an open map, $T(X)$ is an open subspace of Y. As there cannot be a proper open subspace of a normed linear space (corollary 2.16), it follows that $T(X) = Y \implies T$ is onto. □

Remark 6.2. The converse of lemma 6.1 is not true in general. That is, if T is linear and onto, then we cannot say that T is an open map. However, it is true under some special conditions (if T is bounded and X and Y are both Banach - see Open Mapping Theorem).

Lemma 6.2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed linear spaces over $\mathbb{F}$ and $T : X \to Y$ be a linear transformation. Then T is an open map if and only if $0 \in Int(T(B_X(0,1))$.*

Proof. ($\implies$) Assume that T is open map. Then T takes open sets to open sets. We know that $B_X(0,1)$ is open in X. Therefore, $T(B_X(0,1))$ is open in Y. Also, $T(0) = 0$ as T is linear. As $0 \in B_X(0,1)$, we have that $0 \in T(B_X(0,1))$. As $T(B_X(0,1))$ is open, every point (including 0) is an interior point of $T(B_X(0,1))$. Hence, $0 \in \text{Int}(T(B_X(0,1)))$.

($\impliedby$) Assume $0 \in \text{Int}(T(B_X(0,1))$, then $\exists r > 0$ such that $B_Y(0,r) \subseteq (T(B_X(0,1))$
To show that T is an open map, let $G \subseteq X$ be an open subset of X.

Claim: $T(G)$ is open in Y
Let $y \in T(G)$. Then $\exists x \in G$ such that $T(x) = y$. Since G is open, $\exists s > 0$ such that $B_X(x,s) \subseteq G$.

$$\begin{aligned}
&\implies x + sB_X(0,1) \subseteq G \\
&\implies T(x + sB_X(0,1)) \subseteq T(G) \\
&\implies T(x) + sT(B_X(0,1)) \subseteq T(G) \\
&\implies y + sB_Y(0,r) \subseteq y + sT(B_X(0,1)) \subseteq T(G) \\
&\quad [\text{Since } B_Y(0,r) \subseteq T(B_X(0,1))] \\
&\implies B_Y(y, sr) \subseteq T(G)
\end{aligned}$$

$\therefore$ y is an interior point of $T(G)$, and since $y \in T(G)$ is arbitrarily chosen, $T(G)$ is an open set in Y.
As $G \subseteq X$ was an arbitrary open set, we have that T sends every open set in X to open set in Y.
Hence T is an open map.

□

Theorem 6.13 (Open Mapping Theorem). *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces over $\mathbb{F}$. Let $T \in \mathcal{B}(X,Y)$, i.e. T is a continuous (bounded) linear transformation and T is onto. Then T is an open map.*

Proof. To show that T is an open map, it is enough to show that $0 \in \text{Int}(T(B_X(0,1))$ (lemma 6.2).

Observe that $X = \bigcup_{n=1}^{\infty} B_X(0,n)$

$$\implies T(X) = T\left(\bigcup_{n=1}^{\infty} B_X(0,n)\right) = \bigcup_{n=1}^{\infty} T(B_X(0,n)) = \bigcup_{n=1}^{\infty} n\, T(B_X(0,1))$$
$$\implies Y = T(X) = \bigcup_{n=1}^{\infty} n\, T(B_X(0,1)) = \bigcup_{n=1}^{\infty} n\, \overline{T(B_X(0,1))}$$

Since Y is Banach space, by Baire's Category Theorem, Y cannot be written as a countable union of nowhere dense sets. So, at least one of the sets in the union is not nowhere dense $\implies$ there exists a $N \in \mathbb{N}$ such that

$$\left(N\, \overline{T(B_X(0,1))}\right)^{\circ} \neq \phi \implies \left(\overline{T(B_X(0,1))}\right)^{\circ} \neq \phi$$

i.e. $\exists y \in Y$ and $r > 0$ such that

$$B_Y(y,r) \subseteq \overline{T(B_X(0,1))}$$

Observe that,

$$\begin{aligned} B_Y(0,r) &= \frac{1}{2}B_Y(y,r) - \frac{1}{2}B_Y(y,r) \\ &\subseteq \frac{1}{2}\,\overline{T(B_X(0,1))} - \frac{1}{2}\,\overline{T(B_X(0,1))} \\ &\subseteq \overline{T(B_X(0,1))} \end{aligned}$$

Claim: for some $0 < \varepsilon < 1$ such that $B_Y(0,r) \subseteq T(B_X(0,1+\varepsilon))$
Let $\varepsilon > 0$ be given and $y \in B_Y(0,r)$. Since $B_Y(0,r) \subseteq \overline{T(B_X(0,1))} \implies y \in \overline{T(B_X(0,1))}$
Then $\exists x_1 \in B_X(0,1)$ such that

$$\|y - T(x_1)\| < \varepsilon r$$

Take $y_1 = y - T(x_1)$ observe that $y_1 \in B_Y(0,\varepsilon r) \subseteq \overline{T(B_X(0,\varepsilon))}$
Then $\exists x_2 \in B_X(0,\varepsilon)$ such that

$$\|y_1 - T(x_2)\| < \frac{\varepsilon r}{2}$$

Now take $y_2 = y_1 - T(x_2)$, and observe $y_2 \in B_Y\left(0, \frac{\varepsilon r}{2}\right) \subseteq \overline{T\left(B_X\left(0, \frac{\varepsilon}{2}\right)\right)}$
Then $\exists x_3 \in B_X\left(0, \frac{\varepsilon}{2}\right)$ such that

$$\|y_2 - T(x_3)\| < \frac{\varepsilon r}{2^2}$$

where $y_2 = y_1 - T(x_2)$ and $y_1 = y - T(x_1)$, So $y_2 = y - T(x_1) - T(x_2)$, i.e. we have

$$\|y - T(x_1) - T(x_2) - T(x_3)\| < \frac{\varepsilon r}{2^2}$$

If we proceed in the same way, at the nth stage we will have: $\exists x_n \in B_X(0, \frac{\varepsilon}{2^{n-2}})$ such that

$$\|y - T(x_1) - T(x_2) - \cdots - T(x_n)\| < \frac{\varepsilon r}{2^{n-1}} \tag{6.1}$$

Now, since $\forall n \in \mathbb{N},\ x_n \in B_X\left(0, \frac{\varepsilon}{2^{n-2}}\right)$

$$\begin{aligned}
\sum_{n=1}^{\infty} \|x_n\| &= \|x_1\| + \sum_{n=2}^{\infty} \|x_n\| \\
&\leq 1 + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-2}} \\
&\leq 1 + \varepsilon \sum_{n=0}^{\infty} \frac{1}{2^n} \\
&\leq 1 + 2\varepsilon
\end{aligned}$$

And since X is a Banach space, every absolutely convergent series is convergent (theorem 3.1) in X. Here we have $\sum_{n=1}^{\infty} x_n$ is convergent. Let $x := \sum_{n=1}^{\infty} x_n$

$$\begin{aligned}
T(x) = T\left(\sum_{n=1}^{\infty} x_n\right) &= T\left(\lim_{k\to\infty} \sum_{n=1}^{k} x_n\right) \\
&= \lim_{k\to\infty} T\left(\sum_{n=1}^{k} x_n\right) \quad \text{(since } T \text{ is continuous)} \\
&= \lim_{k\to\infty} \left(\sum_{n=1}^{k} T(x_n)\right)
\end{aligned}$$

$$\implies T(x) = \sum_{n=1}^{\infty} T(x_n) = y \text{ (from equation 6.1)}$$

So we have $x \in X$ such that $\|x\| < 1 + 2\varepsilon$ and $y = T(x)$

$$\begin{aligned}
&\implies y \in T(B_X(0, 1+2\varepsilon)) \\
&\implies B_Y(0, r) \subseteq T(B_X(0, 1+2\varepsilon)) \\
&\implies (1+2\varepsilon)\ B_Y\left(0, \frac{r}{1+2\varepsilon}\right) \subseteq (1+2\varepsilon)\ T(B_X(0,1)) \\
&\implies B_Y\left(0, \frac{r}{1+2\varepsilon}\right) \subseteq T(B_X(0,1)) \\
&\implies 0 \in Int(T(B_X(0,1)))
\end{aligned}$$

Hence T is an open map by lemma 6.2. □

Exercise 6.4. Show that we cannot drop the conditions of X and Y being Banach in Open

Mapping Theorem. The reader is highly encouraged to think of some examples.

6.4 Bounded Inverse Theorem & Applications

Theorem 6.14 (Bounded Inverse theorem)**.** *Let X, Y be Banach spaces over $\mathbb{F}$. Let T be a bounded linear transformation ($T \in \mathcal{B}(X, Y)$). If T is bijective, then then T^{-1} is bounded, i.e. $T^{-1} \in \mathcal{B}(X, Y)$.*

Proof. Given $T : X \to Y$ is a bounded, linear, and bijective map. Then by Open Mapping Theorem, T is an open map.
Thus, T is a bijective, continuous (bounded), and open map, which implies that T is a homeomorphism.
Hence, $T^{-1} : Y \to X$ is a continuous linear transformation, i.e. $T \in \mathcal{B}(Y, X)$. □

Corollary 6.15. *Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be Banach spaces (where $\|\cdot\|_1$ and $\|\cdot\|_2$ are arbitrary norms). Assume that $\exists c > 0$ such that $\|x\|_2 \leq c \|x\|_1$, $\forall x \in X$. Then there exists a $d > 0$ such that $\|x\|_1 \leq c \|x\|_2$, $\forall x \in X$.*

Proof. Let us define $\mathcal{I} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ by $\mathcal{I}(x) = x, \ \forall x \in X$
Obviously, the identity map $\mathcal{I}$ is bijective and linear. Since,

$$\|\mathcal{I}(x)\|_2 = \|x\|_2 \leq c \ \|x\|_1$$

We have that $\mathcal{I}$ is a bounded, linear, and bijective map. By Bounded Inverse theorem, we have

$$\mathcal{I}^{-1} \in \mathcal{B}((X, \|\cdot\|_2), (X, \|\cdot\|_1))$$

So (by definition 5.3), there exists some $d > 0$ such that

$$\begin{aligned} \left\|\mathcal{I}^{-1}(x)\right\|_1 &\leq d \ \|x\|_2 \\ \implies \|x\|_1 &\leq d \ \|x\|_2 \end{aligned}$$

Thus, our claim is proved. □

Exercise 6.5. Let $x_0 \in [a, b]$. Show that $(C^1[a, b], \|\cdot\|_{x_0})$ is Banach space, where

$$\|f\|_{x_0} = |f(x_0)| + \|f'\|_\infty, \ \forall f \in C^1[a, b]$$

Exercise 6.6. Let $x_0, y_0 \in [a, b]$ and $f \in C^1[a, b]$. Define the norms $\|\cdot\|_{x_0}$ and $\|\cdot\|_{y_0}$ on $C^1[a, b]$ as
$\|f\|_{x_0} = |f(x_0)| + \|f'\|_\infty$ and $\|f\|_{y_0} = |f(y_0)| + \|f'\|_\infty$ respectively. Show that the norms

$\|\cdot\|_{x_0}$ and $\|\cdot\|_{y_0}$ are equivalent, i.e., there exists some $\alpha, \beta > 0$ such that

$$\alpha \|f\|_{x_0} \leq \|f\|_{y_0} \leq \beta \ \|f\|_{x_0}$$

6.5 Closed Graph Theorem

Exercise 6.7. Let $1 \leq p \leq \infty$ and (X, d) and (Y, ρ) be two metric spaces.
Consider $X \times Y = \{(x, y) : x \in X, \ y \in Y\}$.
Define $D_p : X \times Y \to \mathbb{R}$ by

$$D_p((x, y), (x_1, y_1)) = \begin{cases} (d(x, x_1)^p + \rho(y, y_1)^p)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max\{d(x, x_1), \rho(y, y_1)\}, & p = \infty \end{cases}$$

Check that D_p is a metric on $X \times Y$.

Exercise 6.8. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces over $\mathbb{F}$.
Let $X \times Y = \{(x, y) : x \in X, \ y \in Y\}$.
Let $(x, y), \ (x_1, y_1) \in X \times Y$ and $\alpha \in \mathbb{F}$. Define

$$(x, y) + (x_1, y_1) = (\underbrace{x + x_1}_{\text{addition in } X}, \underbrace{y + y_1}_{\text{addition in } Y})$$

$$\alpha \cdot (x, y) = (\underbrace{\alpha \cdot x}_{\text{multiplication in } X}, \underbrace{\alpha \cdot y}_{\text{multiplication in } Y})$$

Check that $(X \times Y, +, \cdot)$ is a vector space over $\mathbb{F}$.

Remark 6.3. The vector space as defined above in exercise 6.8 is denoted by $X \oplus Y$.

Exercise 6.9. Let $(x, y) \in X \oplus Y$ and $1 \leq p \leq \infty$. Define

$$\|(x, y)\|_p = \begin{cases} (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max\{\|x\|_X, \|y\|_Y\}, & p = \infty \end{cases}$$

Check that $\|\cdot\|_p$ is a norm on $X \oplus Y$.

Exercise 6.10. Prove the following:

1. $(X \oplus Y, \|\cdot\|_p)$ is a Banach space if and only if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.

2. $(X \oplus Y, \|\cdot\|_p)$ is separable if and only if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are separable.

Definition 6.2 (Graph)**.** Let X, Y be two sets and $f : X \to Y$ is a function. The Graph of f is defined and denoted by $G(f) := \{(x, f(x)) \ : \ x \in X\} \ (\subseteq X \times Y)$.

Lemma 6.3. *Let X, Y are normed linear spaces over $\mathbb{F}$ and $T \in B(X,Y)$. Show $(X \oplus Y, \|\cdot\|_\infty)$ is a normed linear space and $G(T) = \{(x, T(x)) \ : \ x \in X\}$ is closed subset in $(X \oplus Y, \|\cdot\|_\infty)$.*

Proof. From exercise 6.9, it follows that that $(X \oplus Y, \|\cdot\|_\infty)$ is a normed linear space.
Claim: $G(T) = \{(x, T(x)) \ : \ x \in X\}$ is a closed subset.
Let $(x_n, T(x_n))$ be a sequence in $G(T)$ such that $(x_n, T(x_n)) \to (x, y)$ in $X \oplus Y$. To show that $G(T)$ is closed, we need to prove that the limit $(x, y) \in G(T)$, i.e. $y = T(x)$.

$$\begin{aligned}
\text{i.e. } \|(x_n, T(x_n)) - (x,y)\|_\infty &\to 0 \text{ as } n \to \infty \\
\|(x_n - x, T(x_n) - y)\|_\infty &\to 0 \text{ as } n \to \infty \\
\text{i.e. } \max\{\|x_n - x\|_X, \|T(x_n) - y\|_Y\} &\to 0 \text{ as } n \to \infty \\
\implies \|x_n - x\|_X \to 0 \text{ and } \|T(x_n) - y\|_Y &\to 0 \text{ as } n \to \infty \\
\implies x_n \to x \text{ and } T(x_n) &\to y \text{ as } n \to \infty \\
\text{Since } T \text{ is continuous, } x_n \to x \implies T(x_n) &\to T(x) \text{ as } n \to \infty \\
&\implies y = T(x)
\end{aligned}$$

Hence, $(x, y) = (x, T(x)) \in G(T)$. So, $G(T)$ is a closed subset of $(X \oplus Y, \|\cdot\|_\infty)$. □

Theorem 6.16 (Closed Graph theorem)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces over $\mathbb{F}$ and $T : X \to Y$ be a linear transformation. If $G(T)$ (Graph of T) is closed in $(X \oplus Y, \|\cdot\|_\infty)$, then T is bounded (continuous), i.e. $T \in \mathcal{B}(X,Y)$.*

Proof. Given that, $G(T)$ is closed in $(X \oplus Y, \|\cdot\|_\infty)$.
Let us define $S : G(T) \to X$ by $S(x, T(x)) = x, \ \forall (x, T(x)) \in G(T)$.
Claim 1: S is a linear transformation.
Let $x_1, \ x_2 \in X$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned}
S((\alpha x_1, \alpha T x_1) + (x_2, T x_2)) &= S((\alpha x_1 + x_2, \alpha T x_1 + T x_2)) \\
&= \alpha x_1 + x_2 \\
&= \alpha S((x_1, T x_1)) + S((x_2, T x_2))
\end{aligned}$$

Claim 2: S is bijective.
$S((x_1, T x_1)) = S((x_2, T x_2)) \implies x_1 = x_2$ (by definition), so S is one-one.
$\forall x \in X$, $(x, Tx) \in G(T)$ is its pre-image, so S is onto.
Observe that,

$$\|S(x, Tx)\|_X = \|x\|_X \leq \max\{\|x\|_X, \|Tx\|_Y\} = \|(x, Tx)\|_\infty \ \forall (x, Tx) \in G(T)$$

Thus, S is a bounded linear and bijective map. As X and Y are Banach, from exercise 6.10, it follows that $X \oplus Y$ is Banach. Since $G(T)$ is a closed subset (see lemma 6.3) of the Banach

space $(X \oplus Y, \|\cdot\|_\infty)$, we have that $G(T)$ is a Banach space.
Now as both $G(T)$ and X are Banach, by Bounded Inverse theorem, we have that $S^{-1}: X \to G(T)$ is a bounded linear transformation.
Since S^{-1} is bounded, there exists a $d > 0$ such that

$$\begin{aligned}
\left\|S^{-1}(x)\right\|_\infty &\leq d \ \|x\|_X, \ \forall x \in X \\
\implies \|(x, T(x))\|_\infty &\leq d \ \|x\|_X, \ \forall x \in X \\
\implies \max\{\|x\|_X, \|T(x)\|_Y\} &\leq d \ \|x\|_X, \ \forall x \in X \\
\implies \|T(x)\|_Y &\leq d \ \|x\|_X, \ \forall x \in X \\
\implies T &\in \mathcal{B}(X, Y)
\end{aligned}$$

Hence, T is a bounded linear transformation.

□

Definition 6.3 (Closed Operator)**.** Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be two Banach spaces over $\mathbb{F}$, and $T: D(T)(\subseteq X) \to Y$ is linear transformation, where $D(T)$ is the domain of T. Then T is called a closed operator if $G(T)$ is closed in $(X \oplus Y, \|\cdot\|_\infty)$.

Remark 6.4. If X is not a Banach space, then the Closed Graph theorem may not hold. For example, take $X = (C^1[0,1], \|\cdot\|_\infty)$ and $Y = (C[0,1], \|\cdot\|_\infty)$. We know from example 3.2 that X is not Banach, and from exercise 3.3 that Y is Banach. Verify:

1. T is a linear transformation $(T(f) = f')$.
2. $G(T)$ is closed in $X \oplus Y$.
3. But T is not bounded (i.e., Closed Graph theorem fails to hold if X is not a Banach space).

Exercise 6.11. Construct a counter-example to show that the Closed Graph theorem does not hold if X is Banach but Y is not Banach.

7 Dual Spaces

Recall the definition of topological dual spaces from definition 5.7.

Theorem 7.1. *A normed linear space X is separable if its dual is separable.*

Proof. Let X be a normed linear space such that its dual X^* is separable. The result is obvious if $X = \{0\}$, since then $\{0\}$ is the countable closed subset, whose closure is X itself. Assume that $X \neq \{0\}$.
We know that every subspace of a separable metric space is separable. So, as X^* is separable, we have $S_{X^*} = \{f \in X^* : \|f\| = 1\}$ is separable. Let $\{f_1, f_2, \ldots, f_n, \ldots\}$ be a countable dense subset of S_{X^*}. Since $\|f_n\| = 1$ $\forall n \in \mathbb{N}$ (as all f_n's are in S_{X^*},

$$\exists x_n \in X \text{ such that } \|x_n\| = 1 \text{ and } |f_n(x_n)| > \frac{1}{2} \ \forall n \in \mathbb{N}$$

(since $\|f_n\| = \sup\{|f_n(x)| : x \in X, \|x\| = 1\}$)

We show that the separable space $Y = span\{x_1, x_2, \ldots\}$ is dense in X.
Suppose Y is not dense in X, i.e., $\overline{Y} \neq X$. Let $y_0 \in X \setminus \overline{Y}$. Then by corollary 6.8, $\exists f \in X^*$ such that $\|f\| = 1$, $f(y) = 0$ $\forall y \in \overline{Y}$, and $f(y_0) = dist(y_0, \overline{Y})$. In particular, $f(x_n) = 0$ $\forall n \in \mathbb{N}$, so that

$$\underbrace{\|f - f_n\|}_{\text{this is the supremum}} \geq |(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2} \ \forall n \in \mathbb{N}$$

$\because \|f\| = 1$, we have $f \in S_{X^*}$. But this contradicts that $\{f_1, f_2 \ldots\}$ is dense in S_{X^*}.
Hence, $\overline{Y} = X$, and X is separable. □

Corollary 7.2. *The dual of a non-separable normed linear space cannot be homeomorphic with a separable space.*

Proof. By the contrapositive of theorem 7.1, if X is not separable, then its dual X^* is not separable. Hence, the dual X^* cannot be homeomorphic with a separable space. □

Remark 7.1. The converse of theorem 7.1 is not true in general. For example, the space $(\ell^1(\mathbb{N}), \|\cdot\|_1)$ is separable (see proposition 4.2), but its dual, $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ (see exercise 7.1) is not separable (see proposition 4.3).

Definition 7.1 (Double dual)**.** Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and X^* be its dual. The space of all bounded linear transformations from X^* to $\mathbb{F}$ is called the double dual of X, and is denoted by X^{**}. That is,

$$X^{**} = \mathcal{B}(X^*, \mathbb{F})$$

Proposition 7.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Then its double dual is a Banach space.*

Proof. We know from theorem 6.11 that $\mathcal{B}(X, Y)$ is Banach if and only if Y is Banach. Also, $\mathbb{F}$ is Banach. Hence, $X^{**} = \mathcal{B}(X^*, \mathbb{F})$ is Banach. $\square$

Remark 7.2. Recall that if X is a normed linear space, then the elements in X^* are linear functionals on X. That is, if $f \in X^*$, then $f : X \to \mathbb{F}$ is a linear functional.

Lemma 7.1. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$. Let $x \in X$. Define $\varphi_x : X^* \to \mathbb{F}$ by*

$$\varphi_x(f) = f(x) \; \forall f \in X^*$$

φ_x is known as the ***evaluation map at*** *x. Then φ_x is a continuous linear functional on X^* (that is, $\phi_x \in X^{**}$) and $\|\varphi_x\| = \|x\|$.*

Proof. (Linearity) For all $f, g \in X^*$ and $\alpha \in \mathbb{F}$,

$$\begin{aligned} \varphi_x(f + g) &= (f + g)(x) = f(x) + g(x) = \varphi_x(f) + \varphi_x(g) \\ \varphi_x(\alpha f) &= (\alpha f)(x) = \alpha \cdot f(x) = \alpha \varphi_x(f) \end{aligned}$$

(Continuity) Let $f \in X^*$. Then

$$|\varphi_x(f)| = |f(x)| \le \|f\|\|x\| \text{ (by lemma 5.1, since } f \text{ is bounded)}$$

$\implies \varphi_x$ is continuous.

(Norm preserving) From the above,

$$\begin{aligned} |\varphi_x(f)| = |f(x)| &\le \|f\|\|x\| \; \forall f \in X^* \\ \implies \left| \varphi_x\left(\frac{f}{\|f\|}\right) \right| &\le \|x\| \; \forall f \in X^* \\ \implies |\varphi_x(f)| &\le \|x\| \; \forall f \in S_{X^*} \\ \implies \|\varphi_x\| &\le \|x\| \end{aligned}$$

By corollary 6.6, $\|\varphi_x\| = \sup\{|\varphi_x(f)| : f \in S_{X^*}\} = \sup\{|f(x)| : f \in S_{X^*}\}$
By corollary 6.3, for any $x \in X \setminus \{0\}$, $\exists f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.
$\implies \|x\| \le \|\varphi_x\|$
Thus, $\|\varphi_x\| = \|x\|$ $\square$

Definition 7.2 (Linear isometry)**.** Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed linear spaces over $\mathbb{F}$. A linear map $T : X \to Y$ is called a **linear isometry** if $\|Tx\| = x \; \forall x \in X$.

Definition 7.3 (Linear homeomorphism)**.** Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed linear spaces over $\mathbb{F}$. A linear map $T : X \to Y$ is called a **linear homeomorphism** if T is bijective and bi-continuous (T and T^{-1} are both continuous).

Definition 7.4 (Linear isometrical isomorphism)**.** Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed linear spaces over $\mathbb{F}$. A linear map $T : X \to Y$ is called a **linear isometric isomorphism** if T is a Linear isometry and T is surjective. We say that X and Y are linearly isometrically isomorphic if $\exists$ a linear isometrical isomorphism between them.

Lemma 7.2 (A linear isometry is injective)**.** *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed linear spaces and $T : X \to Y$ be a linear isometry. Then T is injective.*

Proof. As T is an isometry, we have $\|Tx\| = x \ \forall x \in X$.
Let $x_1, x_2 \in X$ be such that $T(x_1) = T(x_2)$. Then

$$\begin{aligned} T(x_1) - T(x_2) &= 0 \\ \implies T(x_1 - x_2) &= 0 \text{ (by linearity of } T) \\ \implies \|T(x_1 - x_2)\| &= 0 \text{ (by property of } \|\cdot\|) \\ \implies \|x_1 - x_2\| &= 0 \ (\because T \text{ is an isometry}) \\ \implies x_1 &= x_2 \end{aligned}$$

□

Corollary 7.3 (Canonical linear isometry)**.** *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$. Define $\Phi : X \to X^{**}$ by $\Phi(x) = \varphi_x \ \forall x \in X$, where $\varphi_x \in X^{**}$ is as defined in lemma ??. Then Φ is linear and $\|\Phi(x)\| = \|x\| \ \forall x \in X$. The map Φ is called **canonical linear isometry** between X and X^{**}.*

Proof. (Linearity of Φ): Let $x, y \in X$. Then

$$\Phi(x + y) = \varphi_{x+y} = \varphi_x + \varphi_y = \Phi(x) + \Phi(y)$$

because

$$\varphi_{x+y}(f) = f(x + y) = f(x) + f(y) = \varphi_x(f) + \varphi_y(f) = (\varphi_x + \varphi_y)(f) \ \forall f \in X^*$$

and $\Phi(\alpha x) = \varphi_{\alpha x} = \alpha\varphi_x = \alpha\Phi(x)$

(Isometry) We have

$$\|\Phi(x)\| = \|\varphi_x\| = \|x\| \ \forall x \in X \text{ (by lemma 7.1)}$$

□

Definition 7.5 (Reflexive space). A normed linear space $(X, \|\cdot\|)$ is called reflexive if the Canonical linear isometry $\Phi : X \to X^{**}$ is surjective.

7.1 Duals of Some Sequence Spaces

Recall the Sequence Spaces from section 2.1 (page 18). In this section, we will find some of their dual spaces.

Let $z \in \mathbb{C}$. Define $Sgn : \mathbb{C} \to \{a \in \mathbb{C} : |a| = 1\} \cup \{0\}$ by

$$Sgn(z) = \begin{cases} \frac{|z|}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Remark 7.3. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$ and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.
Define $y_j = sgn(x_j)|x_j|^{p/q}\ \forall j = 1, \ldots, n$. Then

$$\begin{aligned} |y_j| &= |x_j|^{p/q}\ \forall j = 1, \ldots, n \\ \implies |y_j|^q &= |x_j|^p\ \forall j = 1, \ldots, n \\ \implies \sum_{j=1}^{n} |y_j|^q &= \sum_{j=1}^{n} |x_j|^p\ \forall j = 1, \ldots, n \\ \implies \|y\|_q^q &= \|x\|_p^p\ \forall j = 1, \ldots, n \end{aligned}$$

Lemma 7.4 (Dual of $(\mathbb{F}^n, \|\cdot\|_p)$). *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\mathbb{F}^n, \|\cdot\|_p)^* \cong (\mathbb{F}^n, \|\cdot\|_q)$. That is, $\exists$ a Linear isometrical isomorphism between $(\mathbb{F}^n, \|\cdot\|_p)^*$ and $(\mathbb{F}^n, \|\cdot\|_q)$. In other words, the dual space of $(\mathbb{F}^n, \|\cdot\|_p)$ for some $1 < p < \infty$ can be identified as $(\mathbb{F}^n, \|\cdot\|_q)$, where q is the Conjugate exponent of p.*

Proof. Let us denote $(\mathbb{F}^n, \|\cdot\|_p)$ by $\ell^p(n)$ and $(\mathbb{F}^n, \|\cdot\|_q)$ by $\ell^q(n)$.
Let $\mathbf{x} = (x_1, \ldots, x_n) \in \ell^q(n) = (\mathbb{F}^n, \|\cdot\|_q)$. Define $\eta_{\mathbf{x}} : \ell^p(n) \to \mathbb{F}$ by

$$\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^{n} y_i x_i\ \forall \mathbf{y} = (y_1, \ldots, y_n) \in \ell^p(n)$$

Clearly, $\eta_{\mathbf{x}}$ is a linear functional. Observe that

$$|\eta_{\mathbf{x}}(\mathbf{y})| = \left|\sum_{i=1}^{n} y_i x_i\right| \leq \|\mathbf{y}\|_p \|\mathbf{x}\|_q \quad \forall \mathbf{y} \in \ell^p(n) \text{ (by Hölder's inequality)}$$

$\implies$ $\eta_{\mathbf{x}}$ is continuous and $\|\eta_{\mathbf{x}}\| \leq \|\mathbf{x}\|_q$
Claim: $\|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$

We know that by corollary 6.6,

$$\|\eta_{\mathbf{x}}\| = \sup \underbrace{\{|\eta_{\mathbf{x}}(\mathbf{y})| : \mathbf{y} \in S_{\ell^p(n)}\}}_{\text{We want to show that } \|\mathbf{x}\|_q \in \text{ here}}$$

We want to show that $\exists \mathbf{y} \in \ell^p(n)$ such that $\eta_{\mathbf{x}}(\mathbf{y}) = \|\mathbf{x}\|_q$

Define $y_j = sgn(x_j)|x_j|^{1/p}\ \forall j = 1, \dots, n$
By remark 7.3, we have $\|\mathbf{y}\|_p^p = \|\mathbf{x}\|_q^q \implies \|\mathbf{y}\|_p = \|\mathbf{x}\|_q^{q/p}$
Now, $\frac{\mathbf{y}}{\|\mathbf{y}\|_p} \in S_{\ell^p(n)}$. Then

$$\begin{aligned}
\eta_{\mathbf{x}}\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_p}\right) = \sum_{i=1}^{n} \frac{y_i}{\|\mathbf{y}\|_p} x_i &= \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{n} Sgn(x_i)|x_i|^{q/p} x_i \\
\implies \left|\eta_{\mathbf{x}}\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_p}\right)\right| &= \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{n} |x_i|^{\frac{q}{p}+1} \\
&= \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{n} |x_i|^q \quad \left(\because \frac{1}{p} + \frac{1}{q} = 1\right) \\
&= \frac{1}{\|\mathbf{y}\|_p} \|\mathbf{x}\|_q^q \\
&= \frac{\|\mathbf{x}\|_q^q}{\|\mathbf{x}\|_q^{q/p}} \\
&= \|\mathbf{x}\|_q^{q-\frac{q}{p}} = \|\mathbf{x}\|_q
\end{aligned}$$

Hence, $\|\mathbf{x}\|_q \in \{|\eta_{\mathbf{x}}(y)| : y \in S_{\ell^p(n)}\}$ and $\|\eta_{\mathbf{x}}\| \leq \|\mathbf{x}\|_q$
$\implies \|\mathbf{x}\| = \sup\{|\eta_{\mathbf{x}}(y)| : y \in S_{\ell^p(n)}\} \implies \|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$

Now, define $\Phi : \ell^q(n) \to \ell^p(n)^*$ by $\mathbf{x} \mapsto \eta_{\mathbf{x}}$, i.e. $\Phi(\mathbf{x}) = \eta_{\mathbf{x}}\ \forall \mathbf{x} \in \ell^q(n)$.
Claim: Φ is linear, isometry, and onto.
Clearly, Φ is linear because $\eta_{\mathbf{x}} \in \ell^p(n)^*$ (i.e., $\eta_{\mathbf{x}}$ is a linear functional on $\ell^p(n)$).

$$\|\Phi(\mathbf{x})\| = \|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q\ \forall \mathbf{x} \in \ell^q(n)$$

Thus, Φ is an isometry.
To show that Φ is onto, we have to show that any linear functional g on $\ell^p(n)$ is of the form $\eta_{\mathbf{x}}$ for some $\mathbf{x} \in \ell^q(n)$. Choose $g \in \ell^p(n)^*$. We have to select $\mathbf{x} \in \ell^q(n)$ such that $\eta_{\mathbf{x}} = g$.
Let $\{e_1, \dots, e_n\}$ be a basis for $\mathbb{F}^n$ and $\mathbf{x} = (g(e_1), g(e_2), \dots, g(e_n))$. Clearly, $\mathbf{x} \in \ell^q(n)$. Then

$$\eta_{\mathbf{x}}(e_i) = x_i = g(e_i)\ \forall i = 1, \dots, n$$

This means that g and $\eta_{\mathbf{x}}$ agree on the basis elements. As every $\mathbf{y} \in \ell^p(n)$ is a linear combination of the basis elements, we have $g \equiv \eta_{\mathbf{x}}$.
Therefore, $(\mathbb{F}^n, \|\cdot\|_p)^* \cong (\mathbb{F}^n, \|\cdot\|_q),\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$ □

Lemma 7.5 (Dual of $(c_{00}(\mathbb{N}), \|\cdot\|_p)$). *The dual of* $(c_{00}(\mathbb{N}), \|\cdot\|_p)$ *is* $(\ell^q(\mathbb{N}), \|\cdot\|_q)$, *where* $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *That is,*

$$(c_{00}(\mathbb{N}), \|\cdot\|_p)^* \cong (\ell^q(\mathbb{N}), \|\cdot\|_q)$$

Proof. We will show that $\exists$ a linear isometrical isomorphism $\Phi : (\ell^q(\mathbb{N}), \|\cdot\|_q) \to (c_{00}(\mathbb{N}), \|\cdot\|_p)^*$.
Let $\mathbf{x} = (x_n) \in \ell^q(\mathbb{N})$. Define $\eta_{\mathbf{x}} : (c_{00}(\mathbb{N}), \|\cdot\|_p) \to \mathbb{F}$ by $\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^{\infty} y_i x_i\ \forall \mathbf{y} = (y_n) \in c_{00}(\mathbb{N})$.
Clearly, $\eta_{\mathbf{x}}$ is linear, so $\eta_{\mathbf{x}} \in (c_{00}(\mathbb{N}), \|\cdot\|_p)^*$. Note that as $\mathbf{y} \in c_{00}(\mathbb{N})$, $\exists N \in \mathbb{N}$ such that $y = (y_1, \ldots, y_N, 0, 0, \ldots)$. Thus,

$$\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^{N} y_i x_i\ \forall \mathbf{y} = (y_1, \ldots, y_n, 0, 0, \ldots) \in c_{00}(\mathbb{N})$$

Now, observe that

$$|\eta_{\mathbf{x}}(\mathbf{y})| = \left|\sum_{i=1}^{N} y_i x_i\right| \leq \left|\sum_{i=1}^{\infty} y_i x_i\right| \leq \|\mathbf{y}\|_p \|\mathbf{x}\|_q, \text{ by Hölder's inequality.}$$

Hence, $\|\eta_{\mathbf{x}}\| \leq \|\mathbf{x}\|_q$

Claim: $\|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$
We know, that $\|\eta_{\mathbf{x}}\| = \sup\{|\eta_{\mathbf{x}}(\mathbf{y})| : \|\mathbf{y}\| = 1,\ \text{or } \mathbf{y} \in S_{c_{00}(\mathbb{N})}\}$
We need to find a $\mathbf{y} \in c_{00}(\mathbb{N})$ such that $\|\mathbf{y}\|_p = 1$ and $\|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$.
Let $\mathbf{y} = (y_1, \ldots, y_N, 0, 0, \ldots)$, where $y_i = Sgn(x_i)|x_i|^{q/p}\ \forall i = 1, \ldots, N$
Clearly, $\mathbf{y} \in c_{00}(\mathbb{N})$, $\frac{\mathbf{y}}{\|\mathbf{y}\|_p} \in S_{c_{00}(\mathbb{N})}$, and $\|\mathbf{y}\|_p^p = \|\mathbf{x}\|_q^q$ by remark 7.3.

$$\begin{aligned}
\eta_{\mathbf{x}}\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_p}\right) &= \sum_{i=1}^{\infty} \frac{1}{\|\mathbf{y}\|_p} Sgn(x_i)|x_i|^{q/p} x_i \\
\implies \left|\eta_{\mathbf{x}}\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_p}\right)\right| &= \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{\infty} |x_i|^{\frac{q}{p}+1} = \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{\infty} |x_i|^q \\
&= \frac{\|\mathbf{x}\|_q^q}{\|\mathbf{y}\|_p} = \|\mathbf{x}\|_1^{q-\frac{q}{p}} = \|\mathbf{x}\|_q
\end{aligned}$$

Hence, $\|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$

Now, define $\Phi : (\ell^q(\mathbb{N}), \|\cdot\|_q) \to (c_{00}(\mathbb{N}), \|\cdot\|_p)^*$ by $\mathbf{x} \mapsto \eta_{\mathbf{x}}\ \forall \mathbf{x} = (x_n) \in (\ell^q(\mathbb{N}), \|\cdot\|_q)$

Clearly, Φ is linear. And,

$$\|\Phi(\mathbf{x})\| = \|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q \ \ \forall \mathbf{x} = (x_n) \in \ell^q(\mathbb{N})$$

$\implies$ Φ is an isometry.

Claim: Φ is onto.
Let $g \in (c_{00}(\mathbb{N}), \|\cdot\|_p)^*$. We have to show that any such g is of the form $\eta_{\mathbf{x}}$ for some $\mathbf{x} \in \ell^q(\mathbb{N})$. That is, we have to choose an $\mathbf{x} \in \ell^q(\mathbb{N})$ such that $g = \eta_{\mathbf{x}}$.
For $e_i = (0, \ldots, 0, \underbrace{1}_{i^{th}}, 0, 0, \ldots)$, Consider $\mathbf{x} = (g(e_1), g(e_2), \ldots)$. Let $\mathbf{y} = (y_1, y_2, \ldots, y_N, 0, 0, \ldots) \in c_{00}(\mathbb{N})$, and $w_N = (g(e_1), \ldots, g(e_N), 0, 0, \ldots)$ be the truncations of $\mathbf{x}$ $\forall N \in \mathbb{N}$.
Claim: $\mathbf{x} \in \ell^q(\mathbb{N})$ and $\eta_{\mathbf{x}} = g$.

$$\begin{aligned} g(y_N) = \eta_{\mathbf{x}}(y_N) = \sum_{j=1}^{\infty} y_j x_j &= \sum_{j=1}^{\infty} Sgn(x_j)|x_j|^{q/p} x_j \\ \implies |g(y_N)| &= \sum_{j=1}^{\infty} |x_j|^{\frac{q}{p}+1} \\ &= \sum_{j=1}^{N} |x_j|^q \\ &= \sum_{j=1}^{N} |g(e_j)|^q = \|w_N\|_q^q \end{aligned}$$

Thus,

$$\begin{aligned} g(y_N) &\leq \|g\| \, \|y_N\|_p \ \ \forall N \in \mathbb{N} \\ \implies \|w_N\|_q^q &\leq \|g\| \, \|y_N\|_p = \|g\| \, \|w_N\|_q^{q/p} \\ \implies \|w_N\|_q^{q-\frac{q}{p}} &\leq \|g\| \ \forall N \in \mathbb{N} \\ \implies \|w_N\|_q &\leq \|g\| \ \forall N \in \mathbb{N} \\ \implies \|\mathbf{x}\|_q &\leq \|g\| \text{ (as } w_N\text{'s are truncations of } \mathbf{x}) \\ \implies \mathbf{x} &\in \ell^q(\mathbb{N}) \end{aligned}$$

For $\mathbf{y} = (y_1, y_2, \ldots, y_N, 0, 0, \ldots) \in c_{00}(\mathbb{N})$, we have $\mathbf{y} = \lim_{n\to\infty} \sum_{i=1}^{n} y_i e_i$
By continuity of $g \in c_{00}(\mathbb{N})^*$,

$$g(y) = \lim_{n\to\infty} \sum_{i=1}^{n} y_i g(e_i) = \sum_{i=1}^{N} y_i g(e_i)$$

But

$$\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^{\infty} y_i x_i = \sum_{i=1}^{N} y_i g(e_i) = g(\mathbf{y})$$

Hence, $\eta_{\mathbf{x}} = g \implies \Phi$ is onto. $\square$

Lemma 7.6 (Dual of $(\ell^p(\mathbb{N}), \|\cdot\|_p)$). *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$(\ell^p(\mathbb{N}), \|\cdot\|_p)^* \cong (\ell^q(\mathbb{N}), \|\cdot\|_q)$$

Proof. We will show that $\exists$ a linear isometrical isomorphism $\Phi : (\ell^q(\mathbb{N}), \|\cdot\|_q) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)^*$.
Let $\mathbf{x} = (x_n) \in \ell^q(\mathbb{N})$. Define $\eta_{\mathbf{x}} : (\ell^p(\mathbb{N}), \|\cdot\|_p) \to \mathbb{F}$ by $\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^{\infty} y_i x_i \; \forall \mathbf{y} = (y_n) \in \ell^p(\mathbb{N})$.
$\implies \eta_{\mathbf{x}}$ is linear and $|\eta_{\mathbf{x}}(\mathbf{y})| \le \|\mathbf{y}\|_p \|\mathbf{x}\|_q \;\; \forall \mathbf{y} \in \ell^p(\mathbb{N})$, by Hölder's inequality.
$\implies \eta_{\mathbf{x}}$ is a continuous linear functional and $\|\eta_{\mathbf{x}}\| \le \|\mathbf{x}\|_q$

Claim: $\|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$
Let $y_j = Sgn(x_j)|x_j|^{q/p} \; \forall j \in \mathbb{N}$, and $\mathbf{y} = (y_j)_{j=1}^{\infty}$
Using remark 7.3, $\|\mathbf{y}\|_p = \|\mathbf{x}\|_q^{q/p}$
As $\mathbf{x} \in \ell^q(\mathbb{N})$, $\|\mathbf{x}\|_q < \infty$, so $\|\mathbf{y}\|_p < \infty \implies \mathbf{y} \in \ell^p(\mathbb{N})$
Observe that $\frac{\mathbf{y}}{\|\mathbf{y}\|_p} \in S_{\ell^p(\mathbb{N})}$

$$\begin{aligned}
\eta_{\mathbf{x}}\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_p}\right) &= \sum_{i=1}^{\infty} \frac{y_i}{\|\mathbf{y}\|_p} x_i = \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{\infty} Sgn(x_i)|x_i|^{q/p} x_i \\
\implies \left|\eta_{\mathbf{x}}\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_p}\right)\right| &= \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{\infty} |x_i|^{\frac{q}{p}+1} \\
&= \frac{1}{\|\mathbf{y}\|_p} \sum_{i=1}^{\infty} |x_i|^q = \frac{\|\mathbf{x}\|_q^q}{\|\mathbf{x}\|_q^{q/p}} = \|\mathbf{x}\|_q
\end{aligned}$$

Hence, $\|\mathbf{x}\|_q \in \{|\eta_{\mathbf{x}}(\mathbf{y})| : \mathbf{y} \in S_{\ell^p(\mathbb{N})}\}$ and $\|\eta_{\mathbf{x}}\| \le \|\mathbf{x}\|_q$
$\implies \|\mathbf{x}\|_q = \sup\{|\eta_{\mathbf{x}}(\mathbf{y})| : \mathbf{y} \in S_{\ell^p(\mathbb{N})}\}$
$\implies \|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$

Now, define $\Phi : (\ell^q(\mathbb{N}), \|\cdot\|_q) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)^*$ by $\Phi(\mathbf{x}) = \eta_{\mathbf{x}} \; \forall \mathbf{x} \in \ell^p(\mathbb{N})$
We will show that Φ is linear, isometry, and onto.
That Φ is linear is trivial to verify.

$$\|\Phi(\mathbf{x})\| = \|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_q$$

Hence, Φ is an isometry.

Let $g \in (\ell^p(\mathbb{N}), \|\cdot\|_p)^*$. To show that Φ is onto, we have to select $\mathbf{x} \in (\ell^q(\mathbb{N}), \|\cdot\|_q)$ such

that $\Phi(\mathbf{x}) = \eta_{\mathbf{x}} = g$.
$\forall i \in \mathbb{N}$, $e_i = (0, \dots, 0, \underbrace{1}_{i^{th}}, 0, \dots) \in \ell^p(\mathbb{N})$. Define $\mathbf{x} = (g(e_1), g(e_2), \dots, g(e_n), \dots)$. For $N \in \mathbb{N}$, define $w_N = (g(e_1), g(e_2), \dots, g(e_N), 0, 0, \dots)$ as the truncations of $\mathbf{x}$.
Claim: $\mathbf{x} \in \ell^q(\mathbb{N})$ and $g = \eta_{\mathbf{x}}$.
From the proof of lemma 7.4, it can be easily verified that $g(\mathbf{y}) = \eta_{\mathbf{x}}(\mathbf{y})$ $\forall \mathbf{y} \in c_{00}(\mathbb{N})$. This is because for $\mathbf{y} \in c_{00}(\mathbb{N})$, we have $\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^{\infty} y_i g(e_i) = \sum_{i=1}^{N} y_i g(e_i)$, and we know that any linear functional g can be written in this form.

Claim: $\|\mathbf{x}\|_q \leq \|g\|$, so that $\mathbf{x} \in \ell^q(\mathbb{N})$.
Define $y_j = Sgn(x_j)|x_j|^{q/p}$ $\forall j \in \mathbb{N}$ and $\mathbf{y} = (y_j)_{j=1}^{\infty}$
Let $z_N = (y_1, y_2, \dots, y_N, 0, 0, \dots)$ be the truncations of $\mathbf{y}$ for all $N \in \mathbb{N}$.
Then

$$\begin{aligned}
g(z_N) = \eta_{\mathbf{x}}(z_N) = \sum_{j=1}^{\infty} y_j x_j &= \sum_{j=1}^{\infty} Sgn(x_j)|x_j|^{q/p} x_j \\
\implies |g(z_N)| &= \sum_{j=1}^{\infty} |x_j|^{\frac{q}{p}+1} \\
&= \sum_{j=1}^{N} |x_j|^q \\
&= \sum_{j=1}^{N} |g(e_j)|^q = \|w_N\|_q^q
\end{aligned}$$

Thus,

$$\begin{aligned}
&g(z_N) \leq \|g\| \, \|z_N\|_p \ \ \forall N \in \mathbb{N} \\
\implies &\|w_N\|_q^q \leq \|g\| \, \|z_N\|_p = \|g\| \, \|w_N\|_q^{q/p} \\
\implies &\|w_N\|_q^{q-\frac{q}{p}} \leq \|g\| \ \forall N \in \mathbb{N} \\
\implies &\|w_N\|_q \leq \|g\| \ \forall N \in \mathbb{N} \\
\implies &\|\mathbf{x}\|_q \leq \|g\| \text{ (as } w_N\text{'s are truncations of } \mathbf{x}) \\
\implies &\mathbf{x} \in \ell^q(\mathbb{N})
\end{aligned}$$

Thus, $\eta_{\mathbf{x}} \in \ell^p(\mathbb{N})^*$ and $\Phi(\mathbf{x}) = g = \eta_{\mathbf{x}}$.
Hence, Φ is a linear isometrical isomorphism. □

Corollary 7.7. *The space $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ is reflexive, i.e. the canonical map $\Phi : (\ell^p(\mathbb{N}), \|\cdot\|_p) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)^{**}$ is linear isometry and onto.*

Proof. Define $\Phi : (\ell^p(\mathbb{N}), \|\cdot\|_p) \to (\ell^p(\mathbb{N}), \|\cdot\|_p)^{**}$ by $\mathbf{x} = (x_n) \mapsto \eta_{\mathbf{x}}$ $\forall x \in \ell^p(\mathbb{N})$, where $\eta_{\mathbf{x}} : (\ell^p(\mathbb{N}), \|\cdot\|_p)^* \to \mathbb{F}$ is as defined in lemma 7.6.

We know from lemma 7.6 that for a q such that $\frac{1}{p}+\frac{1}{q}=1$, $(\ell^p(\mathbb{N}),\|\cdot\|_p)^* \cong (\ell^q(\mathbb{N}),\|\cdot\|_q)$ $\therefore$ $\ell^p(\mathbb{N})^{**}$ can be identified as $\ell^q(\mathbb{N})^*$. We have seen that the map $\Phi : \ell^p(\mathbb{N}) \to \ell^q(\mathbb{N})^*$ defined by $\mathbf{x} = (x_n) \in \ell^p(\mathbb{N}) \to \eta_{\mathbf{x}}$ is linear isometry and onto. Thus, the map from $\ell^p(\mathbb{N})$ to $\ell^p(\mathbb{N})^{**}$ is a surjective linear isometry. $\square$

Exercise 7.1 (Dual of $(\ell^1(\mathbb{N}),\|\cdot\|_1)$)**.** Show that the dual of $(\ell^1(\mathbb{N}),\|\cdot\|_1)$ is $(\ell^\infty(\mathbb{N}),\|\cdot\|_\infty)$, i.e.

$$(\ell^1(\mathbb{N}),\|\cdot\|_1) \cong (\ell^\infty(\mathbb{N}),\|\cdot\|_\infty)$$

Lemma 7.8 (Dual of $(c_0(\mathbb{N}),\|\cdot\|_\infty)$)**.** *The dual of* $(c_0(\mathbb{N}),\|\cdot\|_\infty)$ *is* $(\ell^1(\mathbb{N}),\|\cdot\|_1)$*, i.e.*

$$(c_0(\mathbb{N}),\|\cdot\|_\infty)^* \cong (\ell^1(\mathbb{N}),\|\cdot\|_1)$$

Proof. Define $\Phi : (\ell^1(\mathbb{N}),\|\cdot\|_1) \to (c_0(\mathbb{N}),\|\cdot\|_\infty)^*$ by $\mathbf{x} \mapsto \eta_{\mathbf{x}}$ $\forall \mathbf{x} = (x_n) \in (\ell^1(\mathbb{N}),\|\cdot\|_1)$
Where $\eta_{\mathbf{x}} : (c_0(\mathbb{N}),\|\cdot\|_\infty) \to \mathbb{F}$ is defined by $\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^{\infty} y_i x_i$ $\forall \mathbf{y} = (y_n) \in (c_0(\mathbb{N}),\|\cdot\|_\infty)$

Claim: $\eta_{\mathbf{x}}$ is a bounded linear transformation (i.e., $\eta_{\mathbf{x}} \in c_0(\mathbb{N})^*$) and $\|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_1$.
Clearly, $\eta_{\mathbf{x}}$ is linear. Also, $\eta_{\mathbf{x}}$ is bounded, because

$$|\eta_{\mathbf{x}}(\mathbf{y})| = \left|\sum_{i=1}^{\infty} y_i x_i\right| \leq \|\mathbf{y}\|_\infty \|\mathbf{x}\|_1 \quad \text{(by Hölder's inequality)}$$

$\implies \|\eta_{\mathbf{x}}\| \leq \|\mathbf{x}\|_1$
Claim: $\|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_1$
Let $\varepsilon > 0$. As $\mathbf{x} = (x_n) \in \ell^1(\mathbb{N})$, we know that $\sum_{n=1}^{\infty} |x_n| < \infty$
That is, $\exists N \in \mathbb{N}$ such that $\forall n > N$, we have $\sum_{n=N+1}^{\infty} |x_n| < \infty$
Define $\mathbf{y} = (y_n)$ as

$$y_n = \begin{cases} Sgn(x_n), & n \leq N \\ 0, & n > N \end{cases}$$

That is, $\mathbf{y} = (Sgn(x_1), Sgn(x_2), \ldots, Sgn(x_N), 0, 0, \ldots)$. Clearly, $\mathbf{y} \in c_0(\mathbb{N})$ as $y_n \to 0$ as $n \to \infty$

$$\begin{aligned} |\eta_{\mathbf{x}}(\mathbf{y}) - \|\mathbf{x}\|_1| = \left|\sum_{n=1}^{\infty} y_n x_n - \sum_{n=1}^{\infty} |x_n|\right| &= \left|\sum_{n=1}^{N} Sgn(x_n) x_n - \sum_{n=1}^{\infty} |x_n|\right| \\ &= \left|\sum_{n=1}^{N} |x_n| - \sum_{n=1}^{\infty} |x_n|\right| \\ &= \left|\sum_{n=N+1}^{\infty} |x_n|\right| < \varepsilon \end{aligned}$$

$\implies \exists \mathbf{y} = (y_n) \in c_0(\mathbb{N})$ such that $\eta_{\mathbf{x}}(\mathbf{y}) \to \|\mathbf{x}\|_1$ as $n \to \infty$, and $\|\mathbf{y}\|_\infty = 1$

$\implies \|\eta_{\mathbf{x}}\| = \sup\{|\eta_{\mathbf{x}}(\mathbf{y})| : \mathbf{y} \in c_0(\mathbb{N}),\ \|\mathbf{y}\|_\infty = 1\} = \|\mathbf{x}\|_1$

Thus, $\|\Phi(\mathbf{x})\| = \|\eta_{\mathbf{x}}\| = \|\mathbf{x}\|_1\ \ \forall \mathbf{x} = (x_n) \in \ell^1(\mathbb{N})$
$\implies \Phi$ is an isometry.

We shall now show that Φ is onto. Let $g \in c_0(\mathbb{N})^*$. We want to find an $\mathbf{x} = (x_n) \in \ell^1(\mathbb{N})$ such that $g = \eta_{\mathbf{x}}$.
For $e_i = (0, \ldots, 0, \underbrace{1}_{i^{th}}, 0, \ldots\}$, let $\mathbf{x} = (g(e_1), g(e_2), \ldots)$
Claim: $\mathbf{x} \in \ell^1(\mathbb{N})$
Consider $z_N = (Sgn(g(e_1)), Sgn(g(e_2)), \ldots, Sgn(g(e_N)), 0, 0, \ldots) \in c_0(\mathbb{N})$. As g is bounded, we must have $\sup\{|g(\mathbf{x})| : \mathbf{x} \in S_{c_0(\mathbb{N})}\} = M < \infty$.
In particular, the lim sup taken over $\{z_N\}_{N=1}^\infty$ is finite (as $z_N \in c_0(\mathbb{N})$ and $\|z_N\|_\infty = 1$).
$\therefore \sum_{i=1}^\infty |g(e_i)| = \limsup_{z_N} g(z_N) < \infty \implies \mathbf{x} = (g(e_1), g(e_2), \ldots) \in \ell^1(\mathbb{N})$

Now, for any $\mathbf{y} = (y_1, y_2, \ldots) \in c_0(\mathbb{N})$,

$$g(y) = \lim_{n\to\infty} \sum_{i=1}^n y_i g(e_i) = \sum_{i=1}^\infty y_i g(e_i) \quad \left(\because \sum_{i=1}^\infty |g(e_i)| < \infty\right)$$

and $\eta_{\mathbf{x}}(\mathbf{y}) = \sum_{i=1}^\infty y_i x_i = \sum_{i=1}^\infty y_i g(e_i)$
$\implies g = \eta_{\mathbf{x}}\ \forall \mathbf{y} \in c_0(\mathbb{N})$
$\implies \Phi$ is onto. □

Lemma 7.9 (Dual of $(c(\mathbb{N}), \|\cdot\|_\infty)$). *The dual of* $(c(\mathbb{N}), \|\cdot\|_\infty)$ *is* $(\ell^1(\mathbb{N}), \|\cdot\|_1)$, *i.e.*

$$(c(\mathbb{N}), \|\cdot\|_\infty)^* \cong (\ell^1(\mathbb{N}), \|\cdot\|_1)$$

Proof. Let $X = (c(\mathbb{N}), \|\cdot\|_\infty)$
Claim: For every $f \in X^*$, the sequence $(f(e_n))_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$, and

$$f(\mathbf{x}) = \alpha\beta + \sum_{j=1}^\infty x_j f(e_j)\ \forall \mathbf{x} = (x_n) \in c(\mathbb{N}) \tag{7.1}$$

$$\text{where } \alpha = \lim_{n\to\infty} x_n \text{ (which exists because } \mathbf{x} = (x_n) \in c(\mathbb{N})) \tag{7.2}$$

$$\text{and } \beta = f(e_0) - \sum_{j=1}^\infty f(e_j),\ \text{ where } e_0 = (1, 1, \ldots) \in c(\mathbb{N}) \tag{7.3}$$

Moreover, the map $f \mapsto (\beta, f(e_1), f(e_2), \ldots),\ f \in X^*$ is an onto linear isometry from X^* onto $(\ell^1(\mathbb{N}), \|\cdot\|_1)$, so that $X^* \cong (\ell^1(\mathbb{N}), \|\cdot\|_1)$.

For $f \in X^*$, let $\beta_j = f(e_j)\ \forall j \in \mathbb{N}$ and let $\beta = f(e_0) - \sum_{j=1}^\infty \beta_j$

Observe that $c(\mathbb{N}) = \{\mathbf{u} + \alpha e_0 : \mathbf{u} \in c_0(\mathbb{N}),\ \alpha \in \mathbb{F}\}$
$\because$ Every $\mathbf{u} \in c_0(\mathbb{N})$ can be written as $\mathbf{u} = \sum_{j=1}^{\infty} u_j e_j$, we have

$$f(\mathbf{u}+\alpha e_0) = \alpha f(e_0)+f(\mathbf{u}) = \alpha f(e_0)+f\left(\sum_{j=1}^{\infty} u_j e_j\right) = \alpha f(e_0)+\sum_{j=1}^{\infty} u_j f(e_j)\ \forall \mathbf{u} \in c_0(\mathbb{N}), \alpha \in \mathbb{F},\ f \in X^*$$

$\implies f|_{c_0(\mathbb{N})} = f(\mathbf{u}) = \sum_{j=1}^{\infty} u_j f(e_j) \in c_0(\mathbb{N})^*$
As $(c_0(\mathbb{N}), \|\cdot\|_\infty) \cong (\ell^1(\mathbb{N}), \|\cdot\|_1)$, it follows that $(f(e_1), f(e_2), \ldots) \in \ell^1(\mathbb{N})$
Thus, $(\beta_1, \beta_2, \ldots) \in \ell^1(\mathbb{N})$

Claim: For any $\mathbf{x} \in c(\mathbb{N})$, $f(\mathbf{x}) = \alpha\beta + \sum_{j=1}^{\infty} x_j \beta_j$ (equation 7.1)
Take $\mathbf{x} = \mathbf{u} + \alpha e_0$ for some $\mathbf{u} \in c_0(\mathbb{N})$ and $\alpha \in \mathbb{F}$.
We know,

$$f(\mathbf{u} + \alpha e_0) = \alpha f(e_0) + \sum_{j=1}^{\infty} u_j f(e_j)\ \forall \mathbf{u} = (u_j) \in c_0(\mathbb{N}),\ \alpha \in \mathbb{F},\ f \in c(\mathbb{N})^* \tag{7.4}$$

$$\begin{aligned} \text{If } f(\mathbf{x}) &= \alpha\beta + \sum_{j=1}^{\infty} x_j \beta_j, \\ \implies f(\mathbf{u} + \alpha e_0) &= \alpha\left(f(e_0) - \sum_{j=1}^{\infty} f(e_j)\right) + \sum_{j=1}^{\infty} x_j f(e_j) \\ &= \alpha f(e_0) - \alpha \sum_{j=1}^{\infty} f(e_j) + \sum_{j=1}^{\infty} x_j f(e_j) \\ &= \alpha f(e_0) + \sum_{j=1}^{\infty} (x_j - \alpha) f(e_j) \end{aligned}$$

As $\alpha = \lim_{n\to\infty} x_j$, we have $x_j \to \alpha$, so that $x_j - \alpha \to 0$ as $j \to \infty$
$\implies u_j = x_j - \alpha,\ \mathbf{u} = (u_j) \in c_0(\mathbb{N})$
$\implies f(\mathbf{u} + \alpha e_0) = \alpha f(e_0) + \sum_{j=1}^{\infty} u_j f(e_j)$, which is true by equation 7.4.
Therefore, we have shown that any $f \in c(\mathbb{N})^*$ has the representation in equation 7.1.

Clearly from the above representation,

$$\|f\| \leq |\beta| + \sum_{j=1}^{\infty} |\beta_j|$$

We now show that $\|f\| \geq |\beta| + \sum_{j=1}^{\infty} |\beta_j|$
Let $x_k = (Sgn(\beta_1), Sgn(\beta_2), \ldots, Sgn(\beta_k), Sgn(\beta), Sgn(\beta), \ldots)$

Note that $\alpha = \lim_{k\to\infty} x_k = Sgn(\beta)$ and $\|x_k\|_\infty \leq 1 \; \forall k \in \mathbb{N}$. From equation 7.1,

$$f(x_k) = Sgn(\beta)\beta + \sum_{j=1}^{\infty} Sgn(\beta_j)\beta_j = Sgn(\beta)\beta + \sum_{j=1}^{k} Sgn(\beta_j)\beta_j + \sum_{j=k+1}^{\infty} Sgn(\beta)\beta_j$$

$$\implies |f(x_k)| = |\beta| + \sum_{j=1}^{k} |\beta_j| + |\alpha| \sum_{j=k+1}^{\infty} |\beta_j|$$

As $(\beta_1, \beta_2, \dots) \in \ell^1(\mathbb{N})$, we have

$$|\beta| + \sum_{j=1}^{\infty} |\beta_j| \leq |f(x_k) + |\alpha| \sum_{j=k+1}^{\infty} \beta_j \leq \|f\| + |\alpha| \sum_{j=k+1}^{\infty} |\beta_j| \; \forall k \in \mathbb{N}$$

Letting $k \to \infty$,

$$|\beta| + \sum_{j=1}^{\infty} |\beta_j| \leq \|f\| < \infty \; (\implies \mathbf{y} = (\beta, \beta_1, \beta_2, \dots) \in \ell^1(\mathbb{N}))$$

Thus, we have proved that $\forall f \in X^*$, $\mathbf{y} = (\beta, \beta_1, \beta_2, \dots) \in \ell^1(\mathbb{N})$, $f(\mathbf{x}) = \alpha\beta + \sum_{j=1}^{\infty} x_j \beta_j \; \forall \mathbf{x} \in X$, and $\|f\| = \|\mathbf{y}\|_1$

Conversely, suppose $\mathbf{y} = (\eta, \eta_1, \eta_2, \dots) \in \ell^1(\mathbb{N})$. Let $f : X \to \mathbb{F}$ be defined by $f(x) = \alpha\eta + \sum_{j=1}^{\infty} x_j \eta_j$, where $\alpha = \lim_{n\to\infty} x_n$. By a similar argument, it is seen that $f \in X^*$ and $\|f\| = \|\mathbf{y}\|_1$.

Hence, we have obtained a linear isometry from X^* onto $\ell^1(\mathbb{N})$. □

7.2 Transpose of a Bounded Linear Transformation

Remark 7.4. Let $A \in \mathcal{B}(X, Y)$, i.e. $A : X \to Y$ and $g \in Y^*$, i.e. $g : Y \to \mathbb{F}$.
Then $g \circ A : X \to \mathbb{F}$, and $g \circ A$ is continuous because both g and A are continuous.
$\implies g \circ A \in X^*$
$(g \circ A)(x) = g(Ax) \; \forall x \in X$

Definition 7.6 (Transpose of a bounded linear transformation)**.** Let $A \in \mathcal{B}(X, Y)$.
Define $A^* : Y^* \to X^*$ by $g \mapsto A^*g$, where $A^*g : X \to \mathbb{F}$ is defined by $(A^*g)(x) = g(Ax) \; \forall x \in X$.
That is, $g \in Y^* \mapsto g \circ A \in X^*$ under A^*.
Then A^* is linear and A^* is called the **(conjugate) transpose** of A.

Lemma 7.10. *A^* is a well-defined linear transformation.*

Proof. Suppose $g_1, g_2 \in Y^*$ be such that $A^*g_1 = A^*g_2$. Then $g_1(A(x)) = g_2(A(x)) \; \forall x \in X$
As $A(x) \in Y \; \forall x \in X$, it follows that $g_1 \equiv g_2$. Thus, A^* is well-defined.

Define $T : \mathcal{B}(X,Y) \to \mathcal{B}(Y^*, X^*)$ by $T(A) = A^*$, where $A : X \to Y$ is a bounded linear transformation, and $A^* : Y^* \to X^*$ is the transpose of A, i.e. $A^*g = g \circ A \; \forall g \in Y^*$.

$$\begin{aligned}
T(A+B) &= (A+B)^* \\
\implies (T(A+B))(x) &= (A+B)^*g \\
&= g((A+B)(x)) = g(A(x)+B(x)) \; (\because A, B \in \mathcal{B}(X,Y)) \\
&= g(Ax) + g(Bx) \; (\because g \in Y^* \implies g \text{ is linear}) \\
&= A^* + B^* = T(A) + T(B)
\end{aligned}$$

$$\begin{aligned}
T(\alpha A) &= (\alpha A)^* \\
(\alpha A)^*g &= g(\alpha Ax) = \alpha g(Ax) = \alpha A^* = \alpha T(A)
\end{aligned}$$

□

Theorem 7.3. *Let $A \in \mathcal{B}(X,Y)$. Then*

1. *$A^* \in \mathcal{B}(Y^*, X^*)$, i.e. the transpose of A is also a bounded linear transformation.*
2. *$\|A^*\| = \|A\|$, i.e. the transpose preserves norm.*

Proof. Given that $A \in \mathcal{B}(X,Y)$. By definition and lemma 7.10, we know that $A^* \in L(Y^*, X^*)$.

1. Let $g \in Y^*$. Then

$$\begin{aligned}
\|A^*g\| &= \sup\{|(A^*g)(x)| : x \in S_X\} \ldots (*) \\
&= \sup\{|g(Ax)| : x \in S_X\} \\
&\leq \sup\{\|g\|\|Ax\| : x \in S_X\} \;\; \text{(using lemma 5.1, as } g \text{ is bounded)} \\
&\leq \sup\{\|g\|\|A\|\|x\| : x \in S_X\} \;\; \text{(again by lemma 5.1, as } A \text{ is bounded)} \\
&= \|g\|\|A\| \; \forall g \in Y^* \; (\because x \in S_X \implies \|x\| = 1)
\end{aligned}$$

Hence, A^* is bounded, i.e. $A^* \in \mathcal{B}(Y^*, X^*)$ and $\|A^*\| \leq \|A\|$.

Claim: $\|A\| \leq \|A^*\|$
We know from corollary 6.6 that for $x \in X$, $\|x\| = \sup\{|f(x)| : f \in S_{X^*}\}$

Let $x \in S_X$. Then $Ax \in Y$, so

$$\begin{aligned} \|Ax\| &= \sup\{|g(Ax)| : g \in S_{Y^*}\} \\ &= \sup\{|(A^*g)(x)| : g \in S_{Y^*}\} \\ &\leq \|A^*g\| \quad \text{from } (*) \\ &\leq \|A^*\|\|g\| = \|A^*\| \\ \implies \|Ax\| &\leq \|A^*\| \ \forall x \in S_X \\ \implies \sup_{x \in S_X} \{\|Ax\|\} &\leq \|A^*\| \\ \implies \|A\| &\leq \|A^*\| \end{aligned}$$

Thus, $\|A\| = \|A^*\|$.

□

Corollary 7.4. *Let X and Y be normed linear spaces. Then the transpose map $A \mapsto A^*$ is a linear isometry from $\mathcal{B}(X, Y)$ to $\mathcal{B}(Y^*, X^*)$.*

Proof. Define the map T as in lemma 7.10. Lemma 7.10 says that T is linear, and theorem 7.3 says that T preserves norm, i.e. $\|T(A)\| = \|A^*\| = \|A\| \ \forall A \in \mathcal{B}(X, Y)$. □

Remark 7.5. If $A \in \mathcal{B}(X, Y)$, then

1. $A^* \in \mathcal{B}(Y^*, X^*)$ and $\|A\| = \|A^*\|$
2. $A^{**} \in \mathcal{B}(X^{**}, Y^{**})$ and $\|A^{**}\| = \|A^*\| = \|A\|$

8 Hilbert Spaces

Recall the definition of an inner product (definition 1.4) and inner product space (definition 1.5.

Remark 8.1. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$. The reader can easily verify that

1. $\forall y \in X,\ \langle \cdot, y \rangle : X \to \mathbb{F}$ is a linear functional.
2. $\forall x \in X,\ \langle x, \cdot \rangle : X \to \mathbb{F}$ is an antilinear functional.
3. $\forall y \in X,\ \langle 0, y \rangle = 0$.
4. $\forall x \in X,\ \langle x, 0 \rangle = 0$.
5. Suppose $y \in X$ and $\langle x, y \rangle = 0\ \forall x \in X$. Then $y = 0$.
6. Let $x \in X$. Suppose $\langle x, y \rangle = 0\ \forall y \in X$. Then $x = 0$.

Proposition 8.1. $\langle x, y \rangle + \langle y, x \rangle = 2Re(\langle x, y \rangle)$

Proof. We know that for $z \in \mathbb{C}$, $z + \overline{z} = 2Re(z)$. Therefore,

$$\langle x, y \rangle + \langle y, x \rangle = \langle x, y \rangle + \overline{\langle x, y \rangle} = 2Re(\langle x, y \rangle)$$

□

Lemma 8.1 (Cauchy-Schwarz inequality)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$. Then*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle}\sqrt{\langle y, y \rangle}\ \forall x, y \in X$$

Proof. Let $x, y \in X,\ \alpha \in \mathbb{F}$. Observe:

$$\begin{aligned}
& 0 \leq \langle x - \alpha y, x - \alpha y \rangle \\
\implies & 0 \leq \langle x, x - \alpha y \rangle - \alpha \langle y, x - \alpha y \rangle \text{ (linearity in first slot)} \\
\implies & 0 \leq \langle x, x \rangle - \overline{\alpha}\langle x, y \rangle - \alpha \langle y, x \rangle + \alpha\overline{\alpha}\langle y, y \rangle \text{ (anti-linearity in second slot)} \\
\implies & 0 \leq \langle x, x \rangle - \overline{\alpha}\langle x, y \rangle - \overline{\overline{\alpha}\langle x, y \rangle} + |\alpha|^2 \langle y, y \rangle \\
\implies & 0 \leq \langle x, x \rangle - 2Re(\overline{\alpha}\langle x, y \rangle) + |\alpha|^2 \langle y, y \rangle \ldots (*) \text{ (by proposition 8.1)}
\end{aligned}$$

If $\langle y, y \rangle = 0$, then the last inequality is trivial.
Assume $\langle y, y \rangle > 0$. Take $\alpha = -\frac{\langle x, y \rangle}{\langle y, y \rangle}$

Then from $(*)$,

$$\begin{aligned}
0 &\leq \langle x, x\rangle - 2\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} + \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle^2}\langle y, y\rangle \\
\implies 0 &\leq \langle x, x\rangle\langle y, y\rangle - |\langle x, y\rangle|^2 \\
\implies |\langle x, y\rangle| &\leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}
\end{aligned}$$

□

Exercise 8.1. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}$ and $x, y \in X$. Then show that equality holds in Cauchy-Schwarz inequality if and only if $\exists \alpha \in \mathbb{F}$ such that $x = \alpha y$ (i.e. when x and y are linearly dependent).

Lemma 8.2 (Norm induced by an inner product)**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}$. For $x \in X$, and define*

$$\|x\| = \sqrt{\langle x, x\rangle}$$

Then $\|\cdot\|$ is a norm on X, called the norm induced by inner product $\langle\cdot,\cdot\rangle$.

Proof. $\forall x \in X$, we have $\|x\| = \sqrt{\langle x, x\rangle}$

1. $\|x\| = 0 \iff \sqrt{\langle x, x\rangle} = 0 \iff \langle x, x\rangle = 0 \iff x = 0$
2. $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\bar{\alpha}\langle x, x\rangle} = \sqrt{|\alpha|^2\langle x, x\rangle} = |\alpha|\sqrt{\langle x, x\rangle} = |\alpha|\|x\|$
3. Let $x, y \in X$. Then

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \langle x, x\rangle + 2Re(\langle x, y\rangle) + \langle y, y\rangle \text{ (by proposition 8.1)} \\
&= \|x\|^2 + 2Re(\langle x, y\rangle) + \|y\|^2 \\
&\leq \|x\|^2 + 2\sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle} + \|y\|^2 \text{ (using Cauchy-Schwarz inequality)} \\
&= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2 \\
\implies \|x + y\| &\leq \|x\| + \|y\|
\end{aligned}$$

□

Remark 8.2. We often write the norm induced by an inner product as $\|\cdot\|_2$, because $\|x\|_2 = (\langle x, x\rangle)^{\frac{1}{2}}$

Remark 8.3. From the Cauchy-Schwarz inequality, we have

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$$

Remark 8.4. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{F}$ is a function.

Remark 8.5. $(X \times X, \|\cdot\|_2)$ is a normed linear space. Let $(x, y),\ (a, b) \in X \times X$. Define

$$\langle (x, y), (a, b)\rangle = \langle x, a\rangle + \langle y, b\rangle$$

Check that $\langle\cdot,\cdot\rangle$ is an inner product on $X \times X$, and

$$\|(x, y)\|_2 = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}}$$

is a norm on $X \times X$.

Lemma 8.3. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then*

1. *$\forall y \in X$, the function $\langle\cdot, y\rangle : X \to \mathbb{F}$ is linear and continuous.*
2. *$\forall x \in X$, the function $\langle x, \cdot\rangle : X \to \mathbb{F}$ is anti-linear and continuous.*
3. *$\langle\cdot,\cdot\rangle : X \times X \to \mathbb{F}$ is continuous.*

Proof. 1. It is easy to verify the linearity of $\langle\cdot, y\rangle$ using the properties of an inner product. Fix $y \in X$. Let $(x_n) \subseteq X$ be such that $x_n \to x \in X$. To show that $\langle\cdot, y\rangle$ is continuous, we have to show that $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$. We have

$$|\langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n - x, y\rangle| \le \|x_n - x\|\|y\| \text{ (Cauchy-Schwarz inequality) } \to 0 \text{ as } n \to \infty$$

Hence, $\langle\cdot, y\rangle$ is continuous.

2. The proof is similar and left to the reader as an exercise.

3. Let $(x_n, y_n) \subseteq X \times X$ be such that $(x_n, y_n) \to (x, y) \in X \times X$ as $n \to \infty$.
Claim: $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$

$$\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x, y_n\rangle + \langle x, y_n\rangle - \langle x, y\rangle| \\
&= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\
&\le |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \text{ (by triangle inequality)} \\
&\le \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \text{ (by Cauchy-Schwarz inequality)} \\
&\le L\|x_n - x\| + \|x\|\|y_n - y\| \to 0 \text{ as } n \to \infty
\end{aligned}$$

□

Definition 8.1 (Hilbert space)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}$. We say that $(X, \langle\cdot,\cdot\rangle)$ is Hilbert space if $(X, \|\cdot\|)$ is a Banach Space, where $\|\cdot\|$ is the norm on X induced by the inner product $\langle\cdot,\cdot\rangle$.

Example 8.1 (Some Hilbert spaces)**.** The following are some examples of Hilbert spaces.

1. $(\mathbb{R}^n, +, \cdot)$ is a vector space over $\mathbb{R}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Define

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

   Check that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$.
   For $\mathbf{x} \in \mathbb{R}^n$, define

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} = \|\mathbf{x}\|_2$$

   We know that $(\mathbb{R}^n, \|\cdot\|_2)$ is a Banach space. Hence, $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

2. $(\mathbb{C}^n, +, \cdot)$ is a vector space over $\mathbb{C}$. For $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n$, define

$$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

   Check that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^n$.
   For $\mathbf{z} \in \mathbb{C}^n$,

$$\|\mathbf{z}\| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle} = \left(\sum_{i=1}^{n} |z_i|^2 \right)^{\frac{1}{2}} = \|\mathbf{z}\|_2$$

   We know that $(\mathbb{C}^n, \|\cdot\|_2)$ is a Banach space. Hence, $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

3. $(\ell^2(\mathbb{N}), +, \cdot)$ is a vector space over $\mathbb{F}$. Let $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n) \in \ell^2(\mathbb{N})$, and define $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$. Check that $\langle \cdot, \cdot \rangle$ forms an inner product on $\ell^2(\mathbb{N})$.
   For $\mathbf{x} \in \ell^2(\mathbb{N})$, define

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}} = \|\mathbf{x}\|_2$$

   We know that $(\ell^2(\mathbb{N}), \|\cdot\|_2)$ is a Banach space. Thus, $(\ell^2(\mathbb{N}), \langle \cdot, \cdot \rangle)$ is a Hilbert space.

4. $(C[a, b], +, \cdot)$ is a vector space over $\mathbb{F}$. Let $f, g \in C[a, b]$. Define $\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}dt$. We know that $\langle \cdot, \cdot \rangle$ is an inner product on $C[a, b]$. For $f \in C[a, b]$, define

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \left(\int_a^b |f(t)|^2 \right)^{\frac{1}{2}}$$

   From exercise 3.3, we know that $(C[a, b], \|\cdot\|_2)$ is not a Banach space. Thus, $(C[a, b], \langle \cdot, \cdot \rangle)$ is an inner product space but **not** a Hilbert space.

5. The collection of all $\mathbb{F}$-valued polynomials over $[a, b]$, $(\mathcal{P}[a, b], +, \cdot)$ is a vector space

over $\mathbb{F}$ and is a subspace of $C[a,b]$. $(\mathcal{P}[a,b], \langle\cdot,\cdot\rangle)$ (where the inner product is same as that defined on $C[a,b]$ above) is an inner product space but **not** a Hilbert space because $(\mathcal{P}[a,b], \|\cdot\|_2)$ is not Banach.

6. $(c_{00}(\mathbb{N}), +, \cdot)$ is a vector space over $\mathbb{F}$ and $c_{00}(\mathbb{N}) \subsetneq \ell^2(\mathbb{N})$. Check that $(c_{00}(\mathbb{N}), \langle\cdot,\cdot\rangle)$ is an inner product space, where the inner product $\langle\cdot,\cdot\rangle$ is the same as that defined on $\ell^2(\mathbb{N})$. But by example 3.3 (1), $(c_{00}(\mathbb{N}), \|\cdot\|_2)$ is not a Banach space. Thus, $(c_{00}(\mathbb{N}), \langle\cdot,\cdot\rangle)$ is **not** a Hilbert space.

Remark 8.6. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|\cdot\|$ be the norm induced by $\langle\cdot,\cdot\rangle$. Let $x, y \in X$. Then

$$\|x+y\|^2 = \langle x,x\rangle + 2Re(\langle x,y\rangle) + \langle y,y\rangle$$

This is an useful identity which relates an inner product and the norm induced by it.

Lemma 8.4 (Parallelogram law)**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Let $x, y \in X$. Then*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof. Using remark 8.6, we have

$$\|x+y\|^2 = \|x\|^2 + 2Re(\langle x,y\rangle) + \|y\|^2$$
$$\|x-y\|^2 = \|x\|^2 - 2Re(\langle x,y\rangle) + \|y\|^2$$

Adding these two, we get the desired identity. □

Remark 8.7. If $\|\cdot\|$ is a norm induced by an inner product on X, then $\|\cdot\|$ satisfies the Parallelogram law. For the converse, see the following proposition.

Proposition 8.2. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$. Suppose that $\forall x, y \in X$, the Parallelogram law is satisfied. Then the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$ on X. In fact, the inner products are defined as:*

1. *If $\mathbb{F} = \mathbb{R}$, then for $x, y \in X$,*

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

2. *If $\mathbb{F} = \mathbb{C}$, then for $x, y \in X$,*

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

These are known as ***polarization identities****.*

Proposition 8.3. *The norm $\|\cdot\|_p$ on $(\mathbb{F}^n, \|\cdot\|_p)$ is induced by an inner product if and only if $p = 2$.*

Proof. We know that for $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}^n$, $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is induced by the inner product given by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$.

We will show that $\|\cdot\|_p$ does not satisfy the Parallelogram law for $p \neq 2$, and so it cannot be induced by any inner product.
Let $\mathbf{x} = (1, 1, 0, \ldots, 0)$ and $\mathbf{y} = (1, -1, 0, 0, \ldots, 0) \in \mathbb{F}^n$.
Then $\mathbf{x} + \mathbf{y} = (2, 0, 0, \ldots, 0)$ and $\mathbf{x} - \mathbf{y} = (0, 2, 0, 0, \ldots, 0)$.
So, $\|\mathbf{x} + \mathbf{y}\|_p = 2 = \|\mathbf{x} - \mathbf{y}\|_p$, and $\|\mathbf{x}\|_p = 2^{\frac{1}{p}} = \|\mathbf{y}\|_p$. But,

$$\|\mathbf{x} + \mathbf{y}\|_p^2 + \|\mathbf{x} - \mathbf{y}\|_p^2 = 4 + 4 = 8 \neq 2(\|\mathbf{x}\|_p^2 + \|\mathbf{y}\|_p^2) = 2(2^{\frac{2}{p}} + 2^{\frac{2}{p}}), \text{ if } p \neq 2$$

□

Exercise 8.2. Verify that the norm $\|\cdot\|_p$ on $\ell^p(\mathbb{N})$ is induced by an inner product if and only if $p = 2$.

8.1 Best Approximation & Projection Theorems

Recall the definition of a convex set from definition 2.5. Also, recall the definition of Best approximation property from definition 2.11.

Theorem 8.1 (Best approximation theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and C be a convex and closed set in $\mathcal{H}$. Then for each $h \in \mathcal{H}$, $\exists$ unique $k \in C$ such that*

$$\|h - k\| = dist(h, C) = \inf\{\|h - m\| : m \in C\}$$

$k \in C$ is known as the best approximation of $h \in \mathcal{H}$.

Proof. (Existence of $k \in C$) Let $d = dist(h, C) = \inf\{\|h - m\| : m \in C\}$
$\implies \exists (k_n) \subseteq C$ such that $\|h - k_n\| \to d$ as $n \to \infty \ldots\ldots (*)$
Claim: $\exists k \in C$ such that $k_n \to k$ as $n \to \infty$
We have

$$\|k_n - k_m\|^2 = 2(\|k_n\|^2 + \|k_m\|^2) - \|k_n + k_m\|^2 \text{ (by Parallelogram law)}$$
$$\implies \|k_n - k_m\|^2 = \|(k_n - h) + (k_m - h)^2\| = 2(\|k_n - h\|^2 + \|k_m - h\|^2) - \|2h - (k_n + k_m)\|^2$$
$$\implies \left\|\frac{k_n - k_m}{2}\right\|^2 = \frac{1}{2}(\|k_n - h\|^2 + \|k_m - h\|^2) - \left\|h - \frac{k_n - k_m}{2}\right\|^2$$
(dividing both sides by 4)

Now, observe that as $k_n,\ k_m \in C$ and C is convex, we have $\frac{k_n - k_m}{2} \in C$. By definition, as d is the infimum, we have $\|h - \frac{k_n-k_m}{2}\| \geq d$. Hence, from the last equality, we have

$$\begin{aligned}
\left\|\frac{k_n - k_m}{2}\right\|^2 &\leq \frac{1}{2}(\underbrace{\|k_n - h\|^2}_{\to d^2 \text{ as } n\to\infty} + \underbrace{\|k_m - h\|^2}_{\to d^2 \text{ as } m\to\infty}) - d^2 \\
&\to d^2 - d^2 = 0 \text{ as } m, n \to \infty \\
\implies \|k_n - k_m\| &\to 0 \text{ as } m, n \to \infty
\end{aligned}$$

$\implies (k_n)$ is Cauchy in $\mathcal{H}$.
As a Hilbert space is complete by definition, $\exists k \in \mathcal{H}$ such that $k_n \to k$ as $n \to \infty$
As C is closed in $\mathcal{H}$ and $(k_n) \subseteq C$, we have $k \in C$ (since a closed set contains all its limit points). Thus, we have

$$\begin{aligned}
k_n &\to k \text{ in } C \text{ as } n \to \infty \\
\implies h - k_n &\to h - k \text{ in } \mathcal{H} \\
\implies \|h - k_n\| &\to h - k \text{ as } n \to \infty \ldots (**)
\end{aligned}$$

From $(*)$ and $(**)$, we have $\|h - k\| = d$.
Thus, $\exists k \in C$ such that $\|h - k\| = dist(h, C)$

(Uniqueness) Let $k_1,\ k_2 \in C$ be such that $\|h - k_1\| = \|h - k_2\| = d$
Claim: $k_1 = k_2$

$$\|k_1 - k_2\|^2 = \|k_1 - h + h - k_2\|^2 = 2(\|k_1 - h\|^2 + \|h - k_2\|^2) - \|2h - (k_1 + k_2)\|^2 \text{ (by Parallelogram law)}$$

$$\begin{aligned}
\implies 0 \leq \left\|\frac{k_1 - k_2}{2}\right\|^2 &= \frac{1}{2}(\|k_1 - h\|^2 + \|h - k_2\|^2) - \left\|h - \frac{k_1 + k_2}{2}\right\|^2 \\
&\leq \frac{d^2}{2} + \frac{d^2}{2} - d^2 = 0
\end{aligned}$$

Hence, $\|k_1 - k_2\| = 0 \implies k_1 = k_2$ □

Remark 8.8. Let $\mathcal{H}$ be a Hilbert space and C be a closed convex set in $\mathcal{H}$. Then for each $h \in \mathcal{H}$,$\exists$ unique $k \in C$ such that $\|h - k\| = dist(h, C)$ by theorem 8.1. We can define

$$P_C : \mathcal{H} \to \mathcal{H} \text{ by } P_C(h) = k,\ \ k \in C$$

Definition 8.2 (Orthogonality)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $x, y \in X$. We say that x is orthogonal to y if $\langle x, y\rangle = 0$. We write it as $x \perp y$.

Remark 8.9. If x is orthogonal to y, then y is orthogonal to x.

$$x \perp y \iff y \perp x$$

Example 8.2. 1. Consider $(\mathbb{F}^n, \langle \cdot, \cdot \rangle)$ with the standard inner product.
Let $e_i = (0, \ldots, \underbrace{1}_{i^{th}}, 0, \ldots, 0) \in \mathbb{F}^n, \; 1 \leq i \leq n$
Then $e_i \perp e_j \; \forall i \neq j$

2. For $(\ell^2(\mathbb{N}), \langle \cdot, \cdot \rangle), \; e_i \perp e_j \; \forall i \neq j$.

Theorem 8.2 (Pythagoras' Theorem)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $x, y \in X$ be such that x and y are orthogonal to each other, i.e. $\langle x, y \rangle = 0$. Then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Proof. We have $\langle x, y \rangle = \langle y, x \rangle = 0$.

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&= \langle x, x \rangle + \langle y, y \rangle \\
&= \|x\|^2 + \|y\|^2
\end{aligned}$$

□

Definition 8.3 (Orthogonal complement)**.** Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space an M be a subset of X (not necessarily a subspace). The orthogonal complement of M is defined as

$$M^{\perp} = \{x \in X : \langle x, m \rangle = 0 \; \forall m \in M\}$$

That is, the orthogonal complement of $M \subseteq X$ consists of the elements of X that are orthogonal to all elements of M.

Lemma 8.5. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space an M be a subset of X. Then $M^{\perp}$ is a closed subspace of X.*

Proof. **Claim 1:** $M^{\perp}$ is a subspace of X.
Let $x, y \in M^{\perp}$ and $\alpha \in \mathbb{F}$. Then $\langle x, m \rangle = 0 = \langle y, m \rangle$. Then

$$\langle \alpha x + y, m \rangle = \alpha \langle x, m \rangle + \langle y, m \rangle = 0 \; \forall m \in M$$

$\implies \alpha x + y \in M^{\perp} \implies M^{\perp}$ is a subspace of X.

Claim 2: $M^\perp$ is closed in X. Let $(x_n) \subseteq M^\perp$ such that $x_n \to x$ in X. As $(x_n) \subseteq M^\perp$, we have $\langle x_n, m\rangle = 0\ \forall m \in M,\ \forall n \in \mathbb{N}$. To show that $M^\perp$ is closed, it is enough to show that $x \in M^\perp$, i.e. $\langle x, m\rangle = 0\ \forall m \in M$.
For any $m \in M$,

$$\langle x, m\rangle = \left\langle \lim_{n\to\infty} x_n, m \right\rangle = \lim_{n\to\infty} \langle x_n, m\rangle = 0 \text{ (as } \langle \cdot, m\rangle \text{ is continuous by lemma 8.3 (1))}$$

$\implies x \in M^\perp$ □

Lemma 8.6. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $M_1,\ M_2$ be two subsets of X such that $M_1 \subseteq M_2$. Then $M_2^\perp \subseteq M_1^\perp$.*

Proof. We have

$$M_2^\perp = \{x \in X : \langle x, m_2\rangle = 0\ \forall m_2 \in M_2\}$$
$$M_1^\perp = \{x \in X : \langle x, m_1\rangle = 0\ \forall m_1 \in M_1\}$$

Given that $M_1 \subseteq M_2$. Let $m \in M_2^\perp$. Then $\langle m, m_2\rangle = 0\ \forall m_2 \in M_2$. As $M_1 \subseteq M_2$, we have $\langle m, m_1\rangle = 0\ \forall m_1 \in M_1$.
$\implies m \in M_1^\perp \implies M_2^\perp \subseteq M_1^\perp$ □

Lemma 8.7. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and M be a subset of X. Then $M \subsetneq M^{\perp\perp}$, where $M^{\perp\perp} = (M^\perp)^\perp$.*

Proof. We know that

$$M^\perp = \{x \in X : \langle x, m\rangle = 0\ \forall m \in M\}$$
$$M^{\perp\perp} = \{y \in X : \langle y, x\rangle = 0\ \forall x \in M^\perp\}$$

Let $m \in M$ and $x \in M^\perp$. Then $\langle m, x\rangle = 0 = \langle x, m\rangle$
$\implies \langle m, x\rangle = 0\ \forall x \in M^\perp$ (since x was arbitrary)
$\implies m \in M^{\perp\perp} \implies M \subseteq M^{\perp\perp}$

For $M \neq M^{\perp\perp}$: Take $X = \mathbb{R}^2$ with the standard inner product and $M = \{(1,0)\}$. Then

$$M^\perp = \{\mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{x}, (1,0)\rangle = 0\} = \{(0,y) : y \in \mathbb{R}\}$$
$$M^{\perp\perp} = \{\mathbf{z} \in \mathbb{R}^2 : \langle \mathbf{z}, (0,y)\rangle = 0\ \forall (0,y) \in M^\perp\} = \{(a,0) : a \in \mathbb{R}\}$$

Clearly, $M \subset M^{\perp\perp}$ and so, $M \neq M^{\perp\perp}$. □

Remark 8.10. The equality $M = M^{\perp\perp}$ holds under special conditions: when X is a Hilbert space and M is a closed subspace of X (see Projection Theorem).

Exercise 8.3. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Show that

1. If $M = \{0\}$, then $M^{\perp} = X$.

2. If $M = X$, then $M^{\perp} = \{0\}$.

Theorem 8.3. *Let $\mathcal{H}$ be a Hilbert space and M be a closed subspace of $\mathcal{H}$. Then*

1. *For each $h \in \mathcal{H}$, $\exists$ a unique $m \in M$ such that $\|h - m\| = dist(h, M)$. Moreover, $h - m \in M^{\perp}$.*

2. *Conversely, if $h \in \mathcal{H}$ and $m \in M$ be such that $h - m \in M^{\perp}$, then $\|h - m\| = dist(h, M)$.*

Proof. 1. Given that M is a closed subspace of $\mathcal{H}$. As every subspace is convex, we have that M is a closed and convex set in $\mathcal{H}$. By Best approximation theorem, for each $h \in \mathcal{H}$, $\exists$ a unique $m \in M$ such that $\|h - m\| = dist(h, M)$.

Claim: $h - m \in M^{\perp}$, i.e. $\langle h - m, m' \rangle = 0 \ \forall m' \in M$.

$$\begin{aligned} \|h - m - m'\|^2 &= \|h - m\|^2 - 2Re(\langle h - m, m' \rangle) + \|m'\|^2 \\ \implies \|h - (m + m')\|^2 - \|h - m\|^2 &= \|m'\|^2 - 2Re(\langle h - m, m' \rangle) \end{aligned}$$

As M is a subspace and $m,\ m' \in M$, we have $m + m' \in M$. Also, by assumption, $\|h - m\| = \inf\{\|h - m'\| : m' \in M\}$, and $m + m'$ is one of the elements in M. Therefore, we have $\|h - m\| \leq \|h - (m + m')\|$. Hence,

$$\begin{aligned} &\|h - (m + m')\|^2 - \|h - m\|^2 \geq 0 \\ \implies &\|m'\|^2 - 2Re(\langle h - m, m' \rangle) \geq 0 \ \forall m' \in M \\ &\implies \|m'\|^2 \geq 2Re(\langle h - m, m' \rangle) \ \forall m' \in M \ldots\ldots (*) \end{aligned}$$

Let $\langle h - m, m' \rangle = |\langle h - m, m' \rangle| e^{i\theta}$ (polar form)
For $r > 0$, replace m' by $m'' = re^{i\theta} m'$ in $(*)$. Then

$$\begin{aligned} r^2 \|m'\|^2 &\geq 2Re(\langle h - m,\ re^{i\theta} m' \rangle) \\ \implies r^2 \|m'\|^2 &\geq 2Re(re^{-i\theta} \langle h - m, m' \rangle) \\ &= 2Re(re^{-i\theta} |\langle h - m, m' \rangle| e^{i\theta}) \\ &= 2r |\langle h - m, m' \rangle| \\ \implies r\|m'\|^2 &\geq 2|\langle h - m, m' \rangle| \ \forall r > 0 \\ \implies \langle h - m, m' \rangle &= 0 \ \forall m' \in M \ [\text{since } |a| \leq \varepsilon \ \forall \varepsilon > 0 \implies |a| = 0] \\ \implies h - m &\in M^{\perp} \end{aligned}$$

2. Given $h \in \mathcal{H}$ and $m \in M$ such that $h - m \in M^{\perp}$, i.e. $\langle h - m, m' \rangle = 0 \ \forall m' \in M$
   Claim: $\|h - m\| = dist(h, M)$
   We know that $\|h - m\| \geq dist(h, M)$ by definition, as $dist(h, M)$ is the infimum of $\{\|h - m'\| : m' \in M\}$. Hence, it is enough to show that $\|h - m\| \leq \|h - m'\| \ \forall m' \in M$.

$$\begin{aligned}
\|h - m + m'\|^2 &= \|h - m\|^2 + 2Re(\underbrace{\langle h - m, m' \rangle}_{= \ 0 \ \forall m' \in M}) + \|m'\|^2 \\
&= \|h - m\|^2 + \|m'\|^2 \ \forall m' \in M \\
\implies \|h - m\|^2 &\leq \|h - (m - m')\|^2 \ \forall m' \in M \ (\because \|m'\| \geq 0) \\
\implies \|h - m\| &\leq \|h - (m - m')\| \ \forall m' \in M \\
\implies \|h - m\| &\leq \|h - m'\| \ \forall m' \in M \ (\because M \text{ is a subspace}) \\
\implies \|h - m\| &= dist(h, M)
\end{aligned}$$

$\square$

Theorem 8.4 (Projection Theorem)**.** *Let M be a closed subspace of a Hilbert space $\mathcal{H}$. Then*

1. *$\mathcal{H} = M \oplus M^{\perp}$ (M and $M^{\perp}$ are called complementary subspaces).*

2. *$M = M^{\perp\perp}$*

Proof. Given that $\mathcal{H}$ is a Hilbert space and M is a closed subspace of $\mathcal{H}$.

1. Let $h \in \mathcal{H}$. We know by theorem 8.3 that $\exists$ unique $m \in M$ such that $h - m \in M^{\perp}$. Write

$$h = \underbrace{m}_{\in M} + \underbrace{(h - m)}_{\in M^{\perp}}$$

   Hence, $H = M + M^{\perp}$
   Observe that if $x \in M \cap M^{\perp}$, then $x \in M$ and $x \in M^{\perp} \implies \langle x, x \rangle = 0 \implies x = 0$. Thus, $M \cap M^{\perp} = \{0\}$. Thus, $\mathcal{H} = M \oplus M^{\perp}$.

2. We know that

$$\begin{aligned}
M^{\perp} &= \{x \in \mathcal{H} : \langle x, m \rangle = 0 \ \forall m \in M\} \\
M^{\perp\perp} &= \{y \in \mathcal{H} : \langle y, x \rangle = 0 \ \forall x \in M^{\perp}\}
\end{aligned}$$

   $(M \subset M^{\perp\perp})$: Let $m \in M$ and $x \in M^{\perp}$. Then $\langle m, x \rangle = 0 = \langle x, m \rangle \ \forall x \in M^{\perp}$
   $\langle m, x \rangle = 0 \ \forall x \in M^{\perp} \implies m \in M^{\perp\perp} \implies M \subset M^{\perp\perp}$

   $(M^{\perp\perp} \subset M)$: Let $y \in M^{\perp\perp}$. Then $\exists$ unique $m \in M$ such that $y - m \in M^{\perp}$ (m is the best approximation of y).

By definition of $M^{\perp\perp}$, $\langle y, y-m\rangle = 0$
By definition of $M^{\perp}$, $\langle m, y-m\rangle = 0$
$\implies \langle y-m,\ y-m\rangle = 0 \implies y = m \in M$
$\implies M^{\perp\perp} \subset M$
Thus, $M^{\perp\perp} = M$.

□

Lemma 8.8. *Let $\mathcal{H}$ be a Hilbert space and M be a subspace of $\mathcal{H}$. Then $M^{\perp\perp} = \overline{M}$.*

Proof. ($M^{\perp\perp} \subseteq \overline{M}$) : We know that $M \subseteq \overline{M} \implies \overline{M}^{\perp} \subseteq M^{\perp} \implies M^{\perp\perp} \subseteq \overline{M}^{\perp\perp}$ (using lemma 8.6 twice).
By theorem 8.4, if M is a closed subspace of $\mathcal{H}$, then $M^{\perp\perp} = M$. As $\overline{M}$ is a closed subspace of $\mathcal{H}$, it follows that $\overline{M}^{\perp\perp} = \overline{M}$. Hence, $M^{\perp\perp} \subseteq \overline{M}$.

($\overline{M} \subseteq M^{\perp\perp}$) : We know that $M^{\perp}$ is a closed subspace of $\mathcal{H}$ (lemma 8.5). Hence, $M^{\perp\perp}$ is also a closed subspace of $\mathcal{H}$. By lemma 8.7, $M \subset M^{\perp\perp}$. By monotonicity of topological closure, we have $\overline{M} \subset \overline{M^{\perp\perp}} = M^{\perp\perp}$. Thus, $\overline{M} \subset M^{\perp\perp}$. □

Theorem 8.5. *Let $\mathcal{H}$ be a Hilbert space.*

1. *If $M(\neq \phi) \subseteq \mathcal{H}$, then $M^{\perp\perp} = \overline{span(M)}$.*
2. *Let $M(\neq \phi) \subseteq \mathcal{H}$. If $M^{\perp} = \{0\}$, then $span(M)$ is dense in $\mathcal{H}$.*

Proof. 1. We know that $M^{\perp}$ and $M^{\perp\perp}$ are closed subspaces of $\mathcal{H} \implies M^{\perp} = \overline{M^{\perp}}$ and $M^{\perp\perp} = \overline{M^{\perp\perp}}$.
Also, we know that $M \subseteq M^{\perp\perp}$ (lemma 8.7).

$M^{\perp\perp}$ is a subspace containing $M \implies span(M) \subseteq M^{\perp\perp}$.
By monotonicity of closure, we have $\overline{span(M)} \subseteq \overline{M^{\perp\perp}} = M^{\perp\perp}$.
$\implies \overline{span(M)} \subseteq M^{\perp\perp}$

Now, $M \subseteq \overline{span(M)} \implies \left(\overline{span(M)}\right)^{\perp} \subseteq M^{\perp} \implies M^{\perp\perp} \subseteq \left(\overline{span(M)}\right)^{\perp\perp}$ (using lemma 8.6).
But since $\overline{span(M)}$ is a closed subspace of $\mathcal{H}$, we have by theorem 8.4 that $\left(\overline{span(M)}\right)^{\perp\perp} = \overline{span(M)}$.
$\implies M^{\perp\perp} \subseteq \overline{span(M)}$.

Thus, $M^{\perp\perp} = \overline{span(M)}$

2. Let $M^{\perp} = \{0\} = \{x \in \mathcal{H} : \langle x, m\rangle = 0\ \forall m \in M\}$. Then

$$M^{\perp\perp} = \{y \in \mathcal{H} : \langle y, x\rangle = 0\ \forall x \in M^{\perp}\} = \{y \in \mathcal{H} : \langle y, 0\rangle = 0\} = \mathcal{H}$$

Using (1), we have $\mathcal{H} = M^{\perp\perp} = \overline{span(M)}$
Which means that $span(M)$ is dense in $\mathcal{H}$.

□

Lemma 8.9. *Let $\mathcal{H}$ be a Hilbert space and M be a closed subspace of $\mathcal{H}$. For each $h \in \mathcal{H}$, we know by theorem 8.3 that $\exists$ a unique $m \in M$ such that m is the best approximation of h, and $h - m \in M^{\perp}$. Define*

$$P_M : \mathcal{H} \to \mathcal{H} \text{ by } P_M(h) = m, \ \forall h \in \mathcal{H}$$

That is, every $h \in \mathcal{H}$ is mapped to its best approximation in M under P_M. Then

1. *P_M is a linear transformation.*
2. *On M, P_M acts like identity. That is, $P_M(m) = m \ \forall m \in M$.*
3. *P_M is idempotent, i.e. $P_M^2 = P_M$.*
4. *$\|P_M x\| \leq \|x\| \ \forall x \in \mathcal{H}$ (P_M is continuous).*
5. *$\|P_M\| = 1$.*
6. *$Range(P_M) = M$ and $ker(P_M) = M^{\perp}$.*

Proof. 1. $\forall x, y \in \mathcal{H}$ and $\forall \alpha \in \mathbb{F}$, we show that $P_M(\alpha x + y) = \alpha P_M(x) + P_M(y)$.
Let $x, y \in \mathcal{H}$. Then $\exists$ unique $m_1, \ m_2 \in M$ such that $x - m_1 \in M^{\perp}$ and $y - m_2 \in M^{\perp}$ (using theorem 8.3.
This means that $P_M(x) = m_1$ and $P_M(y) = m_2$.
As $M^{\perp}$ is a subspace, we have $(x + y) - (m_1 + m_2) \in M^{\perp}$.
Thus,

$$P_M(x + y) = m_1 + m_2 \text{ (by definition)} = P_M(x) + P_M(y)$$

Also, for $x \in \mathcal{H}$ and $\alpha \in \mathbb{F}$, $\exists$ unique $m_1 \in M$ such that $x - m_1 \in M^{\perp}$. Or, $\alpha x - \alpha m_1 \in M^{\perp}$.
$\implies P_M(\alpha x) = \alpha m_1 = \alpha P_M(x)$

2. Let $m \in M$. As $m - m = 0 \in M^{\perp}$, we have $P_M(m) = m \ \forall m \in M$.

3. Let $x \in \mathcal{H}$. Note that $P_M(x) \in M$, and P_M acts like identity on M. Therefore,

$$P_M^2(x) = \underbrace{P_M}_{\sim\text{identity}} \underbrace{(P_M(x))}_{\in M} = P_M(x)$$

4. Let $x \in \mathcal{H}$. By the Projection Theorem, we can write $x = m + m'$, where $m \in M$ and $m' \in M^{\perp}$. Note that $\langle m, m' \rangle = 0$. Also, $m' = m' - 0 \in M^{\perp} \implies P_M(m') = 0$.

$$\begin{aligned} P_M(x) = P_M(m + m') &= P_M(m) + P_M(m') \\ &= m + 0 \\ &= m \\ \implies \|P_M x\|^2 = \|m\|^2 &\leq \|m\|^2 + \|m'\|^2 = \|x\|^2 \text{ (by Pythagoras' Theorem)} \\ \implies \|P_M x\| &\leq 1 \end{aligned}$$

5. From (4),

$$\begin{aligned} \|P_M x\| &\leq \|x\| \\ \implies \|P_M\| &\leq 1 \end{aligned}$$

Let $m \in M$ with $\|m\| = 1$ (if $\|m\| = k \in \mathbb{R}^+ \neq 1$, then select $\frac{m}{k} \in M$).
Then $P_M(m) = m$, and

$$\begin{aligned} \|P_M(m)\| &= \|m\| = 1 \\ \implies \|P_M\| &\leq 1 \text{ and } \exists m \in M \text{ such that} \|P_M(m)\| = 1 \\ \implies \|P_M\| &= 1 \end{aligned}$$

6. From the definition of P_M, as P_M sends each $x \in \mathcal{H}$ to its best approximation in M, we have $Range(P_M) \subseteq M$.
Now, let $m \in M$. We have $P_M(m) = m \implies m \in Range(P_M)$
$\implies Range(P_M) = M$.

Let $m' \in M^{\perp}$. Then

$$m' = m' - 0 \in M^{\perp} \implies P_M(m') = 0 \implies m' \in ker(P_M) \implies M^{\perp} \subseteq ker(P_M)$$

To show the reverse inclusion, let $x \in ker(P_M)$. Then $P_M(x) = 0$. By Projection

Theorem, write $x = m + m'$, where $m \in M$ and $m' \in M^{\perp}$. Then

$$\begin{aligned} P_M(x) = P_M(m+m') &= 0 \\ \implies P_M(m) + P_M(m') &= 0 \\ \implies P_M(m) &= 0 \ (\because P_M(m') = 0) \\ \implies m &= 0 \ (\because P_M(m) = m) \end{aligned}$$

$$\implies x = m + m' = m' \in M^{\perp} \implies ker(P_M) \subseteq M^{\perp}$$

Thus, $ker(P_M) = M^{\perp}$.

□

Definition 8.4 (Orthogonal projection)**.** Let $T \in \mathcal{B}(\mathcal{H})$ (i.e., T is a bounded linear operator on a Hilbert space $\mathcal{H}$). We say that T is an **orthogonal projection** if:

1. T is idempotent, i.e. $T^2 = T$
2. $Range(T) \perp ker(T)$, i.e. $x \perp y \ \forall x \in Range(T), \ y \in ker(T)$

Remark 8.11*.* The map P_M defined in lemma 8.9 is a bounded linear operator on $\mathcal{H}$. Further, it is idempotent, and $Range(P_M) = M$, $ker(P_M) = M^{\perp}$, which means that $Range(P_M) \perp ker(P_M)$. Thus, P_M is an orthogonal projection of $\mathcal{H}$ onto M.

Lemma 8.10. *If $T \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection. Then there exists a closed subspace M of $\mathcal{H}$ such that $T = P_M$.*

Proof. We claim that the required closed subspace M is $Range(T)$.

Claim: $M = Range(T)$ is closed.
As T is an orthogonal projection, we have $(ker(T))^{\perp} = Range(T)$. We know by lemma 8.5 that for any subset M of a Hilbert space $\mathcal{H}$, $M^{\perp}$ is a closed subspace of $\mathcal{H}$. Thus, $(ker(T))^{\perp} = Range(T)$ is closed in $\mathcal{H}$.

Claim: $T = P_M$, i.e. $T = P_{Range(T)}$. Define P_M (as in lemma 8.9) by $P_M(x) = m$, if $x - m \in M^{\perp}$.
Here, we have $P_M(x) = y$, if $x - y \in ker(T)$ (since $M^{\perp} = (Range(T))^{\perp} = ker(T)$).
Let for $x \in \mathcal{H}$, $T(x) = y$. Then

$$T(T(x)) = T(y) \implies T(x) = T(y) \implies T(x) - T(y) = 0 \implies T(x-y) = 0 \implies x - y \in ker(T)$$

And, $x - y \in ker(T) \implies P_M(x) = y = T(x)$.
As $x \in \mathcal{H}$ was arbitrary, it follows that $T \equiv P_M$. □

Remark 8.12. From lemma 8.9 and lemma 8.10, it follows that there exists a one-to-one correspondence between the set of all bounded linear operators on a Hilbert space $\mathcal{H}$, and the set of all closed subspaces M of $\mathcal{H}$.

8.2 Dual of a Hilbert Space

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $y \in X$. Define $\eta_y : X \to \mathbb{F}$ by

$$\eta_y(x) = \langle x, y\rangle \ \forall x \in X.$$

Clearly, η_y is linear.
We have $|\eta_y(x)| = |\langle x, y\rangle| \leq \|x\|\|y\| \ \forall x \in X$ (by Cauchy-Schwarz inequality)
$\implies \eta_y$ is continuous $\implies \eta_y \in X^*$. Also, $\|\eta_y\| \leq y$ (from the above inequality). Further,

$$\eta_y\left(\frac{y}{\|y\|}\right) = \left\langle \frac{y}{\|y\|}, y\right\rangle = \frac{1}{\|y\|}\langle y, y\rangle = \frac{\|y\|^2}{\|y\|} = \|y\|$$

$\implies \|\eta_y\| = y$

Therefore, the map $\Phi : X \to X^*$ defined by $y \mapsto \eta_y$ is an anti-linear isometry.

Theorem 8.6 (Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$ and $f : \mathcal{H} \to \mathbb{F}$ be a continuous linear functional. Then there exists a unique $x_0 \in \mathcal{H}$ such that*

$$f(x) = \langle x, x_0\rangle \ \forall x \in \mathcal{H}$$

That is, $f = \eta_{x_0}$ for a unique $x_0 \in \mathcal{H}$. Moreover, we have $\|f\| = \|x_0\|$.

Proof. If $f = 0$, we can simply take $x_0 = 0$, and the proof is done.
Let $f \neq 0$. As f is continuous, by lemma 5.6, $N(f)$ is a closed and proper subspace of $\mathcal{H}$. By Projection Theorem, we can write $\mathcal{H} = N(f) \oplus N(f)^\perp$.

We know that as $f \neq 0$, $\exists h \in \mathcal{H}$ such that $f(h) = 1$ (if $f(h) = \alpha$, then take $\frac{h}{\alpha} \in \mathcal{H}$, so that $f\left(\frac{h}{\alpha}\right) = 1$. By Projection Theorem, we can write $h = h_1 + h_2$, where $h_1 \in N(f)$ and $h_2 \in N(f)^\perp$. Then $f(h_1 + h_2) = 1 \implies f(h_1) + f(h_2) = 1 \implies f(h_2) = 1, h_2 \in N(f)^\perp$.
Hence, choose an $m \in N(f)^\perp$ such that $f(m) = 1$.
By remark 5.17, for all $x \in \mathcal{H}$, we can write

$$x = u_x + \frac{f(x)}{f(m)}m, \text{ for some } u_x \in N(f), \text{ where } f(m) = 1$$

Taking inner product with m both sides, and using the fact that $\langle u_x, m\rangle = 0$ because

$u_x \in N(f)$ and $m \in N(f)^{\perp}$, we have

$$\begin{aligned} \langle x, m\rangle &= \langle u_x, m\rangle + f(x)\langle m, m\rangle \ \forall x \in \mathcal{H} \\ \implies f(x)\langle m, m\rangle &= \langle x, m\rangle \ \forall x \in \mathcal{H} \\ \implies f(x) &= \left\langle x, \frac{m}{\langle m, m\rangle} \right\rangle \end{aligned}$$

Let $x_0 = \frac{m}{\langle m,m\rangle}$, then $f(x) = \langle x, x_0\rangle \ \forall x \in \mathcal{H}$.

(Uniqueness of such x_0): Suppose $\exists x_1,\ x_2 \in \mathcal{H}$ such that

$$\begin{aligned} f(x) = \langle x, x_1\rangle &= \langle x, x_2\rangle \ \forall x \in \mathcal{H} \\ \implies \langle x, x_1 - x_2\rangle &= 0 \ \forall x \in \mathcal{H} \\ \implies x_1 - x_2 &= 0 \\ \implies x_1 &= x_2 \end{aligned}$$

□

Corollary 8.7 (Hilbert space is self-dual)**.** *Let $\mathcal{H}$ be a Hilbert space. Then the map $\Phi : \mathcal{H} \to \mathcal{H}^*$ defined by $x \mapsto \eta_x$ (where η_x is as defined in the beginning of this section) is an anti-linear isometry and onto. That is, $\mathcal{H} \cong \mathcal{H}^*$, or a Hilbert space is its own dual.*

Proof. We have proved in the beginning of this section that the map Φ is an anti-linear isometry. Let $f \in \mathcal{H}^*$. By Riesz Representation Theorem, $\exists$ a unique $x_0 \in \mathcal{H}$ such that $f = \eta_{x_0}$. Hence, Φ is onto. □

8.3 Orthonormal Sets & Orthonormal Bases

Definition 8.5 (Orthonormal set)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $A \subseteq X$. We say that A is orthonormal in X if

1. $\langle x, y\rangle = 0 \ \forall x \neq y,\ x, y \in A$
2. $\langle x, x\rangle = \|x\| = 1 \ \forall x \in A$

Example 8.3 (Orthonormal sets)**.**
1. Consider $(\mathbb{F}^n, \langle\cdot,\cdot\rangle)$ with the standard inner product. Take $e_i = (0, \ldots, \underbrace{1}_{i^{th}}, 0, \ldots, 0) \in \mathbb{F}^n$. The set $A = \{e_i : 1 \leq i \leq n\}$ is orthonormal.
2. For $(\ell^p(\mathbb{N}), \langle\cdot,\cdot\rangle)$, the set $\{e_i = (0, \ldots, 1, 0, \ldots) : i \in \mathbb{N}\}$ is orthonormal.

Definition 8.6 (Orthonormal basis)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. A subset $\mathscr{A} \subseteq X$ is called an orthonormal basis of X if

1. $\mathscr{A}$ is an orthonormal set.

2. $\overline{span(\mathscr{A})} = X$.

Definition 8.7 (Maximal orthonormal set)**.** Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. A set $A \subseteq X$ is called a maximal orthonormal set if

1. A is an orthonormal set.

2. If $\exists$ an orthonormal set $B \subseteq X$ such that $A \subseteq B$, then $A = B$ (i.e., A is maximal).

Lemma 8.11. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space, $X \neq \{0\}$. Then there exists a maximal orthonormal set in X.*

Proof. Let $\mathcal{A} = \{A \subseteq X : A \text{ is an orthonormal set}\}$.
As $X \neq \{0\}$, we have $\mathcal{A} \neq \phi$ because the set $\left\{\frac{x}{\|x\|}\right\}$ for any $x \neq 0$ is inside $\mathcal{A}$.

Let $A, B \in \mathcal{A}$. Define a partial order '$\leq$' on $\mathcal{A}$ as "$A \leq B$ iff $A \subseteq B$". Verify that $(\mathcal{A}, \leq)$ is a POSet.
Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Let $M = \bigcup_{A \in \mathcal{C}} A$. Check that $M \in \mathcal{A}$ and $A \subseteq M \; \forall A \in \mathcal{C}$.
$\implies$ Every chain in $\mathcal{A}$ is bounded above.
$\implies$ By Zorn's lemma, $\mathcal{A}$ has a maximal element. As $\mathcal{A}$ consists of all orthonormal sets in X, we have that the maximal element of $\mathcal{A}$ is the maximal orthonormal set in X. □

Lemma 8.12. *Every orthonormal set in X is linearly independent.*

Proof. Let A be an orthonormal set in X. Let $x_1, \ldots, x_n \in A$ and $c_1, \ldots, c_n \in \mathbb{F}$ be such that $\sum_{i=1}^{n} c_i x_i = 0$. Let $j \in \{1, \ldots, n\}$. Then

$$\left\langle \sum_{i=1}^{n} c_i x_i, x_j \right\rangle = \langle 0, x_j \rangle = 0$$

$$\implies \sum_{i=1}^{n} c_i \underbrace{\langle x_i, x_j \rangle}_{=1 \text{ when } i=j} = c_j = 0$$

Hence, we have $c_j = 0 \; \forall j \in [1 : n]$, which means that A is linearly independent. □

Exercise 8.4. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\mathscr{B} \subseteq X$. Then the following are equivalent:

1. $\mathscr{B}$ is a Maximal orthonormal set in X.

2. $\mathscr{B}$ is an Orthonormal basis for X.

Theorem 8.8. *Let E be an orthonormal set of an inner product space $(X, \langle \cdot, \cdot \rangle)$. Then E is an orthonormal basis if and only if $E^{\perp} = \{0\}$, i.e. for $x \in E, \langle x, u \rangle = 0 \; \forall u \in E \implies x = 0$.*

Proof. We show the contrapositive, i.e. $E^\perp \neq \{0\} \iff E$ is not an orthonormal basis.

($\implies$) Suppose E be an orthonormal set such that $E^\perp \neq \{0\}$. Let $x \neq 0 \in E^\perp$ (i.e. $\langle e, x \rangle = 0 \; \forall e \in E$).
Take $u = \frac{x}{\|x\|}$. Define $\tilde{E} = E \cup \{u\}$.
Note that $\|u\| = 1$, and $\langle e, u \rangle = \left\langle e, \frac{x}{\|x\|} \right\rangle = \frac{1}{\|x\|}\langle e, x \rangle = 0 \; \forall e \in E$
$\implies \tilde{E}$ is an orthonormal set properly containing E.
$\implies E$ is not maximal $\implies E$ cannot be orthonormal basis for X (exercise 8.4).

($\impliedby$) Suppose that E is not an orthonormal basis of X. Then E is not maximal, i.e. $\exists$ an orthonormal set $\tilde{E}$ such that $E \subsetneq \tilde{E}$.
Let $v \in \tilde{E} \setminus E$. Then $v \neq 0$ (since $\tilde{E}$ is orthonormal), and
$\langle v, e \rangle = 0 \; \forall e \in E$ (since all e's are also in $\tilde{E}$, $e \neq v$ as $e \in E$ and $v \notin E$, and $\tilde{E}$ is orthonormal).
$\implies v \neq 0 \in E^\perp \implies E^\perp \neq \{0\}$. □

Proposition 8.4. *An orthonormal basis need not be a basis.*

Proof. Let $X = \ell^2(\mathbb{N})$ and $E = \{e_1, e_2, \ldots\}$ where $e_i = (0, \ldots, \underbrace{1}_{i^{th}}, 0, \ldots)$.
Clearly, E is an orthonormal set.
We know,

$$\langle x, e_i \rangle = x_i \; \forall x \in \ell^2(\mathbb{N}), \; \forall i \in \mathbb{N}$$
$$\implies \text{For } x \in \ell^2(\mathbb{N}), \langle x, e_i \rangle = 0 \; \forall i \in \mathbb{N} \implies x = 0$$
$$\implies E^\perp = \{0\}$$

By theorem 8.8, E is an orthonormal basis for $\ell^2(\mathbb{N})$.
However, E cannot be a basis of $\ell^2(\mathbb{N})$ because $\ell^2(\mathbb{N})$ is complete and E is countable. □

Proposition 8.5. *Let X be a finite dimensional inner product space. Then every orthonormal basis for X is also a basis.*

Proof. Let $\mathscr{A} = \{x_1, \ldots, x_n\}$ be an orthonormal basis for X. Then by definition, we have $\overline{span(\mathscr{A})} = X$. Also, $\mathscr{A}$ is linearly independent (lemma 8.12). If we show that $span(\mathscr{A})$ is closed, then we are done, as then $\overline{span(\mathscr{A})} = span(\mathscr{A}) = X \implies \mathscr{A}$ is a basis.

Suppose $(y_n) \subseteq span(\mathscr{A})$ such that $y_n \to y$ in X. We want to show that $y \in span(\mathscr{A})$, so that $span(\mathscr{A})$ is closed.

As $y_n \in span(\mathscr{A})\ \forall n \in \mathbb{N}$, for each n, $\exists\, (a_{i,n})_{i=1}^m$ such that

$$y_n = \sum_{i=1}^{m} a_{i,n}\ x_i$$

As $\{x_1, \dots, x_n\}$ is an orthonormal basis for X, we have $a_{i,n} = \langle y_n, x_i\rangle$ (see lemma 8.13)

$$\lim_{n\to\infty} a_{i,n} = \lim_{n\to\infty} \langle y_n, x_i\rangle = \left\langle \lim_{n\to\infty} y_n, x_i \right\rangle \text{ (by continuity of } \langle\cdot, x_i\rangle) = \langle y, x_i\rangle$$

$\therefore y = \sum_{i=1}^m a_i x_i$, where $a_i = \langle y, x_i\rangle$
$\implies y \in span(\mathscr{A})$. □

Theorem 8.9. *An orthonormal set in a separable inner product space is countable.*

Proof. Let X be an inner product space and E be an orthonormal set in X. Let X be separable, and $\{a_j\}_{j\in\mathbb{N}}$ be a countable dense subset of X. Assume for the sake of contradiction, that E is uncountable.

For every $u,\ v \in E$ with $u \neq v(\implies \langle u, v\rangle = 0)$,

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 + 2Re(\langle u, v\rangle) = 1 + 1 = 2$$

$\implies \|u - v\| = \sqrt{2}$
$\implies$ We have obtained E, an uncountable subset of X, where the distance between each distinct element (of E) is greater than 1. By theorem 4.1, this implies that X is not separable, which is a contradiction. Thus, E must be countable. □

8.4 Riesz-Fischer Theorems

8.4.1 Finite Dimensional Spaces

Lemma 8.13 (Fourier Representation)**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space ($dim(X) = n$) and $\mathscr{B} = \{x_1, \dots, x_n\}$ be an orthonormal basis for X. Then*

$$x = \sum_{i=1}^{n} \langle x, x_i\rangle x_i,\ \ \forall x \in X$$

Proof. Given that $\mathscr{B} = \{x_1, \dots, x_n\}$ is an orthonormal basis of X. As X is finite dimensional, $\mathscr{B}$ is also a basis for X (proposition 8.5). Let $x \in X$. Then $\exists\ c_1, c_2, \dots, c_n \in \mathbb{F}$ such that $x = \sum_{i=1}^n c_i x_i$.

Let $j \in \{1, \dots, n\}$. Then

$$\langle x, x_j \rangle = \left\langle \sum_{i=1}^{n} c_i x_i, x_j \right\rangle = \sum_{i=1}^{n} c_i \underbrace{\langle x_i, x_j \rangle}_{=1 \text{ when } i=j} = c_j$$

$$\implies x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$

□

Lemma 8.14 (Plancherel's identity)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space ($dim(X) = n$) and $\mathscr{B} = \{x_1, \dots, x_n\}$ be an orthonormal basis for X. Then*

$$\|x\|^2 = \sum_{i=1}^{n} |\langle x, x_i \rangle|^2, \ \forall x \in X$$

Proof. Let $x \in X$. By lemma 8.13, we have $x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$. Now,

$$\begin{aligned}
\|x\|^2 = \langle x, x \rangle &= \left\langle \sum_{i=1}^{n} \langle x, x_i \rangle, \ \sum_{j=1}^{n} \langle x, x_j \rangle x_j \right\rangle \\
&= \sum_{i=1}^{n} \langle x, x_i \rangle \left\langle x_i, \sum_{j=1}^{n} \langle x, x_j \rangle x_j \right\rangle \\
&= \sum_{i=1}^{n} \langle x, x_i \rangle \sum_{j=1}^{n} \overline{\langle x, x_j \rangle} \underbrace{\langle x_i, x_j \rangle}_{=1 \text{ when } i=j} \\
&= \sum_{i=1}^{n} \langle x, x_i \rangle \overline{\langle x, x_i \rangle} \\
&= \sum_{i=1}^{n} |\langle x, x_i \rangle|^2
\end{aligned}$$

□

Lemma 8.15 (Parseval's identity)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space ($dim(X) = n$) and $\mathscr{B} = \{x_1, \dots, x_n\}$ be an orthonormal basis for X. Then*

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, x_i \rangle \overline{\langle y, x_i \rangle} \ \forall x, y \in X$$

Proof. Let $x, y \in X$. By lemma 8.13, we have $x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$ and $y = \sum_{i=1}^{n} \langle y, x_i \rangle x_i$.

Then

$$\begin{aligned}
\langle x, y\rangle &= \left\langle \sum_{i=1}^{n} \langle x, x_i\rangle x_i,\ \sum_{j=1}^{n} \langle y, x_j\rangle x_j \right\rangle \\
&= \sum_{i=1}^{n} \langle x, x_i\rangle \sum_{j=1}^{n} \overline{\langle y, x_j\rangle}\ \underbrace{\langle x_i, x_j\rangle}_{=1 \text{ when } i=j} \\
&= \sum_{i=1}^{n} \langle x, x_i\rangle \overline{\langle y, x_i\rangle}
\end{aligned}$$

□

Theorem 8.10 (Riesz-Fischer). *Let $(X, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space $(dim(X) = n)$ and $\mathscr{B} = \{x_1, \ldots, x_n\}$ be an orthonormal basis for X. Define $T : X \to \mathbb{F}^n$ by*

$$T(x) = (\langle x, x_i\rangle, \langle x, x_2\rangle, \ldots, \langle x, x_n\rangle)\ \ \forall x \in X$$

Then T is a surjective linear isometry. That is, any finite dimensional inner product space X is linearly isometrically isomorphic to $\mathbb{F}^n$ $(X \cong \mathbb{F}^n)$.

Proof. (Linear): Let $x, y \in X$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned}
T(\alpha x + y) &= (\langle \alpha x + y, x_1\rangle, \langle \alpha x + y, x_2\rangle, \ldots, \langle \alpha x + y, x_n\rangle) \\
&= (\alpha\langle x, x_1\rangle + \langle y, x_1\rangle,\ \alpha\langle x, x_2\rangle + \langle y, x_2\rangle,\ \ldots,\ \alpha\langle x, x_n\rangle + \langle y, x_n\rangle) \\
&= \alpha\left(\langle x, x_1\rangle,\ \langle x, x_2\rangle,\ \ldots, \langle x, x_n\rangle\right) + \left(\langle y, x_1\rangle,\ \langle y, x_2\rangle,\ \ldots,\ \langle y, x_n\rangle\right) \\
&= \alpha T(x) + T(y)
\end{aligned}$$

(Isometry): Note that $Tx = (\langle x, x_i\rangle, \langle x, x_2\rangle, \ldots, \langle x, x_n\rangle) \in \mathbb{F}^n$, so

$$\|Tx\|_2 = \left(\sum_{i=1}^{n} |\langle x, x_i\rangle|^2\right)^{\frac{1}{2}} = \|x\|_2 \ \ \text{(by Plancherel's identity)}$$

Hence, T is an isometry.

(Onto): As T is an isometry, T is one-one by lemma 7.2. As X and $\mathbb{F}^n$ are finite dimensional, we have that T is also onto. □

8.4.2 Infinite Dimensional Spaces

Lemma 8.16 (Bessel's inequality). *Let $\mathcal{H}$ be a Hilbert space and $\{x_n : n \in \mathbb{N}\}$ be an orthonormal set in $\mathcal{H}$. Then $\forall x \in \mathcal{H}$,*

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq \|x\|^2$$

Proof. Let $n \in \mathbb{N}$. Observe that

$$\begin{aligned}\left\|x - \sum_{k=1}^{n}\langle x, x_k\rangle x_k\right\|^2 &= \left\langle x - \sum_{k=1}^{n}\langle x, x_k\rangle x_k,\ x - \sum_{k=1}^{n}\langle x, x_k\rangle x_k\right\rangle \\ &= \|x\|^2 - \left\langle x,\ \sum_{l=1}^{n}\langle x, x_l\rangle x_l\right\rangle - \left\langle \sum_{k=1}^{n}\langle x, x_k\rangle x_k,\ x\right\rangle + \left\langle \sum_{k=1}^{n}\langle x, x_k\rangle x_k,\ \sum_{l=1}^{n}\langle x, x_l\rangle x_l\right\rangle \\ &= \|x\|^2 - \sum_{l=1}^{n}\overline{\langle x, x_l\rangle}\langle x, x_l\rangle - \sum_{k=1}^{n}\langle x, x_k\rangle\langle x_k, x\rangle + \sum_{k=1}^{n}\langle x, x_k\rangle\sum_{l=1}^{n}\overline{\langle x, x_l\rangle}\cdot\underbrace{\langle x_k, x_l\rangle}_{=1 \text{ when } k=l} \\ &= \|x\|^2 - 2\sum_{k=1}^{n}|\langle x, x_k\rangle|^2 + \sum_{k=1}^{n}\langle x, x_k\rangle\overline{\langle x, x_k\rangle} \\ &= \|x\|^2 - 2\sum_{k=1}^{n}|\langle x, x_k\rangle|^2 + \sum_{k=1}^{n}|\langle x, x_k\rangle|^2 \\ &= \|x\|^2 - \sum_{k=1}^{n}|\langle x, x_k\rangle|^2 \\ &\geq 0 \text{ (what we started with is a positive quantity)}\end{aligned}$$

$$\implies \sum_{k=1}^{n}|\langle x, x_k\rangle|^2 \leq \|x\|^2\ \forall n \in \mathbb{N}$$

$$\implies \sum_{k=1}^{\infty}|\langle x, x_k\rangle|^2 \leq \|x\|^2$$

□

Lemma 8.17 (Fourier Representation). *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space over $\mathbb{F}$ and $\mathscr{A} = \{x_i : i \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$. Then*

$$x = \sum_{i=1}^{\infty}\langle x, x_i\rangle x_i,\ \forall x \in \mathcal{H}$$

Proof. Let $x \in \mathcal{H}$. Observe that $\{\langle x, x_i\rangle\}_{i=1}^{\infty} \in \ell^2(\mathbb{N})$ by Bessel's inequality. Let $S_n =$

$\sum_{k=1}^{n}\langle x, x_k\rangle x_k,\ n \in \mathbb{N}$. Let $n > m$. Then

$$\begin{aligned}
\|S_n - S_m\|^2 &= \langle S_n - S_m,\ S_n - S_m\rangle \\
&= \langle S_n, S_n\rangle - \langle S_n, S_m\rangle - \langle S_m, S_n\rangle + \langle S_m, S_m\rangle \\
&= \sum_{k=1}^{n}|\langle x, x_k\rangle|^2 - \sum_{k=1}^{m}|\langle x, x_k\rangle|^2 - \sum_{k=1}^{m}|\langle x, x_k\rangle|^2 + \sum_{k=1}^{m}|\langle x, x_k\rangle|^2 \\
&= \sum_{k=m+1}^{n}|\langle x, x_k\rangle|^2 \to 0 \text{ as } n \to \infty
\end{aligned}$$

$\implies$ (S_n) is a Cauchy sequence in $\mathcal{H}$.
$\implies$ $\exists y \in \mathcal{H}$ such that $y = \sum_{k=1}^{\infty}\langle x, x_k\rangle x_k$. It remains to show that $x = y$. Observe that

$$\begin{aligned}
\langle y, x_i\rangle &= \left\langle \lim_{n\to\infty}\sum_{k=1}^{n}\langle x, x_k\rangle x_k,\ x_i\right\rangle \\
&= \lim_{n\to\infty}\left\langle \sum_{k=1}^{n}\langle x, x_k\rangle x_k,\ x_i\right\rangle \quad (\because \langle\cdot, x_i\rangle \text{ is continuous, see lemma 8.3}) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\langle x, x_k\rangle\langle x_k, x_i\rangle \\
&= \sum_{k=1}^{\infty}\langle x, x_k\rangle \underbrace{\langle x_k, x_i\rangle}_{=1 \text{ when } k=i} = \langle x, x_i\rangle \\
\implies &\langle y, x_i\rangle = \langle x, x_i\rangle\ \forall i \in \mathbb{N} \\
\implies &\langle x - y, x_i\rangle = 0\ \forall i \in \mathbb{N} \\
\implies &x - y = 0 \text{ (As } \{x_i\}_{i\in\mathbb{N}} \text{ is an orthonormal basis: using lemma 8.8)} \\
\implies &x = y \\
\implies &x = \sum_{k=1}^{\infty}\langle x, x_k\rangle x_k
\end{aligned}$$

□

Theorem 8.11. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A}$ be an orthonormal basis for $\mathcal{H}$. Then $\mathcal{H}$ is a separable Hilbert space if and only if $\mathscr{A}$ is countable.*

Proof. ($\implies$) Assume that $\mathcal{H}$ is separable. Then every orthonormal set in $\mathcal{H}$ is countable by theorem 8.9. Hence, the orthonormal basis $\mathscr{A}$ is countable.

($\impliedby$) Suppose that $\mathcal{H}$ is a Hilbert space with a countable orthonormal basis $\mathscr{A} = \{e_1, e_2, \dots\}$. Then any $x \in \mathcal{H}$ can be uniquely written as $x = \sum_{i=1}^{\infty} c_i e_i$, where $c_i = \langle x, e_i\rangle$ (using lemma 8.17).
We know that $S = \mathbb{Q} + i\mathbb{Q}$ is a countable dense subset in $\mathbb{C}$.

For every $n \in \mathbb{N}$, consider the following subset of $\mathcal{H}$:

$$A_n = \left\{ \sum_{i=1}^{n} s_i e_i : s_i \in S \ \forall i \in [1:n] \right\}$$

Being a finite union of countable sets, each A_n is countable.
Define $A = \bigcup_{n=1}^{\infty} A_n$
Being a countable union of countable sets, A is countable.

Claim: A is a dense subset of $\mathcal{H}$.
Let $x \in \mathcal{H}$. Then $x = \sum_{i=1}^{\infty} c_i e_i$, where $c_i = \langle x, e_i \rangle \in \mathbb{C}$.
$\because$ This finite sum is convergent in $\mathcal{H}$, fixing $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ such that

$$\left\| \sum_{i=N+1}^{\infty} c_i e_i \right\| < \frac{\varepsilon}{2}$$

As S is dense in $\mathbb{C}$, $\forall i \leq N$, we can find $s_i \in S$ such that

$$|c_i - s_i| < \frac{\varepsilon}{2^{i+1}}$$

Consider the element $x_N = \sum_{i=1}^{N} s_i e_i \in A$

$\sum_{i=N+1}^{\infty} c_i e_i = \underbrace{\sum_{i=1}^{\infty} c_i e_i}_{\in \mathcal{H}} - \underbrace{\sum_{i=1}^{N} c_i e_i}_{\in \mathcal{H}} \in \mathcal{H}$

$$\begin{aligned}
\therefore \|x - x_N\| &= \left\| \sum_{i=1}^{\infty} c_i e_i - \sum_{i=1}^{N} s_i e_i \right\| \\
&= \left\| \sum_{i=1}^{N} (c_i - s_i) e_i + \sum_{i=N+1}^{\infty} c_i e_i \right\| \\
&\leq \left\| \sum_{i=1}^{N} (c_i - s_i) e_i \right\| + \left\| \sum_{i=N+1}^{\infty} c_i e_i \right\| \\
&\leq \sum_{i=1}^{N} |c_i - s_i| + \frac{\varepsilon}{2} \text{ (Parseval's inequality)} \\
&< \sum_{i=1}^{N} \frac{\varepsilon}{2^{i+1}} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$\therefore x_N \in B(x, \varepsilon) \cap A$
As x and $\varepsilon > 0$ are arbitrary, we have that A is dense in $\mathcal{H} \implies \mathcal{H}$ is separable. $\square$

Lemma 8.18 (Plancherel's identity)**.** *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space* *over* $\mathbb{F}$ *and* $\mathscr{A} = \{x_i : i \in \mathbb{N}\}$ *be an orthonormal basis of* $\mathcal{H}$*. Then*

$$\|x\|^2 = \sum_{i=1}^{\infty} |\langle x, x_i \rangle|^2, \ \forall x \in \mathcal{H}$$

Proof. Let $x \in \mathcal{H}$. Then

$$\begin{aligned}
\|x\|^2 = \langle x, x \rangle &= \left\langle \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, \ \sum_{l=1}^{\infty} \langle x, x_l \rangle x_l \right\rangle \\
&= \sum_{k=1}^{\infty} \langle x, x_k \rangle \sum_{l=1}^{\infty} \overline{\langle x, x_l \rangle} \cdot \underbrace{\langle x_k, x_l \rangle}_{=1 \text{ when } k=l} \\
&= \sum_{k=1}^{\infty} \langle x, x_k \rangle \overline{\langle x, x_l \rangle} \\
&= \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2
\end{aligned}$$

□

Lemma 8.19 (Parseval's identity)**.** *Let* $\mathcal{H}$ *be a separable infinite dimensional Hilbert space over* $\mathbb{F}$ *and* $\mathscr{A} = \{x_i : i \in \mathbb{N}\}$ *be an orthonormal basis of* $\mathcal{H}$*. Then*

$$\langle x, y \rangle = \sum_{i=1}^{\infty} \langle x, x_i \rangle \overline{\langle y, x_i \rangle}, \ \forall x, y \in \mathcal{H}$$

Proof. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned}
\langle x, y \rangle &= \left\langle \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i, \ \sum_{l=1}^{\infty} \langle y, x_l \rangle x_l \right\rangle \\
&= \sum_{i=1}^{\infty} \langle x, x_i \rangle \sum_{l=1}^{\infty} \overline{\langle y, x_l \rangle} \cdot \underbrace{\langle x_i, x_l \rangle}_{=1 \text{ when } i=l} \\
&= \sum_{i=1}^{\infty} \langle x, x_i \rangle \overline{\langle y, x_i \rangle}
\end{aligned}$$

□

Theorem 8.12 (Riesz-Fischer Theorem)**.** *Let* $\mathcal{H}$ *be a infinite dimensional separable Hilbert space over* $\mathbb{F}$ *and* $\mathcal{A} = \{x_i : i \in \mathbb{N}\}$ *is an orthonormal basis for* $\mathcal{H}$*. Then the map* $T : \mathcal{H} \to \ell^2(\mathbb{N})$ *defined by* $T(x) = (\langle x, x_n \rangle)_{n=1}^{\infty}$, $\forall x \in \mathcal{H}$ *is linear isometry and isomorphism, i.e.* $\mathcal{H} \cong \ell^2(\mathbb{N})$*.*

Proof. **Claim:** T is linear.

Let $x, y \in \mathcal{H},\ \alpha \in \mathbb{F}$, and

$$\begin{aligned}
T(x+\alpha y) &= (\langle x+\alpha y, x_n\rangle)_{n=1}^{\infty} \\
&= (\langle x, x_n\rangle + \alpha\langle y, x_n\rangle)_{n=1}^{\infty} \quad \text{(Since } \langle\cdot,\cdot\rangle \text{ is linear in first component)} \\
&= (\langle x, x_n\rangle)_{n=1}^{\infty} + (\alpha\langle y, x_n\rangle)_{n=1}^{\infty} \\
&= T(x) + \alpha T(y)
\end{aligned}$$

Hence, T is linear.

Claim: T is an Isometry.
By Plancherel's identity, we have

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2$$

And here, $\|T(x)\|_2 = \|(\langle x, x_n\rangle)_{n=1}^{\infty}\|_2 = \left(\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2\right)^{\frac{1}{2}} = \|x\|_2$. So, T is an isometry.

Claim: T is onto.
Since T is isometry, it follows that T is one-one (lemma 7.2. It remains to show that T is onto.
Let $(\alpha_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N})$, let $x = \sum_{n=1}^{\infty} \alpha_n x_n$, (since $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent in $\mathcal{H}$, which follows from Bessel's inequality)
For any $i, \in \mathbb{N}$

$$\begin{aligned}
\langle x, x_i\rangle &= \left\langle \sum_{n=1}^{\infty} \alpha_n x_n, x_i \right\rangle \\
&= \sum_{n=1}^{\infty} \alpha_n \langle x_n, x_i\rangle \\
&= \alpha_i \quad \text{(since } (x_j)_{j=1}^{\infty} \text{ is orthonormal set)}
\end{aligned}$$

Hence $x = \sum_{n=1}^{\infty} \langle x, x_i\rangle x_n$, which implies ,

$$T(x) = (\langle x, x_i\rangle)_{i=1}^{\infty} = (\alpha_i)_{i=1}^{\infty}$$

So $(\alpha_i)_{i=1}^{\infty} \in \ell^2(\mathbb{N})$ has an pre-image $x \in \mathcal{H}$. So, T is onto map.
Thus T is a linear isometrical isomorphism, and $\mathcal{H} \cong \ell^2(\mathbb{N})$. □

9 Problems

The following are a few problems that the reader may attempt to test their understanding.

1. Let $X = (\ell^1(\mathbb{N}), \|\cdot\|_1)$ be a normed linear space and $U = \{(x_n) \in X : x_n = 0 \; \forall n \geq 3\}$. Let $\alpha,\ \beta \in \mathbb{F}$ and define a linear functional $f : U \to \mathbb{F}$ by $f((x_n)) = \alpha x_1 + \beta x_2 \; \forall (x_n) \in U$. Calculate all the Hahn-Banach extensions of f.

2. Let $(X, \|\cdot\|)$ be a normed linear space and $\{x_1, x_2, \ldots, x_m\}$ be a linearly independent set in X. Let $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{F}^m$. Then show that there is a $f \in X^*$ such that $f(x_i) = \alpha_i \; \forall i \in [1 : m]$.

3. Give an example of a linear transformation $T : (c_0(\mathbb{N}), \|\cdot\|_\infty) \to (c_0(\mathbb{N}), \|\cdot\|_\infty)$ such that $N(T)$ is a closed subspace of $c_0(\mathbb{N})$, but T is not bounded.

4. Let $1 \leq p \leq \infty$ and $X = (\ell^p(\mathbb{N}), \|\cdot\|_p)$. Is $(\mathcal{B}(X), \|\cdot\|_{op})$ separable? Justify your answer.

5. Let $X = (C[0,1], \|\cdot\|_\infty)$ be a normed linear space over $\mathbb{R}$.

   (a) Let $x_0 \in [0,1]$. Define $T_{x_0} : X \to \mathbb{R}$ by $T_{x_0}(f) = f(x_0),\ \forall f \in X$. Show that T is a bounded linear functional and calculate its norm.

   (b) Let $x_0, y_0 \in [0,1]$ with $x_0 \neq y_0$. Show that there is at least one $f \in X$ such that $f(x_0) = 1$ and $f(y_0) = -1$.

   (c) Is X^* separable? Justify your answer.

6. Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and f, g be two linear functionals on X. Show that $N(f) \subseteq N(g)$ if and only if there exists a constant $c \in \mathbb{F}$ such that $g = cf$. ($N(f)$ denotes the null space of f)

7. Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{F}$ and $\phi : X \to \mathbb{F}$ be a linear functional. Define $\|x\|_\phi = \|x\| + |\phi(x)|$, for $x \in X$. Show that $\|\cdot\|_\phi$ is a norm on X. Further, show that $(X, \|\cdot\|_\phi)$ is a Banach space if and only if ϕ is a continuous linear functional.

8. Let $\mathcal{H}$ be a Hilbert space and $\{x_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. Let $(\alpha_n) \in \ell^2(\mathbb{N})$. Show that $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in $\mathcal{H}$.

9. Let $(H_n, \langle\cdot,\cdot\rangle_n)$ be a sequence of inner product spaces over $\mathbb{F}$, and let

$$H = \{(x_n) : x_n \in H_n,\ n \in \mathbb{N}\}$$

   Show that H is an inner product space with the inner product

$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} \langle x_n, y_n \rangle_n$$

Show that H is a Hilbert space if and only if each H_n is a Hilbert space.

10. Define an orthogonal projection on a Hilbert space $\mathcal{H}$. Let

$$\mathcal{A} = \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is an orthogonal projection}\}$$

and

$$\mathcal{M} = \{M \subseteq \mathcal{H} : M \text{ is a closed subspace of } \mathcal{H}\}$$

Then show that there exists a bijective map $h : \mathcal{A} \to \mathcal{M}$.

10 Bibliography

1. Prof. Venku Naidu D, Functional Analysis Lecture notes, Fall 2024, Indian Institute of Technology Hyderabad

2. M. Thamban Nair, Functional Analysis - A First Course, Fourth Printing, PHI Learning Public Limited, Delhi, 2014

3. B.V. Limaye, Functional Analysis, Second edition, New Age International, New Delhi, 1996

4. J.B. Conway, A Course in Functional Analysis, Second edition, Graduate Texts in Mathematics, Vol. 96, Springer, 1990